A Radó theorem for locally solvable structures of co-rank one

J. Hounie

Dedicated to Linda Rothschild

Abstract. We extend the classical theorem of Radó to locally solvable structures of co-rank one. One of the main tools in the proof is a refinement of the Baouendi-Treves approximation theorem that may be of independent interest.

Mathematics Subject Classification (2000). Primary 35F05, 35B60; Secondary 35D99 .

Keywords. Radó theorem, locally integrable structures, local solvability, Baouendi-Treves approximation theorem.

1. Introduction

A classical theorem of Radó, in the form given by Cartan, states that a continuous function defined on an open set of the complex plane which is holomorphic outside the closed set where it vanishes is holomorphic everywhere. This theorem implies easily that the same result also holds for functions of several complex variables. Radó's theorem may be regarded as a theorem about removing singularities of the Cauchy-Riemann operator, but in that theory it is customary to impose additional restrictions on the set outside which the equation holds and is wished to be removed (for instance, the set to be removed may be required to have null capacity or to have null or bounded Hausdorff measure of some dimension). The beauty of the classical result of Radó lies in the fact that the set $u^{-1}(0)$ is removed without any assumption about its size or geometric properties. The theorem was extended by replacing the set $u^{-1}(0)$ by $u^{-1}(E)$ where E is a compact subset of null analytic capacity ([St]) or is a null-set for the holomorphic Dirichlet class ([C2]). A generalization for a more general class of functions was given by Rosay and Stout

Work supported in part by CNPq, FINEP and FAPESP.

[RS] who extended Radó's result to CR functions on strictly pseudoconvex hypersurfaces of $\mathbb{C}^n$ and other extension (in the spirit of removing singularities of CR functions) was given in [A]. For homogeneous solutions of locally solvable vector fields with smooth coefficients, a Radó type theorem was proved in [HT].

In this paper we extend Radó's theorem to homogeneous solutions of locally integrable structures of co-rank one that are locally solvable in degree one. Thus, we deal with an overdetermined system of equations

$$\begin{cases} L_1 u = 0, \\ L_2 u = 0, \\ \cdots\cdots\cdots \\ L_n u = 0, \end{cases} \tag{1.1}$$

where $L_1, \dots, L_n$, $n \geq 1$, are pairwise commuting smooth complex vector fields defined on an open subset of $\mathbb{R}^{n+1}$ and assume that this system of vector fields has local first integrals at every point and it is solvable in the sense that the equation

$$\begin{cases} L_1 u = f_1, \\ L_2 u = f_2, \\ \cdots\cdots\cdots \\ L_n u = f_n, \end{cases} \tag{1.2}$$

can be locally solved for all smooth right hand sides that satisfy the compatibility conditions $L_j f_k = L_k f_j$, $1 \leq j, k \leq n$ (see Section 3 for precise statements). Under these condition it is shown that if u is continuous and satisfies (1.1) outside $u^{-1}(0)$ then it satisfies (1.1) everywhere (Theorem 4.1). This solvability hypothesis can be characterized in terms of the connectedness properties of the fibers of local first integrals ([CT], [CH]) and this characterization is one of the main ingredients in the proof of Theorem 4.1, which is given in Section 4. Another key tool is a refinement of the Baouendi-Treves approximation theorem, which seems to have interest *per se* and it is stated and proved in Section 2 for general locally integrable structures. In Section 5 we apply Theorem 4.1 to obtain a result on uniqueness in the Cauchy problem for continuous solutions with Cauchy data on rough initial surfaces.

2. The approximation theorem

The approximation formula ([BT1], [BT2], [T1], [T2], [BCH]) is of local nature and we will restrict our attention to a locally integrable structure $\mathcal{L}$ defined in an open subset Ω of $\mathbb{R}^N$ over which $\mathcal{L}^\perp$ is spanned by the differentials $dZ_1, \dots, dZ_m$ of m smooth functions $Z_j \in C^\infty(\Omega)$, $j = 1, \dots, m$, at every point of Ω. Thus, if n is the rank of $\mathcal{L}$, we recall that $N = n + m$.

Given a continuous function $u \in C(\Omega)$ we say that u is a homogeneous solution of $\mathcal{L}$ and write $\mathcal{L}u = 0$ if, for every local section L of $\mathcal{L}$ defined on an open subset $U \subset \Omega$,

$$Lu = 0 \quad \text{on } U \text{ in the sense of distributions.}$$

Simple examples of homogeneous solutions of $\mathcal{L}$ are the constant functions and also the functions $Z_1, \dots, Z_m$, since $LZ_j = \langle dZ_j, L\rangle = 0$ because $dZ_j \in \mathcal{L}^\perp$, $j = 1, \dots, m$. By the Leibniz rule, any product of smooth homogeneous solutions is again a homogeneous solution, so a polynomial with constant coefficients in the m functions Z_j, i.e., a function of the form

$$P(Z) = \sum_{|\alpha|\le d} c_\alpha Z^\alpha, \quad \alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m, \quad c_\alpha \in \mathbb{C}, \tag{2.1}$$

is also a homogeneous solution. The classical approximation theorem for continuous functions states that any continuous solution u of $\mathcal{L}u = 0$ can be uniformly approximated by polynomial solutions such as (2.1). More precisely:

Theorem 2.1. *Let $\mathcal{L}$ be a locally integrable structure on Ω and assume that $dZ_1, \dots, dZ_m$ span $\mathcal{L}^\perp$ at every point of Ω. Then, for any $p \in \Omega$, there exist two open sets U and W, with $p \in U \subset \overline{U} \subset W \subset \Omega$, such that*

every $u \in C(W)$ that satisfies $\mathcal{L}u = 0$ on W is the uniform limit of a sequence of polynomial solutions $P_j(Z_1, \dots, Z_m)$:

$$u = \lim_{j\to\infty} P_j \circ Z \quad \text{uniformly in } \overline{U}.$$

In this section we will prove a refinement of the approximation theorem (Theorem 2.2) that we will later use in the proof of the paper's main result. In order to prove this variation, we start by reviewing the main steps in the proof of the classical approximation theorem. The first one is to choose local coordinates

$$\{x_1, \dots, x_m, t_1, \dots, t_n\}$$

defined on a neighborhood of the point p and vanishing at p so that, for some smooth, real-valued functions $\varphi_1, \dots, \varphi_m$ defined on a neighborhood of the origin and satisfying

$$\varphi_k(0,0) = 0, \quad d_x\varphi_k(0,0) = 0, \quad k = 1, \dots, m,$$

the functions Z_k, $k = 1, \dots, m$, may be written as

$$Z_k(x,t) = x_k + i\varphi_k(x,t), \quad k = 1, \dots, m, \tag{2.2}$$

on a neighborhood of the origin. Then we choose $R > 0$ such that if

$$V = \{q : \quad |x(q)| < R, \ |t(q)| < R\}$$

then, on a neighborhood of $\overline{V}$ we have

$$\left\| \left(\frac{\partial\varphi_j(x,t)}{\partial x_k} \right) \right\| < \frac{1}{2}, \quad (x,t) \in \overline{V}, \tag{2.3}$$

where the double bar indicates the norm of the matrix $\varphi_x(x,t) = (\partial\varphi_j(x,t)/\partial x_k)$ as a linear operator in $\mathbb{R}^m$. Modifying the functions φ_k's off a neighborhood of $\overline{V}$ we may assume without loss of generality that the functions $\varphi_k(x,t)$, $k = 1, \dots, m$, are defined throughout $\mathbb{R}^N$, have compact support and satisfy (2.3) everywhere, that is

$$\left\| \left(\frac{\partial \varphi_j(x,t)}{\partial x_k} \right) \right\| < \frac{1}{2}, \quad (x,t) \in \mathbb{R}^N. \tag{2.3$'$}$$

