Liouville's theorem revisited

Ruy Tojeiro

Abstract: Liouville's theorem turns out to be equivalent to a rigidity theorem for isometric immersions of open subsets of Euclidean space of dimension $n \geq 3$ into the light cone of Minkowski space of dimension $(n+2)$. We give a short direct proof of this theorem, thus yielding a simple new proof of Liouville's theorem. Understanding where things go wrong in the case $n = 2$ leads to an interesting characterization of the complex exponential function.

1 Introduction

A fundamental result in conformal geometry is the following well-known theorem of Liouville [**Li**]:

Theorem 1 *Let $f\colon U \to \mathbb{R}^n$ be a conformal map defined on a connected open subset of Euclidean space $\mathbb{R}^n$ of dimension $n \geq 3$. Then $f = L|_U$ is the restriction to U of a similarity or the composition $f = I \circ L|_U$ of such a map with an inversion with respect to a sphere of unit radius.*

The importance of Liouville's theorem may be measured, if not by its strong implications in conformal geometry, by the amount of proofs available in the literature; see e.g. [**H-J**], [**Ja**], [**Ku**], [**Ma**], [**Ne**], [**Sp**] and [**Fr**]. Most of them, including the one for $n = 3$ known as the "classical" proof, split into two parts, in the first of which one proves that a conformal map in dimension $n \geq 3$ has the property that (pieces of) spheres and affine subspaces are carried into (pieces of) spheres or affine subspaces. The proof is then completed by a lemma due to Möbius, according to which this property implies the conclusion of the theorem; see e.g. [**Sp**], v.III, p. 310.

In this article we use a different approach to Liouville's theorem, based on the fact that conformal maps on open subsets of $\mathbb{R}^n$ are in correspondence with isometric immersions of these subsets into the the light cone of Minkowski $(n+2)$-dimensional space. We provide an elementary account of this correspondence and establish the equivalence of Liouville's theorem and a rigidity theorem for such isometric immersions (see Theorem 5 below). Then we give a short direct proof of this theorem. Besides yielding a simple proof of Liouville's theorem, some of the underlying ideas have shown good potential for generalizations; for instance, they have been recently used in [**To**$_1$] and [**To**$_2$] to study conformal immersions into Euclidean space of Riemannian and warped products of Riemannian manifolds.

We have included a section where we discuss R. Nevanlina's proof of Liouville's theorem [**Ne**] in the light of the ideas developed in this article.

Understanding where things go wrong in the case $n = 2$ leads to the following interesting characterization of the complex exponential function:

Theorem 2 *Let $f: U \to \mathbb{R}^2$ be a conformal map defined on the connected open subset $U \subset \mathbb{R}^2$. Assume that one family of coordinate curves is mapped by f into a family of (pieces of) circles or straight lines. Then there exist an inversion I with respect to a circle of unit radius, a similarity L and a composition H of a dilation, a translation and reflections in the coordinate axes and the line $y = x$, such that $f = I \circ L \circ \exp \circ H|_U$, or else f is such a composition with possibly some of its components replaced by the identity map.*

Acknowledgment. We are grateful to the referee for his careful reading of the manuscript as well as for several useful and appropriate comments and remarks. We also would like to acknowledge our indebtedness to the referee of our paper [**To**$_1$], some of whose suggestions have also been useful here.

2 Conformal geometry in the light cone

If $\mathbb{R}^{n+2}$ is endowed with a Lorentz scalar product

$$\langle\langle v, w\rangle\rangle = -v_0w_0 + v_1w_1 + \ldots + v_{n+1}w_{n+1},$$

for $v = (v_0, \ldots, v_{n+1})$ and $w = (w_0, \ldots, w_{n+1})$, then it becomes the $(n+2)$-dimensional *Minkowski space*, and is denoted by $\mathbb{L}^{n+2}$. A vector $v \in \mathbb{L}^{n+2}$ is said to be *space-like*, *light-like* or *time-like* according as $\langle\langle v, v\rangle\rangle > 0$, $\langle\langle v, v\rangle\rangle = 0$ or $\langle\langle v, v\rangle\rangle < 0$, respectively. The same terminology is used for a subspace $V \subset \mathbb{L}^{n+2}$, depending on whether the restriction of $\langle\langle\ ,\ \rangle\rangle$ to V is positive-definite, degenerate (i.e., $V \cap V^\perp \neq \{0\}$) or Lorentzian, respectively. The set of light-like vectors

$$\mathbb{V}^{n+1} = \{p \in \mathbb{L}^{n+2}: \langle\langle p, p\rangle\rangle = 0\}$$

is called the *light cone* of $\mathbb{L}^{n+2}$. The intersection

$$\mathbb{E}^n = \mathbb{E}^n_w = \{p \in \mathbb{V}^{n+1}: \langle\langle p, w\rangle\rangle = 1\}$$

of $\mathbb{V}^{n+1}$ with the affine hyperplane $\langle\langle p, w\rangle\rangle = 1$ is a model of n–dimensional Euclidean space for any $w \in \mathbb{V}^{n+1}$. Namely, fix $p_0 \in \mathbb{E}^n$ and a linear isometry $A: \mathbb{R}^n \to \{p_0, w\}^\perp$. Then the map $\Psi = \Psi_{p_0,w,A}: \mathbb{R}^n \to \mathbb{E}^n \subset \mathbb{L}^{n+2}$ given by

$$x \in \mathbb{R}^n \mapsto p_0 + A(x) - (1/2)|x|^2 w$$

is an isometry, as follows by computing

$$d\Psi(x)X = A(X) - \langle X, x\rangle w \text{ for all } x, X \in \mathbb{R}^n. \quad (1)$$

We call (p_0, w, A) an *admissible triple*. Notice that if (p_0, w, A) and $(\bar{p}_0, \bar{w}, \bar{A})$ are admissible triples, then the linear map given by $T(p_0) = \bar{p}_0$, $T(w) = \bar{w}$ and $T \circ A = \bar{A}$ is in $\mathbb{O}_1(n+2)$, that is, is orthogonal with respect to $\langle\langle\, ,\, \rangle\rangle$, and satisfies $T \circ \Psi_{p_0,w,A} = \Psi_{\bar{p}_0,\bar{w},\bar{A}}$.

We also obtain from (1) that the normal space $T_\Psi M_x^\perp$ of Ψ at any $x \in \mathbb{R}^n$, as an isometric immersion into $\mathbb{L}^{n+2}$, is the time-like plane spanned by $\Psi(x)$ and w. Moreover, denoting by $\bar{\nabla}$ and $\tilde{\nabla}$ the usual derivatives in $\mathbb{R}^n$ and $\mathbb{L}^{n+2}$, respectively, we obtain

$$\begin{aligned} \tilde{\nabla}_Y d\Psi(X) &= d\Psi \bar{\nabla}_Y X + \langle\langle \tilde{\nabla}_Y d\Psi(X), w\rangle\rangle \Psi + \langle\langle \tilde{\nabla}_Y d\Psi(X), \Psi\rangle\rangle w \\ &= d\Psi \bar{\nabla}_Y X - \langle X, Y\rangle w, \end{aligned} \quad (2)$$

hence the second fundamental form $\alpha_\Psi \colon TM \times TM \to T_\Psi^\perp M$ of Ψ is given by

$$\alpha_\Psi(X, Y) := \tilde{\nabla}_Y d\Psi(X) - d\Psi \bar{\nabla}_Y X = -\langle X, Y\rangle w. \quad (3)$$

Here, and throughout the paper, we use the abuse of notation of denoting also by $\tilde{\nabla}$ the pulled-back connection on the pulled-back bundle $\Psi^* T\mathbb{L}^{n+2}$, that is, $\tilde{\nabla}$ will also denote the induced derivative of vector fields in $\mathbb{L}^{n+2}$ "along Ψ" (that is, sections of $\Psi^* T\mathbb{L}^{n+2}$) with respect to vector fields in M.

