COMMUTING INVOLUTIONS WHOSE FIXED POINT SET CONSISTS OF TWO SPECIAL COMPONENTS

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ABSTRACT. Let F^n be a connected, smooth and closed n-dimensional manifold. We call F^n a manifold with *property* $\mathcal{H}$ when it satisfies the following property: if N^m is any smooth and closed m-dimensional manifold with $m > n$ and $T : N^m \to N^m$ is a smooth involution whose fixed point set is F^n, then $m = 2n$. Examples of manifolds with this property are: the real, complex and quaternionic even-dimensional projective spaces RP^{2n}, CP^{2n} and HP^{2n}, and the connected sum of RP^{2n} and any number of copies of $S^n \times S^n$, where S^n is the n-sphere and n is not a power of 2. In this paper we describe the equivariant cobordism classification of smooth actions $(M^m; \Phi)$ of the group Z_2^k on closed and smooth m-dimensional manifolds M^m for which the fixed point set of the action consists of two components K and L with property $\mathcal{H}$, and where $dim(K) < dim(L)$. The description is given in terms of the set of equivariant cobordism classes of involutions fixing $K \cup L$.

1. Introduction

A question that arises in equivariant cobordism is the classification, up to cobordism, of smooth Z_2^k-actions $(M^m; \Phi)$, defined on closed and smooth m-dimensional manifolds M^m, with a given condition on the fixed data of Φ. Here, Z_2^k is considered as the group generated by k commuting involutions $T_1, T_2, ..., T_k$, and the fixed data of Φ is $\eta = \bigoplus_\rho \varepsilon_\rho \to F$, where F is the fixed point set of Φ and $\eta = \bigoplus_\rho \varepsilon_\rho$ is the normal bundle of F in M^m decomposed into eigenbundles ε_ρ with ρ running through the $2^k - 1$ nontrivial irreducible representations of Z_2^k. For example, see [11] (F= the real projective space RP^{2n} and $k = 1$), [1] (F with constant dimension n, dim$(\eta)\geq n$ and $k = 1$), [4; Section 31] (F= a set of isolated points and $k = 2$), [2] (F= the union of two

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real projective spaces and $k = 1$) and [5] (F= a connected and n-dimensional manifold, dim(η)$\geq$ $(2^k - 1)n$ and any k).

An interesting feature of this question is that, in some cases, the classification for $k = 1$ completely determines the corresponding classification for any $k \geq 1$. For example, this happens when F= RP^{2n}, CP^{2n}, HP^{2n} or QP^2 = the real, complex or quaternionic even dimensional projective space, or the Cayley projective plane (see [11] and [3]); this also happens when $F = V^n \cup \{p\}$, where p is a point and V^n is any n-dimensional connected manifold with $n > 0$ (see [6]). In all these cases, the equivariant cobordism classes of Z_2^k-actions fixing F can be represented by a special set of Z_2^k-actions obtained from involutions fixing F. More precisely, this can be placed in the following general setting: let $(W;T)$ be any involution. For each r with $1 \leq r \leq k$, one may construct a special action of Z_2^k on the product $W^{2^{r-1}} = W \times ... \times W$ (2^{r-1} factors), which we denote by $\Gamma_r^k(W;T)$, in the following inductive way: first set $\Gamma_1^1(W;T) = (W;T)$. Taking $k \geq 2$ and supposing by inductive hypothesis one has constructed $\Gamma_{k-1}^{k-1}(W;T)$, define $\Gamma_k^k(W;T) = (W^{2^{k-1}};T_1,T_2,...,T_k)$, where $(W^{2^{k-1}};T_1,T_2,...,T_{k-1}) = (W^{2^{k-2}} \times W^{2^{k-2}};T_1,T_2,...,T_{k-1}) = \Gamma_{k-1}^{k-1}(W;T) \times \Gamma_{k-1}^{k-1}(W;T)$ and T_k acts switching $W^{2^{k-2}} \times W^{2^{k-2}}$. This defines $\Gamma_k^k(W;T)$ for any $k \geq 1$. Next, define $\Gamma_r^k(W;T) = (W^{2^{r-1}};T_1,T_2,...,T_k)$ setting $(W^{2^{r-1}};T_1,T_2,...,T_r) = \Gamma_r^r(W;T)$ and letting $T_{r+1},...,T_k$ act trivially. If F is a connected manifold, one has the *twist involution* $t: F \times F \to F \times F$, given by $t(x,y) = (y,x)$, and $\Gamma_r^k(F \times F;t) = (F^{2^r};T_1,T_2,...,T_k)$, where $(T_1,T_2,...,T_r)$ is the usual twist Z_2^r-action on F^{2^r} which interchanges factors and $T_{r+1},...,T_k$ act trivially. In this special case, we allow r to be zero, setting $\Gamma_0^k(F \times F;t) = (F;T_1,T_2,...,T_k)$, where each T_i is the identity involution.

Now, from a given Z_2^k-action $(M;\Phi)$, $\Phi = (T_1,...,T_k)$, we can obtain a collection of new Z_2^k-actions, described as follows: first, each automorphism $\sigma : Z_2^k \to Z_2^k$ yields a new action given by $(M;\sigma(T_1),...,\sigma(T_k))$; we denote this action by $\sigma(M;\Phi)$. Next, it was shown in [7] that if $(M;\Phi)$ has fixed data $\bigoplus_\rho \varepsilon_\rho \to F$ and one of the eigenbundles ε_θ is isomorphic to $\varepsilon'_\theta \oplus R$, where $R \to F$ is the trivial one-dimensional bundle, then there is an action

$(N;\Psi)$ with fixed data $\bigoplus_\rho \mu_\rho \to F$, where $\mu_\rho = \varepsilon_\rho$ if $\rho \neq \theta$ and $\mu_\theta = \varepsilon'_\theta$. Thus, the iterated process of removing sections may possibly enlarge the set $\{\sigma(M;\Phi), \sigma \in Aut(Z_2^k)\}$. Summarizing, from a given involution $(W;T)$, we obtain a collection of Z_2^k-actions by applying the operations $\sigma\Gamma_r^k$ on $(W;T)$ and next by removing the (possible) sections from the resultant eigenbundles. If $(W;T)$ and $(N;S)$ are Z_2-cobordant, then the actions obtained by removing the same sections from $\sigma\Gamma_r^k(W;T)$ and $\sigma\Gamma_r^k(N;S)$ are Z_2^k-cobordant, and the converse statement of this fact is also true (it suffices to look at the fixed data; see Section 2). Also, if $(W;T)$ fixes F, then every Z_2^k-action of the above collection fixes F.

Now, for a given F, denote by $\mathcal{A}_k(F)$ the set of all equivariant cobordism classes of Z_2^k-actions containing a representative $(M;\Phi)$ with F as fixed point set, and by $\mathcal{A}_k^m(F) \subset \mathcal{A}_k(F)$ the subset of m-dimensional classes. Denote by $\mathcal{B}_k(F) \subset \mathcal{A}_k(F)$ the subset of the classes obtained from $\mathcal{A}_1(F)$ through the above procedure, and by $\mathcal{B}_k^m(F) \subset \mathcal{B}_k(F)$ the subset of m-dimensional classes. For a fixed partition $\tau = \{\tau_1, ..., \tau_p\}$ $(p \geq 1)$ of the set of components of F, write F as the disjoint union $F = F_1 \cup ... \cup F_p$, where each F_i is the union of the members of τ_i. If $[(M_i^m;\Phi_i)] \in \mathrm{B}_k^m(F_i)$, $1 \leq i \leq p$, then $\cup_{i=1}^p (M_i^m;\Phi_i)$ represents a class of $\mathcal{A}_k^m(F)$; denote by $\mathcal{P}_{k,\tau}^m(F) \subset \mathcal{A}_k^m(F)$ the subset of the classes obtained in this way, and set

$$\mathcal{P}_k(F) = \bigcup_m \bigcup_\tau \mathcal{P}_{k,\tau}^m(F) \subset \mathcal{A}_k(F) .$$

Then one asks, in general, for the F for which $\mathcal{A}_k(F) = \mathcal{P}_k(F)$ (this makes precise the statement "*the classification for $k = 1$ completely determines the corresponding classification for any $k \geq 1$* "). Under this setting, the above mentioned results say that this is true for $F = V^n \cup \{p\}$, RP^{2n}, CP^{2n}, HP^{2n} and QP^2 (if F is connected, $\mathcal{P}_k(F) = \mathcal{B}_k(F)$; in the case $F = V^n \cup \{p\}$, also $\mathcal{P}_k(F) = \mathcal{B}_k(F)$, since an involution cannot have precisely one fixed point).

Recently, in [8], we introduced the following concept: given a connected, smooth and closed n-dimensional manifold F^n, we call F^n a manifold with *property $\mathcal{H}$* if every involution $(M^m;T)$ fixing F^n has $m = n$ or $m = 2n$.

This definition was inspired in [4; Theorem 27.7] and [11], where it was shown that RP^{2n} satisfies this property (similar arguments work to show that CP^{2n}, HP^{2n} and QP^2 have this property). The following facts concerning property $\mathcal{H}$ were seen: if F^n has property $\mathcal{H}$, then n is even, F^n is nonbounding, the tangent bundle over F^n does not have sections and F^n cannot be the total space of a nontrivial differentiable fibering of closed manifolds. Also, property $\mathcal{H}$ is not a cobordism invariant, but is a homotopy invariant (an intrinsic characterization of these manifolds is still an open question). In addition, we presented a lot of new examples of manifolds with this property (that is, not homotopy equivalent to the known examples RP^{2n}, CP^{2n}, HP^{2n} and QP^2): all nonbounding 2-dimensional manifolds, all simply connected and nonbounding 4-dimensional manifolds (for example, the connected sum of CP^2 and any number of copies of $S^2 \times S^2$), all nonbounding 8-dimensional manifolds M^8 with $H^1(M^8, Z_2) = 0$ and $H^2(M^8, Z_2) = 0$, all nonbounding 16-dimensional manifolds M^{16} with $H^i(M^{16}, Z_2) = 0$ for $1 \leq i \leq 4$, and the connected sum $RP^{2n}\#(S^n \times S^n)\#...\#(S^n \times S^n)$ where n is not a power of 2. Generalizing [3], where it had been shown that every Z_2^k-action fixing RP^{2n} is equivariantly cobordant to $\sigma\Gamma_r^k(RP^{2n} \times RP^{2n}; t)$ for some $\sigma : Z_2^k \to Z_2^k$ and $1 \leq r \leq k$ (again similar arguments, strongly based on the structure of the real K-theory of these spaces, work for CP^{2n}, HP^{2n} and QP^2), we showed in [8] that the same result is true for any manifold with property $\mathcal{H}$; in other words, $\mathcal{A}_k(F) = \mathcal{P}_k(F) = \mathcal{B}_k(F)$ for any F with property $\mathcal{H}$. The objective of this paper is to show that $\mathcal{A}_k(F) = \mathcal{P}_k(F)$ when F is the disjoint union of two manifolds with property $\mathcal{H}$, $F = K \cup L$, where $dim(K) < dim(L)$. At the end of the paper we give some simple examples of manifolds F for which $\mathcal{A}_k(F) \neq \mathcal{P}_k(F)$ and of pairs K, L for which the explicit computation of $\mathcal{A}_k(K \cup L)$ is obtained from our results.