Modifying also $\mathcal{L}$ off a neighborhood of $\overline{V}$ we may assume as well that the differentials dZ_j, $j = 1, \dots, m$, given by (2.2), span $\mathcal{L}^\perp$ over $\mathbb{R}^N$. Of course, the new structure $\mathcal{L}$ and the old one coincide on V so any conclusion we draw about the new $\mathcal{L}$ on V will hold as well for the original $\mathcal{L}$. The vector fields

$$M_k = \sum_{\ell=1}^{m} \mu_{k\ell}(x,t) \frac{\partial}{\partial x_\ell}, \quad k = 1, \dots, m,$$

characterized by the relations

$$M_k Z_\ell = \delta_{k\ell} \quad k, \ell = 1, \dots, m,$$

and the vectors fields

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{m} \frac{\partial \varphi_k}{\partial t_j}(x,t) M_k, \quad j = 1, \dots, n,$$

are linearly independent and satisfy $L_j Z_k = 0$, for $j = 1, \dots, n$, $k = 1, \dots, m$. Hence, $L_1, \dots, L_n$ span $\mathcal{L}$ at every point while the $N = n + m$ vector fields

$$L_1, \dots, L_n, M_1 \dots, M_m,$$

are pairwise commuting and span $\mathbb{C}T_p(\mathbb{R}^N)$, $p \in \mathbb{R}^N$. Since

$$dZ_1, \dots, dZ_m, dt_1, \dots, dt_n \quad \text{span } \mathbb{C}T^*\mathbb{R}^N$$

the differential dw of a C^1 function $w(x,t)$ may be expressed in this basis. In fact, we have

$$dw = \sum_{j=1}^{n} L_j w \, dt_j + \sum_{k=1}^{m} M_k w \, dZ_k$$

which may be checked by observing that $L_j Z_k = 0$ and $M_k t_j = 0$ for $1 \le j \le n$ and $1 \le k \le m$, while $L_j t_k = \delta_{jk}$ for $1 \le j, k \le n$ and $M_k Z_j = \delta_{jk}$ for $1 \le j, k \le m$ ($\delta_{jk} =$Kronecker delta). At this point, the open set W in the statement of Theorem 2.1 is chosen as any fixed neighborhood of $\overline{V}$ in Ω. That $u \in C(W)$ satisfies $\mathcal{L}u = 0$ is equivalent to saying that it satisfies on W the overdetermined system of equations

$$\begin{cases} L_1 u = 0, \\ L_2 u = 0, \\ \cdots\cdots\cdots \\ L_n u = 0. \end{cases} \tag{2.4}$$

Given such u we define a family of functions $\{E_\tau u\}$ that depend on a real parameter τ, $0 < \tau < \infty$, by means of the formula

$$E_\tau u(x,t) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau[Z(x,t) - Z(x',0)]^2} u(x',0) h(x') \det Z_x(x',0) \, dx'$$

which we now discuss. For $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$ we will use the notation $[\zeta]^2 = \zeta_1^2 + \dots + \zeta_m^2$, which explains the meaning of $[Z(x,t) - Z(x',0)]^2$ in the formula. The function $h(x) \in C_c^\infty(\mathbb{R}^m)$ satisfies $h(x) = 0$ for $|x| \geq R$ and $h(x) = 1$ in a neighborhood of $|x| \leq R/2$ (R was defined right before (2.3)). Since u is assumed to be defined in a neighborhood of $\overline{V}$, the product $u(x',0)h(x')$ is well defined on $\mathbb{R}^m$, compactly supported and continuous. Furthermore, since the exponential in the integrand is an entire function of $(Z_1, \dots, Z_m)$, the chain rule shows that it satisfies the homogenous system of equations (2.4) and the same holds for $E_\tau u(x,t)$ by differentiation under the integral sign. Then Theorem 2.1 is proved by showing that $E_\tau u(x,t) \to u(x,t)$ as $\tau \to \infty$ uniformly for $|x| < R/4$ and $|t| < T < R$ if T is conveniently small. In particular, the set U in the statement of Theorem 2.1 may be taken as

$$\begin{aligned} U &= B_1 \times B_2, \\ B_1 &= \{x \in \mathbb{R}^m : \quad |x| < R/4\}, \\ B_2 &= \{t \in \mathbb{R}^n : \quad |t| < T\}. \end{aligned}$$

Once this is proved, approximating the exponential $e^{-\tau[\zeta]^2}$ (for fixed large τ) by the partial sum of degree k, $P_k(\zeta)$, of its Taylor series on a fixed polydisk that contains the set $\{\sqrt{\tau}(Z(x,t) - Z(x',0)) : |x|, |x'| < R, |t| < R\}$, a sequence of polynomials in $Z(x,t)$ that approximate uniformly $E_\tau u(x,t)$ for $|x| < R/4$ and $|t| < T$ as $k \to \infty$ is easily constructed.

Thus, the main task is to prove that $E_\tau u(x,t) \to u(x,t)$ as $\tau \to \infty$ uniformly on $B_1 \times B_2$ and in order to do that one proves first that a convenient modification of the operator E_τ, to wit,

$$G_\tau u(x,t) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau[Z(x,t)-Z(x',t)]^2} u(x',t) h(x') \det Z_x(x',t)\, dx',$$

converges uniformly to u on $B_1 \times B_2$ as $\tau \to \infty$. This is easy because (2.3′) ensures that the operator $u \mapsto G_\tau u$ is very close to the convolution of u with a Gaussian, which is a well known approximation of the identity. In particular, the uniform convergence $G_\tau \to u$ holds for any continuous u and it is irrelevant at this point whether u satisfies the equation $\mathcal{L}u = 0$ or not. After $G_\tau u \to u$ has been proved, it remains to estimate the difference $R_\tau u = G_\tau u - E_\tau u$ and it is here that the fact that $\mathcal{L}u = 0$ is crucial.