2.1 The space of spheres

Hyperspheres in Euclidean space $\mathbb{R}^n$ have a neat description in its model $\mathbb{E}^n$ (see e.g. [**Be**], Chapter 20, vol. 2): let $S \subset \mathbb{R}^n$ be a hypersphere with (constant) mean curvature h with respect to a unit normal vector field n. Differentiating the map $\rho \colon S \to \mathbb{L}^{n+2}$ given by $\rho(x) = d\Psi(x)n(x) + h\Psi(x)$ and using (2) we get

$$d\rho(X) = d\Psi \bar{\nabla}_X n + h d\Psi(X) = d\Psi(-hX) + h d\Psi(X) = 0,$$

hence ρ is a constant unit space-like vector $v \in \mathbb{L}^{n+2}$ with $\langle\langle \Psi(x), v\rangle\rangle = 0$ for all $x \in S$. It follows that $\Psi(S) = \mathbb{E}^n \cap \{v\}^\perp$, and from now on we write $\Psi(S) = S$ for short. Observe that $S = \mathbb{E}^n \cap \{v\}^\perp$ is an affine hyperplane iff $0 = h = \langle\langle v, w\rangle\rangle$. Notice also that changing the unit normal vector field n by a sign, and hence the corresponding mean curvature h, makes the unit space-like vector v also to change its sign. Thus, unit space-like vectors in $\mathbb{L}^{n+2}$ are in one-to-one correspondence with *oriented* hyperspheres or affine hyperplanes of $\mathbb{R}^n$, hence the space of oriented hyperspheres and affine hyperplanes of $\mathbb{R}^n$ is naturally identified in this way with *de Sitter space* $\mathbb{S}_1^{n+1}$ of all unit space-like vectors of $\mathbb{L}^{n+2}$.

The relative position of two hyperspheres has a simple description in this model: given hyperspheres or affine subspaces $S_i = \mathbb{E}^n \cap \{v_i\}^\perp$, $1 \le i \le 2$, then they intersect transversally, have a unique common point (or are two parallel affine hyperplanes) or do not intersect iff the subspace spanned by v_1 and v_2 is space-like, degenerate or time-like, respectively. In the first case, if n_x^1 and n_x^2 are the unit normal vectors of S_1 and S_2, respectively, at $x \in S_1 \cap S_2$, then $\langle n_x^1, n_x^2 \rangle = \langle\langle v_1, v_2 \rangle\rangle$. In particular, S_1 and S_2 intersect orthogonally iff $\langle\langle v_1, v_2 \rangle\rangle = 0$.

Let $S = \mathbb{E}^n \cap \{v\}^\perp$ be a hypersphere with (Euclidean) center x_0 and radius r, oriented by its inward pointing unit normal vector field $n(x) = (x_0 - x)/r$, with corresponding mean curvature $h = 1/r$. Using (1), we obtain that $v = d\Psi(x)n(x) + h\Psi(x)$, $x \in S$, is given by

$$v = \frac{1}{r}\Psi(x_0) + \frac{r}{2}w. \tag{4}$$

In particular, if $\mathcal{F} = (S^\lambda)_{\lambda \in \Lambda}$ is a family of concentric hyperspheres and $S^\lambda = \mathbb{E}^n \cap \{v^\lambda\}^\perp$ for unit space-like vectors v^λ, then the subspace $V_\mathcal{F} \subset \mathbb{L}^{n+2}$ spanned by all v^λ is a two-dimensional time-like subspace spanned by the light-like vectors w and $\Psi(x_0) \in \mathbb{E}^n$, where x_0 is the common center of all S^λ. Conversely, if $\mathcal{F}$ is a family of hyperspheres whose associated subspace $V_\mathcal{F}$ is a two-dimensional time-like subspace containing w, then $\mathcal{F}$ is a family of concentric hyperspheres whose common center is the point $x_0 \in \mathbb{R}^n$ such that $\Psi(x_0)$ is the unique light-like vector in $\mathbb{E}^n \cap V_\mathcal{F}$.

On the other hand, if $S = \mathbb{E}^n \cap \{v\}^\perp$ is an affine hyperplane oriented by a unit normal vector n, then $v = d\Psi(x)n$, $x \in S$, is given by $v = A(n) - cw$, where $c \in \mathbb{R}$ is the constant value of $\langle n, x \rangle$, $x \in S$. Therefore, for a family $\mathcal{F}$ of parallel affine hyperplanes the corresponding subspace $V_\mathcal{F}$ is a two-dimensional light-like subspace containing w. Conversely, any family $\mathcal{F}$ of hyperspheres or affine hyperplanes having a two-dimensional light-like subspace containing w as its associated subspace $V_\mathcal{F}$ is a family of parallel affine hyperplanes.

2.2 Conformal maps into $\mathbb{R}^n$ as isometric immersions into $\mathbb{V}^{n+1}$

Given a *conformal immersion* $G\colon M \to \mathbb{V}^{n+1}$ with *conformal distortion* $\varphi\colon M \to \mathbb{R}_+^* := \{t \in \mathbb{R} : t > 0\}$ of a Riemannian manifold M, which means that

$$\langle\langle dG(p)X, dG(p)Y \rangle\rangle = \varphi^2 \langle X, Y \rangle \text{ for all } p \in M \text{ and } X, Y \in T_pM,$$

then for any smooth function $\mu\colon M \to \mathbb{R}_+^*$ the map

$$G_\mu\colon M \to \mathbb{V}^{n+1}, \ \ p \mapsto \mu(p)G(p),$$

is also conformal with conformal distortion $\mu\varphi$: since $dG_\mu(X) = d\mu(X)G + \mu dG(X)$, we get

$$\langle\langle dG_\mu(X), dG_\mu(Y) \rangle\rangle = \mu^2 \langle\langle dG(X), dG(Y) \rangle\rangle,$$

because $\langle\langle G, G \rangle\rangle = 0$ whence $\langle\langle dG(X), G \rangle\rangle = 0$ for any $X \in TM$.

In particular, any conformal immersion $f\colon\, M \to \mathbb{R}^n$ with conformal distortion $\varphi\colon M \to \mathbb{R}^*_+$ gives rise to an isometric immersion

$$\mathcal{I}(f) = \mathcal{I}_{p_0,w,A}(f) := (\Psi \circ f)_{\varphi^{-1}}\colon\, M \to \mathbb{V}^{n+1}.$$

Conversely, if $F\colon\, M \to \mathbb{V}^{n+1}$ is an isometric immersion whose image does not intersect the line $\mathbb{R}w := \{tw : t \in \mathbb{R}\}$, define $\mathcal{C}(F) = \mathcal{C}_{p_0,w,A}(F)\colon M \to \mathbb{R}^n$ by

$$\Psi \circ \mathcal{C}(F) = \Pi \circ F,$$

where $\Pi = \Pi_w\colon\, \mathbb{V}^{n+1} \setminus \mathbb{R}_w \to \mathbb{E}^n_w$ is the projection onto $\mathbb{E}^n_w$ given by $\Pi(p) = p/\langle\langle p, w\rangle\rangle$. Since Π is easily checked to be conformal with conformal distortion $\varphi_\Pi(p) = \langle\langle p, w\rangle\rangle^{-1}$, it follows that $\mathcal{C}(F)$ is also conformal with conformal distortion $\varphi_\Pi \circ F = \langle\langle F, w\rangle\rangle^{-1}$.