2. **Preliminaries**

First we need some basic facts about Z_2^k-actions. Given a Z_2^k-action $(M;\Phi)$, $\Phi = (T_1,...,T_k)$, the fixed point set of Φ, F, is a disjoint union of closed submanifolds of M. The normal bundle of F in M, η, decomposes under Φ into the Whitney sum of the subbundles on which Z_2^k acts as one of the irreducible (nontrivial) real representations, which are all one-dimensional and can be described by homomorphisms $\rho : Z_2^k \to Z_2 = \{+1,-1\}$ which are onto: Z_2^k acts on the reals so that $g \in Z_2^k$ acts as multiplication by $\rho(g)$. In other words, $\eta = \bigoplus_\rho \varepsilon_\rho$, where ε_ρ is the subbundle of η on which Z_2^k acts in the fibers as ρ; that is, where each T_j acts as multiplication by $\rho(T_j)$, and where the sum excludes the trivial homomorphism $1 \in Hom(G, Z_2)$. Alternatively, ε_ρ is the normal bundle of F in the fixed point set F_ρ of the subgroup $kernel(\rho)$. Setting $\mathcal{P} = Hom(Z_2^k, Z_2) - \{1\}$, the fixed data of $(M;\Phi)$ can be written as $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$, the fixed set F and a list of $2^k - 1$ vector bundles over it indexed by $\mathcal{P}$. Each s-dimensional component of $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$ can be considered as an element of $\mathcal{N}_s(\prod_{\rho\in\mathcal{P}} BO(n_\rho))$, the bordism of s-dimensional manifolds with a map into a product of classifying spaces $BO(n_\rho)$ for n_ρ-dimensional vector bundles, where n_ρ denotes the dimension of ε_ρ over the component (this is the *simultaneous cobordism* between lists of vector bundles: if $\mathcal{P}$ is any finite set, two lists (indexed by $\mathcal{P}$) of vector bundles over closed n-dimensional manifolds, $(F^n;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$ and $(V^n;\{\mu_\rho\}_{\rho\in\mathcal{P}})$, are *simultaneously cobordant* if there exists a $(n+1)$-dimensional manifold W^{n+1} with boundary $\partial(W^{n+1}) = F^n \cup V^n$ (disjoint union) and a list of vector bundles over W^{n+1}, $(W^{n+1};\{\eta_\rho\}_{\rho\in\mathcal{P}})$, so that each η_ρ restricted to $F^n \cup V^n$ is equivalent to $\varepsilon_\rho \cup \mu_\rho$). According to [10], the equivariant cobordism class of $(M;\Phi)$ is determined by the simultaneous cobordism class of $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$.

For example, if $(W;T)$ is an involution fixing F and if $\eta \to F$ is the normal bundle of F in M, then the fixed data of the Z_2^k-action $\Gamma_r^k(W;T)$ described in Section 1 is $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$, where $\{\varepsilon_\rho\}_{\rho\in\mathcal{P}}$ consists of 2^{r-1} copies of $\eta \to F$,

$2^{r-1}-1$ copies of the tangent bundle $\tau \to F$ and $2^k - 2^r$ copies of the zero-dimensional bundle $0 \to F$. In terms of the representations $\rho \in \mathcal{P}$, $\varepsilon_\rho = 0$ when $H = ker(\rho)$ does not contain all the involutions $T_{r+1}, ..., T_k$ (equivalently, some T_j with $r+1 \leq j \leq k$ acts in ε_ρ as -1), $\varepsilon_\rho = \eta$ when H contains $T_{r+1}, ..., T_k$ and do not contain T_1 (equivalently, each T_j with $r+1 \leq j \leq k$ acts in ε_ρ as 1, and T_1 acts in ε_ρ as -1), and $\varepsilon_\rho = \tau$ when H contains $T_1, T_{r+1}, ..., T_k$ (equivalently, each T_j with $r+1 \leq j \leq k$ and T_1 act in ε_ρ as 1). In particular, for a given F, the fixed data of the Z_2^r-twist Z_2^k-action $(F^{2^r}; T_1, T_2, ..., T_k) = \Gamma_r^k(F \times F; t)$ is $(F; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$, where $\{\varepsilon_\rho\}_{\rho \in \mathcal{P}}$ consists of $2^r - 1$ copies of the tangent bundle $\tau \to F$ and $2^k - 2^r$ copies of the zero bundle $0 \to F$; in terms of $\rho \in \mathcal{P}$, $\varepsilon_\rho = \tau$ when each T_j with $r+1 \leq j \leq k$ acts in ε_ρ as 1, and $\varepsilon_\rho = 0$ for the remaining $\rho \in \mathcal{P}$.

Remark. Suppose that $(F; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$ is the fixed data of a Z_2^k-action $(M; \Phi)$. Denote by $\mathcal{A}$ the set of vector bundles over F which lie in $\{\varepsilon_\rho\}$. Then $(F; \{\varepsilon_\rho\})$ gives a map $\theta : \mathcal{P} \to \mathcal{A}$, and if $\sigma : Z_2^k \to Z_2^k$ is an automorphism, $\sigma(M; \Phi)$ gives rise to a new map $\mathcal{P} \to \mathcal{A}$ which is θ composed with some bijection $\mathcal{P} \to \mathcal{P}$. We note that not every bijection $\mathcal{P} \to \mathcal{P}$ gives a map $\mathcal{P} \to \mathcal{A}$ which necessarily is derived from some automorphism $Z_2^k \to Z_2^k$, since the number of such bijections may be greater than the number of bases of Z_2^k. In particular, we cannot in principle guarantee that all such maps $\mathcal{P} \to \mathcal{A}$ come from Z_2^k-actions. This is not the case, however, when $k = 2$; if $(F; \{\varepsilon_{\rho_1}, \varepsilon_{\rho_2}, \varepsilon_{\rho_3}\})$ is the fixed data of a Z_2^2-action with map $\mathcal{P} = \{\rho_1, \rho_2, \rho_3\} \to \mathcal{A}$, then all of the other possible maps $\mathcal{P} \to \mathcal{A}$ come from Z_2^2-actions, since they are derived from automorphisms $Z_2^2 \to Z_2^2$. Therefore all the assertions made in this paper concerning Z_2^2-actions will be independent of the map $\mathcal{P} \to \mathcal{A}$.

3. Individual cobordism of the bundles of the fixed data

Suppose that K and L are two manifolds with property $\mathcal{H}$ and with $dim(K) < dim(L)$. As mentioned in Section 1, we are concerned with the study of the cobordism classification of the Z_2^k-actions $(M; \Phi)$ for which the fixed point set is $K \cup L$. Let $(K; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}}) \cup (L; \{\mu_\rho\}_{\rho \in \mathcal{P}})$ be the fixed data of Φ. Our first goal is the analysis of the individual cobordism of the eigenbundles ε_ρ and μ_ρ, and in this direction Theorems 3.9 and 3.10 are the central part of this section. In the next section we will be concerned with the final task, the determination of the simultaneous cobordism of the lists $\{\varepsilon_\rho\}_{\rho \in \mathcal{P}}$ and $\{\mu_\rho\}_{\rho \in \mathcal{P}}$.

We need a lot of preliminary results, and the first five such results (Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5) are of general nature. Let $(M; \Phi)$ be a Z_2^k-action with fixed data $(F; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$, and let Ω be a subgroup of $Hom(Z_2^k, Z_2)$. Our first step will be to show that the part of the fixed data of $(M; \Phi)$ given by $(F; \{\varepsilon_\rho\}_{\rho \in \Omega \cap \mathcal{P}})$ can be realized as the fixed data of some subgroup $G \subset Z_2^k$ acting (by restriction) on the fixed point set of the restriction of Φ to some appropriate subgroup $H \subset Z_2^k$. First note that there exists a subgroup $G \subset Z_2^k$ so that the restriction $Hom(Z_2^k, Z_2) \to Hom(G, Z_2)$ maps Ω isomorphically onto $Hom(G, Z_2)$. In fact, consider the natural isomorphism $Z_2^k \to Hom(Hom(Z_2^k, Z_2), Z_2)$ given by $T \to \varphi_T$, where $\varphi_T(\rho) = \rho(T)$ for any $\rho \in Hom(Z_2^k, Z_2)$. Choose a basis $(\rho_1, \rho_2, ..., \rho_r, \xi_1, \xi_2, ..., \xi_{k-r})$ for $Hom(Z_2^k, Z_2)$ so that $(\rho_1, \rho_2, ..., \rho_r)$ is a basis for Ω, and consider $(T_1, T_2, ..., T_r, S_1, S_2, ..., S_{k-r})$ the basis for Z_2^k which corresponds to the dual basis $(\rho_1^*, \rho_2^*, ..., \rho_r^*, \xi_1^*, \xi_2^*, ..., \xi_{k-r}^*)$ of $Hom(Hom(Z_2^k, Z_2), Z_2)$ under the above isomorphism. Evidently, $(\rho_1, \rho_2, ..., \rho_r, \xi_1, \xi_2, ..., \xi_{k-r})$ is the dual basis of $(T_1, T_2, ..., T_r, S_1, S_2, ..., S_{k-r})$. Set $G =< T_1, T_2, ..., T_r >$. Since $\rho_i(T_j) = -1$ if $i = j$ and $\rho_i(T_j) = 1$ if $i \neq j$, one has that $(\rho_{1|G}, \rho_{2|G}, ..., \rho_{r|G})$ is a basis for $Hom(G, Z_2)$, thus the restriction maps Ω isomorphically onto $Hom(G, Z_2)$. Now set $H =< S_1, S_2, ..., S_{k-r} >$, $F_H =$ the fixed point set of H and $\Psi=$ the restriction of Φ to $G \times F_H$. One then has the following

Lemma 3.1. *The fixed data of the G-action $(F_H; \Psi)$ is $(F; \{\mu_{\rho'}\}_{\rho' \in \mathcal{P}'})$, where for each $\rho' \in \mathcal{P}' = Hom(G, Z_2) - \{1\}$ one has $\mu_{\rho'} = \varepsilon_\rho$, where ρ is the unique element of $\Omega \cap \mathcal{P}$ with $\rho_{|G} = \rho'$. In other words, the fixed data of G acting on the fixed set of H is F with the subbundles ε_ρ, $\rho \in \Omega \cap \mathcal{P}$, and in terms* *of* $\mathcal{P}' = Hom(G, Z_2) - \{1\}$*, these subbundles are indexed under the restriction* $\Omega \cap \mathcal{P} \to \mathcal{P}'$*.*