Let B_1 and B_2 be as described before in the outline of the proof of Theorem 2.1 and set $\tilde{B}_1 = \{x \in \mathbb{R}^m : \quad |x| < R\}$. In the version we want to prove u need not be a solution in a neighborhood of $\overline{V}$ but on a smaller open subset of $\tilde{B}_1 \times B_2$. With this notation we have

Theorem 2.2. *Let u be continuous on $\overline{\tilde{B}}_1 \times \overline{B}_2$ and assume that there is an open and connected set ω, $0 \in \omega \subset B_2$, such that*

$$\mathcal{L}u = 0, \quad \text{on } \tilde{B}_1 \times \omega.$$

Then $E_\tau u(x,t) \to u(x,t)$ uniformly on compact subsets of $B_1 \times \omega$ as $\tau \to \infty$.

Remark 2.3. The main point in Theorem 2.2 is that as soon as the equation holds on $\tilde{B}_1 \times \omega$, in order to obtain a set where the approximation holds we do not need to replace ω by a smaller subset, although we must shrink $\tilde{B}_1$ to B_1 and the radius T of B_2 has been initially taken small as compared to the radius R of $\tilde{B}_1$. Under some more restrictive circumstances, we may even avoid taking T is small, as the proof of Theorem 4.1 below shows.

Proof. The formula that defines $E_\tau u(x,t)$ only takes into account the values of $u(x,0)$. It will be enough to prove that $E_\tau u \to u$ uniformly on compact subsets of $B_1 \times \omega$ as $j \to \infty$. The argument that shows in the classical setup that $G_\tau u \to u$ uniformly on $\tilde{B}_1 \times B_2$ applies here word by word, because it only uses the fact that u is continuous on the closure of $\tilde{B}_1 \times B_2$ and it is carried out by freezing $t \in B_2$ and showing that $G_\tau u$ is an approximation of the identity on $\mathbb{R}^m$, uniformly in $t \in B_2$. Hence, the proof is reduced to showing that $R_\tau u = G_\tau u - E_\tau u$ converges uniformly to 0 on compact subsets of $B_1 \times \omega$.

When u satisfies $\mathcal{L}u = 0$ throughout $\tilde{B}_1 \times B_2$, we have the formula

$$R_\tau u(x,t) = \int_{[0,t]} \sum_{j=1}^{n} r_j(x,t,t',\tau)\, dt'_j, \tag{2.5}$$

where

$$r_j(x,t,t',\tau) = \tag{2.6}$$
$$(\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau[Z(x,t)-Z(x',t')]^2} u(x',t') L_j h(x',t') \det Z_x(x',t')\, dx'$$

and $[0,t]$ denotes the straight segment joining 0 to t. This may be shown by writing for fixed ζ and τ

$$g(t') = \tilde{G}_\tau u(\zeta,t') = \int_{\mathbb{R}^m} e^{-\tau[\zeta - Z(x',t')]^2} u(x',t') h(x') \det Z_x(x',t')\, dx'$$

and applying the fundamental theorem of calculus

$$g(t) - g(0) = \int_{[0,t]} \sum_{j=1}^{n} \frac{\partial g}{\partial t'_j}(t')\, dt'_j. \tag{2.7}$$

Then a computation that exploits that $L_j u = 0$, $j = 1, \dots, n$, shows that (see [BCH, p.64] for details)

$$\frac{\partial g}{\partial t'_j}(t') = \tilde{r}_j(\zeta,t',\tau) \tag{2.8}$$

where

$$\tilde{r}_j(\zeta,t',\tau) = \int_{\mathbb{R}^m} e^{-\tau[\zeta - Z(x',t')]^2} u(x',t') L_j h(x',t') \det Z_x(x',t')\, dx'.$$

Hence, (2.7) for $\zeta = Z(x,t)$ shows that $R_\tau u = G_\tau u - E_\tau u$ is given by (2.5) and (2.6). Let's return to the case in which u is only known to satisfy $L_j u = 0$,

$j = 1, \ldots, n$, on $\tilde{B}_1 \times \omega$. We still have, for $t \in \omega$,

$$g(t) - g(0) = \int_{\gamma_t} \sum_{j=1}^{n} \frac{\partial g}{\partial t'_j}(t')\, dt'_j. \tag{2.7$'$}$$

where γ_t denotes a polygonal path contained in ω that joins the origin to t. On the other hand, (2.8) remains valid in the new situation. This is true because its proof depends on integration by parts with respect to x — which can be performed as well on $\tilde{B}_1 \times \omega$— and on local arguments. Thus, we get

$$R_\tau u(x,t) = \int_{\gamma_t} \sum_{j=1}^{n} r_j(x,t,t',\tau)\, dt'_j, \quad (x,t) \in \tilde{B}_1 \times \omega. \tag{2.5$'$}$$

This gives the estimate

$$|R_\tau u(x,t)| \le C|\gamma_t| \max_{1\le j\le n} \sup_{t'\in\omega} |r_j(x,t,t',\tau)|, \quad (x,t) \in B_1 \times \omega.$$

However, due to the fact that the factor $L_j h(x',t')$ vanishes for $|x'| \ge R/2$, we have

$$\left| e^{-\tau[Z(x,t)-Z(x',t')]^2} \right| \le e^{-c\tau}, \quad (x,t) \in B_1 \times B_2,\ |x'| \ge R/2,\ t' \in B_2,$$

for some $c > 0$. This follows, taking account of (2.3$'$), from

$$\begin{aligned} \Re[Z(x,t) - Z(x',t')]^2 &\ge |x - x'|^2 - |\varphi(x,t) - \varphi(x',t')|^2 \\ &\ge \frac{|x-x'|^2}{2} - |\varphi(x',t) - \varphi(x',t')|^2 \\ &\ge c. \end{aligned} \tag{2.9}$$

Note that $|x - x'| \ge R/4$ for $|x'| \ge R/2$ and $|x| \le R/4$, while the term $|\varphi(x',t) - \varphi(x',t')| \le C|t-t'|$ will be small if t and t' are both small which may be obtained by taking T small. Thus

$$|R_\tau u(x,t)| \le C|\gamma_t| e^{-c\tau}, \quad (x,t) \in B_1 \times \omega. \tag{2.10}$$

If $K \subset\subset \omega$, there is a constant C_K such that any $t \in K$ can be reached from the origin by a polygonal line of length bounded by C_K so (2.10) shows that $|R_\tau u(x,t)| \to 0$ uniformly on $B_1 \times K$. □

Corollary 2.4. *Under the hypotheses of the theorem, there is a sequence of polynomial solutions $P_j(Z_1, \ldots, Z_m)$ that converges uniformly to u uniformly on compact subsets of $B_1 \times \omega$ as $j \to \infty$.*