Clearly, we have $\mathcal{C}_{p_0,w,A}(\mathcal{I}_{p_0,w,A}(f)) = f$ and $\mathcal{I}_{p_0,w,A}(\mathcal{C}_{p_0,w,A}(F)) = F$ for any conformal immersion $f\colon\, M \to \mathbb{R}^n$ and for any isometric immersion $F\colon\, M \to \mathbb{V}^{n+1}$ with $F(M) \subset \mathbb{V}^{n+1} \setminus \mathbb{R}w$.

2.3 $T \in \mathbb{O}_1(n+2)$ as a conformal map in $\mathbb{R}^n$

Throughout this subsection we assume that an admissible triple (p_0, w, A) has been fixed and omit the corresponding subscripts for simplicity of notation. By the discussion in the preceding subsection, any $T \in \mathbb{O}_1(n+2)$ gives rise to a conformal map $\mathcal{C}(T \circ \Psi)$ in $\mathbb{R}^n$ (minus the unique point in $(T \circ \Psi)^{-1}(\mathbb{R}w)$ if Tw and w are not colinear). We will show that the conformal maps so obtained are precisely the compositions $I \circ L$ of a similarity and an inversion with respect to a hypersphere (which can always be taken with unit radius). We start with some special cases.

Proposition 3 *The following holds:*

(i) If $R \in \mathbb{O}_1(n+2)$ is the reflection $R(p) = p - 2\langle\langle p, v\rangle\rangle v$ with respect to the hyperplane in $\mathbb{L}^{n+2}$ orthogonal to the unit space-like vector v, with $\langle\langle v, w\rangle\rangle \neq 0$, then

$$\mathcal{C}(R \circ \Psi) = I \tag{5}$$

is the inversion with respect to the hypersphere $S = \mathbb{E}^n \cap \{v\}^\perp$.

(ii) If $G \in \mathbb{O}_1(n+2)$ satisfies $G(w) = \lambda w$ for some $\lambda \in \mathbb{R}^*_+$, then

$$\mathcal{C}(G \circ \Psi) = L \tag{6}$$

for some similarity L of ratio λ. Conversely, given any similarity L of ratio $\lambda \in \mathbb{R}^*_+$ there exists $G \in \mathbb{O}_1(n+2)$ satisfying $G(w) = \lambda w$ such that (6) holds. In particular, isometries of $\mathbb{R}^n$ correspond in this way to the elements of $\mathbb{O}_1(n+2)$ that fix w.

Proof: (i) Writing v as in (4) in terms of the center x_0 and radius r of $S = \mathbb{E}^n \cap \{v\}^\perp$, a straightforward computation yields

$$R \circ \Psi(x) = \Psi(x) - 2\langle\langle \Psi(x), v \rangle\rangle v = \frac{|x - x_0|^2}{r^2}(p_0 + A(I(x)) - \frac{1}{2}|I(x)|^2 w),$$

where

$$I(x) = x_0 + r^2 \frac{(x - x_0)}{|x - x_0|^2}, \quad x \neq x_0,$$

is the inversion with respect to S. Thus $\Pi_w \circ R \circ \Psi = \Psi \circ I$, which gives (5).

(ii) The map $L = \Psi^{-1} \circ \Pi_w \circ T \circ \Psi \colon \mathbb{R}^n \to \mathbb{R}^n$ has conformal distortion λ, hence is a similarity of ratio λ. Thus $\Pi_w \circ T \circ \Psi = \Psi \circ L$, which yields (6). For the converse we use that any similarity L of $\mathbb{R}^n$ of ratio λ is given by $L(x) = \lambda B(x) + x_0$ for some $x_0 \in \mathbb{R}^n$ and some $B \in \mathbb{O}(n)$. Define

$$\bar{p_0} = \frac{1}{\lambda}(p_0 + Ax_0 - \frac{1}{2}|x_0|^2 w)$$

and $\bar{w} = \lambda w$. Then $\bar{p_0}, \bar{w} \in \mathbb{V}^{n+1}$ and $\langle\langle \bar{p_0}, \bar{w} \rangle\rangle = 1$. Moreover, $\bar{A} \colon \mathbb{R}^n \to \mathbb{L}^{n+2}$ given by

$$\bar{A}(x) = A(B(x)) - \langle B(x), x_0 \rangle w$$

is a linear isometry onto $\{\bar{p_0}, \bar{w}\}^\perp$, hence $(\bar{p_0}, \bar{w}, \bar{A})$ is an admissible triple. Let $G \in \mathbb{O}_1(n+2)$ be defined by $G(p_0) = \bar{p_0}$, $G(w) = \bar{w}$ and $G \circ A = \bar{A}$. Then it is easily checked that

$$\Psi(L(x)) = p_0 + A(L(x)) - \frac{1}{2}|L(x)|^2 w = \lambda G(\Psi(x)),$$

which is equivalent to (6). ∎

We now consider the general case.

Proposition 4 *For any $T \in \mathbb{O}_1(n+2)$ there exists a composition $I \circ L$ of a similarity L and an inversion I with respect to a hypersphere of unit radius (possibly with I replaced by the identity map) such that*

$$\mathcal{C}(T \circ \Psi) = I \circ L. \tag{7}$$

Conversely, given any composition $I \circ L$ of a similarity and an inversion, there exists $T \in \mathbb{O}_1(n+2)$ such that (7) holds.

Proof: Define $(\bar{p_0}, \bar{w}, \bar{A})$ by $\bar{p_0} = T(p_0)$, $\bar{w} = T(w)$ and $\bar{A} = T \circ A$. If $\bar{w} = \lambda w$ for some $\lambda \in \mathbb{R}_+^*$, the statement follows from Proposition 3-(ii) (with I replaced by the identity map). Otherwise, consider the reflection $R(p) = p - 2\langle\langle p, v \rangle\rangle v$ determined by the unit space-like vector $v = \langle\langle \bar{w}, w \rangle\rangle^{-1}\bar{w} + (1/2)w$, and let $G \in \mathbb{O}_1(n+2)$ be given by

$$G(w) = R(\bar{w}) = -(1/2)\langle\langle \bar{w}, w \rangle\rangle w, \quad G(p_0) = R(\bar{p_0}) \text{ and } G \circ A = R \circ \bar{A}.$$

Then $R \circ G$ takes w to $\bar{w}$, p_0 to $\bar{p}_0$ and $R \circ G \circ A = \bar{A}$, whence $R \circ G = T$. By Proposition 3-(i), the map $\mathcal{C}(R \circ \Psi) = I$ is an inversion with respect to the hypersphere of unit radius $S = \mathbb{E}^n \cap \{v\}^\perp$, whereas $\mathcal{C}(G \circ \Psi) = L$ is a similarity of ratio $\lambda = -(1/2)\langle\langle \bar{w}, w \rangle\rangle$ by Proposition 3-(ii). Then (7) follows from

$$\Pi_w \circ T \circ \Psi = \Pi_w \circ R \circ G \circ \Psi = \Pi_w \circ R \circ \Psi \circ L = \Psi \circ I \circ L. \tag{8}$$

For the converse, let R and G correspond to I and L by parts (i) and (ii) of Proposition 3, respectively, and set $T = R \circ G$. Then (7) holds, as follows again from (8). ∎

3 A rigidity theorem

In this section we prove the following rigidity result for isometric immersions $F\colon U \subset \mathbb{R}^n \to \mathbb{V}^{n+1}$, $n \geq 3$, and show that it is equivalent to Liouville's theorem.