Proof. See Lemma 3.1 of [8]. $\square$

Lemma 3.2. *Let* $(M;\Phi)$ *be a* Z_2^k*-action with fixed data* $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$*, and take* $\rho_0 \in \mathcal{P}$*. Then there is an automorphism* $\sigma : Z_2^k \to Z_2^k$ *such that, if the fixed data of* $\sigma(M;\Phi)$ *is* $(F;\{\mu_\rho\}_{\rho\in\mathcal{P}})$*, then* $\mu_{\rho_1} = \varepsilon_{\rho_0}$*, where* $\rho_1 : Z_2^k \to Z_2 = \{+1,-1\}$ *is the representation given by* $\rho_1(T_1) = -1$ *and* $\rho_1(T_i) = 0$ *for* $i \geq 2$*.*

Proof. Choose $\tau_2, \tau_3, \cdots, \tau_k$ generating $kernel(\rho_0)$ and $\tau_1 \notin kernel(\rho_0)$. Then the automorphism $\sigma : Z_2^k \to Z_2^k$ defined by $\sigma(T_i) = \tau_i$, $1 \leq i \leq k$, clearly works. $\square$

Again considering a Z_2^k-action $(M;\Phi)$ with fixed data $(F;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$, choose $\rho_0 \in \mathcal{P}$ and let $P \subset M$ be any component of the fixed point set of $H = kernel(\rho_0)$. Denote by F_P the union of the components of F that are contained in P, and take $T \notin H$. Then the involution $(P;T)$ has F_P as fixed point set, with the normal bundle of F_P in P being $\varepsilon_{\rho_0} \to F_P$. Each nontrivial homomorphism from H into Z_2 gives rise to the pair of representations $\theta, \theta' \in \mathcal{P}$, given by $\theta = \theta' =$ the given homomorphism on H, $\theta(T) = 1$ and $\theta'(T) = -1$. One may consider the nontrivial homomorphisms from H into Z_2 as being indexed by the homomorphisms θ, and the part of the fixed data of $(M;\Phi)$ over F_P can be written as $(F_P;\varepsilon_{\rho_0},\{\varepsilon_\theta,\varepsilon_{\theta'}\}_\theta)$. Consider $RP(\varepsilon_{\rho_0}) \to F_P$ the real projective space bundle associated to $\varepsilon_{\rho_0} \to F_P$, and denote by $\xi \to RP(\varepsilon_{\rho_0})$ the line bundle of the double cover $S(\varepsilon_{\rho_0}) \to RP(\varepsilon_{\rho_0})$, $S(\varepsilon_{\rho_0})$ the sphere bundle of ε_{ρ_0}. Then one has the object

$$(RP(\varepsilon_{\rho_0});\xi,\{\varepsilon_\theta \oplus (\xi \otimes \varepsilon_{\theta'})\}_\theta) \quad ,$$

the projective space bundle $RP(\varepsilon_{\rho_0})$ and the list of bundles over it formed by the standard line bundle ξ and the bundles $\varepsilon_\theta \oplus (\xi \otimes \varepsilon_{\theta\prime})$, where here we are suppressing bundle maps. Setting $m_\theta = dim(\varepsilon_\theta \oplus (\xi \otimes \varepsilon_{\theta\prime}))$ and $j = dim(P)$, this object represents an element in the bordism group

$$\mathcal{N}_{j-1}(BO(1) \times \prod_\theta BO(m_\theta)).$$

Lemma 3.3. *The object* $(RP(\varepsilon_{\rho_0}); \xi, \{\varepsilon_\theta \oplus (\xi \otimes \varepsilon_{\theta'})\}_\theta)$*, which is the union of the corresponding objects over each component of* F_P*, bounds as an element of* $\mathcal{N}_{j-1}(BO(1) \times \prod_\theta BO(m_\theta))$.

Proof. This follows from the argument outlined in [6; Section 3; pages 88, 89 and 90] (or in [5; Section 2; pages 107 and 108]), adapted to the situation in which F_P may have several components. □

Lemma 3.4. *Let* F *be a connected and closed manifold,* $\varepsilon \to F$ *a vector bundle over* F *with* $dim(\varepsilon) = dim(F)$ *and* $\xi \to RP(\varepsilon)$ *the usual line bundle. Suppose that* $\{\kappa_\rho\}$ *and* $\{\kappa'_\rho\}$ *are lists of vector bundles over* F*, indexed by the same set* $\mathcal{Q}$*, such that the list* $(RP(\varepsilon); \xi, \{\kappa_\rho \oplus (\xi \otimes \kappa'_\rho)\}_{\rho \in \mathcal{Q}})$ *is a (simultaneous) boundary. Then the list* $(F; \varepsilon, \{\kappa_\rho, \kappa'_\rho\}_{\rho \in \mathcal{Q}})$ *is simultaneously cobordant to the list* $(F; \tau, \{\kappa_\rho, \kappa'_\rho\}_{\rho \in \mathcal{Q}})$*, where* τ *is the tangent bundle of* F.

Proof. The proof is exactly the argument involving characteristic numbers used in the proof of part b) of the lemma of [5; Section 3; page 108]. □

Lemma 3.5. *Let* $(M; \Phi)$ *be a* Z_2^k*-action whose fixed point set consists of two components,* K *and* L*, and let* $(K; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}}) \cup (L; \{\mu_\rho\}_{\rho \in \mathcal{P}})$ *be the fixed data of* Φ*. Suppose that, for every* $\rho \in \mathcal{P}$*,* $dim(\varepsilon_\rho) = dim(K)$ *(in particular,* $dim(M) = 2^k dim(K)$*)and either* $\mu_\rho = 0$ *or* $dim(\mu_\rho) + dim(L) \neq dim(\varepsilon_\rho) + dim(K)$*. Then the list* $(K; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$ *is simultaneously cobordant to the list* $(K; \{\tau_\rho\}_{\rho \in \mathcal{P}})$*, where, for each* $\rho \in \mathcal{P}$*,* τ_ρ *is the tangent bundle of* K.

Proof. If $\mu_\rho = 0$, L is a component of the fixed point set of $kernel(\rho)$, and thus the component of $kernel(\rho)$ containing K does not contain L. If $dim(\mu_\rho) + dim(L) \neq dim(\varepsilon_\rho) + dim(K)$, the component of $kernel(\rho)$ containing K (with dimension $dim(\varepsilon_\rho) + dim(K) = 2dim(K)$) is different from the component of $kernel(\rho)$ containing L (with dimension $dim(\mu_\rho) + dim(L)$). Thus, for every $\rho \in \mathcal{P}$, the component of $kernel(\rho)$ containing K does not contain L. By Lemma 3.3, the list $(RP(\varepsilon_\rho); \xi, \{\varepsilon_\theta \oplus (\xi \otimes \varepsilon_{\theta'})\}_\theta)$ then is a simultaneous boundary for any $\rho \in \mathcal{P}$. By iteratively applying Lemma 3.4 $2^k - 1$ times, one then gets the result. □

Returning to the actions $(M;\Phi)$ with fixed data $(K;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})\cup(L;\{\mu_\rho\}_{\rho\in\mathcal{P}})$, where K and L have property $\mathcal{H}$, set $dim(M)=m$, $dim(K)=p\ <\ dim(L)=q$. To ease the notation, for a vector bundle θ over K (over L), write $\theta\equiv\eta$ when there exists an involution $(W;T)$ fixing $K\cup L$ and θ is the normal bundle of K (of L) in M; if W is not connected, we require that all components of W have the same dimension. Also write 0 for the zero bundle over K (over L) and $\theta\equiv\tau$ when θ is cobordant to the tangent bundle over K (over L).

Lemma 3.6. $(\varepsilon_\rho,\mu_\rho)\equiv(\eta,\eta),(\tau,\tau),(0,0),(\tau,0)$ *and* $(0,\tau))$ *are the possibilities for the cobordism types of the pairs* $(\varepsilon_\rho,\mu_\rho)$, $\rho\in\mathcal{P}$.

Proof. For each $\rho\in\mathcal{P}$, denote by U_ρ and V_ρ the components of the fixed point set of the subgroup $kernel(\rho)$ containing K and L, respectively. Then either $U_\rho=V_\rho$ or $U_\rho\cap V_\rho=\emptyset$. Taking $T\in Z_2^k-ker(\rho)$, in the first case one has that (U_ρ,T) is an involution fixing $K\cup L$, which means that $(\varepsilon_\rho,\mu_\rho)\equiv(\eta,\eta)$. In the second case, (U_ρ,T) and (V_ρ,T) are involutions fixing K and L, respectively. Since K has property $\mathcal{H}$, $\dim(U_\rho)=p$ or $2p$. If $\dim(U_\rho)=p$, $(U_\rho,T)=(K,Id)$ and $\varepsilon_\rho=0$. If $\dim(U_\rho)=2p$, (U_ρ,T) is cobordant to $(K\times K;t)$ and $\varepsilon_\rho\equiv\tau$ (see [1]). Similarly, $\dim(V_\rho)=q$ or $2q$, $\mu_\rho=0$ when $\dim(V_\rho)=q$ and $\mu_\rho\equiv\tau$ when $\dim(V_\rho)=2q$. Thus, $(\varepsilon_\rho,\mu_\rho)\equiv(\tau,\tau),(0,0),(\tau,0)$ or $(0,\tau)$ in the second case (by dimensional reasons, all these types are different, with the exception of (η,η) and $(\tau,0)$ when $dim(L)=2dim(K)$). □

For Z_2^2-actions with fixed data $(K;\{\varepsilon_{\rho_1},\varepsilon_{\rho_2},\varepsilon_{\rho_3}\})\cup(L;\{\mu_{\rho_1},\mu_{\rho_2},\mu_{\rho_3}\})$, we additionaly have the following

Lemma 3.7. *Suppose that* $(\varepsilon_{\rho_i},\mu_{\rho_i})\equiv(\eta,\eta)$ *for at least one* $i\in\{1,2,3\}$. *Then the only possibilities for the cobordism types of the objects* $((\varepsilon_{\rho_1},\varepsilon_{\rho_2},\varepsilon_{\rho_3}),(\mu_{\rho_1},\mu_{\rho_2},\mu_{\rho_3}))$ *are, up to permutations,* $((\eta,\eta,\tau),(\eta,\eta,\tau))$ *and* $((\eta,0,0),(\eta,0,0))$.