□

3. Structures of co-rank one

A smooth locally integrable structure $\mathcal{L}$ of rank $n \geq 1$ defined on an open subset $\Omega \subset \mathbb{R}^{n+1}$ is said to be a structure of co-rank one. Thus $\mathcal{L}^\perp$ is locally spanned by a single function Z that, in appropriate local coordinates $(x, t_1, \dots, t_n)$ centered around a given point, may be written as

$$Z(x,t) = x + i\varphi(x,t), \quad |x| < a, \quad |t| < r,$$

where $\varphi(x,t)$ is smooth, real valued and satisfies $\varphi(0,0) = \varphi_x(0,0) = 0$. Then, $\mathcal{L}$ is locally spanned by the vector fields

$$L_j = \frac{\partial}{\partial t_j} - i\frac{\varphi_{t_j}}{1 + i\varphi_x}\frac{\partial}{\partial x}, \quad j = 1, \dots, n,$$

on $X \doteq (-a,a) \times \{|t| < r\}$. It turns out that $[L_j, L_k] = 0$, $1 \leq j, k \leq n$. Given an open set $Y \subset X$, consider the space of p-forms

$$C^\infty(Y, \bigwedge{}^p) \doteq \{u = \sum_{|J|=p} u_J(x,t) dt_J,\ u_J \in C^\infty(Y)\}$$

as well as the differential complex

$$L : C^\infty(Y, \bigwedge{}^p) \longrightarrow C^\infty(Y, \bigwedge{}^{p+1})$$

defined by

$$L = \sum_{|J|=p} \sum_{j=1}^{n} L_j u_J(x,t)\, dt_j \wedge dt_J.$$

The fact that $L^2 = 0$ ensues from the relations $[L_j, L_k] = 0$, $1 \leq j, k \leq n$.

Definition 3.1. .1 The operator L is said to be solvable at $\omega_0 \in \Omega$ in degree q, $1 \leq q \leq n$, if for every open neighborhood $Y \subset X$ of ω_0 and $f \in C^\infty(Y, \bigwedge^q)$ such that $Lf = 0$, there exists an open neighborhood $Y' \subset Y$ of ω_0 and a $(q-1)$-form $u \in C^\infty(Y', \bigwedge^{q-1})$ such that $Lu = f$ in Y'. If this holds for every $\omega_0 \in \Omega$ we say that the structure $\mathcal{L}$ is locally solvable in Ω in degree q.

If $z_0 = x_0 + iy_0 \in \mathbb{C}$ and $Y \subset X$ we will refer to the set

$$\mathcal{F}(z_0, Y) = \{(x,t) \in Y :\ Z(x,t) = z_0\} = Z^{-1}(z_0) \cap Y$$

as *the fiber of the map* $Z : X \longrightarrow \mathbb{C}$ *over* Y. The local solvability in degree q can be characterized in terms of the homology of the fibers of Z for any degree $1 \leq q \leq n$, as it was conjectured by Treves in [T3]. The full solution of this conjecture took several years (see [CH] and the references therein). In this paper we will only consider locally solvable structures of co-rank one that are locally solvable in degree $q = 1$. For $q = 1$, the geometric characterization of local solvability at the origin means that ([CT],[CH],[MT]), given any open neighborhood X of the origin there is another open neighborhood $Y \subset X$ of the origin such that, for every regular value $z_0 \in \mathbb{C}$ of $Z : X \longrightarrow \mathbb{C}$, either $\mathcal{F}(z_0, Y)$ is empty or else the homomorphism

$$\tilde{H}_0(\mathcal{F}(z_0, Y), \mathbb{C}) \longrightarrow \tilde{H}_0(\mathcal{F}(z_0, X), \mathbb{C}) \qquad (*)_0$$

induced by the inclusion map $\mathcal{F}(z_0, Y) \subset \mathcal{F}(z_0, X)$ is identically zero. We are denoting by $\tilde{H}_0(M, \mathbb{C})$ the 0-th reduced singular homology space of M with complex coefficients. In other words, $(*)_0$ means that there is at most one connected component C_q of $\mathcal{F}(z_0, X)$ that intersects Y. Thus, if $q \in Y$, $z_0 = Z(q)$ is a regular value and C_q is the connected component of $\mathcal{F}(z_0, X)$ that contains q, it follows that

$$\mathcal{F}(z_0, Y) = Y \cap \mathcal{F}(z_0, X) = Y \cap C_q.$$

4. A Radó theorem for structures of co-rank one

Consider a smooth locally integrable structure $\mathcal{L}$ of rank $n \geq 1$ defined on an open subset $\Omega \subset \mathbb{R}^{n+1}$. A function u defined on Ω is a Radó function if

i) $u \in C(\Omega)$ and
ii) satisfies the differential equation $\mathcal{L}u = 0$ on $\Omega \setminus u^{-1}(0)$, in the weak sense.

We say that $\mathcal{L}$ has the Radó property if every Radó function is a homogeneous solution on Ω, i.e., the singular set where u vanishes and where the equation is a priori not satisfied can be removed and the equation $\mathcal{L}u = 0$ holds everywhere.

Theorem 4.1. *Assume that $\mathcal{L}$ is locally solvable in degree 1 in Ω. Then $\mathcal{L}$ has the Radó property.*

The Rad property has a local nature: it is enough to show that a Rad function u satisfies the equation $\mathcal{L}u = 0$ in a neighborhood of an arbitrary point $p \in \Omega$ such that $u(p) = 0$. We may choose local coordinates $x, t_1, \ldots, t_n$ such that $x(p) = t_j(p) = 0$, $j = 1, \ldots, n$, in which a first integral $Z(x, t)$ has the form

$$Z(x, t) = x + i\varphi(x, t), \quad |x| \leq 1, \quad |t| = |(t_1, \ldots, t_n)| \leq 1,$$

where φ is real valued and $\varphi(0, 0) = \varphi_x(0, 0) = 0$. Let us write

$$I = (-1, 1), \quad B = \{t : \ |t| < 1\}, \quad X = I \times B.$$

By Theorem 2.1, we may further assume without loss of generality that any continuous solution of $\mathcal{L}v = 0$ defined on a neighborhood of $|x| \leq 1$, $|t| \leq 1$, is uniformly approximated by $E_\tau v$ for $|x| < a$, $|t| < T$, where a and T are convenient positive small values. Of course, we cannot apply this to our Rad function u since u is not known to satisfy the equation everywhere.

In order to prove that $\mathcal{L}u = 0$ in some neighborhood of the origin we will consider different cases.