Theorem 5 *Let $F\colon U \to \mathbb{V}^{n+1}$ be an isometric immersion of a connected open subset of $\mathbb{R}^n$, $n \geq 3$. Then $F = \Psi_{\bar{p}_0,\bar{w},\bar{A}}|_U$ for some admissible triple $(\bar{p}_0, \bar{w}, \bar{A})$.*

Thus, if an admissible triple (p_0, w, A) is fixed, then Theorem 5 states that any isometric immersion $F\colon U \to \mathbb{V}^{n+1}$ is given by $F = T \circ \Psi_{p_0,w,A}|_U$ for some $T \in \mathbb{O}_1(n+2)$. In other words, the isometric immersion $\Psi_{p_0,w,A}|_U$ is *rigid*, that is, it is unique up to compositions with orthogonal linear transformations of $\mathbb{L}^{n+2}$.

First we prove the equivalence with Liouville's theorem.

3.1 Equivalence between Liouville's theorem and Theorem 5

Let $f\colon U \to \mathbb{R}^n$ be a conformal map on a connected open subset of $\mathbb{R}^n$, $n \geq 3$. Choose some admissible triple (p_0, w, A) and set $F = \mathcal{I}_{p_0,w,A}(f)\colon U \to \mathbb{V}^{n+1}$. Assuming Theorem 5 we obtain that $F = T \circ \Psi_{p_0,w,A}|_U$ for some $T \in \mathbb{O}_1(n+2)$. Then

$$f = \mathcal{C}_{p_0,w,A}(F) = \mathcal{C}_{p_0,w,A}(T \circ \Psi_{p_0,w,A})|_U,$$

and the conclusion of Liouville's theorem follows from Proposition 4.

Conversely, given an isometric immersion $F\colon U \to \mathbb{V}^{n+1}$ of a connected open subset of $\mathbb{R}^n$, $n \geq 3$, set $f = \mathcal{C}_{p_0,w,A}(F)$ for some admissible triple (p_0, w, A). By Liouville's theorem, either $f = L|_U$ is the restriction to U of a similarity or the composition $f = I \circ L|_U$ of such a map with an inversion with respect to a sphere of unit radius. It follows from Proposition 4 that $f = \mathcal{C}_{p_0,w,A}(T \circ \Psi_{p_0,w,A})|_U$ for some $T \in \mathbb{O}_1(n+2)$, hence

$$F = \mathcal{I}_{p_0,w,A}(f) = T \circ \Psi_{p_0,w,A}|_U.$$

3.2 Proof of Theorem 5

The key part of the proof is the following lemma.

Lemma 6 *F is umbilic, that is, there exists a normal vector field $\bar{w}$ such that*

$$\alpha_F(X,Y) = -\langle X,Y\rangle \bar{w}, \ \textit{for all } X,Y \in \mathbb{R}^n. \tag{9}$$

Proof: It suffices to prove that

$$\alpha_F(X,Y) = 0 \ \text{for all } X,Y \in \mathbb{R}^n \text{ with } \langle X,Y\rangle = 0.$$

First notice that the normal space of F at any $x \in U$ is a time-like plane that contains the position vector $F(x)$, as follows by differentiating $\langle\langle F,F\rangle\rangle = 0$, which gives $\langle\langle dF(X),F\rangle\rangle = 0$ for any $X \in \mathbb{R}^n$. Differentiating once more yields

$$\langle\langle \alpha_F(X,Y),F\rangle\rangle = \langle\langle \tilde{\nabla}_X dF(Y),F\rangle\rangle = -\langle X,Y\rangle \tag{10}$$

for all $X,Y \in \mathbb{R}^n$, where $\tilde{\nabla}$ denotes the derivative of $\mathbb{L}^{n+2}$. Now fix $x \in U$ and $X \in \mathbb{R}^n$, denote by $\mathcal{H}$ the affine hyperplane through x orthogonal to X and define $\xi\colon \mathcal{H}\cap U \to \mathbb{L}^{n+2}$ by $\xi = dF(X)$. Then, for any $Y \in \mathbb{R}^n$ orthogonal to X we have from (10) that

$$d\xi(Y) = \alpha_F(X,Y) = \omega_X(Y)F \tag{11}$$

for some one-form ω_X on $\mathcal{H}\cap U$. Regard $\Theta = d\xi$ as a one-form on $\mathcal{H}\cap U$ with values in $\mathbb{L}^{n+2}$. Then, its exterior derivative $d\Theta(Y,Z) = \tilde{\nabla}_Y\Theta(Z) - \tilde{\nabla}_Z\Theta(Y) - \Theta([Y,Z])$ satisfies

$$0 = d\Theta(Y,Z) = d\omega_X(Y,Z)F - \omega_X(Y)dF(Z) - \omega_X(Z)dF(Y).$$

Taking linearly independent vectors $Y,Z \in \{X\}^\perp$ (here we use $n \geq 3$!) and using that $dF(Y)$, $dF(Z)$ and F are linearly independent since F is an immersion and the position vector F is a nonzero normal vector field, we get $\omega_X(Y) = 0 = \omega_X(Z)$. Thus $\omega_X = 0$. ∎

It is now an easy task to complete the proof of Theorem 5. Actually, we argue in two different ways. The second argument is included because it will lead us in the last section to the characterization in Theorem 2 of the exponential function when $n = 2$.

3.2.1 First argument

We first show that $\bar{w}$ is a light-like vector field. This follows from (9) and the Gauss equation of F,

$$\langle\langle \alpha_F(X,X),\alpha_F(Y,Y)\rangle\rangle - \|\alpha_F(X,Y)\|^2 = K(X,Y) = 0,$$

where $K(X,Y)$ denotes the sectional curvature for a two-plane spanned by vectors X and Y. Now, from (9) and (10) we get

$$-\langle X,Y\rangle\langle\langle\bar{w},F\rangle\rangle = \langle\langle\alpha_F(X,Y),F\rangle\rangle = -\langle X,Y\rangle$$

for all $X,Y\in\mathbb{R}^n$, thus $\langle\langle\bar{w},F\rangle\rangle = 1$ everywhere. We show next that $\bar{w}$ is in fact a constant vector field. First, $\tilde{\nabla}_X\bar{w}$ has no tangent component:

$$\langle\langle\tilde{\nabla}_X\bar{w},dF(Y)\rangle\rangle = -\langle\langle\alpha_F(X,Y),\bar{w}\rangle\rangle = 0 \text{ for all } Y\in\mathbb{R}^n.$$

On the other hand, its normal component is $\langle\langle\tilde{\nabla}_X\bar{w},F\rangle\rangle\bar{w}+\langle\langle\tilde{\nabla}_X\bar{w},\bar{w}\rangle\rangle F=0$, as follows by differentiating $\langle\langle\bar{w},\bar{w}\rangle\rangle=0$ and $\langle\langle\bar{w},F\rangle\rangle=1$. We conclude that $F(U)\subset\mathbb{E}^n_{\bar{w}}$. Choosing any $\tilde{p}_0\in\mathbb{E}^n_{\bar{w}}$ and any linear isometry $\tilde{A}\colon\mathbb{R}^n\to\{\tilde{p}_0,\bar{w}\}^\perp$, we obtain that $\Psi^{-1}_{\tilde{p}_0,\bar{w},\tilde{A}}\circ F$ is the restriction to U of an isometry $H\colon\mathbb{R}^n\to\mathbb{R}^n$, hence by Proposition 3-(ii) there exists $G\in\mathbb{O}_1(n+2)$ fixing $\bar{w}$ such that

$$F=\Psi_{\tilde{p}_0,\bar{w},\tilde{A}}\circ H|_U = G\circ\Psi_{\tilde{p}_0,\bar{w},\tilde{A}}|_U = \Psi_{\bar{p}_0,\bar{w},\bar{A}}|_U$$

for $\bar{p}_0=G\tilde{p}_0$ and $\bar{A}=G\circ\tilde{A}$. ∎

Remark 7. After proving Lemma 6 and the fact that the vector field $\bar{w}$ is a constant light-like vector, the conclusion of the theorem could also be obtained from the Fundamental theorem of submanifolds of Minkowski space, applied to F and $\Psi=\Psi_{p_0,w,A}|_U$. In fact, we have a vector bundle isometry $\tau\colon T_\Psi M^\perp\to T_F M^\perp$, given by $\tau(\Psi)=F$ and $\tau(w)=\bar{w}$, that preserves second fundamental forms, because of (3) and (9), and normal connections, for these vanish identically. Thus $F=T\circ\Psi$ for some $T\in\mathbb{O}_1(n+2)$.