Proof. Using Lemma 3.6 and the facts that $p<q$, $p+\sum_{i=1}^{3}dim(\varepsilon_{\rho_i})=q+\sum_{i=1}^{3}dim(\mu_{\rho_i})$ and $p+dim(\varepsilon_{\rho_i})=q+dim(\mu_{\rho_i})$ when $(\varepsilon_{\rho_i},\mu_{\rho_i})\equiv(\eta,\eta)$, all the possibilities are easily showed to be impossible by dimensional reasons, with the exception of the two possibilities listed above. □

In Lemma 3.6, if $U_\rho \cap V_\rho = \emptyset$ for every $\rho \in \mathcal{P}$, the argument shows that the possibilities for $(\varepsilon_\rho, \mu_\rho)$ are $(\tau,\tau),(0,0),(\tau,0)$ and $(0,\tau)$. We will be first concerned with this case; to do this, recall the following

Lemma 3.8. *Suppose $(M;\Phi)$, $\Phi = (T_1,...,T_k)$, is a Z_2^k-action fixing F, where F has property $\mathcal{H}$. Then $(M;\Phi)$ is equivariantly cobordant to $\sigma\Gamma_r^k(F\times F;t)$ for some automorphism $\sigma : Z_2^k \to Z_2^k$ and some $0 \le r \le k$.*

Proof. As mentioned in Section 1, this is the main result of [8]. □

Returning to our previous notations, one then has the following

Theorem 3.9. *Suppose $U_\rho \cap V_\rho = \emptyset$ for every $\rho \in \mathcal{P}$. Then there exist $1 \le r \le k$, $0 \le s \le k$ and automorphisms $\sigma, \sigma' : Z_2^k \to Z_2^k$ so that $(M;\Phi)$ is equivariantly cobordant to $\sigma\Gamma_r^k(K\times K;t) \cup \sigma'\Gamma_s^k(L\times L;t)$.*

Proof. Set M_K and M_L for the components of M containing K and L, respectively. Since $(M-(M_K\cup M_L);\Phi)$ is a Z_2^k-action without fixed points, the main result of [10] says that $(M-(M_K\cup M_L);\Phi)$ bounds as a manifold with Z_2^k-action. Thus we can suppose, without loss of generality, that $M = M_K \cup M_L$. If $M_K \cap M_L = \emptyset$, we obtain the desired result by applying Lemma 3.8 to the actions $(M_K;\Phi)$ and $(M_L;\Phi)$ (and in this case we can have $r=1$ and $s=0$). Therefore we can assume $M = M_K = M_L$ connected (and $m > q > p$). Since

$$dim(K) + \sum_{\rho\in\mathcal{P}} dim(\varepsilon_\rho) = dim(L) + \sum_{\rho\in\mathcal{P}} dim(\mu_\rho),$$

the number of bundles $\varepsilon_\rho \equiv \tau$ is ≥ 2. Let Ω be the subset of $Hom(Z_2^k, Z_2)$ given by $\Omega = \{1\} \cup \{\rho \in \mathcal{P} \,|\, \varepsilon_\rho \equiv \tau\}$. We assert that Ω is a subgroup of $Hom(Z_2^k, Z_2)$. In fact, take $\rho_1, \rho_2 \in \Omega$ with $\rho_1 \ne \rho_2$, and suppose $\varepsilon_{\rho_3} = 0$, where $\rho_3 = \rho_1\rho_2$. Since $\{1,\rho_1,\rho_2,\rho_3\}$ is a subgroup of $Hom(Z_2^k, Z_2)$, by Lemma 3.1 there exist subgroups $G, H \subset Z_2^k$ with G isomorphic to Z_2^2 and $Z_2^k = G \oplus H$, so that the fixed data of the Z_2^2-action obtained by letting G act on the fixed point set F_H of H is $(K;\{\varepsilon_{\rho_1},\varepsilon_{\rho_2},\varepsilon_{\rho_3}\}) \cup (L;\{\mu_{\rho_1},\mu_{\rho_2},\mu_{\rho_3}\})$. Again we can write $F_H = F_K \cup F_L$, where F_K and F_L are, respectively, the components of F_H containing K and L. If $F_K \cap F_L = \emptyset$, the Z_2^2-action $(F_K;\Phi_{|G})$ has fixed data $(K;\{\varepsilon_{\rho_1},\varepsilon_{\rho_2},\varepsilon_{\rho_3}\})$ with $\varepsilon_{\rho_1}\equiv\tau$, $\varepsilon_{\rho_2}\equiv\tau$ and $\varepsilon_{\rho_3}=0$, thus contradicting Lemma 3.8. Therefore $F_K=F_L$ and the Z_2^2-action $(F_K;\Phi_{|G})$ has fixed data $(K;\{\varepsilon_{\rho_1},\varepsilon_{\rho_2},\varepsilon_{\rho_3}\})\cup(L;\{\mu_{\rho_1},\mu_{\rho_2},\mu_{\rho_3}\})$ with at least one μ_{ρ_i} being nonzero. Because $dim(F_K)=dim(K)+dim(\varepsilon_{\rho_1})+dim(\varepsilon_{\rho_2})+dim(\varepsilon_{\rho_3})=3p=dim(L)+dim(\mu_{\rho_1})+dim(\mu_{\rho_2})+dim(\mu_{\rho_3})\geq 2q$ and $q>p$, one necessarily has $dim(L)+dim(\mu_{\rho_1})+dim(\mu_{\rho_2})+dim(\mu_{\rho_3})=2q$ and there is only one μ_{ρ_i} with $\mu_{\rho_i}\equiv\tau$. By Lemma 3.2, there is then an automorphism $\sigma:Z_2^2\to Z_2^2$ such that $(L;\{\mu_{\rho_1},\mu_{\rho_2},\mu_{\rho_3}\})$ is simultaneously cobordant to the fixed data of the Z_2^2-action $\sigma\Gamma_1^2(L\times L;t)$, and hence $(F_K;\Phi_{|G})\cup\sigma\Gamma_1^2(L\times L;t)$ is equivariantly cobordant to a Z_2^2-action with fixed data $(K;\{\varepsilon_{\rho_1},\varepsilon_{\rho_2},\varepsilon_{\rho_3}\})$. This again contradicts Lemma 3.8, which forces $\varepsilon_{\rho_3}\equiv\tau$ and Ω to be a subgroup of $Hom(Z_2^k,Z_2)$. In particular, the number of bundles ε_ρ with $\varepsilon_\rho\equiv\tau$ is 2^r-1 for some $2\leq r\leq k$ (r is the dimension of Ω as Z_2-vector space) and $m=2^rp$. By Lemma 3.1, there exist subgroups $G,H\subset Z_2^k$ with G isomorphic to Z_2^r and $Z_2^k=G\oplus H$, so that the fixed data of G acting on the fixed point set F_H of H is $(K;\{\varepsilon_\rho\}_{\rho\in\Omega\cap\mathcal{P}})\cup(L;\{\mu_\rho\}_{\rho\in\Omega\cap\mathcal{P}})$. As in the previous argument, write $F_H=F_K\cup F_L$. Since $\varepsilon_\rho=0$ for every $\rho\in\mathcal{P}-(\Omega-\{1\})$, $M=F_K=F_L=F_H$. Then one has a Z_2^r-action $(M;\Phi_{|G})$ with fixed data $(K;\{\varepsilon_\rho\}_{\rho\in\Omega\cap\mathcal{P}})\cup(L;\{\mu_\rho\}_{\rho\in\Omega\cap\mathcal{P}})$ where, for each $\rho\in\Omega\cap\mathcal{P}$, $dim(\varepsilon_\rho)=dim(K)$ and either $\mu_\rho=0$ or $dim(\mu_\rho)+dim(L)=2q>2p=dim(\varepsilon_\rho)+dim(K)$. By Lemma 3.5, it follows that the list $(K;\{\varepsilon_\rho\}_{\rho\in\Omega\cap\mathcal{P}})$ is simultaneously cobordant to the list $(K;\{\tau_\rho\}_{\rho\in\Omega\cap\mathcal{P}})$, where, for each $\rho\in\Omega\cap\mathcal{P}$, τ_ρ is the tangent bundle $\tau\to K$. Now choose a basis $(T_1',...,T_r',T_{r+1}'',...,T_k'')$ for Z_2^k so that $(T_1',...,T_r')$ is a basis for G and $(T_{r+1}'',...,T_k'')$ is a basis for H. Consider the automorphism $\varphi:G\to G$ where $\varphi(T_i)=T_i'$ if $1\leq i\leq r$ and $\varphi(T_i)=T_i''$ if $r<i\leq k$, and the Z_2^k-action $\varphi(M^m;\Phi)$. To describe the part of the fixed data of this action over K, note that if $\rho\in\mathcal{P}$ is the trivial homomorphism on H, then $\rho\in\Omega$ and thus $\varepsilon_\rho\equiv\tau$; otherwise, $\rho\notin\Omega$, which means that $\varepsilon_\rho=0$. Since the list $\{\varepsilon_\rho\}_{\rho\in\Omega\cap\mathcal{P}}$ is simultaneously cobordant to the list $\{\tau_\rho\}_{\rho\in\Omega\cap\mathcal{P}}$, the part of the fixed data of $\varphi(M^m;\Phi)$ over K then is simultaneously cobordant to the list $\{\varepsilon_\rho\}_{\rho\in\mathcal{P}}$ given by $\varepsilon_\rho=\tau$ when ρ is the trivial homomorphism on H and $\varepsilon_\rho=0$ otherwise, which

in turn is the fixed data of the Z_2^k-action $\Gamma_r^k(K \times K; t)$. Setting $\sigma = \varphi^{-1}$, one then has that the Z_2^k-action $(M; \Phi) \cup \sigma\Gamma_r^k(K \times K; t)$ is equivariantly cobordant to a Z_2^k-action fixing L. By Lemma 3.8, this action is equivariantly cobordant to $\sigma'\Gamma_s^k(L \times L; t)$ for some automorphism $\sigma' : Z_2^k \to Z_2^k$ and some $1 \leq s \leq k$, which gives the result. □

Remark. Note that, if $(M; \Phi)$ is equivariantly cobordant to $\sigma\Gamma_r^k(K \times K; t) \cup \sigma'\Gamma_s^k(L \times L; t)$, then $dim(M) = 2^r dim(K) = 2^s dim(L)$. Writing $dim(K) = 2^a b$ and $dim(L) = 2^c d$, where b and d are odd, one then has $b = d$. Thus, if $b \neq d$, there is no action of the above type fixing $K \cup L$.

