On p-Laplacian differential inclusions - global existence, compactness properties and asymptotic behavior *

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Abstract

In this work we consider coupled systems of p-laplacian differential inclusions. We obtain the global existence of solutions and good compactness properties in such way that the set of all possible solutions compose a generalized semiflow which has a global attractor for each pair of positive diffusion coefficients. We also prove that the attractors are upper semicontinuous on positive finite diffusion parameters.

Keywords: p-laplacian reaction-diffusion systems differential inclusions attractors upper semicontinuity

1 Introduction

This work is concerning the following system:

$$(S)\begin{cases} \dfrac{\partial u}{\partial t} - div(D_1|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u \in F(u,v) & t>0 \\ \dfrac{\partial v}{\partial t} - div(D_2|\nabla v|^{q-2}\nabla v) + |v|^{q-2}v \in G(u,v) & t>0 \\ \dfrac{\partial u}{\partial n}(t,x) = \dfrac{\partial v}{\partial n}(t,x) = 0 \quad \text{in } \partial\Omega, & t \geq 0 \\ (u(0),v(0)) \text{ in } L^2(\Omega)\times L^2(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, connected and smooth set, $n \geq 1$, $p,q > 2$, $D_1, D_2 \in L^\infty(\Omega)$, $\infty > M \geq D_i(x) \geq \sigma > 0$ q.t.p. in Ω, $i = 1,2$, and F and G are bounded, upper semicontinuous and positively sublinear multivalued operators.

We start by exhibiting a proof of the existence of global solutions for (S) in Section 2.1, following the same steps as in [5] in order to get global existence for a larger class of systems. Then, by compactness arguments, we prove in Section 2.2 that the family of all possible solutions characterize a generalized semiflow ([2]) in $L^2(\Omega)\times L^2(\Omega)$,

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which has a global compact invariant attractor $\mathscr{A}_{(D_1,D_2)}$ (Section 2.3). Additionally we prove in Section 3.2 that the family $\{\mathscr{A}_{(D_1,D_2)}\}$ is upper semicontinuous on (D_1,D_2).

In order to obtain the upper semicontinuity of the attractors $\{\mathscr{A}_{(D_1,D_2)}\}$ it is enough to guarantee uniform estimates and continuity properties of the flow. Uniform estimates for solutions of this kind of systems are generally obtained in a very natural way. On the other hand, although the continuity on initial data and diffusion coefficients seems to be also direct conclusions, it was necessary to adjust a compactness theorem to fit our needs before proving it. Compactness results played an important role in this work. In fact throughout this text we need not only the original form of the Theorem of Baras, which is demonstrated in [11], but also a slightly different version of it that admits initial data in a pre-compact set, and additionally we need to appeal to a more general result, entirely described at Section 3.1, were we obtain, as it is done in the proof of the Theorem of Baras and by supposing almost the same conditions, the compactness of a set of solutions $\{u_\lambda\}$ of different problems, governed by different operators A^λ,

$$\begin{cases} \dfrac{du_\lambda}{dt} + A^\lambda u_\lambda \ni f_\lambda \\ u_\lambda(0) = u_0. \end{cases}$$

There is a huge number of questions we can formulate and investigate after we realize that problems like (S) are reasonably well posed and have its dynamics described by multivalued semigroups which admit attractors and enjoy strong properties of continuity, dissipativity and compactness and the aim of this work is to establish the first steps in the study of the asymptotic behavior of this kind of system.

2 On Attractors for Multivalued Semigroups Defined by Generalized Semiflows

In this section we first give the general framework for studying attractors for differential problems without unicity. For more details we refer the reader to [8]. After this we establish an existence result associating with (S) a generalized semiflow which possesses a global compact attractor.

Definition 2.1. *[2] A **generalized semiflow** $\mathscr{G}$ on X is a family of maps $\varphi : [0,\infty) \to X$ satisfying the conditions:*

(H1) For each $z \in X$ there exists at least one $\varphi \in \mathscr{G}$ with $\varphi(0) = z$.

(H2) If $\varphi \in \mathscr{G}$ and $\tau \geq 0$, then $\varphi^\tau \in \mathscr{G}$, where $\varphi^\tau(t) := \varphi(t+\tau), \forall t \in [0,\infty)$.

(H3) If $\varphi, \psi \in \mathscr{G}$, and $\psi(0) = \varphi(t)$ for some $t \geq 0$, then $\theta \in \mathscr{G}$, where

$$\theta(\tau) \doteq \begin{cases} \varphi(\tau) & \text{for } \tau \in [0,t] \\ \psi(\tau - t) & \text{for } \tau \in (t,\infty) \end{cases}$$

(H4) If $\{\varphi_j\}_{j=1}^\infty \subset \mathscr{G}$ and $\varphi_j(0) \to z$, then there exists a subsequence $\{\varphi_\mu\}$ of $\{\varphi_j\}$ and $\varphi \in \mathscr{G}$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geq 0$.

Definition 2.2. *A* ***multivalued semigroup*** $\{T(t)\}_{t\geq 0}$ ***defined by*** $\mathscr{G}$ *is a family of multivalued operators* $T(t) : P(X) \to P(X)$ *such that, for each* $t \geq 0$,

$$T(t)E \doteq \{\varphi(t);\ \varphi \in \mathscr{G} \text{ with } \varphi(0) \in E\}.$$

Definition 2.3. *We say that there exists a* ***complete orbit*** *through* $x \in X$ *if there is a map* $\psi : \mathbb{R} \to X$ *such that, for any* $s \in \mathbb{R}$, $\psi^s|_{\mathbb{R}^+} \in \mathscr{G}$ *and* $\psi(0) = x$. *In this case, the complete orbit of* ψ *is given by*

$$\gamma(\psi) = Im\ \psi = \{\psi(t), t \in \mathbb{R}\}.$$

We also say that ψ ***is a complete orbit through*** x.

Definition 2.4. *We say that a complete orbit* $\psi : \mathbb{R} \to X$ *is* ***stationary*** *if* $\psi(t) = z$, *for all* $t \in \mathbb{R}$ *for some* $z \in X$. *We set*

$$Z(\mathscr{G}) \doteq \{z \in X;\ \text{there exists a complete orbit } \psi \text{ such that } \psi(t) = z,\ \forall\, t \in \mathbb{R}\}.$$

Definition 2.5. *Given* $E \subset X$, $\varphi \in \mathscr{G}$,

- $\omega(\varphi) \doteq \{z \in X; \varphi(t_j) \to z, t_j \to +\infty\}$.
- $\omega_B(E) \doteq \{z \in X; \exists\, \varphi_j \in \mathscr{G}, \{\varphi_j(0)\} \subset E, \{\varphi_j(0)\} \in B(X)$, *and there is* $\{t_j\} \subset \mathbb{R}^+, t_j \to +\infty$ *with* $\varphi_j(t_j) \to z\}$.
- $\omega(E) \doteq \bigcap_{t\geq 0} \overline{\gamma_t^+(E)}$.

If E *is a bounded subset of* X, *then* $\omega_B(E) = \omega(E)$.

Definition 2.6. *Let be* $A, E \in P(X)$. *We say that* A ***attracts*** E *if for any* $\varepsilon > 0$ *there exists* $\tau = \tau(\varepsilon, E) \geq 0$ *such that* $T(t)E \subset O_\varepsilon(A)$ *for all* $t \geq \tau$, *or equivalently,* $dist(T(t)E, A) \to 0$ *as* $t \to +\infty$.

Definition 2.7. *A subset* $\mathscr{A}$ *is a* ***global B-attractor*** *if it attracts all bounded subsets of* X, *and is a* ***global point attractor*** *if it attracts all points of* X.

Definition 2.8.
(a) $\mathscr{G}$ *is* ***bounded dissipative or B-dissipative*** *if there is a bounded global B-attractor for* $\mathscr{G}$.
(b) $\mathscr{G}$ *is* ***point dissipative*** *if there is a bounded global point attractor for* $\mathscr{G}$.
(c) We say that $\mathscr{G}$ *is* φ***-dissipative*** *if there is a bounded set* B_0 *such that, for any* $\varphi \in \mathscr{G}$, $\varphi(t) \in B_0$ *for all sufficiently large* t.

Remark 2.1. *Bounded dissipative* $\Rightarrow$ *point dissipative* $\Rightarrow$ φ*-dissipative.*
(φ***-dissipativity*** *is called* ***point dissipativity*** *in [2]).*

Definition 2.9. $\mathscr{G}$ *is* ***asymptotically compact*** *if, for any sequence* $\{\varphi_j\} \subset \mathscr{G}$ *with* $\{\varphi_j(0)\} \in B(X)$, *and for any sequence* $\{t_j\}$, $t_j \to +\infty$, *the sequence* $\{\varphi_j(t_j)\}$ *has a convergent subsequence.*

Theorem 2.1. *[2] Let* $\mathscr{G}$ *be a generalized semiflow. If* $\mathscr{G}$ *has a compact invariant global B-attractor, then* $\mathscr{G}$ *is* φ*-dissipative and asymptotically compact. Reciprocally, if* $\mathscr{G}$ *is* φ*-dissipative and asymptotically compact, then* $\mathscr{G}$ *has a compact invariant global B-attractor* $\mathscr{A}$. *The global B-attractor* $\mathscr{A}$ *is unique and given by* $\mathscr{A} = \bigcup_{B\in B(X)} \omega(B) = \omega_B(X)$. *Furthermore* $\mathscr{A}$ *is the maximal compact invariant subset of* X, *and* $\mathscr{A}$ *is minimal among all closed global B-attractors.*

Theorem 2.2. *[8] Let $\mathscr{G}$ be asymptotically compact and φ-dissipative. Then the maximal compact invariant global B-attractor $\mathscr{A}$ of the Theorem 2.1 can be characterized by:*

(i) $\mathscr{A} = \bigcup_{K \in K(X)} \omega(K)$;

(ii) $\mathscr{A}$ is the union of all complete bounded orbits in X;

(iii) $\mathscr{A}$ is the union of all complete precompact orbits in X;

(iv) $\mathscr{A}$ is the maximal invariant bounded set in X.