Case 1

Assume that $\nabla_t \varphi(0, 0) \neq 0$, say $\partial\varphi(0, 0)/\partial t_1 \neq 0$ (this is the simple elliptic case). Then, replacing the coordinate function t_1 by φ and leaving $t_2, \ldots t_n$ unchanged, we obtain a local change of coordinates defined in a small ball centered at the origin. In the new coordinates we have $\varphi(x,t) \equiv t_1$. Then the system (2.4) becomes

$$\begin{cases} \left(\dfrac{\partial}{\partial t_1} - i\dfrac{\partial}{\partial x}\right) u = 0, \\ \qquad\qquad \dfrac{\partial}{\partial t_2} u = 0, \\ \qquad \cdots\cdots\cdots\cdots \\ \qquad\qquad \dfrac{\partial}{\partial t_n} u = 0. \end{cases}$$

Take an arbitrary point $p = (x_0, \tau_1, \tau')$, $\tau' = (\tau_2, \ldots, \tau_n)$ such that $u(p) \neq 0$ and a cube Q centered at p that does not intersect the zero set of u. Choosing Q sufficiently small, we may approximate u uniformly on Q by polynomials in the first integral $Z = x + it_1$. Then, for fixed t', the restricted function $u_{t'}(t_1, x) = u(x, t_1, t')$ is a holomorphic function of $x + it_1$ on a slice of Q. Keeping $\tau' = t_0'$ fixed and varying (x_0, τ_1), it turns out that $u_{t_0'}(t_1, x) = u(x, t_1, t_0')$ is a holomorphic function of $x + it_1$ outside its zero set so, by the classical Rad theorem it is a holomorphic everywhere. Similarly, keeping (x_0, τ_1) fixed and letting t' vary, we see that the function $t' \mapsto u(x_0, \tau_1, t')$ is locally constant on the set $\{t' : u(x_0, \tau_1, t') \neq 0\}$, thus constant on its connected components. The continuity of u then shows that $t' \mapsto u(x_0, \tau_1, t')$ is constant for fixed (x_0, τ_1). Hence, u is independent of $t' = (t_2, \ldots, t_n)$ and the restricted function $u_{t_0'}(t_1, x) = u(x, t_1, t_0')$ is a holomorphic function of $x + it_1$ so u satisfies $\mathcal{L}u = 0$ in a neighborhood of the origin.

We already know that $\varphi_x(0,0) = 0$ and in view of Case 1 we will assume from now on that

$$\nabla_{x,t}\varphi(0,0) = 0. \tag{4.1}$$

We recall that, for $z_0 = x_0 + iy_0 \in \mathbb{C}$ and $Y \subset X$, the set

$$\mathcal{F}(z_0, Y) = \{(x,t) \in Y : \ Z(x,t) = z_0\} = Z^{-1}(z_0) \cap Y$$

is referred to as *the fiber of the map* $Z : X \longrightarrow \mathbb{C}$ *over* Y. To deal with the next cases, the following lemma will be important; its proof is a consequence of the classical approximation theorem.

Lemma 4.2. *The Rad function u is constant on the connected components of the fibers of Z over X.*

Proof. Let $q \in X$. Assume first that $u(q) \neq 0$. Then u is a homogeneous solution of $\mathcal{L}$ on a neighborhood of q and by a standard consequence of Theorem 2.1 applied at the point q, u must be constant on the fibers of some first integral Z_1 of $\mathcal{L}$ defined on a sufficiently small neighborhood of q. The germs of the fibers at q are invariant objects attached to $\mathcal{L}$ and do not depend on the particular first integral, i.e., replacing Z_1 by Z, there exists a neighborhood W of q such that u is constant on $\mathcal{F}(z_0, W)$, $z_0 = Z(q) = x_0 + iy_0$. Let $\{x_0\} \times \mathcal{C}_q$ be the connected component of $\mathcal{F}(z_0, X) \setminus u^{-1}(0)$ that contains q and denote by $\{x_0\} \times \mathcal{C}_q^{\#}$ the connected component of $\mathcal{F}(z_0, X)$ that contains q, so $\mathcal{C}_q \subset \mathcal{C}_q^{\#}$. We have seen that u is locally constant on $\{x_0\} \times \mathcal{C}_q$, hence it assumes a constant value $c \neq 0$ on $\{x_0\} \times \mathcal{C}_q$. This implies that u cannot vanish at any point on the closure of $\{x_0\} \times \mathcal{C}_q$ in X and therefore $\mathcal{C}_q$ is both open and closed in $\mathcal{C}_q^{\#}$ so $\mathcal{C}_q = \mathcal{C}_q^{\#}$. Hence, $\{x_0\} \times \mathcal{C}_q$ is a connected component of $\mathcal{F}(z_0, X)$ and the lemma is proved in this case.

Assume now $u(q) = 0$ and let $\{x_0\} \times \mathcal{C}_q$ be the connected component of $\mathcal{F}(z_0, X)$ that contains q. We will show that $\{x_0\} \times \mathcal{C}_q \subset u^{-1}(0)$. Suppose there exists a point $q_1 \in \{x_0\} \times \mathcal{C}_q \setminus u^{-1}(0) \neq \emptyset$. By our previous reasoning, u would assume a constant value $c \neq 0$ on the connected component $\{x_0\} \times \mathcal{C}_{q_1}$ of $\mathcal{F}(z_0, X)$ that contains q_1. Since $q_1 \in \{x_0\} \times (\mathcal{C}_{q_1} \cap \mathcal{C}_q)$ we should have $\mathcal{C}_{q_1} = \mathcal{C}_q$ and consequently $u(q) = c \neq 0$, a contradiction. Hence, u vanishes identically on $\mathcal{F}(z_0, X)$. □

At this point we will exploit the assumption that $\mathcal{L}$ is locally solvable in degree one. By the geometric characterization of locally solvable structures of co-rank one, we know that given any open neighborhood X of the origin there is another open neighborhood $Y \subset X$ of the origin such that, for every regular value $z_0 \in \mathbb{C}$ of $Z : X \longrightarrow \mathbb{C}$, either $\mathcal{F}(z_0, Y)$ is empty or else the homomorphism

$$\tilde{H}_0(\mathcal{F}(z_0, Y), \mathbb{C}) \longrightarrow \tilde{H}_0(\mathcal{F}(z_0, X), \mathbb{C}) \qquad (*)_0$$

induced by the inclusion map $\mathcal{F}(z_0, Y) \subset \mathcal{F}(z_0, X)$ is identically zero. As mentioned at the end of Section 3, this implies that if $q \in Y$, $z_0 = Z(q)$ is a regular value and C_q is the connected component of $\mathcal{F}(z_0, X)$ that contains q, it follows that

$$\mathcal{F}(z_0, Y) = Y \cap \mathcal{F}(z_0, X) = Y \cap C_q.$$

By Lemma 4.2 u is constant on C_q. This shows that u is constant on the regular fibers $\mathcal{F}(z_0, Y)$ of Z over Y but, using Sard's theorem, a continuity argument shows that u is constant on all fibers $\mathcal{F}(z_0, Y)$, whether regular or not. Hence, after restricting u to $\overline{Y}$, we may write

$$u(x, t) = U \circ Z(x, t), \quad (x, t) \in \overline{Y}, \qquad (4.2)$$

with $U \in C^0(Z(\overline{Y}))$. Once $U(x, y)$ has been defined, (4.2) will still hold if we replace Y by a smaller neighborhood of the origin. Thus, redefining I and B as $I = (-a, a)$, $B = \{|t| \leq T\}$, with $0 < a < 1$, $0 < T < 1$ conveniently small, we may assume from the start that

$$u(x, t) = U \circ Z(x, t), \quad (x, t) \in \overline{X}.$$

Shrinking I and B further if necessary we may also assume, recalling (4.1), that

$$|\nabla_{x,t}\varphi(x, t)| \leq \frac{1}{24} \quad \text{for } (x, t) \in \overline{X}. \qquad (4.3)$$

