A GENERALIZATION OF THE RUDIN-CARLESON THEOREM

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Abstract. We prove a generalization of the Rudin-Carleson theorem for homogeneous solutions of locally solvable real analytic vector fields.

1. Introduction

It is well known that if a continuous function f on the closure of the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ is holomorphic on D and vanishes on a subset E of the boundary ∂D of positive Lebesgue measure, then $f \equiv 0$. Conversely, Rudin [R] and Carleson [C] independently proved that if $E \subset \partial D$ is a closed set of Lebesgue measure zero, and if g is a continuous function on E, then there is $f \in C(\overline{D})$, holomorphic on D such that f agrees with g on E. Refinements, new proofs, and some generalizations of the Rudin-Carleson theorem were given in the works [B],[D],[G], and [O]. For applications to peak interpolation manifolds for holomorphic functions on domains in $\mathbb{C}^n$ see [Bh], [Na], [R2] and the references in these works. In a recent paper [BH1], we proved the following generalization of the Rudin-Carleson theorem for a class of real analytic complex vector fields:

Theorem A. *Let D be the unit disc and let L be a nonvanishing real analytic vector field defined on a neighborhood U of $\overline{D}$ satisfying the Nirenberg-Treves condition $(\mathcal{P})$. Assume that L does not have a relatively compact orbit in U. Let $E \subset \partial D$ be a closed set with Lebesgue measure zero and assume that $g \in C(\partial D)$ is constant on the intersection $\gamma \cap \partial D$ whenever γ is a one-dimensional orbit of L. Then there is $h \in C(\overline{D})$ satisfying:*

$$Lh = 0 \quad \text{in } D,$$
$$h(z) = g(z), \quad z \in E,$$
$$\sup_{z\in\partial D} |h(z)| \le \sup_{z\in E} |g(z)|.$$

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We recall that condition $(\mathcal{P})$ is a geometric condition that characterizes the class of locally solvable vector fields. This condition and the notion of orbits will be reviewed in Section 2. Observe that a continuous solution u of $Lu = 0$ is constant on any one-dimensional orbit of L and this explains why g is assumed to be constant on the sets $\gamma \cap \partial D$ in Theorem A.

In [BH1] we gave examples that showed that in general, if a real analytic, locally solvable vector field has a compact orbit, it may not have the Rudin-Carleson property. The main goal of this article is to characterize those locally solvable, real analytic vector fields L with compact orbits which exhibit the Rudin-Carleson property. This characterization is given in terms of the conformal type of a one-sided tubular neighborhood of each closed orbit endowed with the natural holomorphic structure induced by L (see Section 2 and Theorem 2.1 for the precise formulation). Section 3 is devoted to the proof of this result and Section 4 presents various examples. In Section 5, we show that for any smooth vector field, local solvability is a necessary and sufficient condition for the validity of the Rudin-Carleson property in arbitrary small neighborhoods of a point in an open set. In the final section we briefly discuss the relationship between the Rudin-Carleson theorem and the F. and M. Riesz theorem in the spirit of [B].

2. Preliminaries and statement of the main result

Let $L = X+iY$ be a smooth vector field on an open set Ω in $\mathbb{C}$ where X and Y are real vector fields. Assume that L is nonzero at each point of Ω. It is well known that condition $(\mathcal{P})$ can be expressed in terms of the orbits of the pair of vector fields $\{X, Y\}$ in the sense of Sussmann ([S]). Two points belong to the same orbit of $\{X, Y\}$ in Ω if they can be joined by a continuous, piecewise differentiable curve such that each piece is an integral curve of X or Y. Since X and Y are assumed to have no common zeros, the orbits of L in Ω are immersed submanifolds of Ω of dimension one or two; moreover, the two-dimensional orbits are open subsets of Ω. Let $\mathcal{O} \subset \Omega$ be a two-dimensional orbit of L in Ω and consider $X \wedge Y \in C^\infty(\Omega; \bigwedge^2(T(\Omega)))$. Since $\bigwedge^2(T(\Omega))$ has a global nonvanishing section $e_1 \wedge e_2$, $X \wedge Y$ is a real multiple of $e_1 \wedge e_2$ and this gives a meaning to the requirement that $X \wedge Y$ does not change sign on any two-dimensional orbit $\mathcal{O}$ of $\{X, Y\}$ in Ω. The vector field L satisfies condition $(\mathcal{P})$ at $p \in \Omega$ if there is a disc $U \subset \Omega$ centered at p such that $X \wedge Y$ does not change sign on any two-dimensional orbit of L in U.

Suppose now $L = X + iY$ is a locally solvable, real analytic vector field in a neighborhood U of $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. If L is a multiple of a real vector field, then by Lemma 2.1 in [BH1], it has the Rudin-Carleson property, and so we will assume throughout that L is not a multiple of a real vector field. If ∂D is an orbit of L then any solution in D that is continuous on $\overline{D}$ will be constant on ∂D and so we will assume that

$$\partial D \text{ is not an orbit.} \tag{2.1}$$

By real analyticity and the assumption that it is not a multiple of a real vector field, L has a finite number of one-dimensional orbits in D. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ denote all the closed orbits of L in D. Observe that these orbits are real analytic, Jordan curves. We divide these closed orbits into two types. A closed orbit $\mathcal{C}_j$ is a type I orbit if it is not enclosed in any other closed orbit. There may however be closed orbits in the precompact component of $D\backslash\mathcal{C}_j$ in D, i.e., closed orbits enclosed by $\mathcal{C}_j$. The remaining closed orbits will be referred to as type II orbits. Suppose now Ω is a connected open subset of a two-dimensional orbit of L in D. Since L satisfies condition $(\mathcal{P})$ and is real analytic, for each $p \in \Omega$, there is a neighborhood U_p of p, a real analytic function $Z : U_p \to \mathbb{C}$ such that $LZ = 0$, $dZ \neq 0$ and Z is a homeomorphism. If $Z' : U'_p :\to \mathbb{C}$ is also another such function on a neighborhood U'_p of p, then $Z' \circ Z^{-1} : Z(U_p \cap U'_p) \to \mathbb{C}$ is holomorphic. In other words, L induces a Riemann surface structure on Ω which we will denote by (Ω, L).

For each type I closed orbit $\mathcal{C}_j$, we fix a one-sided tubular neighborhood Ω_j of $\mathcal{C}_j$ that lies in the non precompact component of $D\backslash\mathcal{C}_j$ such that Ω_j does not intersect any one-dimensional orbit of L. We will also assume that the Ω_j are pairwise disjoint. The following theorem provides a necessary and sufficient condition for L to have the Rudin-Carleson property.

Theorem 2.1. *Let D be the unit disc and let L be a nonvanishing real analytic vector field defined on a neighborhood U of $\overline{D}$ satisfying the Nirenberg-Treves condition $(\mathcal{P})$ and (2.1). Let $E \subset \partial D$ be a closed set with Lebesgue measure zero. Then the following are equivalent:*

(1) *For each type I closed orbit $\mathcal{C}_j$, the Riemann surface (Ω_j, L) is conformal to the punctured disc with the standard structure.*
(2) *For every $g \in C(\partial D)$ that is constant on the intersection $\gamma \cap \partial D$ whenever γ is a one-dimensional orbit of L, there is $h \in C(\overline{D})$ satisfying:*

$$Lh = 0 \quad \text{in } D,$$
$$h(z) = g(z), \quad z \in E,$$
$$\sup_{z\in\partial D} |h(z)| \leq \sup_{z\in E} |g(z)|.$$

Remark 2.2. *In Section 4 we will give examples of real analytic, locally solvable vector fields with compact orbits with and without the Rudin-Carleson property.*

3. Proof of Theorem 2.1

We will consider two cases:

Case 1. Assume that L has a single type I compact orbit $\mathcal{C}$ contained in D. Let γ_j, $j = 1, \dots, n$ denote the one-dimensional, noncompact orbits of L in D. Consider the complement in D of the noncompact one-dimensional orbits, $D^\sharp = D \setminus \bigcup_{j=1}^{n} \gamma_j$ and consider the connected components D_k, $k = 0 \dots, N$ of $D^\sharp$. We now fix k and study the boundary of D_k. Note that if $p \in D \cap \partial D_k$ then $p \in \gamma_j$ for some $j \geq 1$. Therefore, if V is a sufficiently small disc centered at p, $V \setminus \gamma_j$ is a disjoint union of two domains V_1 and V_2 with $V \cap D_k = V_1$. It follows that D_k has a real analytic boundary near p. We suppose from now on that $p \in \partial D_k \cap \partial D$. We will consider different possibilities. If the orbit of L at p is two-dimensional, by the real analyticity of L, we can find a disc B centered at p such that $B \cap D \subset D_k$. This means that near p, ∂D_k consists of $\partial D \cap B$ in this case. Assume next that the orbit at p is one-dimensional. If L is transversal to ∂D at p, then the orbit γ_j through p divides a disc W centered at p into two connected pieces W_1 and W_2 with $W_1 \cap D = W \cap D_k$. Thus near p, D_k has a piecewise real analytic boundary consisting of two curves that intersect at p. Suppose now that L is tangent to ∂D at p. Let γ_j continue to denote the one-dimensional orbit through p. By the real analyticity, in a small disc V centered at p, if $\gamma_j^\flat = \gamma_j \cap V$, there are three possibilities:

(i) Assume $\gamma_j^\flat \subset D \cup \{p\}$. Then since $p \in \partial D_k$, either $\gamma_j^\flat \subset \partial D_k$ or $\gamma_j^\flat \setminus \{p\}$ contains a subarc with an endpoint at $\{p\}$ that bounds ∂D_k (in the second case, replace $\gamma_j^\flat$ by this subarc and call it $\gamma_j^\flat$). Near each $q \in \gamma_j^\flat$, D_k lies on one side of $\gamma_j^\flat$. Hence near p, either $\partial D_k = \gamma_j^\flat$, or ∂D_k consists of $\gamma_j^\flat$ and a subarc of ∂D with one endpoint at p.

(ii) Suppose $\gamma_j^\flat \cap \overline{D} = \{p\}$. Near p, each side of $\gamma_j^\flat$ is contained in distinct two-dimensional orbits. It follows that for some neighborhood V' of p, $V' \cap D \subset D_k$ and so ∂D_k near p equals $V' \cap \partial D$.

(iii) Assume $\gamma_j^\flat = \gamma^+ \cup \gamma^-$, where $\gamma^- \subset D$, and $\gamma^+ \cap D = \emptyset$. Again each side of $\gamma_j^\flat$ is contained in a two-dimensional orbit and so near p, ∂D_k consists of γ^- and an arc in ∂D with p as an endpoint.

We thus see that ∂D_k is piecewise real analytic consisting of a finite number of curves each of which is either an arc of some γ_j, $1 \leq j \leq n$, and the endpoints of this arc belong to the intersection of γ_j with ∂D, or an arc in ∂D with endpoints contained in the intersection of ∂D with a pair of one-dimensional orbits $\gamma_j \cup \gamma_{j'}$, $1 \leq j < j' \leq n$. Note also that, whatever $0 \leq k \leq N$, $\partial D_k \cap \partial D$ contains an open arc. Let D_0 be the connected component that contains the closed orbit $\mathcal{C}$. Notice that $D_0 \setminus \mathcal{C}$ has two connected components, a simply connected one, that we may call the interior of $\mathcal{C}$ and denote it by Ω_0, and an annular one that we will denote by Ω, so $\Omega = D_0 \setminus \overline{\Omega}_0$.

Assume (1) in the theorem holds for the Riemann surface (Ω_1, L) where Ω_1 is the one-sided annular neighborhood of $\mathcal{C}$ in Ω. We may assume that Ω_1 is bounded by $\mathcal{C}$ and a real analytic, simple closed Jordan curve Σ. Consider the Riemann surface (Ω, L). Because the fundamental group of Ω is the integers, by Theorem IV.6.1 in [FK], this Riemann surface is conformal to either the punctured disc, an annulus of the form $\{z : a < |z| < b\}$ for some $a, b > 0$, or the punctured plane where each one is equipped with the standard structure. Since $\partial\Omega$ intersects ∂D on an arc where the orbit of L is two-dimensional, there is an open set Ω_2 such that $\Omega_2 \setminus \Omega$ has nonemtpy interior and Ω_2 is contained in an orbit of L in U of dimension two. This shows that (Ω, L) is a prolongable Riemann surface (we recall that a Riemann surface S is called prolongable if there exists a Riemann surface S' and an injective holomorphic map $f : S \longrightarrow S'$ with $f(S)$ not dense in S'). In particular, (Ω, L) cannot be conformal to the punctured plane.

Next if (Ω, L) is conformal to an annulus of the form $\{z : a < |z| < b\}$ for some $a, b > 0$, let $F : (\Omega, L) \longrightarrow \{z : a < |z| < b\}$ be a conformal map. Since Σ divides Ω into two components, it follows that $F(\Omega_1)$ equals one of the components of $\{z : a < |z| < b\} \setminus F(\Sigma)$. This contradicts the assumption that (Ω_1, L) is conformal to the punctured disc. It follows that (Ω, L) is conformal to the punctured disc. Let $F : (\Omega, L) \longrightarrow \Delta \setminus \{0\}$ be a conformal map. We will now use the arguments in [BH1] to show that F extends continuously to $\partial\Omega \setminus \mathcal{C}$, and this extension is injective and preserves sets of null Lebesgue measure on the part of $\partial\Omega \setminus \mathcal{C}$ that is disjoint from the one-dimensional orbits. We will also show that F has a continuous extension to $\overline{\Omega}$ which is injective away from the one-dimensional orbits, and that maps distinct one-dimensional orbits to distinct single points.

Let $z_0 \in \partial\Omega$. Assume first that $z_0 \in \partial D$ and that z_0 is not contained in a one-dimensional orbit of L. Suppose z_k is a sequence in Ω that converges to z_0. If the sequence $F(z_k)$ does not have a limit, then it clusters at least at two points on $\partial\Delta \setminus \{0\}$. Without loss of generality we may assume $p_k = F(z_{2k})$ converges to v and $q_k = F(z_{2k+1})$ converges to w where v and w are two points on the boundary of $\Delta \setminus \{0\}$. Let T_1 and T_2 be two continuous arcs in Δ such that T_1 contains the p_k and ends at v while T_2 contains the q_k and ends at w. We may assume that $\mathrm{dist}(T_1, T_2) > c$ for some $c > 0$. Let Z be a first integral which is a homeomorphism from a disc U' about z_0 to a neighborhood of the origin and mapping $U' \cap \Omega$ onto V. Let $G = F \circ Z^{-1}$. Let $S_j = F^{-1}(T_j)$, $j = 1, 2$. For $r > 0$ small, let C_r be the intersection of the circle of radius r centered at 0 with the region V. Observe that if r is small enough, say $r \leq r_0$ for some $r_0 > 0$, $Z^{-1}(C_r)$ intersects both S_1 and S_2 since these sets are connected and both accumulate at z_0.

Let $C_r = \{re^{i\theta} : \theta_1(r) < \theta < \theta_2(r)\}$. Let $C'_r = G(C_r)$. Observe that C'_r contains points of both T_1 and T_2 since $Z^{-1}(C_r)$ intersects both S_1 and S_2. Since G is holomorphic, it follows that

$$c < \ell(C'_r) = \int_{\theta_1(r)}^{\theta_2(r)} |G'(re^{i\theta})| r \, d\theta.$$

Applying the Schwarz inequality we get:

$$\frac{c^2}{r} < 2\pi \int_{\theta_1(r)}^{\theta_2(r)} |G'(re^{i\theta})|^2 r \, d\theta$$

which in turn leads to the contradiction that

$$\infty = c^2 \int_0^{r_0} \frac{dr}{r} \, dr < 2\pi \int_0^{r_0} \int_{\theta_1(r)}^{\theta_2(r)} |G'(re^{i\theta})|^2 r \, d\theta \, dr < \pi.$$

It follows that $F(z_k)$ has a limit and therefore F extends continuously upto the point z_0. If $F(z_0) = 0$, then F will map a neighborhood of z_0 in ∂D to 0 which would force F to be constant since by the analyticity of L and (2.1), except possibly at a finite number of points, ∂D is noncharacteristic for L. It follows that $F(z_0) \in \partial\Delta$.

We wish to prove the same property for F^{-1} away from the finite number of points which as we will see later are the images of the one-dimensional orbits under F. This would be equivalent to showing that F is locally one-to-one at the boundary points z_0 as above which we will next show.