2.1 On Existence of Global Solutions

The existence theorem we present in this section can be obtained through a simple reformulation of the main result in [5], where the authors have already announced it could be applied to p-laplacian operators. Here we only put their theorem in an abstract form in such way we can apply it to a class of subdifferentials operators. The necessary changes to accomplish this are pointed below. We consider the system

$$(P) \begin{cases} u_t + Au \in F(u,v) \\ v_t + Bv \in G(u,v) \\ u(0) = u_0, v(0) = v_0. \end{cases}$$

where A and B are monotone operators of subdifferential type defined in a real Hilbert space H.

Definition 2.10. *A strong solution [weak solution] of (P) is a pair (u,v) satisfying: $u,v \in C([0,T];H)$ for which there exists $f,g \in L^1(0,T;H)$, $f(t) \in F(u(t),v(t))$, $g(t) \in G(u(t),v(t))$ a.e. in $(0,T)$, and such that (u,v) is a strong solution [weak solution] (see Definition 3.1 and Theorem 3.4 in [3]) over $(0,T)$ to the system (P_1) below:*

$$(P_1) \begin{cases} u_t + Au = f \\ v_t + Bv = g \\ u(0) = u_0, v(0) = v_0 \end{cases}$$

For the purpose of getting the existence result, the main tool needed is the following fixed point theorem:

Theorem 2.3. *[1, 10] Let K be a nonempty and weakly compact subset in a real Hilbert space H and let $E: K \to P(K)$ be such that for each $u \in K$, $E(u)$ is closed and convex. If the graph of E is weakly $\times$ weakly sequentially closed, then E has at least one fixed point, i.e., there exists at least one element $u \in K$ such that $u \in E(u)$.*

Definition 2.11. *Let U be a topological space. A mapping $G: U \to P(H)$ is called upper semicontinuous at $u \in U$, if*

(i) $G(u)$ is nonempty, bounded, closed and convex.

(ii) For each open subset D in H satisfying $G(u) \subset D$, there exists a neighborhood V of u, such that $G(v) \subset D$, for each $v \in V$.

If G is upper semicontinuous at each $u \in U$, then it is called upper semicontinuous on U.

Definition 2.12. *Let M be a Lebesgue measurable subset in $\mathbb{R}^q$, $q \geq 1$. By a* selection *of $E : M \to P(H)$ we mean a function $f : M \to H$ such that $f(y) \in E(y)$ a.e. $y \in M$, and we denote by Sel E the set*

$$Sel\ E \doteq \{f,\ f : M \to H \text{ is a measurable selection of } E\}.$$

Now we can state:

Theorem 2.4. *Let A and B be univalued operators which are subdifferentials of convex, proper and l.s.c. non negative maps, ψ_A, ψ_B, respectively, defined in a real Hilbert space H, with $\psi_A(0) = \psi_B(0) = 0$. Also suppose each one A and B generates a compact semigroup, and let $F, G : H \times H \to P(H)$ upper semicontinuous and bounded multivalued maps. Then given a bounded subset $B_0 \subset H \times H$, there exists $T_0 > 0$ such that for each $(u_0, v_0) \in B_0$ there exists at least one strong solution (u, v) of (P) defined on $[0, T_0]$.*

Proof. Let $u_0, v_0 \in H$ and $m,\ r > 0$ be such that $\| u_0 \|_H + 1 \leq m$, $\| v_0 \|_H + 1 \leq m$ and $\max\{\| w_1 \|_H, \| w_2 \|_H\} \leq m \Rightarrow \max\{\| z \|_H, \| \bar{z} \|_H\} \leq r$, $\forall\, (z, \bar{z}) \in F(w_1, w_2) \times G(w_1, w_2)$. We can suppose, without loosing generality, that $r > 1$. Let $T_0 > 0$ be such that $T_0\, r^2 \leq 1$ and consider the set

$$K = \{(f,g); f, g \in L^2(0, T_0; H), \| f(t) \|_{L^2(0,T_0;H)} \leq r, \| g(t) \|_{L^2(0,T_0;H)} \leq r\}.$$

K is nonempty and weakly compact in $L^2(0, T_0; H) \times L^2(0, T_0; H)$. We define

$$P_{T_0} : K \to C([0, T_0]; H) \times C([0, T_0]; H)$$

by $P_{T_0}(f, g) = (u, v)$, where (u, v) is the unique solution on $[0, T_0]$ of (P_1). It is easy to verify that

$$\| u(t) \|_H \leq m \qquad \text{and} \qquad \| v(t) \|_H \leq m$$

for all $t \in [0, T_0]$. The Fixed Point Theorem 2.3 can be applied to the operator

$$\varphi : K \to P(K)$$

$$(f, g) \mapsto \text{Sel } F(u, v) \times \text{Sel } G(u, v)$$

where $(u, v) = P_{T_0}(f, g)$. The only fact we have to guarantee is that the graph of φ, $Graf(\varphi)$, is weakly $\times$ weakly sequentially closed in K, but it follows from theorem 3.4, [3], Theorem 2.3.3, [11], Proposition 3.6, [3], and Theorem 3.3, [5]. So we can conclude that there exists $(f, g) \in K$ such that $(f, g) \in \varphi(f, g)$, and consequently $(u, v) = P_{T_0}(f, g)$ is a weak solution of (P_1). Since $(f, g) \in K \subset L^2(0, T_0; H) \times L^2(0, T_0; H)$, the Theorem 3.6, [3], guarantees that $(u, v) = P_{T_0}(f, g)$ is in fact a strong solution of (P_1). □

In order to get global solutions we impose suitable conditions on terms F and G.

Definition 2.13. *The pair (F,G) of maps $F,G : H\times H \to P(H)$, which takes bounded subsets of $H\times H$ into bounded subsets of H, is called* ***positively sublinear*** *if there exist $a>0$, $b>0$, $c>0$ and $m_0>0$ such that for each $(u,v)\in H\times H$ with $\|u\|>m_0$ or $\|v\|>m_0$ for which either there exists $f_0\in F(u,v)$ satisfying $\langle u,f_0\rangle>0$ or there exists $g_0\in G(u,v)$ with $\langle v,g_0\rangle>0$, we have both*

$$\|f\| \le a\|u\|+b\|v\|+c$$

and

$$\|g\| \le a\|u\|+b\|v\|+c$$

for each $f\in F(u,v)$ and each $g\in G(u,v)$.

Let $T>0$ arbitrary. By standard arguments one can proof that each solution of (P) which is defined on $[0,T_0)$, $T_0\le T$ can be extended up to $[0,T]$ if (F,G) are supposed to be positively sublinear operators in $H\times H$. The proof is the very same of that in [5].

2.2 The Generalized Semiflow Associated with (P)

Once we know that system (P) admits global solution for each initial data in $L^2(\Omega)\times L^2(\Omega)$ we now intend to associate a generalized semiflow with (P).

Let $D(u_0,v_0)$ be the set of all solutions of (P) with initial data (u_0,v_0) and consider $\mathbb{G}\doteq\bigcup_{(u_0,v_0)\in H\times H} D(u_0,v_0)$. We claim that $\mathbb{G}$ is a generalized semiflow in $L^2(\Omega)\times L^2(\Omega)$. Hypotheses $(H1)$, $(H2)$ and $(H3)$ in Definition 2.1 are immediately checked. The upper semicontinuity condition $(H4)$ can be obtained from a slightly different version of Theorem 2.3.3, [11], proved by the same arguments, where initial conditions are allowed to vary in a precompact subset $\{(u_{0n},v_{0n})\}\subset H\times H$. Consider the following PVI:

$$(P_{f_n})\begin{cases} \dfrac{du_n}{dt}+Au_n \ni f_n \\ u_n(0)=u_{0_n} \end{cases}$$

where A is a maximal monotone operator in a Hilbert space H, $u_{0_n}\in H$ and $f_n\in K\subset L^1(0,T;H)$. When varying f_n and u_{0_n} we obtain a family of problems and so a family of solutions. Define

$$M(K)\doteq\{u_n; u_n \text{ is the only weak solution of } (P_{f_n}),\ \text{with } (f_n,u_{0_n})\in K\times H\}.$$

The following theorem establishes conditions under which the set $M(K)$ enjoys some compactness property.