Write

$$M(x) = \sup_{|t|\le T} \varphi(x,t)$$
$$m(x) = \inf_{|t|\le T} \varphi(x,t)$$

so

$$Z([-a,a]\times \overline{B}) = \big\{x+iy: \quad |x|\le a,\ m(x)\le y\le M(x)\big\}.$$

The functions $M(x)$ and $m(x)$ are continuous and we may write

$$\{x\in(-a,a): \quad m(x)<M(x)\} = \bigcup_{j=1}^{N} I_j$$

where $I_j \subset (-a,a)$ is an open interval for $1\le j\le N\le\infty$. We also write $\mathcal{N} = (-a,a)\setminus \bigcup_{j=1}^{N} I_j$ and

$$D_j = \{x+iy: \quad x\in I_j,\ m(x)<y<M(x)\}, \quad 1\le j\le N.$$

Case 2

We suppose now that $0 = Z(0,0)\in\bigcup_j D_j$, i.e., we will assume that $0\in I_j$ for some j (that we may take as $j=1$) and $m(0)<0=\varphi(0,0)<M(0)$. If $(0,0)$ is in the interior of $u^{-1}(0)$, it is clear that $\mathcal{L}u$ vanishes in a neighborhood of $(0,0)$ as we wish to prove, so we may assume that u does not vanish identically in any neighborhood of $(0,0)$. Hence, there are points $q=(x_0,t_0)\in I_1\times B$ such that $z_0 \doteq Z(q) = x_0+i\varphi(x_0,t_0)\doteq x_0+iy_0\in D_1$ and $u(q)\ne 0$. Let $\{x_0\}\times\mathcal{C}_q$ be the connected component of $\mathcal{F}(z_0, I_1\times B)$ that contains q. Cover $\overline{\mathcal{C}_q}$ with a finite number L of balls of radius $\delta>0$ centered at points of $\mathcal{C}_q$ and call ω_δ the union of these balls. Notice that any two points in ω_δ can be joined by a polygonal line γ of length $|\gamma|\le(L+2)\delta$. Thus, ω_δ is a connected open set that contains $\mathcal{C}_q$ and, since $m(x_0)<y_0<M(x_0)$, there is no restriction in assuming that q has been chosen so that $\varphi(x_0,t)$ assumes on ω_δ some values which are larger than y_0 as well as some values which are smaller than y_0. Indeed, consider a smooth curve $\gamma(s):[0,1]\longrightarrow\overline{B}$ such that for some $0<s_1<1$, $\varphi(x_0,\gamma(0))=m(x_0)$, $\varphi(x_0,\gamma(s_1))=y_0$, $\varphi(x_0,\gamma(1))=M(x_0)$ and $\varphi(x_0,\gamma(s))\le y_0$ for any $0\le s\le s_1$. Having fixed $\gamma(s)$, we may choose a largest $s_1\in(0,1)$ with that property. This means that there are points $s>s_1$ arbitrarily close to s_1 such that $\varphi(x_0,\gamma(s))>y_0$. Let $[s_0,s_1]$ be the connected component of

$$\{s\in[0,1]: \ \varphi(x_0,\gamma(s))=y_0\}$$

that contains s_1 (note that $0<s_0\le s_1<1$). Set $q=(x_0,\gamma(s_0))$, $q'=(x_0,\gamma(s_1))$ and notice that $\gamma(s)\in\mathcal{C}_q$ for $s_0\le s\le s_1$ by connectedness so $q'\in\mathcal{C}_q$. For any $\varepsilon>0$ there exist $\varepsilon_0,\varepsilon_1\in(0,\varepsilon)$ such that $\gamma(s_0-\varepsilon_0)$ and $\gamma(s_1+\varepsilon_1)$ are points in ω_δ on which $\varphi(x_0,t)$ takes values respectively smaller and larger than y_0.

Next, for small $\delta>0$, we consider the approximation operator $E_\tau u$ on $(x_0-\delta,x_0+\delta)\times B\doteq I_\delta\times B$ (with initial trace taken at $t=t_0$) and we wish to prove that $E_\tau u$ converges to u uniformly on $I_\delta\times\omega_\delta$, after choosing $\delta>0$ sufficiently small to ensure that u does not vanish on $I_\delta \times \omega_\delta$ and therefore satisfies the equation $\mathcal{L}u = 0$ there. This is proved almost exactly as Theorem 2.2. The main difference is that here we do not want to shrink the ball B, so in order to prove the crucial estimate (2.9) we use instead (4.3) to show that the oscillation of φ on ω_δ is $\leq \delta/6$. In fact, for $|x' - x_0| \leq \delta$ and t, $t' \in \omega_\delta$, we have

$$|\varphi(x', t) - \varphi(x', t')| \leq |\varphi(x', t) - y_0| + |y_0 - \varphi(x', t')|.$$

Given $t \in \omega_\delta$, there exists $t_\bullet \in \mathcal{C}_q$ such that $|t - t_\bullet| < \delta$. Then

$$\begin{aligned} |\varphi(x', t) - y_0| = |\varphi(x', t) - \varphi(x_0, t_\bullet)| &\leq \frac{|x' - x_0| + |t - t_\bullet|}{24} \\ &\leq \frac{\delta}{12}. \end{aligned}$$

Similarly, $|y_0 - \varphi(x', t')| \leq \delta/12$, so

$$|\varphi(x', t) - \varphi(x', t')| \leq \frac{\delta}{6}, \quad |x' - x_0| \leq \delta, \quad t, t' \in \omega_\delta.$$

Thus, we obtain

$$\frac{|x - x'|^2}{2} - |\varphi(x', t) - \varphi(x', t')|^2 \geq \frac{\delta^2}{16} - \frac{\delta^2}{36} \geq c > 0$$

for $|x - x_0| \leq \delta/4$ and $|x' - x_0| \geq \delta/2$. The arguments in the proof of Theorem 2.2 allow us to show that $E_\tau u \to u$ uniformly on $I_\delta \times \omega_\delta$. As a corollary, we find a sequence of polynomials $P_j(z)$ that converge uniformly to U on $Z(I_\delta \times \omega_\delta)$ which is a neighborhood of z_0 because $t \mapsto \varphi(x_0, t)$ maps ω_δ onto an open interval that contains y_0. Therefore we conclude that $U(z)$ is holomorphic on a neighborhood of z_0.