3.2.2 Second argument

Applying Lemma 6 for the coordinate vector fields we obtain

$$\frac{\partial^2 F}{\partial u_j\partial u_i}(u)=0 \text{ for all } u\in U \text{ and for all } i,j=1,\dots,n \text{ with } i\neq j, \tag{12}$$

hence

$$\langle\langle\frac{\partial F}{\partial u_i}(u),\frac{\partial F}{\partial u_j}(v)\rangle\rangle=0 \text{ for all } u,v\in U \text{ and for all } i,j=1,\dots,n \text{ with } i\neq j. \tag{13}$$

By the connectedness of U, it clearly suffices to prove the statement for the restriction of F to an arbitrary product $C=\Pi_{j=1}^n I_j\subset U$ of open intervals $I_j\subset\mathbb{R}$. Define linear subspaces $W_i\subset\mathbb{L}^{n+2}$ by

$$W_i=\operatorname{span}\left\{\frac{\partial F}{\partial u_i}(u)\ :\ u\in C\right\}$$

for $i = 1, \ldots, n$. Then (13) implies that $W_1, \ldots, W_n$ are mutually orthogonal. Since they clearly have dimension at least two and $2n > n + 2$ (here we use $n \geq 3$), they can not all be non-degenerate subspaces. Thus, we can assume that $W_1, \ldots, W_k$ are degenerate while $W_{k+1}, \ldots, W_n$ are non-degenerate subspaces for some $k \leq n$. Then there exists a light-like line L_0 such that $W_i \cap W_i^{\perp} = L_0$ for $i = 1, \ldots, k$. Choose a second, distinct light-like line L_1 orthogonal to $W_{k+1}, \ldots, W_n$ and set $\hat{W}_i = W_i \cap L_1^{\perp}$, so that $\hat{W}_i = W_i$ for $i > k$. Then $\hat{W}_i$ is a space-like subspace for each $i = 1, \ldots, n$, whose dimension is at least 1 for $i \leq k$ and at least 2 otherwise, which implies that the subspace $L_0 \oplus \hat{W}_1 \oplus \cdots \oplus \hat{W}_n \oplus L_1$ has dimension at least $2 + k + 2(n - k)$. Since this can not exceed $n + 2$ we get $k = n$. Thus, we have a decomposition

$$\mathbb{L}^{n+2} = L_0 \oplus \hat{W}_1 \oplus \cdots \oplus \hat{W}_n \oplus L_1,$$

and corresponding projections $P_i \colon \mathbb{L}^{n+2} \to \hat{W}_i$. For $i = 1, \ldots, n$, we have that $P_i \circ F$ is constant on $C_i = \Pi_{j \neq i} I_j$ while the component of F in L_1 is constant. Fix $u^0 = (u_1^0, \ldots, u_n^0) \in C$ and define $F_i \colon I_i \to \hat{W}_i$ by $F_i = P_i \circ F \circ j_i^{u^0}$, where $j_i^{u^0} \colon I_i \to C$ denotes the inclusion of I_i into C given by $u_i \mapsto (u_1^0, \ldots, u_i, \ldots, u_n^0)$. Then we may choose unit space-like vectors $v_1, \ldots, v_n$ spanning $\hat{W}_1, \ldots, \hat{W}_n$, respectively, such that $F_i(u_i) = (u_i + a_i)v_i$ for some $a_i \in \mathbb{R}$, $1 \leq i \leq n$, and

$$F = \tilde{p}_0 + \sum_{i=1}^{k} F_i \circ \pi_i + \langle\langle F, \tilde{p}_0 \rangle\rangle \bar{w}, \tag{14}$$

where $\tilde{p}_0$ is a light-like constant vector in L_1 and $\bar{w} \in L_0$ is chosen so that $\langle\langle \tilde{p}_0, \bar{w} \rangle\rangle = 1$. From (14) and $\langle\langle F, F \rangle\rangle = 0$ we get $2\langle\langle F, \tilde{p}_0 \rangle\rangle = -\sum_{i=1}^{k} \langle\langle F_i \circ \pi_i, F_i \circ \pi_i \rangle\rangle$. Let $\tilde{A}$ be the linear isometry of $\mathbb{R}^n$ onto $(L_0 \oplus L_1)^{\perp}$ that takes e_i to v_i, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, and let H denote the translation in $\mathbb{R}^n$ by the vector $(a_1, \ldots, a_n)$. Then it follows from (14) that $F = \Psi_{\tilde{p}_0, \bar{w}, \tilde{A}} \circ H|_U$. By Proposition 3-($i$), there exists $G \in \mathbb{O}_1(n + 2)$ fixing $\bar{w}$ such that

$$F = G \circ \Psi_{\tilde{p}_0, \bar{w}, \tilde{A}}|_U = \Psi_{\bar{p}_0, \bar{w}, \bar{A}}|_U$$

for $\bar{p}_0 = G\tilde{p}_0$ and $\bar{A} = G \circ \tilde{A}$. ∎

4 Comments on Nevanlina's proof

One of the most elementary proofs of Liouville's theorem available in the literature is the one by R. Nevanlina [**Ne**], which also appears with some modifications in several textbooks [**Be**], [**dC**], [**DFN**], [**Bl**]. It goes roughly as follows. First, it is shown that for any pair of orthogonal vectors $e_i, e_j \in \mathbb{R}^n$ it holds that

$$\rho \, d^2 f(e_j, e_i) + d\rho(e_i) df(e_j) + d\rho(e_j) df(e_i) = 0, \tag{15}$$

where $\rho = \varphi^{-1}$ is the inverse of the conformal distortion φ of f, and $d^2 f$ denotes the $\mathbb{R}^n$-valued symmetric bilinear map such that $d^2 f(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}) = \frac{\partial^2 f}{\partial u_i \partial u_j}$. We do not go into details on how (15) is derived, as we shall soon indicate an alternate way of proving it. We just mention that it follows from a tricky computation relying on a useful (but mysterious!) fact known as the *Braid Lemma* (cf. [**Be**], p.224), which states that a trilinear map that is symmetric on the first two variables and skew-symmetric on the last two must vanish. The next step is to differentiate (15) to obtain

$$\begin{aligned} d^2\rho(e_k, e_i) df(e_j) + d\rho(e_i) d^2 f(e_k, e_j) + d^2\rho(e_k, e_j) df(e_i) \\ + d\rho(e_j) d^2 f(e_k, e_i) + d\rho(e_k) d^2 f(e_i, e_j) + \rho\, d^3 f(e_k, e_i, e_j) = 0 \end{aligned}$$

for all pairwise orthogonal vectors $e_i, e_j, e_k \in \mathbb{R}^n$, where now $d^3 f$ denotes the $\mathbb{R}^n$-valued symmetric trilinear map such that $d^3 f(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}) = \frac{\partial^3 f}{\partial u_i \partial u_j \partial u_k}$. Then, observing that the sum of the last five terms is symmetric in k and j one concludes that the same must hold for the first term:

$$d^2\rho(e_k, e_i) df(e_j) = d^2\rho(e_j, e_i) df(e_k).$$

Since $df(e_j)$ and $df(e_k)$ are linearly independent vectors for $j \neq k$, it follows that

$$d^2\rho(e_i, e_j) = 0, \text{ for all } e_i, e_j \text{ with } \langle e_i, e_j \rangle = 0. \tag{16}$$

This implies that there exists a smooth function σ such that

$$d^2\rho(x)(u, v) = \sigma(x)\langle u, v\rangle \text{ for all } u, v \in \mathbb{R}^n, \tag{17}$$

and one further differentiation then shows that σ is constant. The second part of the proof proceeds by determining ρ explicitly through integration of (17), which leads to the conclusion of the theorem after some more work.