Theorem 2.5. *Let H be a real Hilbert space, $A:\mathscr{D}(A)\subset H\to P(H)$ be a maximal monotone operator in H which generates a compact semigroup, $\{u_{0_n}\}\subset\overline{\mathscr{D}(A)}$, with $u_{0_n}\to u_0$ in H, and $K=\{f_n; n\in\mathbb{N}\}$ is a uniformly integrable subset in $L^1(0,T;H)$, then the set $M(K)$ is relatively compact in $C([0,T];H)$.*

Now, consider $\{\varphi_n\} \subset \mathbb{G}$ with $\varphi_n(0) \to z$ as $n \to +\infty$. Then $\varphi_n = (u_n, v_n)$, $\varphi_n(0) = (u_{0_n}, v_{0_n})$, $z = (u_0, v_0)$, and for each n, (u_n, v_n) satisfies the problem

$$(P_n) \begin{cases} \dfrac{du_n}{dt} + Au_n = f_n & \text{in } (0,T) \\ \dfrac{dv_n}{dt} + Bv_n = g_n & \text{in } (0,T) \\ u_n(0) = u_{0_n}, v_n(0) = v_{0_n} \end{cases}$$

Let $K \doteq \{f_n; n \in \mathbb{N}\}$, $\widetilde{K} \doteq \{g_n; n \in \mathbb{N}\}$, where $f_n \in$ Sel $F(u_n, v_n)$ and $g_n \in$ Sel $G(u_n, v_n)$, $n \in \mathbb{N}$. Using the positive sublinearity of the pair (F, G) we can show that the solutions $\{(u_n, v_n),\ n \in \mathbb{N}\}$ are uniformly bounded on $[0,T]$. Since F, G are bounded operators, we conclude that the sets K and $\widetilde{K}$ are uniformly bounded on $[0,T]$, and so, uniformly integrable in $L^1(0,T;H)$. Then from Theorem 2.5 we have that $M(K)$ and $M(\widetilde{K})$ are relatively compact in $C([0,T];H)$. Then (H4) follows from Theorem 3.3, [5], and Proposition 3.6, [3]. So, we obtain the following

Theorem 2.6. *If $F, G : H \times H \to P(H)$ are bounded multivalued maps, upper semicontinuous and (F,G) is positively sublinear, then $\mathbb{G}$ is a generalized semiflow on $H \times H$.*

2.3 On Existence of Attractors

Let $T > 0$, $B \subset H \times H$, $\mathbb{G}$ the generalized semiflow associated with (P), and $P_T(f,g) = (u,v)$ as in the proof of Theorem 2.4, and

$$\widetilde{K}(B) \doteq \{(f,g) \in \operatorname{Sel} F(u,v) \times \operatorname{Sel} G(u,v); (u,v) = P_T(f,g), \text{ and } (u(0),v(0)) \in B\}.$$

The same arguments used to assure the global existence of solutions shows that, if B is a bounded subset of $H \times H$, then $\widetilde{K}(B)$ is a bounded subset of $L^1(0,T;H) \times L^1(0,T;H)$, i. e., there is a constant $C \geq 0$ such that $\|f\|_{L^1(0,T;H)} + \|g\|_{L^1(0,T;H)} \leq C$, $\forall\ (f,g) \in \tilde{K}(B)$.

Theorem 2.7. *If the generalized semiflow $\mathbb{G}$ associated with (P) is eventually bounded then $\mathbb{G}$ is asymptotically compact.*

Proof. Let $\{\varphi_j\} \subset \mathbb{G}$ with $\{\varphi_j(0)\}$ bounded in $H \times H$, and $\{\varphi_j(t_j)\}$ a sequence in $H \times H$ with $t_j \to +\infty$. We want to show that $\{\varphi_j(t_j)\}$ has a convergent subsequence. From definition $\varphi_j = (u_j, v_j)$, $\varphi_j(0) = (u_j(0), v_j(0)) \in H \times H$. As $t_j \to +\infty$ we can suppose $t_j \geq 1$, $\forall j \in \mathbb{N}$ and as $\mathbb{G}$ is a generalized semiflow $\widetilde{\varphi}_j = \varphi_j^{t_j-1} = (u_j^{t_j-1}, v_j^{t_j-1}) \in \mathbb{G}$. Then for each $j \in \mathbb{N}$ there exists $f_j,\ g_j \in L^1(0,1;H)$, $f_j \in$ Sel $F(u_j^{t_j-1}, v_j^{t_j-1})$, $g_j \in$ Sel $G(u_j^{t_j-1}, v_j^{t_j-1})$, and $(u_j^{t_j-1}, v_j^{t_j-1}) = P_1(f_j, g_j)$.

Let $K_i = \pi_i(\widetilde{K}(\{\widetilde{\varphi}_j(0)\}))$ and $M(K_i)(t) = \{u_j^{t_j-1}(t), j \in \mathbb{N}\}$, $i = 1,2$, $t \in [0,1]$, and $\{S(t),\ t \geq 0\}$ the compact semigroup generated by A on H. Once $\mathbb{G}$ is eventually bounded, $\{\widetilde{\varphi}_j(0)\} = \{\varphi_j(t_j - 1)\}$ is a bounded subset of $H \times H$ if j is big enough.

Now, let $h > 0$ be such that $1 - h \in [0,1]$. We define $T_h : M(K_1)(1) \to H$ by setting $u_j^{t_j-1}(1) \mapsto S(h)u_j^{t_j-1}(1-h)$. Since $M(K_1)(1-h)$ is a bounded subset of H, then T_h is a compact operator. Also we have

$$\| S(h)u_j^{t_j-1}(1-h)-u_j^{t_j-1}(1) \| \leq \int_{1-h}^{1} \| f_j(s) \| \, ds, \ \forall\, j \in \mathbb{N}.$$

As $K_1 = \{f_j, j \in \mathbb{N}\}$ is a bounded subset of H, K_1 is uniformly integrable in $L^1(0,1;H)$, so we have $\lim_{h\to 0} T_h = I$, uniformly in $M(K_1)(1)$. Therefore the map $I : M(K_1)(1) \to M(K_1)(1)$ is a compact operator and then, $M(K_1)(1)$ is relatively compact in H. The same arguments show that $M(K_2)(1)$ is relatively compact in H, therefore $\{\varphi_j(t_j)\}$ has a convergent subsequence in $H \times H$. □

Therefore, according to Theorem 2.1, in order to assure the existence of a compact invariant global B-attractor for (P), it is enough to impose conditions on F, G to guarantee that the generalized semiflow $\mathbb{G}$ defined by (P) is B-dissipative and so, eventually bounded and $\varphi-$dissipative.

Now, to obtain enough dissipativity properties, we restrict ourselves to systems where A and B are $p-$laplacian operators on $H = L^2(\Omega,\mathbb{R})$ under homogeneous Neumann boundary conditions. First of all, we properly define the operators A and B.

Let $V \doteq W^{1,p}(\Omega)$. Consider the operator $A_1 : V \to V^*$ given by

$$A_1u(v) \doteq \int_\Omega D_1(x)|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx + \int_\Omega |u(x)|^{p-2}u(x)v(x)dx$$

for each $u \in V$. Then, according Example 2.3.7, [3], we have that the operator A, the realization of A_1 at $H = L^2(\Omega)$ given by

$$\begin{cases} \mathscr{D}(A) \doteq \{u \in V; A_1u \in H\} \\ A(u) = A_1u, \text{ if } u \in \mathscr{D}(A), \end{cases} \tag{1}$$

is a maximal monotone operator in H. Besides, it is not difficult to see that there are constants $w_1 = w_1(n,\sigma) > 0$, $w_2 = w_2(n,p,M) > 0$, $c_1 \doteq 0 \in \mathbb{R}$ and $p > 2$ such that for all $u \in V$ the following two conditions hold:

$$\langle A_1u,u\rangle_{V^*,V} \geq w_1 \| u \|_V^p + c_1,$$

$$\| A_1u \|_{V^*} \leq w_2 \| u \|_V^{p-1} < w_2(\| u \|_V^{p-1} + 1).$$

So we can conclude that $\overline{\mathscr{D}(A)} = H = L^2(\Omega)$ and the operator $A : \mathscr{D}(A) \subset H \to H$ generates a compact semigroup, [4]. We still observe that A is the subdifferential of φ_1, where $\varphi_1 : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\varphi_1(u) : \begin{cases} \dfrac{1}{p}\Big[\displaystyle\int_\Omega D_1(x)|\nabla u|^p dx + \int_\Omega |u|^p dx\Big], & u \in W^{1,p}(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

From now on we write $-div(D_1|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u$ to mean $A(u)$ as we have just defined.

Now we want to prove $B-$dissipativity for the following system:

$$(P_N)\begin{cases} \dfrac{\partial u}{\partial t} - div(D_1|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u \in F(u,v) & \text{in } (0,T)\times\Omega \\ \dfrac{\partial v}{\partial t} - div(D_2|\nabla v|^{q-2}\nabla v) + |v|^{q-2}v \in G(u,v) & \text{in } (0,T)\times\Omega \\ \dfrac{\partial u}{\partial n}(t,x) = \dfrac{\partial v}{\partial n}(t,x) = 0 & \text{in } (0,T)\times\partial\Omega \\ u(0,x) = u_0(x), v(0,x) = v_0(x) & \text{in } \Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 1$, $p,q > 2$, $D_1, D_2 \in L^\infty(\Omega)$ and $D_i(x) \geq \sigma > 0$ almost everywhere in Ω, $i = 1,2$, u_0, $v_0 \in L^2(\Omega)$ F,G are bounded, upper semicontinuous and positively sublinear multivalued operators in $L^2(\Omega)\times L^2(\Omega)$, $\frac{\partial u}{\partial n} \doteq D_1|\nabla u|^{p-2}\langle \nabla u, \overrightarrow{\eta}\rangle$, and $\overrightarrow{\eta}$ is the outward unit normal vector. Observe that we always deal with the problem (P_N) like an initial value problem, and the Neumann boundary condition appears, in a weak sense, in the definition of the perturbed $p-$laplacian operator.