Summing up, we have proved that the continuous function $U(z)$ is holomorphic on

$$D_1 = \{x + iy : \quad x \in I_1, \; m(x) < y < M(x)\}$$

except at the points where U vanishes. By the classical theorem of Rad, U is holomorphic everywhere in D_1, in particular it is holomorphic in a neighborhood of $z = 0$ and, since $u = U \circ Z$, this implies that the equation $\mathcal{L}u = 0$ is satisfied in a neighborhood of $(x, t) = (0, 0)$.

Case 3

This is the general case and we make no restrictive assumption about the central point $p = (0, 0)$, in particular, any of the inequalities $m(0) \leq \varphi(0, 0) \leq M(0)$ may become an equality. It follows from the arguments in Case 2 that U is holomorphic on

$$D_j = \{x + iy : \quad x \in I_j, \; m(x) < y < M(x)\}, \quad 1 \leq j \leq N.$$

This already shows that $\mathcal{L}u = 0$ on $Z^{-1}(D_j)$ which, in general, is a proper subset of $I_j \times B$. To see that u is actually a homogeneous solution throughout $I_j \times B$ we apply Mergelyan's theorem: for fixed j, there exists a sequence of polynomials $P_k(z)$ that converges uniformly to U on $\overline{D}_j$. Thus, $\mathcal{L}(P_j \circ Z) = 0$ and $P_j(Z(x, t)) \to u(x, t)$

uniformly on $I_j \times B$, so $\mathcal{L}u = 0$ on $I_j \times B$ for every $1 \le j \le N$. Thus, we conclude that $\mathcal{L}u = 0$ on $\big((-a,a) \setminus \mathcal{N}\big) \times B$. The vector fields (2.4) may be written as

$$L_j = \frac{\partial}{\partial t_j} + \lambda_j \frac{\partial}{\partial x}, \quad \lambda_j = -\frac{i\varphi_{t_j}}{1 + i\varphi_x}, \quad j = 1, \dots, n.$$

To complete the proof, we wish to show that

$$\int_{I \times B} u(x,t) L_j^t \psi(x,t)\, dx dt = 0$$

for any $\psi(x,t) \in C_0^\infty(I \times B)$. For each $0 < \epsilon < 1$, choose $\psi_\epsilon(x) \in C^\infty(-a,a)$ such that

1. $0 \le \psi_\epsilon(x) \le 1$;
2. $\psi_\epsilon(x) \equiv 1$ when $\operatorname{dist}(x, \mathcal{N}) \ge 2\epsilon$ and $\psi_\epsilon(x) \equiv 0$ for $\operatorname{dist}(x, \mathcal{N}) \le \epsilon$;
3. for some $C > 0$ independent of $0 < \epsilon < 1$, $|\psi_\epsilon'(x)| \le C\epsilon^{-1}$.

Since u is a solution on $\big(I \setminus \mathcal{N}\big) \times B$,

$$\begin{aligned} 0 &= \int_{I \times B} u(x,t) L_j^t (\psi_\epsilon(x) \psi(x,t))\, dx dt \\ &= \int_{I \times B} u(x,t) \psi_\epsilon(x) L_j^t(\psi)(x,t)\, dx dt - \\ &\qquad\qquad \int_{I \times B} u(x,t) \lambda_j(x,t) \psi(x,t) \psi_\epsilon'(x)\, dx dt. \end{aligned}$$

Observe that since $\lambda_j(x,t) \equiv 0$ for $(x,t) \in \mathcal{N} \times B$ and $\psi_\epsilon'(x)$ is supported in the set $\{x \in \mathbb{R} : \ \epsilon \le \operatorname{dist}(x, \mathcal{N}) \le 2\epsilon\}$,

$$\lim_{\epsilon \to 0} \int_{I \times B} h(x,t) \lambda_j(x,t) \psi(x,t) \psi_\epsilon'(x)\, dx dt = 0,$$

while

$$\lim_{\epsilon \to 0} \int_{I \times B} u(x,t) \psi_\epsilon(x) L^t \psi(x,t)\, dx dt = \int_{I \times B} u(x,t) L^t \psi(x,t)\, dx dt.$$

It follows that $\int_{I \times B} u(x,t) L^t \psi(x,t)\, dx dt = 0$ and hence $\mathcal{L}u = 0$ holds on $I \times B$. $\square$

5. An application to uniqueness

The Radó property can be used to give uniqueness in the Cauchy problem for continuous solutions without requiring any regularity for the initial "surface". Let $\mathcal{L}$ be smooth locally integrable structure of co-rank one defined on an open subset Ω of $\mathbb{R}^{n+1}$, $n \ge 1$, and let $U \subset \Omega$ be open. We denote by ∂U the set boundary points of U relative to Ω, i.e., $p \in \partial U$ if and only if $p \in \Omega$ and for every $\varepsilon > 0$, the ball $B(p, \varepsilon)$ contains both points of U and points of $\Omega \setminus U$. The orbit of $\mathcal{L}$ through the point p is defined as the orbit of p in the sense of Sussmann [Su] with respect to the set of real vector fields $\{X_\alpha\}$, with $X_\alpha = \Re L_\alpha$, where $\{L_\alpha\}$ is the set of all local smooth sections of $\mathcal{L}$ (we refer to [T1], [T2] and [BCH, Ch 3] for more information on orbits of locally integrable structures).

Definition 5.1. We say that ∂U is weakly noncharacteristic with respect to $\mathcal{L}$ if for every point $p \in \partial U$ the orbit of $\mathcal{L}$ through p intersects $\Omega \setminus \overline{U}$.

Recall that if $\Sigma = \partial U$ is a C^1 hypersurface, Σ is said to be noncharacteristic at $p \in \Sigma$ if $\Re L|_p$ is not tangent to Σ for some local smooth section of $\mathcal{L}$. This implies that the orbit of $\mathcal{L}$ through p must exit $\overline{U}$, so for regular surfaces the notion of "noncharacteristic" at every point is stronger than that of "weakly noncharacteristic". Similarly, if Σ is an orbit of $\mathcal{L}$ of dimension n that bounds some open set U, it will be a smooth hypersurface that fails to be noncharacteristic at every point and also fails to be weakly noncharacteristic. On the other hand, it is easy to give examples of a regular hypersurface Σ that bounds U and is characteristic precisely at one point p while the orbit through p eventually exits $\overline{U}$. In this case Σ will be weakly noncharacteristic although it fails to be noncharacteristic at every point.