In spite of its simplicity, in that nothing "from outside" is used in the proof but clever computations, it is hard to grasp the geometry behind it. The following discussion may help to shed some light on the geometrical meaning of some of those computations. First, we have

$$d^2 f(e_i, e_j) = \bar{\nabla}_{e_j} df(e_i) = df \hat{\nabla}_{e_j} e_i,$$

where $\hat{\nabla}$ is the Levi-Civita connection of the metric induced by f and e_i is also regarded as a constant vector field along U. Then (15) follows from the relation between the Levi-Civita connections of conformal metrics (recalling that $\langle e_i, e_j \rangle = 0$):

$$0 = \rho \bar{\nabla}_{e_j} e_i = \rho \hat{\nabla}_{e_j} e_i + d\rho(e_i) e_j + d\rho(e_j) e_i.$$

Now let $F = \mathcal{I}_{p_0,w,A}(f) = \rho \bar{f}$ be as in Subsection 2.2, with $\bar{f} = \Psi \circ f$. Then

$$d^2F(e_i, e_j) = d^2\rho(e_i, e_j)\bar{f} + d\rho(e_i)d\bar{f}(e_j) + d\rho(e_j)d\bar{f}(e_i) + \rho d^2\bar{f}(e_i, e_j).$$

On the other hand, using (2) we get $d^2\bar{f}(e_i, e_j) = \tilde{\nabla}_{e_j} d\bar{f}(e_i) = d\Psi(d^2 f(e_i, e_j))$, hence

$$\begin{aligned} d^2F(e_i, e_j) &= \rho^{-1} d^2\rho(e_i, e_j)F + d\Psi \left(d\rho(e_i)df(e_j) + d\rho(e_j)df(e_i) + \rho\, d^2 f(e_i, e_j) \right) \\ &= \rho^{-1} d^2\rho(e_i, e_j)F. \end{aligned}$$

This gives an explicit expression for the one-form ω_X in (11):

$$\omega_X(Y) = \rho^{-1} d^2\rho(X, Y) \text{ for all } Y \in \mathbb{R}^n \text{ orthogonal to } X, \tag{18}$$

hence (16) is equivalent to the vanishing of ω_X for every $X \in \mathbb{R}^n$.

Another important remark for us concerns the geometrical meaning of (15): if $\mathcal{H}_i$ denotes an affine hyperplane orthogonal to e_i, then (15) amounts to saying that $S_i = f(\mathcal{H}_i \cap U)$ is a piece of a hypersphere or affine hyperplane in $\mathbb{R}^n$. Namely, assuming e_i of unit length, a unit normal vector field to S_i is $N_i = \rho\, df(e_i)$, thus (15) is equivalent to

$$\bar{\nabla}_{e_j} N_i = d\rho(e_j)df(e_i) + \rho d^2 f(e_i, e_j) = -d\rho(e_i)df(e_j), \tag{19}$$

which just says that all principal curvatures of S_i are equal to $d\rho(e_i)$. In particular, (16) expresses the fact that such principal curvatures are constant along S_i.

Applying this observation for $e_i = \partial/\partial u_i$, $1 \le i \le n$, we see that the geometric content of (12) (for $F = \mathcal{I}_{p_0,w,A}(f)$) is that f maps the pieces in U of all coordinate hyperplanes $u_i = a_i \in \mathbb{R}$, $1 \le i \le n$, to pieces of hyperspheres or affine hyperplanes. This suggests us an alternate, more geometrical argument to conclude the proof of Liouville's theorem after having (16). It relies on the following characterization of n mutually orthogonal families of hyperspheres or affine hyperplanes in $\mathbb{R}^n$, $n \ge 3$:

Proposition 8 *Let n families of hyperspheres or affine hyperplanes in $\mathbb{R}^n$, $n \ge 3$, each of which with at least two elements, have the property that every member of one family be orthogonal to every member of all of the others. Then either they are orthogonal families of parallel affine hyperplanes, or there exists an inversion that maps them into such families.*

Assuming Proposition 8 for a while, the proof of Liouville's theorem is then completed as follows: composing f with an inversion as in Proposition 8 we end up, possibly after a further composition with an orthogonal linear map, with a conformal map that takes coordinate hyperplanes into coordinate hyperplanes with respect to the same coordinate. But a map $g\colon U \to \mathbb{R}^n$ that takes coordinate hyperplanes into coordinate hyperplanes with respect to the same coordinate must clearly be of the form

$$g(x_1, \ldots, x_n) = (g_1(x_1), \ldots, g_n(x_n))$$

for some smooth functions of one variable $g_1, \ldots, g_n$. If, in addition, g is a conformal map, then we get from $|\partial g/\partial x_i| = |\partial g/\partial x_j|$ at any point of U that $g_i(x_i) = \pm\lambda x_i + a_i$ for some $\lambda, a_i \in \mathbb{R}$, $1 \le i \le n$, that is, up to a translation and reflections in the coordinate hyperplanes the map g is a dilation by λ. The conclusion follows.

4.1 Proof of Proposition 8

The following simple proof of Proposition 8 is a good illustration of the usefulness of the model of Euclidean space of Section 2 to study problems of a conformal nature. Let $\mathcal{F}_i = (S_i^\lambda)_{\lambda\in\Lambda}$, $1 \le i \le n$, be families of hyperspheres or affine hyperplanes in $\mathbb{R}^n$ as in the statement. Write $S_i^\lambda = \mathbb{E}^n \cap \{v_i^\lambda\}^\perp$ for $S_i^\lambda \in \mathcal{F}_i$ and unit space-like vectors v_i^λ, $1 \le i \le n$. For each $i = 1, \dots, n$, let $V_i \subset \mathbb{L}^{n+2}$ be the subspace spanned by the vectors v_i^λ, $\lambda \in \Lambda$. Then the assumption on the families $\mathcal{F}_i$, $1 \le i \le n$, amounts to saying that $V_i \subset V_j^\perp$ for $i, j = 1, \dots, n$ with $i \ne j$. On the other hand, the fact that $\mathcal{F}_i$ has more than one element implies that the dimension of V_i is at least two. As in Subsection 3.2.2, it follows that there exists a light-like line ℓ such that $V_i \cap V_i^\perp = \ell$ for $i = 1, \dots, n$. Choose a distinct light-like line $\tilde{\ell}$ and set $\hat{V}_i = V_i \cap \tilde{\ell}^\perp$. Then each $\hat{V}_i$ is a one-dimensional space-like subspace and we obtain a decomposition

$$\mathbb{L}^{n+2} = \ell \oplus \hat{V}_1 \oplus \cdots \oplus \hat{V}_n \oplus \tilde{\ell}.$$

Now we consider two possible cases, according as w belongs to ℓ or not. In the former case, each $\mathcal{F}_i$ is a family of affine hyperplanes parallel to the affine hyperplane $\mathbb{E}^n \cap \hat{V}_i^\perp$, $1 \le i \le n$. If $w \notin \ell$, choose $\zeta \in \ell$ with $\langle\langle \zeta, w \rangle\rangle = 1$ and let I be the inversion in $\mathbb{R}^n$ determined by the reflection $R \in \mathbb{O}_1(n+2)$ with respect to the hyperplane in $\mathbb{L}^{n+2}$ orthogonal to the unit space-like vector $v = \zeta + (1/2)w$, so that $w = -2T(\zeta)$. Arguing as before with $\hat{V}_i$ replaced by $T(\hat{V}_i)$, $1 \le i \le n$, we obtain that I takes the families $\mathcal{F}_i$ into mutually orthogonal families of parallel affine hyperplanes as in the preceding case.