In this particular case (P_N) the same conditions imposed on F,G to get global existence of solutions are enough to prove that the generalized semiflow $\mathbb{G}_N$ defined by (P_N) is B-dissipative. We have the following:

Theorem 2.8. *Let $F,G : H\times H \to P(H)$ bounded, upper semicontinuous and positively sublinear operators. There exists a bounded set B_0 in $H\times H$ and $t_0 > 0$ such that for any $\varphi \in \mathbb{G}_N$, $\varphi(t) \in B_0$, $\forall\, t \geq t_0$. Then, in particular, the generalized semiflow $\mathbb{G}_N$ defined by (P_N) is B-dissipative.*

Proof. In fact, if $\varphi = (u,v) \in \mathbb{G}_N$ is a solution of (P_N) then there exists a pair $(f,g) \in$ Sel $F(u,v)\times$ Sel $G(u,v)$, $f,g \in L^1(0,T;H)$ for each $T > 0$ such that u,v satisfy the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - div(D_1|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = f & \text{in } (0,T)\times\Omega \\ \dfrac{\partial v}{\partial t} - div(D_2|\nabla v|^{q-2}\nabla v) + |v|^{q-2}v = g & \text{in } (0,T)\times\Omega \\ \dfrac{\partial u}{\partial n}(t,x) = \dfrac{\partial v}{\partial n}(t,x) = 0 & \text{in } (0,T)\times\partial\Omega \\ u(0,x) = u_0(x), v(0,x) = v_0(x) & \text{in } \Omega, \end{cases}$$

Multiplying the first equation by u, the other one by v and supposing, without loosing generality that $p \geq q$, we obtain

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|u(t)\|_H^2 & \leq & -\delta\|u(t)\|_H^p + \langle f(t),u(t)\rangle \\ \dfrac{1}{2}\dfrac{d}{dt}\|v(t)\|_H^2 & \leq & -\widetilde{\delta}\|v(t)\|_H^q + \langle g(t),v(t)\rangle \end{cases}$$

where $\delta,\widetilde{\delta}$ are positive real numbers depending on $D_1, D_2, \sigma, \Omega, p, q$. From Definition

2.13, Cauchy-Schwartz and Young's inequality it follows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|u(t)\|_H^2+\|v(t)\|_H^2) &\leq -C_1(\|u(t)\|_H^q+\|v(t)\|_H^q)+C_2\\
&= -C_1(\|u(t)\|_H^{2q/2}+\|v(t)\|_H^{2q/2})+C_2\\
&\leq -C_1\frac{1}{2^{q/2}}(\|u(t)\|_H^2+\|v(t)\|_H^2)^{q/2}+C_2,
\end{aligned}$$

where C_1, C_2 are positive real numbers depending on $D_1, D_2, \sigma, \Omega, p, q$.

We can conclude the proof appealing to Lemma 5.1, [9]. □

From Theorems 2.1, 2.7 and 2.8 we conclude that $\mathbb{G}_N$ has a compact invariant global $B-$attractor $\mathscr{A}^N_{(D_1,D_2)}$. The global $B-$attractor $\mathscr{A}^N_{(D_1,D_2)}$ is unique and given by

$$\mathscr{A}^N_{(D_1,D_2)} = \bigcup_{B\in B(H\times H)} \omega(B) = \omega_B(H\times H).$$

Furthermore $\mathscr{A}^N_{(D_1,D_2)}$ is the maximal compact invariant subset of $H\times H$, and is minimal among all closed global $B-$attractors. According to Theorem 2.2, we also have that $\mathscr{A}^N_{(D_1,D_2)}$ is the union of all complete bounded orbits in $H\times H$.

3 Dependence on Parameters

In this section we consider the problem

$$(P_{N_\lambda})\begin{cases}
\dfrac{\partial u_\lambda}{\partial t} - div(D_1^\lambda|\nabla u_\lambda|^{p-2}\nabla u_\lambda)+|u_\lambda|^{p-2}u_\lambda \in F(u_\lambda,v_\lambda) & \text{in} \quad (0,T)\times\Omega\\
\dfrac{\partial v_\lambda}{\partial t} - div(D_2^\lambda|\nabla v_\lambda|^{q-2}\nabla v_\lambda)+|v_\lambda|^{q-2}v_\lambda \in G(u_\lambda,v_\lambda) & \text{in} \quad (0,T)\times\Omega\\
\dfrac{\partial u_\lambda}{\partial n}(t,x) = \dfrac{\partial v_\lambda}{\partial n}(t,x) = 0 & \text{in} \quad (0,T)\times\partial\Omega\\
u_\lambda(0,x) = u_{0,\lambda}(x), v_\lambda(0,x) = v_{0,\lambda}(x) & \text{in} \quad \Omega
\end{cases}$$

where $u_{0,\lambda}, v_{0,\lambda} \in L^2(\Omega)$, $D_1^\lambda, D_2^\lambda \in L^\infty(\Omega)$, $D_i^\lambda(x) \geq \sigma > 0$ almost everywhere in Ω, $i=1,2$, $\lambda\in[0,\lambda_0]$ and $D_i^\lambda \to D_i^0$, in $L^\infty(\Omega)$ as $\lambda\to 0$, $i=1,2$. Ω, p,q and F,G are supposed to satisfy the same conditions as in Section 2.2. For each $\lambda\in[0,\lambda_0]$ we can associate a generalized semiflow $\mathbb{G}_\lambda$ which has a global compact invariant B-attractor $\mathscr{A}_\lambda$, as it is proved in Section 2.3.

For the purpose of obtaining the upper semicontinuity of $\{A_\lambda\}_{\lambda\in[0,\lambda_0]}$ on $\lambda=0$ we only need to guarantee that the associated flow enjoys properties of continuity and solutions of (P_{N_λ}) can be uniformly estimated. For getting the continuity of the flow, we first announce a compactness result. After this we show that the conditions imposed on (P_{N_λ}) are sufficient to guarantee uniform estimates and then we obtain the upper semicontinuity of the attractors.

In this section we denote $A^{D_1^\lambda}(\theta) \doteq -div(D_1^\lambda|\nabla\theta|^{p-2}\nabla\theta)+|\theta|^{p-2}\theta$, and analogously $B^{D_2^\lambda}(\theta) \doteq -div(D_2^\lambda|\nabla\theta|^{q-2}\nabla\theta)+|\theta|^{q-2}\theta$; $S^{D_1^\lambda}$ the semigroup generated by $A^{D_1^\lambda}$ and T_λ the multivalued semigroup defined by $\mathbb{G}_\lambda$.

3.1 A Compactness Argument

Let Λ be a metric space and consider the following Cauchy problem:

$$(P_\lambda)\begin{cases} \dfrac{du_\lambda}{dt} + A^\lambda u_\lambda \ni f_\lambda \\ u_\lambda(0) = u_0 \in H \end{cases}$$

where $f_\lambda \in L^1(0,T;H)$, $\forall \lambda \in \Lambda$, A^λ is a maximal monotone operator in a Hilbert space H, $A^\lambda = \partial\phi^\lambda$ is the subdifferential of some convex, proper and lower semi continuous non negative map $\phi^\lambda : H \to \mathbb{R}$, and suppose $\overline{\cap_{\lambda\in\Lambda} D(\phi^\lambda)} = H$ and that for each $u \in \cap_{\lambda\in\Lambda} D(\phi^\lambda)$ there exists a constant $k(u) > 0$ such that $\phi^\lambda(u) \leq k(u)$, $\forall\, \lambda \in \Lambda$. Let $\{S^\lambda(t)\}$ be the semigroup generated by A^λ in H.