Using the terminology of Section 1, Theorem 3.9 says that, under the hypotheses in question, $(M; \Phi)$ belongs to $\mathcal{P}_{k,\tau}^m(K \cup L) \subset \mathcal{P}_k(K \cup L)$, where τ is the partition $\{\{K\}, \{L\}\}$. Therefore we can assume from now that there exists at least one $\rho \in \mathcal{P}$ for which $U_\rho = V_\rho$. Writing as before $M = M_K \cup M_L$, one has in this case $M = M_K = M_L$ connected because $M_K \supset U_\rho = V_\rho \subset M_L$. The following result closes the task proposed in this section.

Theorem 3.10. *Let $(M; \Phi)$ be a Z_2^k-action with fixed data $(K; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}}) \cup (L; \{\mu_\rho\}_{\rho \in \mathcal{P}})$ as above. Write $\mathcal{P}_1 = \{\rho \in \mathcal{P} \ / \ (\varepsilon_\rho, \mu_\rho) \equiv (\eta, \eta)\}$, $\mathcal{P}_2 = \{\rho \in \mathcal{P} \ / \ (\varepsilon_\rho, \mu_\rho) \equiv (\tau, \tau)\}$ and $\mathcal{P}_3 = \{\rho \in \mathcal{P} \ / \ (\varepsilon_\rho, \mu_\rho) = (0, 0)\}$. Then $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$, $\mathcal{P}_1$ has 2^{r-1} elements and $\mathcal{P}_2$ has $2^{r-1} - 1$ elements for some $1 \leq r \leq k$ (consequently, $\mathcal{P}_3$ has $2^k - 2^r$ elements). Further, there is an automorphism $\sigma : Z_2^k \to Z_2^k$ such that the pairs $(\varepsilon_\rho, \mu_\rho)$ are indexed by $\mathcal{P}$ following the pattern of an action of the type $\sigma\Gamma_r^k(W; T)$, where $(W; T)$ is an involution fixing $K \cup L$ with W connected.*

Proof. By hypothesis, there is at least one $\rho_0 \in \mathcal{P}$ for which $U_{\rho_0} = V_{\rho_0}$, and $(\varepsilon_{\rho_0}, \mu_{\rho_0}) \equiv (\eta, \eta)$. For any $\rho \in \mathcal{P} - \{\rho_0\}$, we apply Lemma 3.1 to the subgroup $\{1, \rho_0, \rho, \rho_0\rho\} \subset Hom(Z_2^k, Z_2)$ to get that there exist subgroups $G, H \subset Z_2^k$ with G isomorphic to Z_2^2 and $Z_2^k = G \oplus H$, so that the fixed data of the Z_2^2-action obtained by letting G act on the fixed point set F_H of H is $(K; \{\varepsilon_{\rho_0}, \varepsilon_\rho, \varepsilon_{\rho_0\rho}\}) \cup (L; \{\mu_{\rho_0}, \mu_\rho, \mu_{\rho_0\rho}\})$. By the argument outlined before Lemma 3.1, $H \subset kernel(\rho_0)$; writing as before $F_H = F_K \cup F_L$, one then has

$U_{\rho_0} = V_{\rho_0} \subset F_H$, and thus $F_H = F_K = F_L$ is connected. By Lemma 3.7, one then has the following possibilities:

i) $(\varepsilon_\rho, \mu_\rho) = (0,0)$ and $(\varepsilon_{\rho_0\rho}, \mu_{\rho_0\rho}) = (0,0)$;

ii) $(\varepsilon_\rho, \mu_\rho) = (\eta, \eta)$ and $(\varepsilon_{\rho_0\rho}, \mu_{\rho_0\rho}) = (\tau, \tau)$; or

iii) $(\varepsilon_\rho, \mu_\rho) = (\tau, \tau)$ and $(\varepsilon_{\rho_0\rho}, \mu_{\rho_0\rho}) = (\eta, \eta)$.

Therefore this gives $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$; further, this gives that the rule $\rho \to \rho_0\rho$ determines a bijection between $\mathcal{P}_1$ and $\mathcal{P}_2 \cup \{1\}$, with inverse given by the same rule. We assert that $\mathcal{P}_2 \cup \{1\}$ is a subgroup of $Hom(Z_2^k, Z_2)$. In fact, take $\rho_1, \rho_2 \in \mathcal{P}_2$ with $\rho_1 \neq \rho_2$. Arguing as before, Lemma 3.1 gives a Z_2^2-action $(F_K \cup F_L; \psi)$ with fixed data $(K; \{\varepsilon_{\rho_1}, \varepsilon_{\rho_2}, \varepsilon_{\rho_1\rho_2}\}) \cup (L; \{\mu_{\rho_1}, \mu_{\rho_2}, \mu_{\rho_1\rho_2}\})$. Suppose $F_K = F_L$, and set $x = dim(\varepsilon_{\rho_1\rho_2})$, $y = dim(\mu_{\rho_1\rho_2})$. If $(\varepsilon_{\rho_1\rho_2}, \mu_{\rho_1\rho_2}) \equiv (\eta, \eta)$, the Z_2^2-action $(F_K \cup F_L; \psi)$ contradicts Lemma 3.7 because $(\tau, \tau) \neq (\eta, \eta)$. Since $3p + x = 3q + y$ and $p < q$, $x > y$ and thus $x \neq 0$. This means that we cannot have $(\varepsilon_{\rho_1\rho_2}, \mu_{\rho_1\rho_2}) \equiv (\tau, \tau)$, $(0, \tau)$ or $(0, 0)$, and it remains the possibility $(\varepsilon_{\rho_1\rho_2}, \mu_{\rho_1\rho_2}) \equiv (\tau, 0)$. In this case, $(F_K \cup F_L; \psi)$ satisfies the hypotheses of Lemma 3.5, and thus $(\varepsilon_{\rho_1}, \varepsilon_{\rho_2}, \varepsilon_{\rho_1\rho_2})$ is simultaneously cobordant to (τ_K, τ_K, τ_K), where $\tau_K \to K$ is the tangent bundle of K. It follows that the Z_2^2-action $(F_K \cup F_L; \psi) \cup \Gamma_2^2(K \times K; t)$ is equivariantly cobordant to a Z_2^2-action with fixed data $(L; \{\mu_{\rho_1}, \mu_{\rho_2}, \mu_{\rho_1\rho_2}\})$, which contradicts Lemma 3.8. Hence $F_K \cap F_L = \emptyset$, and we can apply Lemma 3.8 to the Z_2^2-actions $(F_K; \psi)$ and $(F_L; \psi)$ to conclude that $(\varepsilon_{\rho_1\rho_2}, \mu_{\rho_1\rho_2}) \equiv (\tau, \tau)$, thus showing the assertion. It follows that $\mathcal{P}_2$ has $2^{r-1} - 1$ elements and, consequently, $\mathcal{P}_1$ has 2^{r-1} elements for some $2 \leq r \leq k$.

To prove the last statement of the theorem, first note that $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \{1\}$ is a subgroup of $Hom(Z_2^k, Z_2)$ isomorphic to Z_2^r: if $\rho_1, \rho_2 \in \mathcal{P}_1$, then $\rho_0\rho_1$ and $\rho_0\rho_2$ belong to the subgroup $\mathcal{P}_2 \cup \{1\}$ and $(\rho_0\rho_1)(\rho_0\rho_2) = \rho_1\rho_2 \in \mathcal{P}_2 \cup \{1\}$. If $\rho_1 \in \mathcal{P}_1$ and $\rho_2 \in \mathcal{P}_2$, ρ_2 can be written as $\rho_2 = \rho_1'\rho_0$ with $\rho_1' \in \mathcal{P}_1$, and $\rho_1\rho_1' \in \mathcal{P}_2 \cup \{1\}$ by the previous argument; thus $\rho_1\rho_2 = \rho_0(\rho_1\rho_1') \in \mathcal{P}_1$. In this way, Lemma 3.1 gives a decomposition $Z_2^k = G \oplus H$, with G isomorphic to Z_2^r and with the Z_2^r-action $(F_H; \Phi_{|G})$ having fixed data $(K; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}_1 \cup \mathcal{P}_2}) \cup (L; \{\mu_\rho\}_{\rho \in \mathcal{P}_1 \cup \mathcal{P}_2})$. By dimensional reasons, $F_H = F_K \cup F_L = F_K = F_L = M$, and thus each element of H acts in M as the identity involution. Applying again Lemma

3.1 to the action $(M;\Phi_{|G})$ and the subgroup $\mathcal{P}_2\cup\{1\}\subset\mathcal{P}_1\cup\mathcal{P}_2\cup\{1\}$, one obtains a decomposition $G=G_1\oplus G_2$ and the action $(F_{G_2};\Phi_{|G_1})$ with fixed data $(K;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}_2})\cup(L;\{\mu_\rho\}_{\rho\in\mathcal{P}_2})$. Since G_1 is isomorphic to Z_2^{r-1}, G_2 is isomorphic to Z_2; denote by S the generator of G_2 and by $\nu\to F_{G_2}=F_S$ the normal bundle of F_{G_2} in M. One knows that S acts as -1 in ν. Since

$$\nu_{|K\cup L}=(\bigoplus_{\rho\in\mathcal{P}_1}\varepsilon_\rho\to K)\cup(\bigoplus_{\rho\in\mathcal{P}_1}\mu_\rho\to L)\ ,$$

one then has that S acts as -1 in ε_ρ and in μ_ρ for every $\rho\in\mathcal{P}_1$. Evidently, S acts as 1 in $F_{G_2}=F_S$ and thus S acts as 1 in ε_ρ and in μ_ρ for every $\rho\in\mathcal{P}_2$. Choose $\tau_{r+1},\tau_{r+2},...,\tau_k$ generating H (one may have $H=\{1\}$, and in this case this step is dispensable) and $f_2,f_3,...,f_r$ generating G_1. Consider the automorphism $\varphi:Z_2^k\to Z_2^k$ given by $\varphi(T_1)=S$, $\varphi(T_i)=f_i$ if $2\le i\le r$ and $\varphi(T_i)=\tau_i$ if $r+1\le i\le k$. Then in the fixed data of the Z_2^k-action $\varphi(M;\Phi)$ the pairs $(\varepsilon_\rho,\mu_\rho)$ are indexed by $\mathcal{P}$ following the pattern of an action of the type $\Gamma_r^k(W;T)$, where $(W;T)$ is an involution fixing $K\cup L$ with W connected. Setting $\sigma=\varphi^{-1}$, one then gets the result. $\square$