We claim that at each hypocomplex point $z_0 \in \partial\Omega$ as above, the function F extends to be a homeomorphism up to z_0. To see this, first assume that L is transversal to ∂D at z_0. In this case, after contracting U about z_0, $Z(U \cap \partial D)$ is a real analytic piece of the boundary of V through the origin. The function G is holomorphic on V, continuous up to the boundary near the origin, and sends a real analytic boundary piece of V through 0 into the boundary of the disc Δ. By the Schwarz reflection principle, G extends as a holomorphic function in a neighborhood of the origin which in turn leads to a real analytic extension of F past z_0. Suppose now $z_1 \in \partial\Omega$ is another hypocomplex point where L is transversal to ∂D and assume that $F(z_0) = F(z_1) = w$. Then since F extends as a solution past both z_0 and z_1, and L is hypocomplex at these points, the extended F is an open map and hence there are neighborhoods U_0, U_1, W of z_0, z_1 and w respectively such that $F(U_0) = F(U_1) = W$. Moreover, because F is extended using the reflection principle, we may assume that $F(U_0 \cap \Omega) = W \cap \Delta = F(U_1 \cap \Omega)$. But this contradicts the injectivity of F on Ω. Hence $F(z_0) \neq F(z_1)$. Recall that since ∂D is not an orbit of L, there are only a finite number of points on ∂D where L is not transversal to ∂D. Suppose now z_2, z_3 are two points in $\partial\Omega \cap \partial D$ where $F(z_2) = F(z_3)$ and assume that L is transversal to ∂D at z_2 and tangent to ∂D at z_3. We have seen that there is a neighborhood U_2 of z_2 in $\overline{D}$ where F is one-to-one and such that $F(U_2)$ is a neighborhood of $F(z_2)$ in $\overline{\Delta}$. But then, if $z \in \Omega$ and is sufficiently close to z_3, $F(z) \notin F(U_2)$, contradicting the continuity of F at z_3. Therefore, $F(z_2) \neq F(z_3)$. Finally, suppose z_4 and z_5 are two points in $\partial\Omega \cap \partial D$ where L is hypocomplex and assume L is tangent to ∂D at both points and $F(z_4) = F(z_5) = w_0$. Since there are only a finite number of such points in ∂D, there is an open arc I in ∂D containing z_4 in its interior and consisting of hypocomplex points such that z_4 is the only point where L is not transversal to ∂D. Since F is one-to-one on $I\backslash\{z_4\}$, $F(I)$ is an open arc in $\partial\Delta$ containing w_0 in its interior. There is also a similar arc J with z_5 in its interior and we may assume that I and J are disjoint. But then this would contradict the injectivity of F on $I \cup J\backslash\{z_4, z_5\}$ and so we must have $F(z_4) \neq F(z_5)$. Hence F can be extended as a homeomorphism up to the part of the boundary of Ω that is disjoint from the one-dimensional orbits. It is also real analytic past all but a finite number of the points that don't lie in the one-dimensional orbits.

Assume next that $z_0 \in \partial\Omega \cap D \cap \gamma_j$ for some $j \geq 1$ and write, for simplicity of notation, $\gamma_j = \Gamma$. Write $L = X + iY$ with X and Y real vector fields. Replacing L, if necessary, by a convenient nonvanishing multiple of L we may assume that Γ is a closed integral curve of X joining two points A and B that belong to ∂D. Since Y vanishes on Γ, it vanishes identically on any integral curve of X that contains Γ (by analyticity). We may consider an integral curve Γ_1 of X that extends Γ past both endpoints A and B, so that Γ_1 is a one-dimensional orbit of L in a neighborhood U of $\overline{D}$ with endpoints in $U \setminus \overline{D}$. In a tubular neighborhood V of Γ_1 we may choose coordinates that rectify the flow of X and in which L has a canonical form. More precisely, we may choose local coordinates (x, t), so that V is expressed as $|x| \leq 1$, $|t| \leq 2$, $x(p) = t(p) = 0$, $x(A) = x(B) = 0$, $t(A) = 1$, $t(B) = -1$ and L has the form

$$L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial t},$$

with $t \mapsto b(x,t) \geq 0$ and not identically zero for $0 < x \leq 1$, and $b(0,t) \equiv 0$, $-2 \leq t \leq 2$. The intersection of Ω with V is described by

$$\Omega \cap V = \{(x,t) : \ 0 < x \leq 1, \ \beta(x) < t < \alpha(x)\}.$$

Here, $\alpha(x)$, $\beta(x)$ are continuous on $[0,1]$ and analytic on $(0,1]$, $\alpha(0) = 1$, $\beta(0) = -1$ and their graphs are contained in $\partial D \cap \partial\Omega$. By restricting F to $\Omega \cap V$, we obtain an injective map $F(x,t)$ from $\Omega \cap V$ into D. We may assume that F has already been extended as a homeomorphism from

$$\{(x,t):\ 0<x\le 1,\ \beta(x)\le t\le \alpha(x)\}$$

into $\overline{D}$. Hence, $F(x,t)$ maps the graphs $t=\alpha(x)$, $t=\beta(x)$, $0<x<1$, into some open arcs $\widehat{A'C'}$, $\widehat{B'D'}\subset \partial D$. Consider the vertical segment $T_\varepsilon=\{\varepsilon\}\times[\beta(\varepsilon),\alpha(\varepsilon)]$, $0<\varepsilon<1$, that is mapped by F into a curve $F(T_\varepsilon)$ contained in D that joins two boundary points $A'_\varepsilon \doteq F(\varepsilon,\alpha(\varepsilon))\in \widehat{A'C'}$, $B'_\varepsilon \doteq F(\varepsilon,\beta(\varepsilon))\in \widehat{B'D'}$. Notice that $A'_\varepsilon\to A'$ and $B'_\varepsilon\to B'$ as $\varepsilon\to 0$. The length of $F(T_\varepsilon)$ is

$$\ell(F(T_\varepsilon))=\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)}|F_t(\varepsilon,t)|\,dt=\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)}\big(U_t^2(\varepsilon,t)+V_t^2(\varepsilon,t)\big)^{1/2}\,dt$$

with $F=\Re F+i\Im F=U+iV$. Since $LF=0$, i.e., $U_t=bV_x$, $V_t=-bU_x$, we have

$$\begin{aligned}\ell(F(T_\varepsilon))^2&=\left(\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)}b(\varepsilon,t)\big(U_x^2(\varepsilon,t)+V_x^2(\varepsilon,t)\big)^{1/2}\,dt\right)^2\\&\le\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)}b(\varepsilon,t)\,dt\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)}b(\varepsilon,t)\big(U_x^2(\varepsilon,t)+V_x^2(\varepsilon,t)\big)\,dt\\&\le\int_{-2}^{2}b(\varepsilon,t)\,dt\,I(\varepsilon)\le C\varepsilon\,I(\varepsilon).\end{aligned}\tag{2.2}$$

On the other hand,

$$\begin{aligned}\int_0^1 I(\varepsilon)\,d\varepsilon&=\int_0^1\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)}b(\varepsilon,t)\big(U_x^2(\varepsilon,t)+V_x^2(\varepsilon,t)\big)\,dtd\varepsilon\\&=\int_0^1\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)}\det\frac{\partial(U,V)}{\partial(\varepsilon,t)}\,dtd\varepsilon=\operatorname{area}(F(\Omega\cap V))<\pi.\end{aligned}$$

Since the integral on the left hand side is finite, we see that the product $\varepsilon I(\varepsilon)$ cannot remain bounded below by a positive constant in any neighborhood of the origin. In other words, there is a sequence $\varepsilon_j\searrow 0$ such that $\varepsilon_j I(\varepsilon_j)\searrow 0$ and (2.2) shows that $\ell(F(T_{\varepsilon_j}))\to 0$. Hence, $|A'_{\varepsilon_j}-B'_{\varepsilon_j}|\to 0$ and we conclude that $A'=B'$. Notice that the region

$$F\big(\{(x,t):\ 0<x<\varepsilon_j,\ \beta(x)<t<\alpha(x)\}\big)$$

is bounded by the closed curved made of three arcs, to wit, the circular arc from A'_{ε_j} to A', the circular arc from $B'=A'$ to B'_{ε_j} and the curve $F(T_{\varepsilon_j})$ that joins B'_{ε_j} to A'_{ε_j}. It is therefore easy to see that the diameter of that region tends to zero as $j\to\infty$, so given $r>0$ we may find j_0 such that

$$F\big(\{(x,t):\ 0<x<\varepsilon_j,\ \beta(x)<t<\alpha(x)\}\big)\subset\Delta(A',r),\quad j\ge j_0.$$

This shows that, if we extend F to $\{0\}\times[-1,1]$ by setting $F(0,t)=A'$, $-1\le t\le 1$, we obtain a continuous extension.