Lemma 3.1. *For each $u_0 \in H$ fixed, $\{S^\lambda(.)u_0\}_{\lambda\in\Lambda} \subset C([0,T];H)$ is equicontinuous at $t_0 = 0$.*

Proof. In fact, let $h > 0$. We first suppose $u_0 \in \cap_{\lambda\in\Lambda} D(\phi^\lambda)$. From Theorem 2.1.1, [11],

$$\|S^\lambda(h)u_0 - u_0\|_H \leq 3\|u_0 - J_h^\lambda u_0\|_H, \text{ where } J_h^\lambda = (I + h\partial\phi^\lambda)^{-1}.$$

Also, from Proposition 2.11, [3], $\frac{1}{2h}\|u_0 - J_h^\lambda u_0\|_H^2 \leq \phi^\lambda(u_0) \leq k(u_0)$, $\forall\, \lambda \in \Lambda$. Then

$$\|S^\lambda(h)u_0 - u_0\|_H \leq 6hk(u_0),\ \forall\, \lambda \in \Lambda,$$

and it guarantees the equicontinuity of the family $\{S^\lambda(.)u_0\}_{\lambda\in\Lambda}$ at $t_0 = 0$. Now let $u_0 \in H$, and consider a sequence $\{u_0^n\} \subset \cap_{\lambda\in\Lambda} D(\phi^\lambda)$ such that $u_0^n \overset{n\to\infty}{\longrightarrow} u_0$. Given $\varepsilon > 0$, let $N_0 \in \mathbb{N}$ such that $\|u_0^{N_0} - u_0\|_H \leq \frac{\varepsilon}{6}$. Then

$$\begin{aligned} \|S^\lambda(h)u_0 - u_0\|_H &\leq \|S^\lambda(h)u_0 - S^\lambda(h)u_0^{N_0}\|_H + \|S^\lambda(h)u_0^{N_0} - u_0^{N_0}\|_H \\ &+ \|u_0^{N_0} - u_0\|_H \leq 2\|u_0^{N_0} - u_0\|_H + 6hk(u_0^{N_0}),\ \forall\, \lambda \in \Lambda. \end{aligned}$$

and we get the result. □

Let $K \doteq \{f_\lambda,\ \lambda \in \Lambda\}$ be an uniformly integrable subset of $L^1(0,T;H)$ and let

$$M(K) \doteq \{u_\lambda; u_\lambda \text{is the only weak solution of } (P_\lambda) \text{ on } [0,T], \lambda \in \Lambda\}.$$

We have

Theorem 3.1. *If for each $t \in [0,T]$, the set $M(K)(t) \doteq \{u_\lambda(t); u_\lambda \in M(K)\}$ is relatively compact in H, then $M(K)$ is relatively compact in $C([0,T];H)$.*

This result can be proved by using Lemma 3.1 and following exactly the same steps as in Theorem 2.3.1, [11].

Theorem 3.2. *Let us suppose that $K \doteq \{f_\lambda,\ \lambda \in \Lambda\}$ is an uniformly integrable subset of $L^1(0,T;H)$. If for each $t \in (0,T]$ and $h > 0$ such that $t - h \in (0,T]$, the operator $T_h : M(K)(t) \to H$ defined by $T_h u_\lambda(t) = S^\lambda(h)u_\lambda(t-h)$ is compact, then $M(K) = \{u_\lambda; \lambda \in \Lambda\}$ is relatively compact in $C([0,T];H)$.*

Proof. The proof follows Theorem 2.3.3, [11]. For each $t \in (0,T]$ and $h > 0$ with $t-h \in (0,T]$ we define $T_h : M(K)(t) \to H$ by $T_h u_\lambda(t) = S^\lambda(h)u_\lambda(t-h)$. Using the hypothesis we have that T_h is a compact operator. From Lemma 2.3.1, [11] and hypothesis on K, $\lim_{h\to 0} T_h = I$ uniformly on $M(K)(t)$. Therefore the identity operator $I : M(K)(t) \to M(K)(t)$ is a compact operator and then since that $M(K)(t)$ is a bounded set in H we have that $M(K)(t)$ is relatively compact for each $t \in (0,T]$. Note that $M(K)(0) = \{u_0\}$ is relatively compact in H. We get the result from Theorem 3.1. □

3.2 Upper Semicontinuity of Attractors

For each $\lambda \in \Lambda := [0,\lambda_0]$ let $\mathscr{A}_\lambda$ be the compact invariant global B-attractor associated with the generalized semiflow $\mathbb{G}_\lambda$ defined by (P_{N_λ}). Our objective in this section is prove that the family of attractors $\{A_\lambda\}_{\lambda\in\Lambda}$ is upper semicontinuous at $\lambda = 0$. To accomplish that we appeal to Theorem 1.2 in [7] which is rewritten below in generalized semiflow context:

Theorem 3.3. *Let Λ be a metric space, λ_1 be a non-isolated point and let $\{\mathbb{G}_\lambda\}_{\lambda\in\Lambda}$ a family of generalized semiflows in the Banach space X satisfying:*
(i) For each $\lambda \in \Lambda$, $\mathbb{G}_\lambda$ has a compact and invariant global B-attractor $\mathscr{A}_\lambda$ and $\bigcup_{\lambda\in\Lambda} \mathscr{A}_\lambda \in B(X)$;
(ii) The multivalued map defined by $\mathbb{G}_\lambda$, $\lambda \mapsto T_\lambda(t)(\mathscr{A})$, $\mathscr{A} \doteq \overline{\bigcup_{\lambda\in\Lambda} \mathscr{A}_\lambda}$, is w-upper semicontinuous at λ_1 for large t, i.e., there exists $t_0 > 0$ such that for each $t \geq t_0$ fixed, given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $T_\lambda(t)(\mathscr{A}) \subset O_\varepsilon(T_{\lambda_1}(t)(\mathscr{A}))$, $\forall\, \lambda \in O_\delta(\lambda_1)$.
Then $dist(\mathscr{A}_\lambda, \mathscr{A}_{\lambda_1}) \to 0$, as $\lambda \to \lambda_1$.

We have the following:

Lemma 3.2. *If (u_λ, v_λ) is a solution of (P_{N_λ}), then there exist positive numbers r_0 and a constant $t_0 > 0$ such that $\|(u_\lambda(t), v_\lambda(t))\|_{H\times H} \leq r_0$, for each $t \geq t_0$ and $\lambda \in \Lambda$.*

Proof. In fact, if (u_λ, v_λ) satisfies (P_{N_λ}), then there exists $(f_\lambda, g_\lambda) \in F(u_\lambda, v_\lambda) \times G(u_\lambda, v_\lambda)$ such that

$$(P^1_\lambda)\begin{cases} \dfrac{\partial u_\lambda}{\partial t} - div(D^\lambda_1 |\nabla u_\lambda|^{p-2}\nabla u_\lambda) + |u_\lambda|^{p-2}u_\lambda = f_\lambda \\ \dfrac{\partial v_\lambda}{\partial t} - div(D^\lambda_2 |\nabla v_\lambda|^{q-2}\nabla v_\lambda) + |v_\lambda|^{q-2}v_\lambda = g_\lambda \end{cases}$$

The very same arguments employed in the proof of Theorem 2.8 can also be applied here observing that the constants $\delta, \widetilde{\delta}, C_1, C_2$ can be uniformly chosen for $\lambda \in \Lambda$, once $D^\lambda_i(x) \geq \sigma > 0$ almost everywhere in Ω, $i = 1,2$, $\forall\, \lambda \in [0,\lambda_0]$ and $D^\lambda_i \to D^0_i$, in $L^\infty(\Omega)$ as $\lambda \to 0$, and so there is a positive constant $M_i > 0$ such that $\|D^\lambda_i\|_{L^\infty(\Omega)} \leq M_i, \forall\, \lambda \in \Lambda$. Thus there are positive numbers r_0 and t_0 such that $\|(u_\lambda(t), v_\lambda(t))\|_{H\times H} \leq r_0$, for each $t \geq t_0$ and $\lambda \in \Lambda$. □

Remark 3.1. *The constants r_0 and t_0 in Lemma 3.2 are independents of the initial values and can be chosen uniformly on $[0,\lambda_0]$.*

Remark 3.2. *If (u_λ, v_λ) is a solution of (P_{N_λ}), then there exists a positive number $K = K(u_{0,\lambda}, v_{0,\lambda}, t_0)$ such that $\|(u_\lambda(t), v_\lambda(t))\|_{H\times H} \le K$, $\forall\, t \in [0,t_0]$. If the initial values are all in a bounded set of $H\times H$ or if $u_\lambda(0) = u_0$, $v_\lambda(0) = v_0$, $\forall\, \lambda \in [0,\lambda_0]$, then K is uniform on $[0,\lambda_0]$, i.e., we have that $\|(u_\lambda(t), v_\lambda(t))\|_{H\times H} \le K$, for each $\lambda \in [0,\lambda_0]$ and for each $t \ge t_0$. In this case we can consider $t_0 = 0$ in Lemma 3.2.*

Lemma 3.3. *There exists a bounded set B_0 in $H\times H$ such that $\mathscr{A}_\lambda \subset B_0$, $\forall\, \lambda \in [0,\lambda_0]$.*

Proof. Let $(x_\lambda, y_\lambda) \in \mathscr{A}_\lambda$. Since $\mathscr{A}_\lambda = T_\lambda(t_0)\mathscr{A}_\lambda$, where T_λ is the multivalued semiflow defined by $\mathbb{G}_\lambda$. Then, from Lemma 3.2, we have $\|(x_\lambda, y_\lambda)\|_{H\times H} \le r_0$. □

Thus, the family $\{\mathbb{G}_\lambda\}_{\lambda\in[0,\lambda_0]}$ satisfies condition (i) in Theorem 3.3. Now by using Lemma 3.2 and the fact that F and G are bounded, we can repeat the arguments used in [6], Lemma 2.2 for each equation in (P_λ^1) to obtain:

Lemma 3.4. *If $\varphi_\lambda \doteq (u_\lambda, v_\lambda) \in \mathbb{G}_\lambda$, then there exist positive constants $k > 0$ and $t_1 > t_0$, independents of λ, such that*

$$\|\varphi_\lambda(t)\|_{W^{1,p}(\Omega)\times W^{1,q}(\Omega)} = \|u_\lambda(t)\|_{W^{1,p}(\Omega)} + \|v_\lambda(t)\|_{W^{1,q}(\Omega)} < k,$$

for every $t \ge t_1$ and $\lambda \in [0,\lambda_0]$, where t_0 is the positive constant in the Lemma 3.2.