4. **Simultaneous cobordism of the fixed data**

As we have seen, a Z_2^k-action $(M;\Phi)$ as in Theorem 3.10 gives rise to a set of 2^{r-1} involutions $(U_\rho\cup V_\rho;T_\rho)$, $\rho\in\mathcal{P}_1$, where $dim(U_\rho)=dim(V_\rho)$, $T_\rho\notin kernel(\rho)$ and the fixed point set of T_ρ is $K\cup L$. These involutions determine, in turn, a subset of the set of all equivariant cobordism classes of involutions fixing $K\cup L$, $\mathcal{A}_1(K\cup L)$. In order to better understand this subset, one needs to know some additional facts about $\mathcal{A}_1(K\cup L)$. In general, for a given F which is not a boundary, $\mathcal{A}_1(F)$ may be empty (take for instance $F=S^n\cup\{point\}$, S^n the n-sphere with $n\neq 1,2,4$ and 8 [4; Theorem 27.6]) and it is always finite (this follows from the strengthened Boardman 5/2-theorem of [1]). Denote by R^j (by R) the trivial j-dimensional (one-dimensional) vector bundle over any base space. An element $[(W^n;T)]\in\mathcal{A}_1(F)$ is determined by the cobordism class of the normal bundle $\eta\to F$, which is an union of bundles over the components of F. Among all the bundles over F cobordant to η, there will be a greatest natural number $p \geq 0$ for which η is cobordant to a bundle $\kappa \oplus R^p \to F$ (p is the same over all the components). Now one knows from [4; Theorem 26.4] that if $\eta \to F$ is the fixed data of an involution and $\eta \to F$ is equivalent to $\eta' \oplus R \to F$, then $\eta' \to F$ also is the fixed data of an involution. It follows that there are involutions $(\overline{W}^{n-p+i}; T)$ fixing F for which the normal bundle of F in $\overline{W}^{n-p+i}$ is $\kappa \oplus R^i$ for $0 \leq i \leq p$, with $(\overline{W}^{n-p+p}; T)$ cobordant to $(W^n; T)$. Further, one knows how to add additional trivial bundles to the normal bundle of an involution. One may form

$$\Gamma(W^n; T) = (\frac{S^1 \times W^n}{-1 \times T}; \text{conjugation} \times 1) \ .$$

The fixed point set of this involution consists of a copy of F with normal bundle $\eta \oplus R \to F$ and a copy of W^n with normal bundle $R \to W^n$. If W^n bounds as a manifold, $R \to W^n$ bounds as a bundle and $\Gamma(W^n; T)$ is cobordant to an involution with fixed data $\eta \oplus R \to F$. This procedure of removing sections and adding trivial bundles provides a subset of $\mathcal{A}_1(F)$ with representatives having fixed data $\kappa \oplus R^i \to F$, $0 \leq i \leq p + t$, $t \geq 0$, where, in the process of adding trivial bundles, t is the first number for which the involution $(\overline{W}^{n+t}; T)$ has the underlying manifold $\overline{W}^{n+t}$ nonbounding, and $\mathcal{A}_1(F)$ is a disjoint union of subsets of this type.

Now, if $(W; T)$ has fixed data $\eta \to F$, then the fixed data of a Z_2^k-action obtained by removing sections from the fixed data of an action of the type $\sigma\Gamma_r^k(W; T)$ has 2^{r-1} eigenbundles obtained by removing sections from η. Thus, the corresponding 2^{r-1} involutions (fixing F) belong to the same subset of $\mathcal{A}_1(F)$. Also this fixed data has $2^{r-1} - 1$ eigenbundles obtained by removing sections from the tangent bundle of F; in particular, if some component of F has property $\mathcal{H}$, the tangent bundle over this component has no sections (see [8]), and thus no section can be removed from these $2^{r-1} - 1$ eigenbundles over F. By comparison, our next (and final) task then is to show that the 2^{r-1} involutions $(U_\rho \cup V_\rho; T_\rho)$, $\rho \in \mathcal{P}_1$, given by Theorem 3.10, belong to the same subset of $\mathcal{A}_1(K \cup L)$, and that the list $(K; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$ $((L; \{\mu_\rho\}_{\rho \in \mathcal{P}}))$, with the description in terms of individual cobordism given by Theorem 3.10, is simultaneously cobordant to a list over K (over L) as above described.

This will be done putting together Lemma 3.4 with the following

Lemma 4.1. *Let F be a connected and closed manifold, $\varepsilon \to F$ a vector bundle over F with $dim(\varepsilon) = dim(F)$ and $\xi \to RP(\varepsilon)$ the usual line bundle. Suppose that $\{\kappa_\rho\}$ and $\{\kappa'_\rho\}$ are lists of vector bundles over F, indexed by the same set $\mathcal{Q}$, such that the list $(RP(\varepsilon); \xi, \{\kappa_\rho \oplus (\xi \otimes \kappa'_\rho)\}_{\rho\in\mathcal{Q}})$ is a (simultaneous) boundary. Choose any $\rho_0 \in \mathcal{Q}$, and set $p = dim(\kappa_{\rho_0})$, $q = dim(\kappa'_{\rho_0})$. Then the list $(F; \varepsilon, \kappa_{\rho_0}, \kappa'_{\rho_0}, \{\kappa_\rho, \kappa'_\rho\}_{\rho\in\mathcal{Q}-\{\rho_0\}})$ is simultaneously cobordant to the list $(F; \varepsilon, \kappa_{\rho_0}, \kappa_{\rho_0} \oplus R^{q-p}, \{\kappa_\rho, \kappa'_\rho\}_{\rho\in\mathcal{Q}-\{\rho_0\}})$ when $p \le q$, and simultaneously cobordant to the list $(F; \varepsilon, \kappa'_{\rho_0} \oplus R^{p-q}, \kappa'_{\rho_0}, \{\kappa_\rho, \kappa'_\rho\}_{\rho\in\mathcal{Q}-\{\rho_0\}})$ when $p \ge q$.*

Proof. Set $n = dim(F) = dim(\varepsilon)$, $m_\rho = dim(\kappa_\rho)$ and $n_\rho = dim(\kappa'_\rho)$ for each $\rho \in \mathcal{Q}$; in particular, $m_{\rho_0} = p$ and $n_{\rho_0} = q$. One lets

$$\begin{aligned} W(F) &= 1 + w_1 + w_2 + \cdots + w_n, \\ W(\varepsilon) &= 1 + \theta_1 + \theta_2 + \cdots + \theta_n, \\ W(\kappa_\rho) &= 1 + u_1^\rho + u_2^\rho + \cdots + u_{m_\rho}^\rho \quad \text{and} \\ W(\kappa'_\rho) &= 1 + v_1^\rho + v_2^\rho + \cdots + v_{n_\rho}^\rho \end{aligned}$$

be the Stiefel-Whitney classes of F, ε, κ_ρ and κ'_ρ. Letting $c \in H^1(RP(\varepsilon); Z_2)$ be the first Stiefel-Whitney class of the line bundle ξ, one knows that the Stiefel-Whitney class of $RP(\varepsilon)$ is

$$W(RP(\varepsilon)) = (1 + w_1 + ... + w_n).\{(1+c)^n + \theta_1(1+c)^{n-1} + ... + \theta_{n-1}(1+c) + \theta_n\}\ ,$$

the Stiefel-Whitney class of ξ is $W(\xi) = 1 + c$, and the Stiefel-Whitney class of the bundle $\kappa_\rho \oplus (\xi \otimes \kappa_{\rho'})$ is

$$W(\kappa_\rho \oplus (\xi \otimes \kappa_{\rho'})) = (1 + u_1^\rho + u_2^\rho + \cdots + u_{m_\rho}^\rho).\{(1+c)^{n_\rho} + v_1^\rho(1+c)^{n_\rho-1} + \\ ... + v_{n_\rho-1}^\rho(1+c) + v_{n_\rho}^\rho\}\ .$$

Because $(RP(\varepsilon); \xi, \{\kappa_\rho \oplus (\xi \otimes \kappa'_\rho)\}_{\rho\in\mathcal{Q}})$ is a boundary, any class of dimension $2n-1$ given by a product of classes $w_i(RP(\varepsilon))$, c and $w_j(\kappa_\rho \oplus (\xi \otimes \kappa_{\rho'}))$ gives a zero characteristic number for $RP(\varepsilon)$. We will apply this using certain special classes, which are polynomials in the above classes, and were introduced in [9]. Specifically, for any r, one lets

$$W[r] = \frac{W(RP(\varepsilon))}{(1+c)^{n-r}} \quad \text{and} \quad W_\rho[r] = \frac{W(\kappa_\rho \oplus (\xi \otimes \kappa_{\rho'}))}{(1+c)^{n_\rho - r}}\ .$$

That is,

$$W[r] = (1+w_1+...+w_n)\{(1+c)^r+\theta_1(1+c)^{r-1}+...+\theta_r+\frac{\theta_{r+1}}{1+c}+...+\frac{\theta_n}{(1+c)^{n-r}}\}$$

and

$$W_\rho[r] = (1+u_1^\rho+...+u_{m_\rho}^\rho)\{(1+c)^r+v_1^\rho(1+c)^{r-1}+...+v_r^\rho+\frac{v_{r+1}^\rho}{1+c}+...+\frac{v_{n_\rho}^\rho}{(1+c)^{n_\rho-r}}\} .$$

For these classes, one has the following special properties (see [9]):

$$W[r]_{2r} = w_r c^r + \text{terms with smaller } c \text{ powers},$$
$$W[r]_{2r+1} = (w_{r+1}+\theta_{r+1})c^r + \text{terms with smaller } c \text{ powers},$$
$$W[r]_{2r+2} = \theta_{r+1}c^{r+1} + \text{terms with smaller } c \text{ powers},$$

and in the same way,

$$W_\rho[r]_{2r} = u_r^\rho c^r + \text{terms with smaller } c \text{ powers},$$
$$W_\rho[r]_{2r+1} = (u_{r+1}^\rho+v_{r+1}^\rho)c^r + \text{terms with smaller } c \text{ powers},$$
$$W_\rho[r]_{2r+2} = v_{r+1}^\rho c^{r+1} + \text{terms with smaller } c \text{ powers}.$$

For a sequence $\omega = (i_1, ..., i_s)$ of natural numbers, one lets $|\omega| = i_1 + ... + i_s$, and for $w = 1 + w_1 + ... + w_p$, one lets $w_\omega = w_{i_1}...w_{i_s}$ be the product of the classes w_i.