4.2 Remarks on the regularity

One final remark on Liouville's theorem concerns the amount of regularity that the map f must have in order for its conclusion to be true. The existing proofs in the literature usually hold for C^3 maps. The result is known for C^1 maps, but the proof is much harder [**Ha**]. Nevanlina's proof requires f to be of class C^4: this is needed for the conclusion that the map σ in (17) be constant. Our proof also needs the C^4 assumption: the argument used in Lemma 6 to prove that the one-form ω_X in (11) vanishes depends on the map ξ being C^2; this is equivalent to F being C^3, which in turn amounts to f being C^4. However, our proof can be made into a C^3 proof if we replace this argument by the one just explained in Nevanlina's proof which derives (16) from (15), and take into account that (16) is equivalent to the vanishing of ω_X for every $X \in \mathbb{R}^n$, as pointed out after (18). Notice that also Nevanlina's proof becomes valid for C^3 maps if we replace its second part (after having (16)) by the geometrical argument proposed in this section.

5 The case $n = 2$

In our proof of Theorem 5 in Subsection 3.2 we have indicated that the assumption that $n \geq 3$ is essential to prove that the one-form ω_X in (11) vanishes for all $X \in \mathbb{R}^n$. As a consequence, this assumption is also needed to derive (12), which is equivalent to the vanishing of $\omega_{\partial/\partial u_i}$ for all coordinate vector fields $\partial/\partial u_i$, $1 \leq i \leq n$. Our discussion in the previous section then makes clear the additional condition that f must satisfy in order for this to hold when $n = 2$: it must map each coordinate curve of one family (and hence of both) to a piece of a straight line or circle. This follows from (19), which for $n = 2$ shows that the curvature of a coordinate curve $x = x_0$ is (up to sign) $\partial\rho/\partial x$, and hence the vanishing of $\omega_{\partial/\partial x}$, that is, of $\partial^2\rho/\partial x\partial y$, is precisely the condition for this curvature to be constant.

5.1 Proof of Theorem 2

Set $F = \mathcal{I}_{p_0,w,A}(f)\colon U \to \mathbb{V}^3$ for some admissible triple (p_0, w, A). By the discussion in the preceding paragraph, under the assumptions of Theorem 2 we may proceed exactly as in Subsection 3.2.2: define linear subspaces $W_1, W_2 \subset \mathbb{L}^4$ by

$$W_1 = \operatorname{span}\left\{\frac{\partial F}{\partial x}(x,y) \,:\, (x,y) \in C\right\} \quad \text{and} \quad W_2 = \operatorname{span}\left\{\frac{\partial F}{\partial y}(x,y) \,:\, (x,y) \in C\right\},$$

where $C = I \times J$ is a product of intervals contained in U. Then we have as before that $W_1 \subset W_2^\perp$, which now leads to two possibilities: either there exists a light-like line L such that $W_1 \cap W_1^\perp = L = W_2 \cap W_2^\perp$, or one of W_1 and W_2, say, W_1, is a time-like two-dimensional subspace and W_2 is its space-like two-dimensional orthogonal complement. In the former case, we are exactly in the situation of Subsection 3.2.2, so we arrive at the conclusion of Liouville's theorem: f is either the restriction $f = L|_U$ of a similarity or the composition $f = I \circ L|_U$ of such map with an inversion with respect to a circle of unit radius.

Now assume that W_1 is a time-like two-dimensional subspace. Writing $F = (F_1, F_2)$ according to the decomposition $\mathbb{L}^4 = W_1 \oplus W_2$, we obtain (by looking at the definitions of W_1 and W_2) that F_1 and F_2 depend only on x and y, respectively, and thus they define unit speed curves in W_1 and W_2, respectively. Moreover, from

$$0 = \langle\!\langle F, F\rangle\!\rangle = \langle\!\langle F_1, F_1\rangle\!\rangle + \langle\!\langle F_2, F_2\rangle\!\rangle$$

it follows that there exists $c > 0$ such that $\langle\!\langle F_1, F_1\rangle\!\rangle = -c^2$ and $\langle\!\langle F_2, F_2\rangle\!\rangle = c^2$. Choosing orthonormal bases $\{z_1, z_2\}$ of W_1, with $\langle\!\langle z_1, z_1\rangle\!\rangle = -1$, and $\{z_3, z_4\}$ of W_2 we obtain

$$F = c(\cosh((\pm x+x_0)/c)z_1+\sinh((\pm x+x_0)/c)z_2+\cos((\pm y+y_0)/c)z_3+\sin((\pm y+y_0)/c)z_4)$$

for some $x_0, y_0 \in \mathbb{R}$. Hence $F = c(G \circ H|_U)$, where H is a composition of a dilation by $1/c$, reflections in the coordinate axis and a translation by $(x_0, y_0) \in \mathbb{R}^2$, and

$$G(x,y) = \cosh(x)z_1 + \sinh(x)z_2 + \cos(y)z_3 + \sin(y)z_4.$$

Set $\bar{w} = -z_1 - z_2$, $\bar{p}_0 = (z_1 - z_2)/2$ and let $\bar{A}\colon \mathbb{R}^2 \to W_2$ be the linear isometry that takes e_1 to z_3 and e_2 to z_4. Then we can write G as

$$G(x, y) = e^{-x}\Psi_{\bar{p}_0,\bar{w},\bar{A}}(\exp(x, y)),$$

where exp denotes the complex exponential function. Now let $T \in O_1(4)$ be given by $T(w) = \bar{w}$, $T(p_0) = \bar{p}_0$, and $T \circ A = \bar{A}$. By Proposition 4,

$$\mathcal{C}_{p_0,w,A}(\Psi_{\bar{p}_0,\bar{w},\bar{A}}) = \mathcal{C}_{p_0,w,A}(T \circ \Psi_{p_0,w,A}) = I \circ L$$

for some similarity L and some inversion I with respect to a circle of unit radius. Thus,

$$f = \mathcal{C}_{p_0,w,A}(F) = \mathcal{C}_{p_0,w,A}(c\,G \circ H|_U) = \mathcal{C}_{p_0,w,A}(\Psi_{\bar{p}_0,\bar{w},\bar{A}}) \circ \exp \circ H|_U = I \circ L \circ \exp \circ H|_U.$$

Had we assumed instead W_2 to be time-like, we would get the same conclusion with x and y interchanged, thus f would be given as before after a reflection in the line $y = x$.