Remark 3.3. *If $(u_\lambda, v_\lambda) \in \mathbb{G}_\lambda$ with initial values all in a bounded set of $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$ or if $u_\lambda(0) = u_0$, $v_\lambda(0) = v_0$, $\forall\, \lambda \in [0,\lambda_0]$, then there exists a positive constant $\widetilde{K}$ such that $\|(u_\lambda(t), v_\lambda(t))\|_{W^{1,p}(\Omega)\times W^{1,q}(\Omega)} \le \widetilde{K}$, $\forall\, \lambda \in [0,\lambda_0]$ and $\forall\, t \in [0,t_1]$. In this case we can consider $t_1 = 0$ in Lemma 3.4.*

As a consequence of Lemma 3.4 we have that $\cup_{\lambda\in\Lambda}\mathscr{A}_\lambda$ is a bounded subset of $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$ and so we can conclude the following:

Lemma 3.5. *$\mathscr{A} \doteq \overline{\cup_{\lambda\in\Lambda}\mathscr{A}_\lambda}$ is a compact subset of $H\times H$.*

Theorem 3.4. *The map $\lambda \mapsto T_\lambda(t)(\mathscr{A})$ is w-upper semicontinuous at $\lambda_1 \doteq 0$ for each $t > 0$.*

Proof. Suppose, on contrary, that there exists a number $t_0 > 0$ such that the map $\lambda \mapsto T_\lambda(t_0)(\mathscr{A})$ is not w-upper semicontinuous at $\lambda_1 \doteq 0$. So, there exists a γ-neighborhood $O_\gamma(T_0(t_0)(\mathscr{A}))$ such that for each $n \in \mathbb{N}$ there exists $0 \le \lambda_n < \min\{\lambda_0, \frac{1}{n}\}$ and $\xi_{\lambda_n} \in T_{\lambda_n}(t_0)(\mathscr{A})$ with $\xi_{\lambda_n} \notin O_\gamma(T_0(t_0)(\mathscr{A}))$. (Note that $\lambda_n \to \lambda_1 = 0$ as $n \to +\infty$). Then $\xi_{\lambda_n} = \varphi_{\lambda_n}(t_0) = (u_{\lambda_n}(t_0), v_{\lambda_n}(t_0))$, $\varphi_{\lambda_n} \in \mathbb{G}_{\lambda_n}$, $\varphi_{\lambda_n}(0) \in \mathscr{A}$.

It is enough to show that there is a subsequence $\{\xi_{\lambda_{n_k}}\}$ of $\{\xi_{\lambda_n}\}$ with $\xi_{\lambda_{n_k}} \to \xi_0 \in T_0(t_0)(\mathscr{A})$, and so we obtain a contradiction.

In fact, we have that $\varphi_{\lambda_n} = (u_{\lambda_n}, v_{\lambda_n})$ is a solution of (P_{N_λ}) with $(u_{\lambda_n}(0), v_{\lambda_n}(0)) \in \mathscr{A}$. So, there exists $f_{\lambda_n}, g_{\lambda_n} \in L^1(0,T;H)$, with

$$f_{\lambda_n}(t) \in F(u_{\lambda_n}(t), v_{\lambda_n}(t)),\ g_{\lambda_n}(t) \in G(u_{\lambda_n}(t), v_{\lambda_n}(t))\ a.e. \text{ in } (0,T),$$

and such that $(u_{\lambda_n}, v_{\lambda_n})$ is a weak solution over $(0,T)$ of the system $(P^1_{\lambda_n})$ below:

$$
(P^1_{\lambda_n})\left\{\begin{array}{ll}
\dfrac{\partial u_{\lambda_n}}{\partial t} - div(D_1^{\lambda_n}|\nabla u_{\lambda_n}|^{p-2}\nabla u_{\lambda_n}) + |u_{\lambda_n}|^{p-2}u_{\lambda_n} = f_{\lambda_n} & \text{in } (0,T) \\
\dfrac{\partial v_{\lambda_n}}{\partial t} - div(D_2^{\lambda_n}|\nabla v_{\lambda_n}|^{q-2}\nabla v_{\lambda_n}) + |v_{\lambda_n}|^{q-2}v_{\lambda_n} = g_{\lambda_n} & \text{in } (0,T) \\
u_{\lambda_n}(0) = u_{0,\lambda_n}, v_{\lambda_n}(0) = v_{0,\lambda_n} &
\end{array}\right.
$$

We can suppose $t_0 \in (0,T)$. As $\mathscr{A}$ is compact $(u_{\lambda_n}(0), v_{\lambda_n}(0)) \to (u_0, v_0) \in \mathscr{A}$. Let $u_{\lambda_n}(\cdot) \doteq I(u_{0,\lambda_n})f_{\lambda_n}(\cdot)$, $v_{\lambda_n}(\cdot) \doteq I(v_{0,\lambda_n})g_{\lambda_n}(\cdot)$ and $z_{\lambda_n}(\cdot) \doteq I(u_0)f_{\lambda_n}(\cdot)$ and $w_{\lambda_n}(\cdot) \doteq I(v_0)g_{\lambda_n}(\cdot)$ be the solutions of the problems

$$
(P_{f_{\lambda_n},u_0})\left\{\begin{array}{l}
\dfrac{\partial z_{\lambda_n}}{\partial t} - div(D_1^{\lambda_n}|\nabla z_{\lambda_n}|^{p-2}\nabla z_{\lambda_n}) + |z_{\lambda_n}|^{p-2}z_{\lambda_n} = f_{\lambda_n} \\
z_{\lambda_n}(0) = u_0
\end{array}\right.
$$

and

$$
(P_{g_{\lambda_n},v_0})\left\{\begin{array}{l}
\dfrac{\partial w_{\lambda_n}}{\partial t} - div(D_2^{\lambda_n}|\nabla w_{\lambda_n}|^{q-2}\nabla w_{\lambda_n}) + |w_{\lambda_n}|^{q-2}w_{\lambda_n} = g_{\lambda_n} \\
w_{\lambda_n}(0) = v_0,
\end{array}\right.
$$

respectively.

From $(P^1_{\lambda_n})$ we obtain

$$\frac{1}{2}\parallel u_{\lambda_n}(t)\parallel_H^2 \leq \frac{1}{2}\parallel u_{0,\lambda_n}\parallel_H^2 + \int_0^t \langle f_{\lambda_n}(s), u_{\lambda_n}(s)\rangle_H ds.$$

As $\{u_{0,\lambda_n}\}$ is a convergent sequence there exists a positive constant R such that $\parallel u_{0,\lambda_n}\parallel_H^2 \leq R^2$. Thus,

$$\frac{1}{2}\parallel u_{\lambda_n}(t)\parallel_H^2 \leq \frac{1}{2}R^2 + \int_0^t \langle f_{\lambda_n}(s), u_{\lambda_n}(s)\rangle_H ds.$$

From Gronwall inequality and hypotheses on F, G we obtain that there are positive constants α, β, γ and C such that

$$
\begin{array}{rcl}
\parallel u_{\lambda_n}(t)\parallel_H & \leq & C + \gamma T \\
 & + & \int_0^t [\alpha \parallel u_{\lambda_n}(s)\parallel_H + \beta \parallel v_{\lambda_n}(s)\parallel_H] ds.
\end{array}
$$

So, there exists a positive constant M independent of t such that

$$\parallel u_{\lambda_n}(t)\parallel_H \leq M + \int_0^t [\alpha \parallel u_{\lambda_n}(s)\parallel_H + \beta \parallel v_{\lambda_n}(s)\parallel_H] ds.$$

Analogously, there exists a positive constant $\widetilde{M}$ independent of t such that

$$\parallel v_{\lambda_n}(t)\parallel_H \leq \widetilde{M} + \int_0^t [\beta \parallel u_{\lambda_n}(s)\parallel_H + \alpha \parallel v_{\lambda_n}(s)\parallel_H] ds.$$

Adding these two inequalities and denoting by $N = M + \widetilde{M}$ and $\rho = \alpha + \beta$ we have

$$\parallel u_{\lambda_n}(t)\parallel_H + \parallel v_{\lambda_n}(t)\parallel_H \leq N + \rho \int_0^t [\parallel u_{\lambda_n}(s)\parallel_H + \parallel v_{\lambda_n}(s)\parallel_H] ds$$

and by Gronwall-Bellman inequality follows that

$$\| u_{\lambda_n}(t) \|_H + \| v_{\lambda_n}(t) \|_H \leq Ne^{\rho T}$$

for all $t \in [0,T]$, and for all $n \in \mathbb{N}$.

As F and G carry bounded sets of $H \times H$ in bounded sets of H there exists $L > 0$ such that

$$\| f_{\lambda_n}(t) \|_H \leq L \qquad \text{e} \qquad \| g_{\lambda_n}(t) \|_H \leq L$$

for all $t \in [0,T]$, and for all $n \in \mathbb{N}$.

Let $K \doteq \{f_{\lambda_n}; n \in \mathbb{N}\}$, $\widetilde{K} \doteq \{g_{\lambda_n}; n \in \mathbb{N}\}$, $M(K) \doteq \{z_{\lambda_n}; n \in \mathbb{N}\}$ and $M(\widetilde{K}) \doteq \{w_{\lambda_n}; n \in \mathbb{N}\}$. Once K and $\widetilde{K}$ are bounded sets, it is easy to see they are uniformly integrable subsets.