Then given sequences $\omega = (i_1, ..., i_s)$, $\omega' = (j_1, ..., j_t)$, $\omega_\rho = (i_1^\rho, ..., i_{s_\rho}^\rho)$ and $\omega'_\rho = (j_1^\rho, ..., j_{t_\rho}^\rho)$ for each $\rho \in \mathcal{Q}$, and a natural number $1 \leq r \leq \text{maximum}\{p, q\}$ with

$$|\omega| + |\omega'| + \sum_\rho |\omega_\rho| + \sum_\rho |\omega'_\rho| + r = n \ ,$$

one may form the class

$$X = (\textstyle\prod_{i\in\omega} W[i]_{2i}).(\prod_{i\in\omega'} W[i-1]_{2i}).$$
$$\textstyle\prod_{\rho\neq\rho_0}\{(\prod_{i\in\omega_\rho} W_\rho[i]_{2i}).(\prod_{i\in\omega'_\rho} W_\rho[i-1]_{2i})\}.$$
$$(\textstyle\prod_{i\in\omega_{\rho_0}} W_{\rho_0}[i]_{2i}).(\prod_{i\in\omega'_{\rho_0}} W_{\rho_0}[i-1]_{2i}).W_{\rho_0}[r-1]_{2r-1} \ .$$

Since X has dimension $2n-1$ and is a polynomial in the characteristic classes of $RP(\varepsilon)$, ξ and $\kappa_\rho \oplus (\xi \otimes \kappa_{\rho'})$, it gives the zero characteristic number $X[RP(\varepsilon)]$ for $RP(\varepsilon)$. From the properties above listed, one has

$$\textstyle\prod_{i\in\omega} W[i]_{2i} = W(F)_\omega.c^{|\omega|} + \text{terms with smaller } c \text{ powers},$$
$$\textstyle\prod_{i\in\omega'} W[i-1]_{2i} = W(\varepsilon)_{\omega'}.c^{|\omega'|} + \text{terms with smaller } c \text{ powers},$$
$$\textstyle\prod_{i\in\omega_\rho} W_\rho[i]_{2i} = W(\kappa_\rho)_{\omega_\rho}.c^{|\omega_\rho|} + \text{terms with smaller } c \text{ powers},$$
$$\textstyle\prod_{i\in\omega'_\rho} W_\rho[i-1]_{2i} = W(\kappa'_\rho)_{\omega'_\rho}.c^{|\omega'_\rho|} + \text{terms with smaller } c \text{ powers and}$$
$$W_{\rho_0}[r-1]_{2r-1} = (u_r^{\rho_0} + v_r^{\rho_0}).c^r + \text{terms with smaller } c \text{ powers}.$$

It follows that X has the form

$X = W(F)_\omega.W(\varepsilon)_{\omega'}.\prod_{\rho\neq\rho_0}(W(\kappa_\rho)_{\omega_\rho}.W(\kappa'_\rho)_{\omega'_\rho}).W(\kappa_{\rho_0})_{\omega_{\rho_0}}.W(\kappa'_{\rho_0})_{\omega'_{\rho_0}}.$
$(u_r^{\rho_0}+v_r^{\rho_0}).c^{n-1}$ + terms with smaller c powers.

Now if a term of dimension $2n-1$ involves a power of c less than $n-1$, it necessarily has a factor of dimension greater than n coming from the cohomology of F, which is zero. Also one knows that $H^*(RP(\varepsilon);Z_2)$ is the free $H^*(F;Z_2)$ module on $1, c, c^2, ..., c^{n-1}$. Therefore

$$0 = X[RP(\varepsilon)] = W(F)_\omega.W(\varepsilon)_{\omega'}.\prod_{\rho\neq\rho_0}(W(\kappa_\rho)_{\omega_\rho}.W(\kappa'_\rho)_{\omega'_\rho}). W(\kappa_{\rho_0})_{\omega_{\rho_0}}.W(\kappa'_{\rho_0})_{\omega'_{\rho_0}}.(u_r^{\rho_0}+v_r^{\rho_0}).c^{n-1}[RP(\varepsilon)] = W(F)_\omega.W(\varepsilon)_{\omega'}. \prod_{\rho\neq\rho_0}(W(\kappa_\rho)_{\omega_\rho}.W(\kappa'_\rho)_{\omega'_\rho}).W(\kappa_{\rho_0})_{\omega_{\rho_0}}.W(\kappa'_{\rho_0})_{\omega'_{\rho_0}}.(u_r^{\rho_0}+v_r^{\rho_0})[F]$$

and thus

$$W(F)_\omega.W(\varepsilon)_{\omega'}.\prod_{\rho\neq\rho_0}(W(\kappa_\rho)_{\omega_\rho}.W(\kappa'_\rho)_{\omega'_\rho}).W(\kappa_{\rho_0})_{\omega_{\rho_0}}.W(\kappa'_{\rho_0})_{\omega'_{\rho_0}}.u_r^{\rho_0}[F] = W(F)_\omega.W(\varepsilon)_{\omega'}.\prod_{\rho\neq\rho_0}(W(\kappa_\rho)_{\omega_\rho}.W(\kappa'_\rho)_{\omega'_\rho}).W(\kappa_{\rho_0})_{\omega_{\rho_0}}.W(\kappa'_{\rho_0})_{\omega'_{\rho_0}}.v_r^{\rho_0}[F] .$$

This says that any class $u_r^{\rho_0}$ (any class $v_r^{\rho_0}$) in a characteristic number of $(F;\varepsilon,\kappa_{\rho_0},\kappa'_{\rho_0},\{\kappa_\rho,\kappa'_\rho\}_{\rho\in\mathcal{Q}-\{\rho_0\}})$ can be replaced by $v_r^{\rho_0}$ (by $u_r^{\rho_0}$) without changing the value of the characteristic number. In particular, if $p \leq q$ and $p < r \leq q$, any class $v_r^{\rho_0}$ can be replaced by the zero class, and thus $(F;\varepsilon,\kappa_{\rho_0},\kappa'_{\rho_0}, \{\kappa_\rho,\kappa'_\rho\}_{\rho\in\mathcal{Q}-\{\rho_0\}})$ and $(F;\varepsilon,\kappa_{\rho_0},\kappa_{\rho_0}\oplus R^{q-p},\{\kappa_\rho,\kappa'_\rho\}_{\rho\in\mathcal{Q}-\{\rho_0\}})$ have the same characteristic numbers. Similarly, $(F;\varepsilon,\kappa_{\rho_0},\kappa'_{\rho_0},\{\kappa_\rho,\kappa'_\rho\}_{\rho\in\mathcal{Q}-\{\rho_0\}})$ and $(F;\varepsilon,\kappa'_{\rho_0}\oplus R^{p-q},\kappa'_{\rho_0},\{\kappa_\rho,\kappa'_\rho\}_{\rho\in\mathcal{Q}-\{\rho_0\}})$ have the same characteristic numbers when $p \geq q$, and the result follows. □

Theorem 4.2. *Let $(M;\Phi)$ be a Z_2^k-action as in Theorem 3.10. Consider $1 \leq r \leq k$ and $\sigma : Z_2^k \to Z_2^k$ the natural number and the automorphism given by Theorem 3.10. Then there exists an involution $(W;T)$ fixing $K \cup L$ with W connected such that $(M;\Phi)$ is equivariantly cobordant to a Z_2^k-action obtained by removing sections from the fixed data of $\sigma\Gamma_r^k(W;T)$. In other words, $(M;\Phi)$ belongs to $\mathcal{B}_k^m(K\cup L) \subset \mathcal{P}_k(K\cup L)$.*

Proof. The proof is continuation of the proof of Theorem 3.10. For each $\rho \in \mathcal{P}_2$, $U_\rho \cap V_\rho = \emptyset$. Then we can iteratively apply Lemmas 3.3 and 3.4 on the components U_ρ and V_ρ, $\rho \in \mathcal{P}_2$, to conclude that the list $(K;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$ (the list $(L;\{\mu_\rho\}_{\rho\in\mathcal{P}})$) is simultaneously cobordant to the list $(K;\{\varepsilon_\rho\}_{\rho\in\mathcal{P}_1\cup\mathcal{P}_3},\{\tau_\rho^K\}_{\rho\in\mathcal{P}_2})$ (to the list $(L;\{\mu_\rho\}_{\rho\in\mathcal{P}_1\cup\mathcal{P}_3},\{\tau_\rho^L\}_{\rho\in\mathcal{P}_2})$), where each τ_ρ^K, $\rho \in \mathcal{P}_2$, is the tangent bundle of K (each τ_ρ^L, $\rho\in\mathcal{P}_2$, is the tangent bundle of L). For each $\rho \in \mathcal{P}_1$, set $n_\rho = dim(\varepsilon_\rho)$ and $m_\rho = dim(\mu_\rho)$. Choose $\rho_0 \in \mathcal{P}_1$ so that $n_{\rho_0} \leq n_\rho$ for any $\rho \in \mathcal{P}_1$; since $p + n_\rho = q + m_\rho$ when $\rho \in \mathcal{P}_1$, also $m_{\rho_0} \leq m_\rho$ for any $\rho \in \mathcal{P}_1$. Taking any $\rho \in \mathcal{P}_1$ with $\rho \neq \rho_0$, one knows that $\overline{\rho} = \rho\rho_0 \in \mathcal{P}_2$ (see the proof of Theorem 3.10), which implies that $U_{\overline{\rho}} \cap V_{\overline{\rho}} = \emptyset$. Note that, if $H = kernel(\overline{\rho})$, then $\rho_{|H} = \rho_{0_{|H}}$, and if $T \notin H$, then either $\rho(T) = 1$ and $\rho_0(T) = -1$, or $\rho(T) = -1$ and $\rho_0(T) = 1$. This means that, in the argument outlined before Lemma 3.3, ρ and ρ_0 are paired with respect to $\overline{\rho}$; that is, with the notation of Lemma 3.3, $\{\rho, \rho_0\} = \{\theta, \theta'\}$ for some θ. Then we can iteratively apply Lemmas 3.3 and 4.1 on the (disjoint) components $U_{\overline{\rho}}$ and $V_{\overline{\rho}}$, where, for each $\rho \in \mathcal{P}_1 - \{\rho_0\}$, $\overline{\rho} = \rho\rho_0$, to conclude that the list $(K; \{\varepsilon_\rho\}_{\rho \in \mathcal{P}_1 \cup \mathcal{P}_3}, \{\tau_\rho^K\}_{\rho \in \mathcal{P}_2})$ is simultaneously cobordant to the list $(K; \{\varepsilon_{\rho_0} \oplus R^{n_\rho - n_{\rho_0}}\}_{\rho \in \mathcal{P}_1}, \{\tau_\rho^K\}_{\rho \in \mathcal{P}_2}, \{\varepsilon_\rho\}_{\rho \in \mathcal{P}_3})$ (to conclude that the list $(L; \{\mu_\rho\}_{\rho \in \mathcal{P}_1 \cup \mathcal{P}_3}, \{\tau_\rho^L\}_{\rho \in \mathcal{P}_2})$ is simultaneously cobordant to the list $(L; \{\mu_{\rho_0} \oplus R^{m_\rho - m_{\rho_0}}\}_{\rho \in \mathcal{P}_1}, \{\tau_\rho^L\}_{\rho \in \mathcal{P}_2}, \{\mu_\rho\}_{\rho \in \mathcal{P}_3})$). Now choose $\rho_1 \in \mathcal{P}_1$ with $n_{\rho_1} \geq n_\rho$ for every $\rho \in \mathcal{P}_1$; then also $m_{\rho_1} \geq m_\rho$ for every $\rho \in \mathcal{P}_1$. Since there is $\rho \in \mathcal{P}_1$ for which $U_\rho = V_\rho$ (that is, with $dim(\mu_\rho) > 0$), $m_{\rho_1} > 0$ and thus $U_{\rho_1} = V_{\rho_1}$. For $T \notin kernel(\rho_1)$, the involution $(U_{\rho_1}; T)$ then has fixed data $(K; \varepsilon_{\rho_1}) \cup (L; \mu_{\rho_1})$, and thus $(U_{\rho_1}; T)$ is equivariantly cobordant to an involution $(W; T)$ with W connected and with fixed data $(K; \varepsilon_{\rho_0} \oplus R^{n_{\rho_1} - n_{\rho_0}}) \cup (L; \mu_{\rho_0} \oplus R^{m_{\rho_1} - m_{\rho_0}})$. The fixed data of the Z_2^k-action $\sigma\Gamma_r^k(W; T)$ then is $(K; \{\varepsilon'_\rho\}_{\rho \in \mathcal{P}_1}, \{\tau_\rho^K\}_{\rho \in \mathcal{P}_2}, \{\varepsilon_\rho\}_{\rho \in \mathcal{P}_3}) \cup (L; \{\mu'_\rho\}_{\rho \in \mathcal{P}_1}, \{\tau_\rho^L\}_{\rho \in \mathcal{P}_2}, \{\mu_\rho\}_{\rho \in \mathcal{P}_3})$, where, for each $\rho \in \mathcal{P}_1$, $\varepsilon'_\rho = \varepsilon_{\rho_0} \oplus R^{n_{\rho_1} - n_{\rho_0}}$ and $\mu'_\rho = \mu_{\rho_0} \oplus R^{m_{\rho_1} - m_{\rho_0}}$. Thus the fixed data of $(M; \Phi)$ is simultaneously cobordant to a list obtained by removing sections from the fixed data of $\sigma\Gamma_r^k(W; T)$, and the theorem is proved. □