Finally, we need to consider the continuous extendability up to points z_0 that are in $\partial\Omega\cap\partial D\cap\gamma_j$, for some $j\ge 1$. Such a point will be in $D_r=\{z:|z|<r\}$, $r>1$, so it must belong to Ω_r (the analog of Ω). Reasoning as above we see that a conformal map $F_r:\Omega_r\longrightarrow\Delta\setminus\{0\}$ extends continuously upto z_0. This in turn leads to the continuity of F since we could have taken it to be the restriction of F_r to Ω. Observe next that $F(\Sigma)$ is a simple closed curve that disconnects Δ and with 0 in its interior. Since Ω_1 is conformal to $\Delta\setminus\{0\}$, we conclude that as $p\to\mathcal{C}$, $F(p)\to 0$, so F has a unique continuous extension up to $\mathcal{C}$ and $F(\mathcal{C})=\{0\}$. Summing up, F can be continuously extended up to $\Omega\cup\partial D_0$, this extension is injective on the pieces of D_0 made up of subarcs of ∂D and sends each of the pieces made up of one-dimensional orbits γ_j, $j\ge 1$, to points $A_j\in\partial\Delta$, in particular $F(\partial D_0)\subset\partial\Delta$. Moreover, it takes sets $X\subset\partial\mathcal{D}_0$ of Lebesgue measure zero into subsets of $\partial\Delta$ of Lebesgue measure zero.

To consider the Rudin-Carleson problem, we are given a closed set $E\subset\partial D$ with $|E|=0$ ($|\cdot|$ denoting the Lebesgue measure in ∂D) and we may assume without loss of generality that the intersections $\gamma_j\cap\partial D\subset E$, $1\le j\le n$. We are also given a continuous function $g\in C(E)$ which is constant on each $\gamma_j\cap\partial D\subset E$, $j=1,\dots,n$. Since $F:\partial D_0\longrightarrow\partial\Delta$ is continuous and preserves sets of Lebesgue measure zero, $\tilde{E}\doteq F(E)$ is a closed set in $\partial\Delta$ of measure zero. The fact that F maps the γ_j to single points implies that there is $\tilde{g}\in C(\tilde{E})$ such that $\tilde{g}\circ F=g$ on the set E. By the Rudin-Carleson theorem, there is a holomorphic function $\tilde{h}$ on Δ which is continuous on $\overline{\Delta}$, agrees with $\tilde{g}$ on $\tilde{E}$ and

$$\sup_{z\in\partial\Delta}|\tilde{h}(z)|\le\sup_{z\in\tilde{E}}|\tilde{g}(z)|$$

Set $h_0=\tilde{h}\circ F$. Then h_0 is continuous on $\overline{\Omega}\cup\partial D_0$ and $Lh_0=0$ in Ω. If γ_j, $j\ge 1$ is a one-dimensional orbit of L that is a piece of ∂D_0, we know that $F(\gamma_j)=A_j$, so $h_0(\gamma_j)=\tilde{h}(A_j)=\tilde{g}(A_j)$. This shows that h_0 agrees with g on the intersection

$$\overline{D}_0\cap E.$$

Finally, we may extend h_0 continuously to all of $\overline{D_0}$ by declaring that h_0 assumes the value $\tilde{h}(0)$ on the closed orbit $\mathcal{C}$ as well as in the interior of $\mathcal{C}$. Thus h_0 is continuous on $\overline{D_0}$ and it is easy to see that $Lh_0=0$ on D_0. The construction of h_0 also shows that

$$\sup_{z\in\partial D_0}|h_0(z)|\le\sup_{z\in E}|g(z)|.$$

For the components D_j, $j\ge 1$, using the method of proof of the main theorem in [BH1], we may find continuous functions $F_{D_j}\in C(\overline{D}_j)$ such that $LF_{D_j} = 0$ in D_j, $F_{D_j}(\overline{D}_j) = \overline{\Delta}$ (where Δ is a copy of the unit disc in $\mathbb{C}$) and F_{D_j} is injective on D_j and on the portion of ∂D_j that lies in ∂D and is disjoint from the one-dimensional orbits. The remaining part of ∂D_j is made up of subarcs C_ℓ of some γ_j, $1 \le j \le n$, and each subarc C_ℓ is mapped by F_{D_j} into a single point $A_\ell \in \partial\Delta$, with $A_\ell \ne A_{\ell'}$ for $\ell \ne \ell'$. Moreover, we may use these functions F_{D_j} $j \ge 1$, and h_0, to obtain a continuous function $h \in C(\overline{D})$ such that $Lh = 0$ in D, and that agrees with g on E. Furthermore,

$$\sup_{\overline{D}} |h(z)| \le \sup_E |g(z)|.$$

Thus (2) in the theorem is satisfied.

Suppose now (2) is satisfied. Consider the structure (Ω_1, L). This structure can not be conformal to the punctured plane since it is prolongable. Assume $F : (\Omega_1, L) \longrightarrow \{z : a < |z| < b\}$ is conformal for some $a, b > 0$. The methods in [BH1] show that F extends as a homeomorphism up to the boundary piece Σ and we may assume that it maps Σ onto $\{w : |w| = b\}$. It then follows that as $z \to \mathcal{C}$, $F(z) \longrightarrow \{w : |w| = a\}$. Let $h \in C(\overline{D})$ be a solution of L in D. There exists $\tilde{h}$ holomorphic on $\{z : a < |z| < b\}$ such that $h = \tilde{h} \circ F$ on Ω_1. Since h is continuous up to $\mathcal{C}$ and is constant on $\mathcal{C}$, $\tilde{h}$ extends continuously up to $\{w : |w| = a\}$ and is constant on this circle. It follows that h is constant on Ω. But then h would be constant on arcs of ∂D that intersect $\partial\Omega$. This contradicts the validity of (2). Hence (1) holds.

Case 2. Assume that L has at least two compact orbits of type I. Let $\gamma_1, \dots, \gamma_m$ be the one-dimensional orbits in D and write $D \setminus \cup_{j=1}^m \gamma_j$ as a union $\bigcup_{i=1}^N W_i$ of components. Assume that (1) in the theorem holds. As before for each i, ∂W_i is piecewise real analytic consisting of arcs of the γ_j or ∂D. For each i, let $\mathcal{C}_1^i, \dots, \mathcal{C}_{n_i}^i$ be all the type I compact orbits in W_i. For each $t = 1, \dots, n_i$, let D_t^i be the relatively compact region in D bounded by $\mathcal{C}_t^i$ and let Ω_t^i denote the one-sided tubular neighborhood of $\mathcal{C}_t^i$ that is disjoint from $\overline{D_t^i}$. We may assume that the Ω_t^i are pairwise disjoint and $\partial\Omega_t^i = \mathcal{C}_t^i \cup \Sigma_t^i$ for some analytic, closed Jordan curves Σ_t^i. When (1) in the theorem holds, for each i and t, there is a conformal map $Z_t^i : \Omega_t^i \to \Delta \setminus \{0\}$. Moreover, as we saw in Case 1, we can extend each Z_t^i continuously to $\overline{D_t^i}$ by setting it to be zero there. Fix i and consider the component W_i. Consider the equivalence relation on W_i such that the equivalence classes $[z]$ are

(1) single points $[z] = \{z\}$ if $z \notin \bigcup_{t=2}^{n_i} \overline{D_t^i}$,
(2) $[z] = \overline{D_t^i}$ if $z \in \overline{D}_t^i$ and $2 \le t \le n_i$.

In other words, for $t \ge 2$, we collapse $\overline{D_t^i}$ to a single class. For each $t \ge 2$, fix once for all a point $z_t \in \overline{D_t^i}$. We denote by $\widehat{W_i} = W_i/\sim$ the quotient space with its natural topology and we will define a conformal structure on $\widehat{W_i} \setminus \overline{D_1^i}$. We need to define an atlas of local holomorphic charts. If $[z] = \{z\}$, $z \notin \bigcup_{t=1}^{n_i} \overline{D_t^i}$, is a class of type (1) we may take a local first integral of L that is a homeomorphism on a neighborhood of z that does not intersect $\bigcup_{t=1}^{n_i} \overline{D_t^i}$. In the case of the $[z_t]$ $(2 \leq t \leq n_i)$, we choose the neighborhood as $[z_t] \cup [\Omega_t^i] = [z_t] \cup \Omega_t^i$ and the holomorphic coordinate will be given by Z_t^i on Ω_t^i and maps $[z_t]$ to zero. These charts are holomorphically related on their overlaps and turn $\widehat{W_i} \setminus \overline{D_1^i}$ into a Riemann surface. Observe that $\widehat{W_i} \setminus \overline{D_1^i}$ has the integers as its fundamental group and it is conformal to the punctured disc, due to the assumption that this is so for the structure (Ω_1^i, L). Given g as in (2) in the theorem, we may now reason as in case (1) to solve the Rudin-Carleson problem in $\widehat{W_i}$ which also solves the same problem in W_i by composition with the quotient map $W_i \longrightarrow \widehat{W_i}$. The solutions on the W_i can then be glued together to lead to a solution on D and hence (2) holds. Conversely if (2) in the theorem holds, the arguments used in case (1) show that (1) has to hold.