Given $t \in (0,T]$ and $h > 0$ such that $t - h \in (0,T]$, consider the operator $T_h : M(K)(t) \to H$ defined by $T_h z_{\lambda_n}(t) = S^{\lambda_n}(h) z_{\lambda_n}(t-h)$.

Statement 1: The operator $T_h : M(K)(t) \to H$ is compact.
In fact, let $\mathscr{B}$ be a bounded subset of $M(K)(t)$. Consider the set

$$S \doteq \{S^{\lambda_n}(h) z_{\lambda_n}(t-h); n \in \mathbb{N} \text{ such that } z_{\lambda_n}(t) \in \mathscr{B}\}.$$

Our goal is to prove that S is relatively compact in H. As $W^{1,p}(\Omega) \subset\subset H \doteq L^2(\Omega)$, it is enough show that S is a bounded set in $W^{1,p}(\Omega)$. Let $z_{\lambda_n}(t) \in M(K)(t)$. We define $v_{\lambda_n} : [t-h, t] \to H$ by $v_{\lambda_n}(\tau) = S^{\lambda_n}(\tau - (t-h)) z_{\lambda_n}(t-h)$, for all $\tau \in [t-h, t]$.

Observe that v_{λ_n} is the only solution of the problem

$$\begin{cases} \dfrac{dv_{\lambda_n}}{d\tau}(\tau) + \partial \varphi^{D_1^{\lambda_n}}(v_{\lambda_n}(\tau)) = 0 & t-h \leq \tau \leq t \\ v_{\lambda_n}(t-h) = z_{\lambda_n}(t-h) & \end{cases}$$

where

$$\varphi^{D_1^{\lambda_n}}(v) \doteq \begin{cases} \dfrac{1}{p}\left[D_1^{\lambda_n} \displaystyle\int_\Omega |\nabla v|^p dx + \int_\Omega |v|^p dx\right], & v \in W^{1,p}(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \frac{d}{d\tau}\varphi^{D_1^{\lambda_n}}(v_{\lambda_n}(\ell)) &= \left\langle \partial \varphi^{D_1^{\lambda_n}}(v_{\lambda_n}(\ell)), \frac{\partial v_{\lambda_n}}{\partial \tau}(\ell) \right\rangle = -\left\langle \frac{\partial v_{\lambda_n}}{\partial \tau}(\ell), \frac{\partial v_{\lambda_n}}{\partial \tau}(\ell) \right\rangle \\ &= -\| \frac{\partial v_{\lambda_n}}{\partial \tau}(\ell) \|_H^2 \leq 0, \text{ a.e. in } [t-h, t]. \end{aligned}$$

Then, calculating the integral from $t-h$ to t, we obtain

$$\frac{1}{p}\| v_{\lambda_n}(t) \|_{W^{1,p}}^p \equiv \varphi^{D_1^{\lambda_n}}(v_{\lambda_n}(t)) \leq \varphi^{D_1^{\lambda_n}}(v_{\lambda_n}(t-h)) = \varphi^{D_1^{\lambda_n}}(z_{\lambda_n}(t-h)).$$

Thus, our work is show that $\{\varphi^{D_1^{\lambda_n}}(z_{\lambda_n}(t-h))\}$ is a bounded set, since

$$S = \{v_{\lambda_n}(t); n \in \mathbb{N} \text{ such that } z_{\lambda_n}(t) \in \mathscr{B}\}.$$

We already knows that there exists $L > 0$ such that

$$\| f_{\lambda_n}(\tau) \|_H \leq L \text{ and } \| g_{\lambda_n}(\tau) \|_H \leq L, \text{ for all } 0 \leq \tau \leq T \text{ and for all } n \in \mathbb{N},$$

and z_{λ_n} and w_{λ_n} satisfy

$$\| z_{\lambda_n}(\ell) \|_{L^2(\Omega)} \leq \| u_0 \|_{L^2(\Omega)} + \int_0^\ell \| f_{\lambda_n}(\tau) \|_{L^2(\Omega)} \, d\tau, \ \forall \, \ell \in [0,T]$$

and

$$\| w_{\lambda_n}(\ell) \|_{L^2(\Omega)} \leq \| v_0 \|_{L^2(\Omega)} + \int_0^\ell \| g_{\lambda_n}(\tau) \|_{L^2(\Omega)} \, d\tau, \ \forall \, \ell \in [0,T].$$

Then,

$$\| z_{\lambda_n}(\ell) \|_{L^2(\Omega)} \leq \| u_0 \|_{L^2(\Omega)} + L.T, \ \forall \, \ell \in [0,T] \text{ and } \forall \, n \in \mathbb{N}$$

and

$$\| w_{\lambda_n}(\ell) \|_{L^2(\Omega)} \leq \| v_0 \|_{L^2(\Omega)} + L.T, \ \forall \, \ell \in [0,T] \text{ and } \forall \, n \in \mathbb{N}.$$

Since $t-h > 0$ there exists $r > 0$ such that $t-h > r$. Using the Uniform Gronwall Lemma, we can show that there exists $\widetilde{r_1} > 0$ such that

$$\varphi^{D_1^{\lambda_n}}(z_{\lambda_n}(\theta + r)) \leq \widetilde{r_1}, \ \forall \, n \in \mathbb{N} \text{ and } \theta > 0.$$

In particular, considering $\theta = t - h - r$, we have

$$\varphi^{D_1^{\lambda_n}}(z_{\lambda_n}(t-h)) \leq \widetilde{r_1}, \ \forall \, n \in \mathbb{N}.$$

This conclude the proof of Statement 1. □

Then we have by Theorem 3.2 that the set $M(K)$ is relatively compact in $C([0,T];H)$ and so there exists $z \in C([0,T];H)$ and there exists a subsequence $\{z_{\lambda_n}(\cdot)\}$ such that $z_{\lambda_n} \to z$ in $C([0,T];H)$.

As each z_{λ_n} is a solution of $(P_{f_{\lambda_n},u_0})$, then z_{λ_n} verify

$$\frac{1}{2} \| z_{\lambda_n}(t) - \theta \|^2 \leq \frac{1}{2} \| z_{\lambda_n}(s) - \theta \|^2 + \int_s^t \langle f_{\lambda_n}(\tau) - y_{\lambda_n}, z_{\lambda_n}(\tau) - \theta \rangle d\tau \tag{2}$$

for all $\theta \in \mathscr{D}(A^{D_1^{\lambda_n}}) \subseteq W^{1,p}(\Omega) \subset H$ and $y_{\lambda_n} = A^{D_1^{\lambda_n}}(\theta)$ and for all $0 \leq s \leq t \leq T$.

Analogously, we can show that there exists $w \in C([0,T];H)$ and there exists a subsequence $\{w_{\lambda_n}(\cdot)\}$ such that $w_{\lambda_n} \to w$ em $C([0,T];H)$, verify

$$\frac{1}{2} \| w_{\lambda_n}(t) - \theta \|^2 \leq \frac{1}{2} \| w_{\lambda_n}(s) - \theta \|^2 + \int_s^t \langle g_{\lambda_n}(\tau) - y_{\lambda_n}, w_{\lambda_n}(\tau) - \theta \rangle d\tau \tag{3}$$

for all $\theta \in \mathscr{D}(B^{D_2^{\lambda_n}}) \subseteq W^{1,q}(\Omega) \subset H$ and $y_{\lambda_n} = B^{D_2^{\lambda_n}}(\theta)$ and for all $0 \leq s \leq t \leq T$.

As $\| f_{\lambda_n}(\tau) \|_H \leq L$ and $\| g_{\lambda_n}(\tau) \|_H \leq L$, for all $0 \leq \tau \leq T$ and for all $n \in \mathbb{N}$, we conclude that there exists a positive constant $\widetilde{L}$ such that

$$\| f_{\lambda_n} \|_{L^2(0,T;H)} \leq \widetilde{L} \qquad \text{and} \qquad \| g_{\lambda_n} \|_{L^2(0,T;H)} \leq \widetilde{L}, \quad \text{for all } n \in \mathbb{N}.$$

As $L^2(0,T;H)$ is a reflexive Banach space there are $f,g \in L^2(0,T;H)$ and subsequences, which we do not relabel, $\{f_{\lambda_n}\}$ and $\{g_{\lambda_n}\}$ such that $f_{\lambda_n} \rightharpoonup f$ and $g_{\lambda_n} \rightharpoonup g$ in $L^2(0,T;H)$. Consequently $f_{\lambda_n} \rightharpoonup f$ and $g_{\lambda_n} \rightharpoonup g$ in $L^1(0,T;H)$.