Examples and final remarks.

1. Let F be the space consisting of p isolated points. It is well known (see [4]) that $\mathcal{A}_k(F) = \emptyset$ for $p = 1$ and any $k \geq 1$, and $\mathcal{A}_1(F) = \emptyset$ for any $p \geq 1$ odd. However, there exist Z_2^k-actions fixing F for each $k \geq 2$ and $p \geq 3$ odd. In fact, consider the Z_2^k-actions $(RP^2; T_1, T_2, ..., T_k)$ and $(S^2; T'_1, T'_2, ..., T'_k)$, where $S^2 \subset R^3$ is the standard 2-sphere in the 3-dimensional euclidean space, $T_1[x_0, x_1, x_2] = [-x_0, x_1, x_2]$, $T_2[x_0, x_1, x_2] = [x_0, -x_1, x_2]$, $T'_1(x, y, z) = (-x, -y, z)$, $T_i = Id$ for

$i \geq 3$ and $T_i' = Id$ for $i \geq 2$, where Id means the identity map. The first action fixes three points and the second fixes two points. Write $p = 3+2t$, $t \geq 0$. Then the disjoint union of $(RP^2; T_1, T_2, ..., T_k)$ and t copies of $(S^2; T_1', T_2', ..., T_k')$ is a Z_2^k-action fixing p points. This means that $\mathcal{A}_k(F) \neq \emptyset$ for $k \geq 2$ and $p \geq 3$ odd. On the other hand, if $p \geq 3$ is odd and $\tau = \{\tau_1, ..., \tau_s\}$ $(s \geq 1)$ is a partition of the set of components of F, then some τ_{i_0} necessarily has an odd number of components. Thus, if F_{i_0} is the subspace of F corresponding to τ_{i_0}, $\mathcal{A}_1(F_{i_0}) = \emptyset$ and, consequently, $\mathcal{B}_k(F_{i_0}) = \emptyset$ and $\mathcal{P}_k(F) = \emptyset$ for every $k \geq 2$.

2. Suppose F a n-dimensional $(n \geq 1)$, connected and closed manifold having Euler characteristic $\chi(F)$ odd (for example, a cartesian product of any number of copies of even dimensional projective spaces). Let $\eta \to F$ be any s-dimensional vector bundle over F with $s \geq 3$ odd, and set $p : RP(\eta) \to F$ for the projection map. For $k \geq 2$, consider $E \subset (RP(\eta))^{2^k}$, $E = \{(x_1, x_2, ..., x_{2^k}) \in RP(\eta))^{2^k} \ / \ p(x_1) = p(x_2) = ... = p(x_{2^k})\}$. Then E is a closed $(n+2^k(s-1))$-dimensional submanifold of $(RP(\eta))^{2^k}$. On E one has the fiberwise twist Z_2^k-action $(E; \Phi)$ fixing a diagonal copy of $RP(\eta)$ and with fixed data $(RP(\eta); \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$, where each ε_ρ is the $(s-1)$-dimensional vector bundle $\mu \to RP(\eta)$ tangent along the fiber. Since $RP(\eta)$ is connected, $\mathcal{P}_k(RP(\eta)) = \mathcal{B}_k(RP(\eta))$. Because $dim(\varepsilon_\rho) > 0$ for every ρ, if $(E; \Phi) \in \mathcal{B}_k(RP(\eta))$ then $(E; \Phi)$ necessarily is, up to automorphisms of Z_2^k, equivariantly cobordant to $\Gamma_k^k(W; T)$ for some involution $(W; T)$ fixing $RP(\eta)$. But then $\mu \to RP(\eta)$ is obtained by removing n sections from some $(n+s-1)$-dimensional vector bundle $\theta \to RP(\eta)$ cobordant to the tangent bundle of $RP(\eta)$. Thus $w_{n+s-1}(\theta) = w_{n+s-1}(RP(\eta)) = 0$, where w_{n+s-1} denotes the top-dimensional Stiefel-Whitney class. Now if $W(F) = 1 + w_1 + ... + w_n$, $W(\eta) = 1 + v_1 + ... + v_s$ and $W(\xi) = 1 + c$ are the Stiefel-Whitney classes of F, η and the usual line bundle ξ, one has

$$W(RP(\eta)) = (1 + w_1 + ... + w_n)((1+c)^s + v_1(1+c)^{s-1} + ... + v_{s-1}(1+c) + v_s))$$

with $c^s + v_1c^{s-1} + ... + v_{s-1}c + v_s = 0$. Then

$$w_{n+s-1}(RP(\eta)) = w_n(\sum_{t=0}^{s-1} \binom{s-t}{s-t-1} c^{s-t-1}v_t) \ .$$

Because s is odd and by dimensional reasons one then gets $w_{n+s-1}(RP(\eta)) = w_nc^{s-1}$. Since

$$w_nc^{s-1}[RP(\eta)] = w_n[F] \equiv \chi(F) \bmod 2 \quad \text{(see [4; Lemma 27.2])},$$

this gives a contradiction, and $\mathcal{A}_k(RP(\eta)) \neq \mathcal{P}_k(RP(\eta))$.

3. Together with Theorems 3.9 and 4.2, the explicit determination of $\mathcal{A}_1(K \cup L)$ provides the explicit determination of $\mathcal{A}_k(K \cup L)$, for every $k \geq 2$. For example, in the recent paper [12], we obtained the cobordism classification of involutions whose fixed point set is $RP^2 \cup RP^{2n}$ for every $n > 1$. Therefore this gives, up to cobordism, all Z_2^k-actions with this specific fixed set. The details concerning the explicit description of $\mathcal{A}_k(RP^2 \cup RP^{2n})$ for $k \geq 2$ can be seen in [12; pages 41, 42 and 43]. As other simple examples, consider $K \cup L = RP^6 \cup CP^4$, $RP^2 \cup CP^4$ and $CP^2 \cup RP^6$. Denote by $\lambda \to RP^n$ and $\nu \to CP^m$ the canonical real and complex line bundles over RP^n and CP^m, respectively. A routine calculation of characteristic numbers based on [4; Theorem 28.1] shows that $\mathcal{A}_1(RP^6 \cup CP^4) = \emptyset$, $\mathcal{A}_1(RP^2 \cup CP^4) = \emptyset$ and $\mathcal{A}_1(CP^2 \cup RP^6) = \{[(W^7;T)], [\Gamma(W^7;T)], [\Gamma^2(W^7;T)]\}$, where $(W^7;T)$ is an involution with fixed data $(\nu \oplus R \to CP^2) \cup (\lambda \to RP^6)$ (this means that $\mathcal{A}_1(CP^2 \cup RP^6)$ has only one subset of the type previously described). Thus, up to cobordism, the possible Z_2^k-actions fixing either $RP^6 \cup CP^4$ or $RP^2 \cup CP^4$ are those given by Theorem 3.9; because of the remark after Theorem 3.9, $\mathcal{A}_k(RP^6 \cup CP^4) = \emptyset$, and $\mathcal{A}_k(RP^2 \cup CP^4)$ consists of classes whose representatives are of the form $\sigma\Gamma_r^k(RP^2 \times RP^2; twist) \cup \sigma'\Gamma_s^k(CP^4 \times CP^4; twist)$, where $k \geq 2$, $2 \leq r \leq k$ and $s = r - 2$.

Concerning $K \cup L = CP^2 \cup RP^6$, again by dimensional reasons, no action is given by Theorem 3.9. Then every Z_2^k-action fixing $CP^2 \cup RP^6$ is cobordant to a Z_2^k-action obtained by removing sections from the fixed data of an action of the type $\sigma\Gamma_r^k(\Gamma^2(W^7;T))$. For example, up to automorphisms of Z_2^2, $\mathcal{A}_2(CP^2 \cup$

RP^6) consists of the classes $[\Gamma_i^2(\Gamma^j(W^7;T))]$, $i=1,2$, $j=0,1,2$, and three more classes obtained by removing sections from $\Gamma_2^2(\Gamma^2(W^7;T))$ (with fixed data $(\nu\oplus R^3,\nu\oplus R^2,\tau)\cup(\lambda\oplus R^2,\lambda\oplus R,\tau)$, $(\nu\oplus R^3,\nu\oplus R,\tau)\cup(\lambda\oplus R^2,\lambda,\tau)$ and $(\nu\oplus R^2,\nu\oplus R,\tau)\cup(\lambda\oplus R,\lambda,\tau)$). By applying all the possible automorphisms on these representatives, we obtain a total of 36 classes for $\mathcal{A}_2(CP^2\cup RP^6)$.

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