4. Some examples

The motivation for Examples 1 and 2 below comes from [BM].

Example 1. Consider a one-form ω expressed in polar coordinates as

$$\omega = e^{i\theta}\left(dr + ir(1-4r^2)h(r^2)d\theta\right)$$

where $h(t)$ is real analytic on $\mathbb{R}$, and $h(0) = 1$. We can express ω as

$$\omega = A(z,\overline{z})dz + B(z,\overline{z})d\overline{z}$$

where

$$2A(z,\overline{z}) = 1 + (1-4r^2)h(r^2) \text{ and } B(z,\overline{z}) = e^{2i\theta}\left(1 - (1-4r^2)h(r^2)\right).$$

The condition that $h(0) = 1$ ensures that ω is a real analytic form in the plane. Let L be a real analytic, non vanishing vector field in the plane such that $\langle \omega, L\rangle = 0$. Observe that the only one-dimensional orbit of L is given by

$$\gamma = \left\{(r,\theta) : r = \frac{1}{2}\right\}.$$

It is easy to see that L is elliptic away from γ and hence L is locally solvable everywhere in the plane. Let $\frac{-1}{2h(\frac{1}{4})} = a + ib$. By separation of variables, in the region $\Omega = \{z : \frac{1}{2} < |z| < 1\}$, one gets a solution of L of the form

$$Z(r,\theta) = \left(r - \frac{1}{2}\right)^{a+ib} e^{E(r)+i\theta},$$

where $E(r)$ is a real analytic function. If h is chosen so that $a > 0$, then the structure induced by L on Ω is conformal to the standard one on a punctured disc as can be seen by using the injective solution $Z(r,\theta)$. By Theorem 2.1, such an L will have the Rudin-Carleson property. On the other hand, if h is chosen so that $a = 0$, then L will not have the Rudin-Carleson property.

Example 2. Example 1 can be modified to get a locally solvable, real analytic vector field with several orbits that are concentric circles (see [BM]). We will now give an example with two compact orbits which are not contained in each other.

Set

$$\begin{aligned} X &= (3-x^2)(x^2-1)\frac{\partial}{\partial x} - 2x(x^2-2)y\frac{\partial}{\partial y}, \\ Y &= -x(x^2-2)y\frac{\partial}{\partial x} + 2x^2(1-y^2)\frac{\partial}{\partial y}, \qquad (x,y) \in \mathbb{R}^2. \end{aligned}$$

The vector field X has four critical points, $(\pm 1, 0)$, $(\pm\sqrt{3}, 0)$ on which Y does not vanish, so the complex vector field $L = X + iY$ has no zeros. We have

$$\begin{aligned} X \wedge Y &= 2(3-x^2)(x^2-1)x^2(1-y^2) - 2x^2(x^2-2)^2y^2 \\ &= 2x^2\big(1 - y^2 - (2-x^2)^2\big)\partial_x \wedge \partial_y \end{aligned}$$

which means that X and Y are linearly dependent if and only if

$$x = 0 \quad \text{or} \quad (x^2-2)^2 + y^2 = 1.$$

Thus the analytic set $X \wedge Y = 0$ has 3 connected components that are analytic curves, two Jordan curves $\mathcal{C}_1$ and $\mathcal{C}_2$ (each one of them is the mirror image of the other with respect to the y-axis) plus the y-axis. Call Ω_j the interior domain bounded by $\mathcal{C}_j$, $j = 1, 2$. Notice that X and Y are tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ and since L never vanishes, $\mathcal{C}_1$ and $\mathcal{C}_2$ are orbits of L. For instance, to check that X is tangent to $\mathcal{C}_j$ observe that the gradient of $(x^2-2)^2 + y^2$ is proportional to $Z = (x^2-2)2x\partial_x + y\partial_y$ while

$$\begin{aligned} X \cdot Z &= (3-x^2)(x^2-1)(x^2-2)2x - 2x(x^2-2)y^2 \\ &= -2x(x^2-2)\big(-4x^2 + 3 + y^2 + x^4\big) \\ &= -2x(x^2-2)\big((x^2-2)^2 + y^2 - 1\big) \end{aligned}$$

and the last factor of the right hand side vanishes on C_j, $j = 1, 2$. Since X and Y are linearly independent on Ω_1 and Ω_2, both are 2-orbits. Set $\Omega_3 = \mathbb{R} \setminus \big(\overline{\Omega}_1 \cup \overline{\Omega}_1\big)$. Then X and Y are linearly independent on $\Omega_3^+ \doteq \Omega_3 \cap \{x > 0\}$ and $\Omega_3^- \doteq \Omega_3 \cap \{x < 0\}$ and since X is transversal to the y-axis it is easy to see that Ω_3 is an orbit. Thus, $X \wedge Y$ vanishes neither on Ω_1 nor on Ω_2 and vanishes but does not change sign on Ω_3, showing that L satisfies condition $(\mathcal{P})$.

Example 3. We will now describe a method to produce locally solvable vector fields L with a large number of closed 1-dimensional orbits. Suppose we are given a locally solvable vector field L_0 defined on $\mathbb{R}^2$ with closed 1-dimensional orbits C_j bounding disjoint 2-dimensional orbits Ω_j, $j = 1, \dots k$ on which L_0 is elliptic. Assume that these orbits are contained on the half plane $x > 0$ and no noncompact 1-dimensional orbit intersects $x > 0$ (what happens for $x < 0$ is irrelevant). We will also assume that L_0 is elliptic at any point of the y-axis, that X_0 is transversal to the y-axis and that $\{x > 0\} \setminus \bigcup_{j=1}^{k} \overline{\Omega}_j$ is a 2-dimensional orbit of L_0 on $\{x > 0\}$. We may write $L_0 = X_0 + iY_0$ where

$$X_0 = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$$
$$Y_0 = c(x,y)\frac{\partial}{\partial x} + d(x,y)\frac{\partial}{\partial y}$$

are real vector fields. Mimicking the way one obtains the Mizohata vector field out of the Cauchy-Riemann vector field, consider the 2-fold transformation $\Phi(x,y) = (x^2, y)$. This leads us to define new vector fields

$$X_1 = a(x^2,y)\frac{\partial}{\partial x} + 2xb(x^2,y)\frac{\partial}{\partial y}$$
$$Y_1 = c(x^2,y)\frac{\partial}{\partial x} + 2xd(x^2,y)\frac{\partial}{\partial y}$$

Restriction of Φ to the half-planes $x > 0$ and $x < 0$ is a diffeomorphism that pulls back $\big(X_0 + iY_0\big)|_{\{x>0\}}$ to a multiple of $L_1 = X_1 + iY_1$. Then each one of the k 1-orbits of L_0 contained in $x > 0$ is mapped by Φ^{-1} into a couple of 1-orbits of L_1, generating $2k$ 1-orbits for L_1. Furthermore, the restriction of L_1 to either $x > 0$ or $x < 0$, satisfies $(\mathcal{P})$. By the hypothesis made on L_0, $X_0 \wedge Y_0$ is, say, positive for $x = 0$ and this implies —writing $X_1 \wedge Y_1 = \beta(x,y)\partial_x \wedge \partial_y$— that $\beta(x,y) = x\gamma(x,y)$ with $\gamma(0,y) > 0$ and since β does not change sign on the complement of the closure of the bounded 2-orbits, it follows that $\gamma > 0$ everywhere on the complement of the closure of the bounded 2-orbits. Hence $\beta(x,y)$ will change sign across the y-axis which is contained in a 2-orbit. We now consider the vector field

$$L_2 = X_1 + ixY_1 \doteq X_1 + iY_2.$$

Since Y_2 is a nonvanishing multiple of Y_1 for $x \neq 0$, the 1-orbits of L_2 and L_1 are the same, in particular L_2 has exactly $2k$ closed 1-orbits. On the other hand, $X_1 \wedge Y_2 = x^2\gamma(x,y)\partial_x \wedge \partial_y$ will not change sign on any 2-orbit, because γ does not change sign on any 2-orbit.