Theorem 5.2. *Let $\mathcal{L}$ be a smooth locally integrable structure of co-rank one defined on an open subset Ω of $\mathbb{R}^{n+1}$, $n \geq 1$. Assume that $\mathcal{L}$ is locally solvable in degree one in Ω and that ∂U is weakly noncharacteristic. If u is continuous on $U \cup \partial U$, satisfies $\mathcal{L}u = 0$ on U in the weak sense and vanishes identically on ∂U, then there is an open set V, $\partial U \subset V \subset \Omega$ such that u vanishes identically on $V \cap U$.*

Proof. The proof is standard. Define $w \in C(\Omega)$ by extending u as zero on $\Omega \setminus (U \cup \partial U)$ (the continuity of w follows from the fact that u vanishes on ∂U). Then w is a Radó function and by Theorem 4.1 the equation $\mathcal{L}w = 0$ holds in Ω. By a classical application of the Baouendi-Treves approximation theorem the support of a homogeneous solution is $\mathcal{L}$-invariant, i.e., $S \doteq \operatorname{supp} w$ may be expressed as a union of orbits of $\mathcal{L}$ in Ω and the same holds for its complement, $\Omega \setminus S \supset \Omega \setminus \overline{U} \neq \emptyset$ (note that since ∂U is weakly noncharacteristic U is not dense in Ω). The fact that ∂U is weakly noncharacteristic implies that the union V of all the orbits of $\mathcal{L}$ that intersect $\Omega \setminus \overline{U}$ is an open set that contains ∂U on which w vanishes identically. $\square$

Example. Consider in $\mathbb{R}^3$, where the coordinates are denoted by t_1, t_2, x, the function

$$Z(x,t) = x + i\, a(x)(t_1^2 + t_2^2)/2.$$

Here $a(x)$ is a smooth real function that is not real analytic at any point and vanishes exactly once at $x = 0$. Then $Z(x,t)$ is a global first integral of the system of vector fields

$$L_1 = \frac{\partial}{\partial t_1} - \frac{i\, t_1 a(x)}{1 + ia'(x)(t_1^2 + t_2^2)/2} \frac{\partial}{\partial x}$$
$$L_2 = \frac{\partial}{\partial t_2} - \frac{i\, t_2 a(x)}{1 + ia'(x)(t_1^2 + t_2^2)/2} \frac{\partial}{\partial x}$$

which span a structure $\mathcal{L}$ of co-rank one. This structure is locally solvable in degree one due to the fact that any nonempty fiber of Z over $\mathbb{R}^3$, $\mathcal{F}(x_0 + iy_0, \mathbb{R}^3)$, is either a circle contained in the hyperplane $x = x_0$ if $x_0 \neq 0$ or the whole hyperplane $x = 0$ if $x_0 = 0$, thus a connected set. If

$$U = \{(x, t_1, t_2) \in \mathbb{R}^3 : \quad t_1^3 > x\}$$

it follows that ∂U is weakly noncharacteristic with respect to $\mathcal{L}$ so theorem 5.2 can be applied in this situation. The choice of $a(x)$ also prevents the use of Holmgren's theorem even at noncharacteristic points.

Consider now a discrete set $D \subset U$ such that every point in ∂U is an accumulation point of D and set $U_1 = U \setminus D$. We have that $\partial U_1 = \partial U \cup D$ is not regular but it is still weakly noncharacteristic, so if a continuous function u satisfies

$$\begin{aligned} L_1 u &= 0 \quad \text{on } U_1 \\ L_2 u &= 0 \quad \text{on } U_1 \\ u &= 0 \quad \text{on } \partial U_1 \end{aligned}$$

then u must vanish identically on a neighborhood V of ∂U_1 and since $\mathbb{R}^3$ is the union of three orbits of $\mathcal{L}$, namely, $x > 0$, $x = 0$ and $x < 0$, it is apparent that u vanishes identically. Notice that L_1 and L_2 are Mizohata type vector fields and they are not locally solvable when considered individually, so uniqueness in the Cauchy problem for this example does not follow from the results in [HT].

References

[A] H. Alexander, *Removable sets for CR functions,* Several complex variables (Stockholm, 1987/1988), Math. Notes, **38**, Princeton Univ. Press, Princeton, NJ, (1993), 1–7.

[BT1] M. S. Baouendi and F. Treves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields,* Ann. of Math. **113** (1981), 387–421.

[BT2] M. S. Baouendi and F. Treves, *A local constancy principle for the solutions of certain overdetermined systems of first order linear partial differential equations,* Math. Analysis and Applications, Part A, Advances in Math. Supplementary Studies **7A** (1981), 245–262.

[BCH] S. Berhanu, P. Cordaro and J. Hounie, *An Introduction to Involutive Structures*, Cambridge University Press, 2007. 432 p.

[C1] E. M. Chirka, *Radó's theorem for CR-mappings of hypersurfaces,* (Russian. Russian summary) Mat. Sb. 185 (1994), **6**, 125–144; translation in Russian Acad. Sci. Sb. Math. **82** (1995), 243–259.

[C2] E. M. Chirka, *A generalization of Radó's theorem,* Ann. Polonici Math. **84**, (2003), 109–112.

[CH] P. D. Cordaro and J. Hounie, *Local solvability for a class of differential complexes,* Acta Math. **187** (2001), 191–212.

[CT] P. D. Cordaro and F. Treves, *Homology and cohomology in hypo-analytic structures of the hypersurface type*, J. Geometric Analysis, **1** (1991), 39–70.

[HT] J. Hounie and J. Tavares, *Radó's theorem for locally solvable vector fields,* Proc. Amer. Math. Soc. **119** (1993), 829–836.

[MT] G. Mendoza and F. Treves *Local solvability in a class of overdetermined systems of linear PDE,* Duke Math. J. **63** (1991), 355–377.

[RS] J-P. Rosay, and E. Stout, *Radó's theorem for C-R functions,* Proc. Amer. Math. Soc. **106** (1989), 1017–1026.

[St] E. L. Stout, *A generalization of a theorem of Radó,* Math. Ann. **177** (1968), 339–340.

[Su] H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions,* Trans. Amer. Math. Soc. **180** (1973), 171–188.

[T1] F. Treves, *Approximation and representation of functions and distributions annihilated by a system of complex vector fields,* École Polytech., Centre de Math., Palaiseau, France, (1981).

[T2] F. Treves, *Hypo-analytic structures,* Princeton University Press, (1992).

[T3] F. Treves, *On the local solvability and local integrability of systems of vector fields*, Acta Math. **151**, 1983, 1–38.

J. Hounie
Departamento de Matemática
Universidad Federal de São Carlos
São Carlos, SP, 13565-905
Brasil
e-mail: hounie@dm.ufscar.br