5.2 Remarks on Theorem 2

Although we have not been able to find Theorem 2 explicitly stated in the literature, it is very likely that it is not new. In fact, browsing through the monumental treatise by Darboux [**Da**], we have found some interesting related results that lead to an alternate proof, which we briefly sketch below.

First we recall that one-parameter families of curves that are the images by a conformal map in the plane of the family of coordinate curves $x = x_0$ or $y = y_0$ are referred to in the classical literature as *isothermal families* of plane curves. Isothermal families and their orthogonal trajectories admit a neat characterization in terms of their curvatures (cf. [**Da**], vol. III, p. 154, eq.(36)), which implies that orthogonal trajectories of isothermal families all of whose members are (pieces of) straight lines or circles must also have the same property. Thus, starting with a map f as in Theorem 2 and taking the images by f of the families of coordinate lines, we end up with two one-parameter families of straight lines and circles, every member of each family being orthogonal to every member of the other. Then we also find in [**Da**] (cf. vol I, p. 228) the following two-dimensional version of Proposition 8 asking to come into play:

Proposition 9 *Let two families of straight lines and circles, each of which with at least two elements, have the property that every member of one family be orthogonal to every member of the other. Then either they are orthogonal families of parallel lines, or one of them is a family of concentric circles and the other a family of straight lines through the common center, or there exists an inversion that maps them into families of one of those two types.*

Using this, a proof of Theorem 2 readily follows: composing our conformal map f with an inversion I given by Proposition 9, and then (working locally) with the complex log function in case $I \circ f$ maps the the coordinate curves into families of straight lines and circles of the second type, we end up, possibly after a further composition with a reflection in the line $y = x$, with a conformal map that takes coordinate curves into coordinate curves with respect to the same coordinate. Then we can argue exactly as in the paragraph preceding Subsection 4.1 to conclude that such a map is, up to a translation and reflections in the coordinate axes, a dilation by a nonzero constant.

5.2.1 Proof of Proposition 9

The proof of Proposition 9 serves as a final illustration of the ideas in Section 2. Let $\mathcal{F}_i = (S_i^\lambda)_{\lambda \in \Lambda}$, $1 \leq i \leq 2$, be families of straight lines and circles as in Proposition 9. Write $S_i^\lambda = \mathbb{E}^2 \cap \{v_i^\lambda\}^\perp$ for $S_i^\lambda \in \mathcal{F}_i$ and unit space-like vectors v_i^λ, $1 \leq i \leq 2$. Let $V_i \subset \mathbb{L}^4$ be the subspace spanned by the vectors v_i^λ, $1 \leq i \leq 2$. Then the assumption on $\mathcal{F}_1$ and $\mathcal{F}_2$ amounts to saying that $V_1 \subset V_2^\perp$. On the other hand, the fact that $\mathcal{F}_i$ has more than one element implies the dimension of V_i to be at least two. Then either there exists a light-like line L such that $V_1 \cap V_1^\perp = L = V_2 \cap V_2^\perp$, or one of V_1 or V_2, say, V_1, is a time-like plane and V_2 is its (space-like) orthogonal complement. In the former case, arguing exactly as in the proof of Proposition 8, we conclude that, up to an inversion in $\mathbb{R}^2$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are orthogonal families of parallel straight lines. Now assume that V_1 is a time-like plane. If $w \in V_1$, then $\mathcal{F}_1$ is a family of concentric circles, as discussed in Subsection 2.1. Otherwise, choose one of the two vectors in $\mathbb{E}^2 \cap V_1$, say ζ. Notice that these two vectors represent precisely the two common points of all the elements of $\mathcal{F}_2$. Now consider the inversion I in $\mathbb{R}^2$ determined by the reflection $T \in \mathbb{O}_1(4)$ with respect to the hyperplane in $\mathbb{L}^4$ orthogonal to the unit space-like vector $v = \zeta + (1/2)w$. In other words, I is the inversion with respect to the sphere of unit radius centered at the point $z \in \mathbb{R}^2$ such that $\Psi(z) = \zeta$. Then $T(W_1)$ is a time-like plane containing $w = -2T(\zeta)$, thus the family $I(\mathcal{F}_1)$ of images by I of elements of $\mathcal{F}_1$ is a family of concentric circles, since it has $T(W_1)$ as associated subspace. It follows that $I(\mathcal{F}_2)$ is a family of straight lines through the common center of the circles of $\mathcal{F}_1$. ∎

5.3 A final remark on the case $n = 2$.

To conclude, we observe that if a conformal map $f\colon U \to \mathbb{R}^2$ as in Theorem 2 has the property that *every* segment of straight line contained in U is mapped by f to a piece of circle or straight line, then it is given as in the statement of Liouville's theorem. For, by the discussion in the previous section, under this assumption Lemma 6 holds for $F = \mathcal{I}_{p_0,w,A}(f)\colon U \to \mathbb{V}^3 \subset \mathbb{L}^4$, and hence the remaining of the proof of Theorem 5 also applies.

References

[**Be**] BERGER, M: *Geometry I,* Universitext, Springer-Verlag, Berlin et al., 1980.

[**Bl**] BLAIR, D: *Inversion Theory and Conformal Mapping,* Student Math. Library, AMS, 2000.

[**Da**] DARBOUX, G.: *Leçons sur la théorie des surfaces* (Reprinted by Chelsea Pub. Co., 1972), Paris 1914.

[**dC**] DO CARMO, M. P.: *Riemannian Geometry,* Birkhäuser, Boston et al., 1992.

[**DFN**] DUBROVIN, B. A., FOMENKO, A. T., NOVIKOV, S. P.: *Modern Geometry, Methods and Applications,* Part I, 2nd ed., Springer-Verlag, Berlin et al., 1992.

[**Fr**] FRANCES, C.: Une preuve du Théorème de Liouville en Géométrie conforme dans le cas analytique, *Enseign. Math.* **49** (2003), 95-100.

[**Ha**] HARTMAN, P.: On isometries and on a theorem of Liouville, *Math. Z.* **69** (1958), 202–210.

[**H-J**] HERTRICH-JEROMIN, U.: *Introduction to Möbius differential geometry*, London Math. Lect. Notes Series, vol. 300, Cambridge Univ. Press, Cambridge, 2003.

[**Ja**] JACOBOWITZ, H.: Two notes on conformal geometry, *Hokkaido Math. J.* **20** (1991), 313–329.

[**Ku**] KULKARNI, R.: Conformal structures and Möbius structures, *Aspects of Math.* E 12, Vieweg, Braunschweig, 1988.

[**Li**] LIOUVILLE, J.: Extension au cas des trois dimensions de la question du tracé géographique, Note VI in the Appendix to G. Monge, *Application de l'Analyse à la Géometrie*, 5th ed., Bachelier, Paris, 1850, pp. 609-616.

[**Ma**] MATSUMOTO, S.: Foundations of flat conformal structure. In *Aspects of Low-Dimensional Manifolds,* 167-261, Adv. Stud. Pure Math. **20**, Tokyo, 1992.

[**Ne**] NEVANLINA, R.: On differential mappings. In *Analytic Functions,* pages 3-9, Princeton University Press, Princeton, 1973.

[**Sp**] SPIVAK, M.: *A Comprehensive Introduction to Differential Geometry.* Second edition. Publish or Perish, Wilmington, 1979.

[**To**$_1$] TOJEIRO, R.: Isothermic submanifolds of Euclidean space. *J. reine angew. Math.* **598** (2006), 1-24.

[**To**$_2$] TOJEIRO, R.: Conformal immersions of warped products. Preprint.

Universidade Federal de São Carlos
Via Washington Luiz km 235
13565-905 – São Carlos – Brazil
e-mail: tojeiro@dm.ufscar.br