For instance, by a translation to the right of the vector field $X + iY$ defined in Example 2, we may obtain an L_0 satisfying the required hypothesis with $k = 2$ and duplicate the number of closed 1-orbits to 4. This process can be continued.

Example 4. We will next describe a locally solvable, real analytic vector field with one compact and one noncompact orbit. Let $\rho_1(x,y) = x^2+y^2-1$ and $\rho_2(x,y) = x-3$. Define the real vector fields X and Y by

$$X = 4(1-x^2)\rho_2(x,y)^2\frac{\partial}{\partial x} - 2y\rho_2(x,y)(2x\rho_2(x,y)+\rho_1(x,y))\frac{\partial}{\partial y}$$

and

$$Y = -2y\rho_2(x,y)(2x\rho_2(x,y)+\rho_1(x,y))\frac{\partial}{\partial x}+(2x\rho_2(x,y)+\rho_1(x,y))^2\frac{\partial}{\partial y}.$$

Let $L = X+iY$. If $\rho_2(a,b) = 0$, then $Y(a,b) = \rho_1(a,b)^2\frac{\partial}{\partial y} = (8+b^2)\frac{\partial}{\partial y} \neq 0$. Suppose $\rho_2(a,b) \neq 0$. Then if $X(a,b) = 0$, either $|a| = 1$ and $b = 0$ or $|a| = 1$ and $2a\rho_2(a,b)+\rho_1(a,b) = 0$. Suppose first $|a| = 1$ and $b = 0$. Then

$$Y(a,b) = (2a\rho_2(a,b)+\rho_1(a,b))^2\frac{\partial}{\partial y} = (2-6a)^2\frac{\partial}{\partial y} \neq 0.$$

On the other hand if $|a| = 1$ and $2a\rho_2(a,b)+\rho_1(a,b) = 0$, then

$$0 = 2a\rho_2(a,b)+\rho_1(a,b) = 2-6a+b^2$$

and the latter equals zero when $|a| = 1$ only if $(a,b) = (1,-2)$ or $(a,b) = (1,2)$. Thus we see that the vector field L is nonzero away from these two points. We have: $X(\rho_1) = 0 = Y(\rho_1)$ on the set where $\rho_1 = 0$ and $X(\rho_2) = 0 = Y(\rho_2)$ on the set where $\rho_2 = 0$. It follows that the circle $\gamma_1 = \{(x,y) : \rho_1(x,y) = 0\}$ and the line $\gamma_2 = \{(x,y) : \rho_2(x,y) = 0\}$ are one-dimensional orbits of $L = X+iY$. Let Ω be a bounded, simply connected region containing $\{(x,y) : x^2+y^2 \leq 1\} \cup \{(3,y) : -3 \leq y \leq 3\}$ and such that $\Omega \cap \gamma_2$ is connected. We choose Ω so that the two points $(1,-2)$ and $(1,2)$ are not in Ω. We have

$$X \wedge Y = 4\rho_2(x,y)^2(2x\rho_2(x,y)+\rho_1(x,y))^2(1-x^2-y^2)\partial_x \wedge \partial_y.$$

Observe that the set $\sigma = \{(x,y) : 2x\rho_2(x,y)+\rho_1(x,y) = 0\}$ is a circle which intersects γ_1 at two points and is disjoint from γ_2. Since

$$X(2x\rho_2+\rho_1) = 24(1-x^2)\rho_2^2(x-1) - 4y^2\rho_2(2x\rho_2+\rho_1),$$

we see that X is transversal to σ except at the points $(1,2)$, $(1,-2)$ which are not in Ω. It follows that in Ω, the vector field L has 2 one-dimensional orbits, namely, γ_1 and $\gamma_2 \cap \Omega$, and three two-dimensional orbits: $\{(x,y) : x^2+y^2 < 1\}$, $\{(x,y) \in \Omega : x^2+y^2 > 1,\ x < 3\}$, and $\{(x,y) \in \Omega : x^2+y^2 > 1,\ x > 3\}$. Observe also that L is real analytic and locally solvable in Ω.

5. A local version of the Rudin-Carleson property

The next result characterizes those locally integrable, smooth vector fields which satisfy a local version of the Rudin-Carleson theorem.

Theorem 5.1. *Let L be a smooth vector field satisfying condition $(\mathcal{P})$ in an open set D. For each point $p \in D$ there is a neighborhood U_p such that if $Q \subset U_p$ is a rectangle, $E \subset \partial Q$ is a closed set with Lebesgue measure zero and $g \in C(\partial Q)$ is constant on the fibers of a first integral Z, then there is $h \in C(\overline{Q})$ satisfying:*

$$Lh = 0 \text{ in } Q, \quad h(z) = g(z)\ \forall z \in E \text{ and } sup\,|h| \leq 2\, sup\,|g|.$$

Conversely, given a locally integrable smooth vector field L on D, if there is a neighborhood W_p of each point $p \in D$ such that for every rectangle $Q \subset W_p$, and every closed set $E \subset \partial Q$ of Lebesgue measure zero and $g \in C(\partial Q)$ constant on the fibers of a first integral Z, there is $h \in C(\overline{Q})$ satisfying: $Lh = 0$ in Q, $h(z) = g(z)\ \forall z \in E$, then L satisfies condition $(\mathcal{P})$ in D.

Proof. Suppose L is a smooth vector field satisfying condition $(\mathcal{P})$. Then it is well known that it is locally integrable (see Theorem 3.2 in [T]). We may assume that in a rectangle $Q = (-A, A) \times (0, T)$, $L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x}$ and $Z(x,t) = x + i\varphi(x,t)$ is a first integral of L, with φ real-valued. The local solvability of L implies that for each $x \in (-A, A)$,

$$\text{the function} \quad t \longmapsto \varphi(x,t) \quad \text{is monotonic on } (0,T).$$

Suppose the set E and the function $g \in C(\partial Q)$ are as in the theorem. Let Ω be the union of the two-dimensional orbits of L in Q. We can write

$$\Omega = \bigcup_{j=1}^{N} (a_j, b_j) \times (0, T)$$

where $N \leq \infty$ and the union is a disjoint union. For each j, let $Q_j = (a_j, b_j) \times (0, T)$. For each j, set $E_j = ([a_j, b_j] \times \{0, T\} \cap E) \cup (\{a_j, b_j\} \times \{0, T\})$. Since L satisfies condition $(\mathcal{P})$, the set $Z(\overline{Q_j})$ is a simply connected set whose boundary is a rectifiable, simple closed curve. In particular, by a version of the Riemann mapping theorem, the classical Rudin-Carleson theorem applies to $Z(\overline{Q_j})$. Fix j. We can find points $p, q \in [a_j, b_j] \times \{0, T\}$ such that the oscillation of g on the set $[a_j, b_j] \times \{0, T\}$,

$$\mathrm{osc}_{[a_j,b_j]\times\{0,T\}}(g) = |g(p) - g(q)|.$$

Let G_j be continuous on $\overline{Z(Q_j)}$, holomorphic on the interior $Z(Q_j)$ such that $G_j(Z(x,t)) = g(x,t) - g(p)$ for $(x,t) \in E_j$ and

$$\sup_{Z(Q_j)} |G_j| \leq \sup_{[a_j,b_j]\times\{0,T\}} |g - g(p)|.$$

This is possible by the Rudin-Carleson theorem since g is constant on the fibers of Z. Observe that

$$\operatorname{osc}_{Z(Q_j)} G_j \leq 2 \operatorname{osc}_{[a_j,b_j]\times\{0,T\}}(g).$$

Define $F_j(z) = G_j(z) + g(p)$. Then F_j is continuous on $\overline{Z(Q_j)}$, holomorphic on the interior $Z(Q_j)$, $F_j(Z(x,t)) = g(x,t)$ for $(x,t) \in E_j$ and

$$\operatorname{osc}_{Z(Q_j)} F_j \leq 2 \operatorname{osc}_{[a_j,b_j]\times\{0,T\}}(g).$$

Define now

$$h(x,t) = \begin{cases} F_k(Z(x,t)), & \text{if } (x,t) \in \overline{Q_k} \\ g(x,0), & \text{if } (x,t) \notin \cup_i [a_i,b_i] \times [0,T]. \end{cases}$$