Statement 2: $u_{\lambda_n} \to z$ and $v_{\lambda_n} \to w$ in $C([0,T];H)$. Moreover, $f(t) \in F(z(t),w(t))$ and $g(t) \in G(z(t),w(t))$ a.e. in $[0,T]$.
In fact, let $t \in [0,T]$. We have

$$\| u_{\lambda_n}(t) - z(t) \|_H \leq \| u_{\lambda_n}(t) - z_{\lambda_n}(t) \|_H + \| z_{\lambda_n}(t) - z(t) \|_H .$$

Then,

$$\begin{aligned} \sup_{t\in[0,T]} \| u_{\lambda_n}(t) - z(t) \|_H &\leq \sup_{t\in[0,T]} \| I(u_{0,\lambda_n}) f_{\lambda_n}(t) - I(u_0) f_{\lambda_n}(t) \|_H \\ &+ \sup_{t\in[0,T]} \| z_{\lambda_n}(t) - z(t) \|_H \\ &\leq \| u_{0,\lambda_n} - u_0 \|_H + \sup_{t\in[0,T]} \| z_{\lambda_n}(t) - z(t) \|_H \to 0 \end{aligned}$$

as $n \to +\infty$. Therefore $u_{\lambda_n} \to z$ in $C([0,T];H)$. Analogously we can show that $v_{\lambda_n} \to w$ in $C([0,T];H)$.

So, by Theorem 3.3 in [5], $f(t) \in F(z(t),w(t))$ and $g(t) \in G(z(t),w(t))$ a.e. in $[0,T]$. □

Now observe that since

$$f_{\lambda_n} \rightharpoonup f \quad \text{in} \quad L^2(0,T;H)$$

implies that

$$f_{\lambda_n} \rightharpoonup f \quad \text{in} \quad L^2(s,t;H), \ \forall\, 0 \leq s \leq t \leq T;$$

and $z_{\lambda_n} \to z$ in $C([0,T];H)$ implies that $z_{\lambda_n} \to z$ in $C([s,t];H)$ and consequently

$$z_{\lambda_n} \to z \quad \text{em} \quad L^2(s,t;H), \ \forall\, 0 \leq s \leq t \leq T;$$

then

$$\langle f_{\lambda_n} - h, z_{\lambda_n} - \theta \rangle_{L^2(s,t;H)} \to \langle f - h, z - \theta \rangle_{L^2(s,t;H)}$$

for all $\theta, h \in H$.

Now consider $\overline{\theta} \in D(A^{D_1^0}) \subset W^{1,p}(\Omega) \subset H$ and let be $\overline{h} \doteq A^{D_1^0}(\overline{\theta}) \in H$. We consider $y_{\lambda_n} \doteq A^{D_1^{\lambda_n}}(\overline{\theta}) = -div(D_1^{\lambda_n} |\nabla\overline{\theta}|^{p-2}\nabla\overline{\theta}) + |\overline{\theta}|^{p-2}\overline{\theta}$. Note that $\mathscr{D}(A^{D_1^{\lambda_n}}) = D(A^{D_1^0}), \ \forall\, n \in \mathbb{N}$. We already knows by (2) that holds

$$\begin{aligned} \frac{1}{2} \| z_{\lambda_n}(t) - \overline{\theta} \|^2 &\leq \frac{1}{2} \| z_{\lambda_n}(s) - \overline{\theta} \|^2 + \int_s^t \langle f_{\lambda_n}(\tau) - y_{\lambda_n}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle d\tau \\ &= \frac{1}{2} \| z_{\lambda_n}(s) - \overline{\theta} \|^2 + \int_s^t \langle f_{\lambda_n}(\tau) - \overline{h} + \overline{h} - y_{\lambda_n}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle d\tau \\ &= \frac{1}{2} \| z_{\lambda_n}(s) - \overline{\theta} \|^2 + \int_s^t \langle f_{\lambda_n}(\tau) - \overline{h}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle d\tau \\ &+ \int_s^t \langle \overline{h} - y_{\lambda_n}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle d\tau. \end{aligned} \tag{4}$$

Statement 3: $\int_s^t \langle \overline{h} - y_{\lambda_n}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle d\tau \to 0$ as $n \to +\infty$.

In fact, consider the function $B^n(x) \doteq D_1^0(x) - D_1^{\lambda_n}(x)$. Since we are supposing that $\|B^n\|_{L^\infty(\Omega)} = \|D_1^0 - D_1^{\lambda_n}\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$, from Hölder's Inequality, we obtain for each $\tau \geq 0$, that yields the following inequality

$$\begin{aligned}
\langle \overline{h} - y_{\lambda_n}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle &= \langle \overline{h}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle - \langle y_{\lambda_n}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle \\
&= \langle B^n |\nabla \overline{\theta}|^{p-2} \nabla \overline{\theta}, \nabla(z_{\lambda_n}(\tau)) - \nabla \overline{\theta} \rangle \\
&\leq \|B^n\|_{L^\infty(\Omega)} \int_\Omega |\nabla \overline{\theta}|^{p-2} |\nabla \overline{\theta}| |\nabla(z_{\lambda_n}(\tau)) - \nabla \overline{\theta}| dx \\
&\leq \|B^n\|_{L^\infty(\Omega)} \Big[\int_\Omega |\nabla \overline{\theta}|^{p-1} |\nabla(z_{\lambda_n}(\tau))| dx + \int_\Omega |\nabla \overline{\theta}|^p dx \Big] \\
&\leq \|B^n\|_{L^\infty(\Omega)} \Big(\|\nabla \overline{\theta}\|_p \|\nabla(z_{\lambda_n}(\tau))\|_p + \|\nabla \overline{\theta}\|_p^p \Big),
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Using the Lemma 3.4 and the Remark 3.3, we have that

$$\Big(\|\nabla \overline{\theta}\|_p \|\nabla(z_{\lambda_n}(\tau))\|_p + \|\nabla \overline{\theta}\|_p^p \Big)$$

is uniformly bounded for $\lambda_n \in [0, \lambda_0]$ and $\tau \geq 0$, that is, there is a positive constant C, independent of $\lambda_n \in [0, \lambda_0]$, such that

$$\Big(\|\nabla \overline{\theta}\|_p \|\nabla(z_{\lambda_n}(\tau))\|_p + \|\nabla \overline{\theta}\|_p^p \Big) \leq C, \ \forall\ \tau \geq 0 \text{ and } \lambda_n \in [0, \lambda_0].$$

Then

$$\int_s^t \langle \overline{h} - y_{\lambda_n}, z_{\lambda_n}(\tau) - \overline{\theta} \rangle d\tau \leq \int_s^t \|B^n\|_{L^\infty(\Omega)} C d\tau \leq \|B^n\|_{L^\infty(\Omega)} CT \to 0$$

as $n \to +\infty$. □

Taking the limit in (4) as $n \to +\infty$, we obtain

$$\frac{1}{2} \parallel z(t) - \overline{\theta} \parallel^2 \leq \frac{1}{2} \parallel z(s) - \overline{\theta} \parallel^2 + \int_s^t \langle f(\tau) - \overline{h}, z(\tau) - \overline{\theta} \rangle d\tau + \varepsilon, \ \forall\ \varepsilon > 0.$$

So,

$$\frac{1}{2} \parallel z(t) - \overline{\theta} \parallel^2 \leq \frac{1}{2} \parallel z(s) - \overline{\theta} \parallel^2 + \int_s^t \langle f(\tau) - \overline{h}, z(\tau) - \overline{\theta} \rangle d\tau$$

for all $\overline{\theta} \in D(A^{D_1^0})$ and $\overline{h} \doteq A^{D_1^0}(\overline{\theta})$ and for all $0 \leq s \leq t \leq T$.

In the same way, we can show that

$$\frac{1}{2} \parallel w(t) - \overline{\theta} \parallel^2 \leq \frac{1}{2} \parallel w(s) - \overline{\theta} \parallel^2 + \int_s^t \langle g(\tau) - \overline{h}, w(\tau) - \overline{\theta} \rangle d\tau$$

for all $\overline{\theta} \in D(B^{D_2^0})$ and $\overline{h} \doteq B^{D_2^0}(\overline{\theta})$ and for all $0 \leq s \leq t \leq T$.

So $(z, w) \in \mathbb{G}_0$ with $(z(0), w(0)) = (u_0, v_0) \in \mathscr{A}$. Then,

$$(z(t), w(t)) \in T_0(t)(\mathscr{A}), \ \forall\ t \geq 0.$$

Thus, defining $\xi_0 \doteq (z(t_0), w(t_0)) \in T_0(t_0)(\mathscr{A})$, we obtain

$$\begin{aligned} \| \xi_{\lambda_n} - \xi_0 \|_{H\times H} &= \| u_{\lambda_n}(t_0) - z(t_0) \|_H + \| v_{\lambda_n}(t_0) - w(t_0) \|_H \\ &\leq \sup_{\tau\in[0,T]} \| u_{\lambda_n}(\tau) - z(\tau) \|_H + \sup_{\tau\in[0,T]} \| v_{\lambda_n}(\tau) - w(\tau) \|_H \to 0, \end{aligned}$$

as $n \to +\infty$, which is a contradiction, and so we conclude that the map

$$[0,\lambda_0] \ni \lambda \mapsto T_\lambda(t)(\mathscr{A})$$

is w-upper semicontinuous on $\lambda_1 \doteq 0$ for each $t > 0$. □

Therefore, the family $\{\mathbb{G}_\lambda\}_{\lambda\in[0,\lambda_0]}$ satisfies the condition (ii) of the Theorem 3.3. Therefore, we obtain immediately, by Theorem 3.3, the following result:

Theorem 3.5. *The family of attractors $\{\mathscr{A}_\lambda\}_{\lambda\in[0,\lambda_0]}$ associated with the problem (P_{N_λ}) is upper semicontinuous on $\lambda_1 = 0$.*

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