Observe that $h(x,t) = g(x,t)$ for $(x,t) \in E$. We will show next that h is continuous on $\overline{Q}$. Clearly h is continuous on Ω. Suppose $(x_0,t_0) \in \overline{Q}$ and $x_0 \notin \cup_i (a_i,b_i)$. Let $(x_k,t_k) \to (x_0,t_0)$. Suppose (x_{k_i},t_{k_i}) is any subsequence. If there is an infinite subset (y_m,t_m) of this subsequence which is disjoint from Ω, then $h(y_m,t_m) = g(y_m,0)$ and so by the continuity of g, $h(y_m,t_m) \to g(x_0,0) = h(x_0,t_0)$. If there is no such subsequence, without loss of generality, we may assume that for each k_i, there is k_i' such that $x_{k_i} \in [a_{k_i'}, b_{k_i'}]$. Assume first that $x_0 \notin \cup_{j=1}^N \{a_j,b_j\}$. Then $h(x_{k_i},t_{k_i}) = F_{k_i'}(Z(x_{k_i},t_{k_i}))$ and

$$\begin{aligned} |h(x_{k_i},t_{k_i}) - h(a_{k_i'},0)| &= |F_{k_i'}(Z(x_{k_i},t_{k_i})) - F_{k_i'}(Z(a_{k_i'},0))| \\ &\leq 2 \operatorname{osc}_{[a_{k_i'},b_{k_i'}]\times\{0,T\}}(g). \end{aligned}$$

Observe that $|a_{k_i'} - b_{k_i'}| \to 0$ because we are assuming that $x_0 \notin \cup_{j=1}^N [a_j,b_j]$. Since $g(a_{k_i'},0) = g(a_{k_i'},T), g(b_{k_i'},0) = g(b_{k_i'},T)$, and $|a_{k_i'} - b_{k_i'}| \to 0$, the oscillation $\operatorname{osc}_{[a_{k_i'},b_{k_i'}]\times\{0,T\}}(g)$ goes to zero as $k_i \to \infty$, and hence since

$$h(a_{k_i'},0) = g(a_{k_i'},0) \to g(x_0,0),$$

it follows that $h(x_{k_i},t_{k_i}) \to h(x_0,0) = h(x_0,t_0)$. We have shown that if $x_0 \notin \cup_{j=1}^N \{a_j,b_j\}$, and $(x_k,t_k) \to (x_0,t_0)$, then every subsequence of $h(x_k,t_k)$ has a further subsequence that converges to $h(x_0,t_0)$. It follows that h is continuous at (x_0,t_0) whenever $x_0 \notin \cup_{j=1}^N \{a_j,b_j\}$. Suppose now $x_0 \in \cup_{j=1}^N \{a_j,b_j\}$. Without loss of generality, assume $x_0 = a_i$ for some i. Then clearly h is continuous from the right (in x) at (x_0,t_0). If $(x_j,t_j) \to (x_0,t_0)$ with each $x_j < x_0$, we can consider subsequences of $h(x_j,t_j)$ as before to conclude that $h(x_j,t_j) \to h(x_0,t_0)$. We have thus shown that h is continuous on $\overline{Q}$.

We will next show that $Lh = 0$ in Q. Let $\psi(x,t) \in C_0^\infty(Q)$. Fix a two dimensional orbit $Q_j = (a_j,b_j) \times (0,T)$. For each sufficiently small

$\epsilon > 0$, let $\psi_\epsilon(x) \in C_0^\infty(a_j, b_j)$ such that $\psi_\epsilon(x) \equiv 1$ on $(a_j + \epsilon, b_j - \epsilon)$ and for some constant C independent of ϵ, $|\psi'_\epsilon(x)| \le C\,\epsilon^{-1}$. From the definition of h, it is clear that $Lh = 0$ in Q_j. We therefore have

$$
\begin{aligned}
0 &= \int_{Q_j} hL^t(\psi_\epsilon(x)\psi(x,t))\,dxdt \quad (\text{since } \psi_\epsilon(x)\psi(x,t) \in C_0^\infty(Q_j)) \\
&= \int_{Q_j} h(x,t)\psi_\epsilon(x)L^t\psi(x,t)\,dxdt \\
&\qquad - \int_{Q_j} h(x,t)a(x,t)\psi'_\epsilon(x)\psi(x,t)\,dxdt.
\end{aligned} \tag{5.1}
$$

Clearly, as $\epsilon \to 0$,

$$
\int_{Q_j} h(x,t)\psi_\epsilon(x)L^t\psi(x,t)\,dxdt \to \int_{Q_j} h(x,t)L^t\psi(x,t)\,dxdt. \tag{5.2}
$$

The function $\psi'_\epsilon(x)$ is supported on a set of measure at most 2ϵ and on the support of this function, $a(x,t) = O(\epsilon)$. Since $\psi'_\epsilon(x) = O(\epsilon^{-1})$, it follows that when $\epsilon \to 0$,

$$
\int_{Q_j} h(x,t)a(x,t)\psi'_\epsilon(x)\psi(x,t)\,dxdt \to 0. \tag{5.3}
$$

From (5.1)-(5.3), we conclude that

$$
\int_{Q_j} hL^t\psi(x,t)\,dxdt = 0
$$

and hence

$$
\int_{\Omega} hL^t\psi(x,t)\,dxdt = 0 \tag{5.4}
$$

where by definition, Ω was the union of the two-dimensional orbits of L in Q. Recall that $L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x}$. Let

$$
\mathcal{N} = \{x \in (-A, A) : a(x,t) \equiv 0,\ 0 \le t \le T\}
$$

and set

$$
\tilde{\mathcal{N}} = \{x \in \mathcal{N} : \frac{\partial a}{\partial x}(x,t) \equiv 0,\ 0 \le t \le T\}.
$$

The implicit function theorem implies that the set $\mathcal{N}\backslash\tilde{\mathcal{N}}$ is a countable set. Therefore, using this and (5.4), we have

$$\begin{aligned}\int_Q hL^t\psi(x,t)\,dxdt &= \int_0^T\int_{\mathcal{N}} h(x,t)L^t\psi(x,t)\,dxdt\\ &= \int_0^T\int_{\tilde{\mathcal{N}}} h(x,t)L^t\psi(x,t)\,dxdt\\ &= \int_{\tilde{\mathcal{N}}}\int_0^T h(x,t)L^t\psi(x,t)\,dtdx\\ &= -\int_{\tilde{\mathcal{N}}} h(x,0)\left(\int_0^T \frac{\partial\psi}{\partial t}(x,t)\,dt\right)dx\\ &\text{(since } h(x,t)\equiv h(x,0) \text{ for } x\in\mathcal{N})\\ &= 0.\end{aligned}$$

It follows that $Lh = 0$ in Q.

Conversely, suppose the locally integrable vector field L satisfies the Rudin-Carleson property for every smooth subdomain of a neighborhood of the origin. Let $Z(x,t) = x + i\varphi(x,t)$ be a first integral of L near the origin such that

$$\left|\frac{\partial\varphi}{\partial x}(x,t)\right| \le \frac{1}{2}. \tag{5.5}$$

Assume L does not satisfy property $(\mathcal{P})$. Then we may assume that for some A, $T > 0$, $x_0 \in (-A, A)$ and $0 < t_0 < T$,

$$\varphi(x_0,0) < \varphi(x_0,t_0), \quad \text{and} \quad \varphi(x_0,t_0) > \varphi(x_0,T). \tag{5.6}$$

By changing t_0 if necessary, and choosing T close enough to t_0, we may also assume that:

$$\varphi(x_0,t) \le \varphi(x_0,t_0) \quad \forall t\in[0,T] \quad \text{and } \varphi(x_0,0) < \varphi(x_0,T) < \varphi(x_0,t_0). \tag{5.7}$$

Let $\delta > 0$ such that

$$\varphi(x,0) < \varphi(x,T) < \varphi(x,t_0) \quad \text{whenever } |x-x_0| \le \delta. \tag{5.8}$$

We will reason in the rectangle $Q = [x_0-\delta, x_0+\delta] \times [0,T]$. Let $x_k \in [x_0-\delta, x_0+\delta]$ be a sequence converging to x_0. Let $E \subset \partial Q$ be a closed set with measure zero containing the sequence $\{(x_k,T)\}$. Choose $g \in C(E)$ such that

$$g(x_k,T) = 0\ \forall k \quad \text{and } g(p)\neq 0 \text{ for some } p\in E\cap\{(x,T): |x-x_0|<\delta\}. \tag{5.9}$$

Suppose now $h(x,t) \in C(Q)$, $Lh = 0$ in the interior of Q, and $h = g$ on the set E. Estimate (5.5) allows us to use the Baouendi-Treves approximation theorem ([BT], [BCH, p.53]) to produce a sequence of entire functions H_k such that $H_k(Z(x,t)) \to h(x,t)$ uniformly on Q. In particular, by (5.8), there is a connected open neighborhood V of the set $\{Z(x,T) : |x - x_0| \leq \delta\}$ on which the sequence $H_k(z)$ will converge to a holomorphic function H. Since $H(Z(x_k,T)) = 0$ for every k and $Z(x_k,T) \to Z(x_0,T) \in V$, we must have $H \equiv 0$ on V. But this contradicts the assumption that $h(p) = g(p) \neq 0$. Therefore, L does not have the Rudin-Carleson property on Q.

6. A link with the F. and M. Riesz theorem

In [B] Bishop proved an abstract theorem which permits a generalization of the Rudin-Carleson theorem to some situations where a version of the F. and M. Riesz theorem is valid. Bishop's theorem has been a key tool in the study of peak-interpolation sets for $A(\Omega)$ where Ω is a bounded domain in $\mathbb{C}^n$ (typically strictly pseudoconvex) and $A(\Omega)$ is the algebra of holomorphic functions on Ω that are continuous up to the boundary (see [Bh], [R2], [Na] and the references therein). We state here a strengthened version from [G] of the theorem proved in [B]:

Theorem 6.1. *(Theorem 12.5 in [G]) Let $C(X)$ be the uniformly-normed Banach space of all continuous complex-valued functions on a compact Hausdorff space X. Let B be a closed subspace of $C(X)$. Let $B^\perp$ consist of all (finite, complex-valued, Baire) measures μ on X such that $\int f\, d\mu = 0$ for all f in B. Let $\hat{\mu}$ be the regular Borel extension of the Baire measure μ. Let S be a closed subset of X with the property that $\hat{\mu}(T) = 0$ for every Borel subset T of S and every μ in $B^\perp$. Let f be a continuous complex-valued function on S and Δ a positive continuous function on X such that $|f(x)| \leq \Delta(x)$ for all x in S. Then there exists F in B with $|F(x)| \leq \Delta(x)$ for all x in X and $F(x) = f(x)$ for all x in S.*

If $X = \mathbb{T}$ equals the unit circle and B denotes the space of continuous functions on $\mathbb{T}$ which are restrictions of functions holomorphic on the unit disc D and continuous on the closure $\overline{D}$, then a measure μ on $\mathbb{T}$ is in $B^\perp$ if and only if it is the boundary value of a holomorphic function on D. By the F. and M. Riesz theorem, it follows that any $\mu \in B^\perp$ is absolutely continuous with respect to Lebesgue measure and so the preceding theorem implies the Rudin-Carleson theorem. The classical F. and M. Riesz theorem was generalized for solutions of locally integrable vector fields in the paper [BH2]. However, unlike the holomorphic case, there are two reasons why we cannot use the F. and M. Riesz property of a vector field together with Theorem 6.1 to deduce the Rudin-Carleson property. Given a vector field L in a neighborhood of $\overline{D}$, let $\mathcal{A}$ denote the subspace of $C(\partial D)$ which are restrictions of functions $u \in C(\overline{D})$ that satisfy $Lu = 0$ in D. In general, $\mathcal{A}$ is not a closed subspace of $C(\partial D)$. Moreover, if $\mu \in \mathcal{A}^{\perp}$, it may not be the boundary value of a solution of L in D. For example, if $M = \frac{\partial}{\partial y} + iy\frac{\partial}{\partial x}$ is the Mizohata vector field and μ is a measure on $\mathbb{T}$ which is of the form $\mu = \delta_{(0,1)} - \delta_{(0,-1)}$ where δ_p denotes the Dirac mass at p, then $\int_{\mathbb{T}} h\, d\mu = 0$ for every $h \in C(\overline{D})$ that satisfies $Mh = 0$ on D. Such a measure cannot be the boundary value of a solution of M. Thus for a general vector field, a measure that is orthogonal to the boundary values of continuous solutions may not be a boundary value of a solution and in fact, it may not be absolutely continuous with respect to Lebesgue measure. If a vector field L satisfies the hypotheses of Theorem 2.1, we have the following:

Corollary 6.2. *Suppose L is a vector field as in Theorem 2.1 defined on a neighborhood U of $\overline{D}$ and satisfying the equivalent conditions* (1), (2) *in the theorem. Let $\mathcal{A}$ denote the algebra of continuous functions h on $\overline{D}$ satisfying the equation $Lh = 0$ in D. Let μ be a complex Baire measure defined on ∂D with the property that*

$$\int_{\partial D} h\, d\mu = 0$$

for every $h \in \mathcal{A}$. If a closed set $E \subseteq \partial D$ has Lebesgue measure zero and it is disjoint from the one-dimensional orbits of L in U, then $\mu(E) = 0$.

Proof. Let F be a closed subset of E. Let P be a positive continuous function on ∂D such that:

(1) $P \equiv 1$ on F.
(2) For any $y \notin F$, $P(y) < 1$.

An application of Theorem 6.1 in the proof of Theorem 2.1 shows that there is $h \in \mathcal{A}$ that equals 1 on F and satisfies $|h(p)| < 1$ for $p \notin F$. By hypothesis, for each positive integer n, we have $\int h^n\, d\mu = 0$. Letting $n \to \infty$, we are led to conclude that $\mu(F) = 0$. By the regularity of the measure μ, it follows that $\mu(E) = 0$.

Let $L = \frac{\partial}{\partial y} + ix\frac{\partial}{\partial x}$. This vector field is locally solvable and the y-axis is a one-dimensional orbit. Therefore, if $u \in C(\overline{D})$ satisfies $Lu = 0$ in D, then it is constant on the $y-$axis. It follows that if $\mu = \delta_{(0,1)} - \delta_{(0,-1)}$, then $\int_{\partial D} u\, d\mu = 0$ for all such solutions. Note that L satisfies the hypotheses of Theorem 2.1 since it has no compact orbits. This example shows that in Corollary 6.2, the set E has to be disjoint from the one-dimensional orbits.

References

[BT] M. S. Baouendi and F. Treves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. of Math. **113** (1981), 387–421.

[BCH] S. Berhanu, J. Hounie, and P. Cordaro, *An introduction to involutive structures*, Cambridge University Press, 2008.

[BH1] S. Berhanu and J. Hounie, *A Rudin-Carleson theorem for planar vector fields, preprint* (2007).

[BH2] S. Berhanu and J. Hounie, *An F. and M. Riesz theorem for planar vector fields*, Math. Ann. **320** (2001), 463–485.

[BM] S. Berhanu and A. Meziani, *Global properties of a class of planar vector fields of infinite type*, Commun. in Partial Differential Equations **22** (1997), 99–142.

[Bh] G. Bharali, *On peak-interpolation manifolds for $A(\Omega)$ for convex domains in $\mathbb{C}^n$*, Transactions of the AMS **356** (2004), 4811–4827.

[B] E. Bishop, *A general Rudin-Carleson theorem*, Proceedings of the AMS **13** (1962), 140–143.

[C] L. Carleson, *Representations of continuous functions*, Math. Z. **66** (1957), 447–451.

[D] R. Doss, *Elementary proof of the Rudin-Carleson and the F. and M. Riesz theorems*, Proceedings of the AMS **82** (1981), 599–602.

[FK] H. Farkas and I. Kra, *Riemann Surfaces*, —2nd ed., Springer Verlag, 1991.

[G] T. Gamelin, *Uniform algebras (second edition)*, Chelsea Pub. Comp., 1984.

[Na] A. Nagel, *Smooth zero sets and interpolation sets for some algebras of holomorphic functions on strictly pseudoconvex domains*, Duke Math. J. **43** (1976), 323–348.

[NT] L. Nirenberg and F. Treves, *Solvability of a first order linear partial differential equation*, Comm. Pure Appl. Math. **16** (1963), 331–351.

[O] D. Oberlin, *A Rudin-Carleson theorem for uniformly convergent Taylor series*, Michigan Math. J. **27** (1980), 309–313.

[R] W. Rudin, *Boundary values of continuous analytic functions*, Proceedings of the AMS **7** (1956), 808–811.

[R2] W. Rudin, *Peak-interpolation sets of class C^1*, Pacific J. Math. **75** (1978), 267-279.

[S] H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Transactions of the AMS **180** (1973), 171–188.

[T] F. Treves, *Approximation and representation of functions and distributions annihilated by a system of complex vector fields*, Center Math. Ecole Polytechnique, Palaiseau, France (1981).

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