

# The vectorial Ribaucour transformation for submanifolds and applications

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## Abstract

In this paper we develop the vectorial Ribaucour transformation for Euclidean submanifolds. We prove a general decomposition theorem showing that under appropriate conditions the composition of two or more vectorial Ribaucour transformations is again a vectorial Ribaucour transformation. An immediate consequence of this result is the classical permutability of Ribaucour transformations. Our main application is an explicit local construction of all Euclidean submanifolds with flat normal bundle. Actually, this is a particular case of a more general result. Namely, we obtain a local explicit construction of all Euclidean submanifolds carrying a parallel flat normal subbundle, in particular of all those that carry a parallel normal vector field. Finally, we describe all submanifolds carrying a Dupin principal curvature normal vector field with integrable conullity, a concept that has proven to be crucial in the study of reducibility of Dupin submanifolds.

An explicit construction of all submanifolds with flat normal bundle of the Euclidean sphere carrying a holonomic net of curvature lines, that is, admitting principal coordinate systems, was given by Ferapontov in [8]. The author points out that his construction “resembles” the vectorial Ribaucour transformation for orthogonal systems developed in [11]. The latter provides a convenient framework for understanding the permutability properties of the classical Ribaucour transformation.

This paper grew out as an attempt to better understand the connection between those two subjects, as a means of unraveling the geometry behind Ferapontov’s construction. This has led us to develop a vectorial Ribaucour transformation for Euclidean submanifolds, extending the transformation in [11] for orthogonal coordinate systems. It turns out that any  $n$ -dimensional submanifold with flat normal bundle of  $\mathbb{R}^{n+m}$  can be obtained by applying a vectorial Ribaucour transformation to an orthogonal coordinate system in an  $n$ -dimensional subspace of  $\mathbb{R}^{n+m}$ . This yields the following explicit local construction of *all*  $n$ -dimensional submanifolds with flat normal bundle of  $\mathbb{R}^{n+m}$ . Notice that carrying a principal coordinate system is not required.

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**Theorem 1.** *Let  $\varphi_1, \dots, \varphi_m$  be smooth real functions on an open simply connected subset  $U \subset \mathbb{R}^n$  satisfying*

$$[\text{Hess } \varphi_i, \text{Hess } \varphi_j] = 0, \quad 1 \leq i, j \leq m,$$

*and let  $\mathcal{G}: U \rightarrow M_{n \times m}(\mathbb{R})$  be defined by  $\mathcal{G} = (\nabla \varphi_1, \dots, \nabla \varphi_m)$ . Then for any  $x \in U$  there exists a smooth map  $\Omega: V \rightarrow \text{Gl}(\mathbb{R}^m)$  on an open subset  $V \subset U$  containing  $x$  such that  $d\Omega = \mathcal{G}^t d\mathcal{G}$  and  $\Omega + \Omega^t = \mathcal{G}^t \mathcal{G} + I$ . Moreover, the map*

$$f = \begin{pmatrix} \text{id} + \mathcal{G}\Omega^{-1}\varphi \\ \Omega^{-1}\varphi \end{pmatrix}$$

*with  $\varphi^t = (\varphi_1, \dots, \varphi_m)$  defines, at regular points, an immersion  $f: V \rightarrow \mathbb{R}^{n+m}$  with flat normal bundle.*

*Conversely, any isometric immersion  $f: M^n \rightarrow \mathbb{R}^{n+m}$  with flat normal bundle can be locally constructed in this way.*

The case of submanifolds of the sphere can be easily derived from the preceding result and the observation that any such submanifold arises as the image of a unit parallel normal vector field to a submanifold with flat normal bundle of Euclidean space (see Corollary 19). In this way we recover Ferapontov's result for the holonomic case (see Theorem 20), thus proving his guess correct.

Theorem 1 is actually a particular case of a more general result. In fact, we obtain a similar local explicit construction (see Theorem 18) of all isometric immersions  $\tilde{f}: \tilde{M}^{n+m} \rightarrow \mathbb{R}^{n+m+p}$  carrying a parallel flat normal subbundle of rank  $m$ , in particular of all those that carry a parallel normal vector field, starting with an isometric immersion  $f: M^n \rightarrow \mathbb{R}^{n+p}$  and a set of Codazzi tensors  $\Phi_1, \dots, \Phi_m$  on  $M^n$  that commute one with each other and with the second fundamental form of  $f$ . We refer the reader to [1] for results of a global nature on such isometric immersions, with strong implications for the submanifold geometry of orbits of orthogonal representations.

By putting together the preceding result with Theorem 8 of [6], we obtain an explicit construction (see Theorem 22) in terms of the vectorial Ribaucour transformation of all Euclidean submanifolds that carry a Dupin principal curvature normal vector field with integrable conullity (see Section 7 for the precise definitions), a concept that has proven to be crucial in the study of reducibility of Dupin submanifolds (see [6]).

A key feature of the Ribaucour transformation for submanifolds (in particular, orthogonal systems) is its permutability property. Namely, given two Ribaucour transforms of a submanifold, there is, generically, a fourth submanifold that is a simultaneous Ribaucour transform of the first two, giving rise to a *Bianchi quadrilateral*.

More generally, for any integer  $k \geq 2$  we define a *Bianchi  $k$ -cube* as a  $(k+1)$ -tuple  $(\mathcal{C}_0, \dots, \mathcal{C}_k)$ , where each  $\mathcal{C}_i$ ,  $0 \leq i \leq k$ , is a family of submanifolds with exactly  $\binom{k}{i}$  elements, such that every element of  $\mathcal{C}_1$  is a Ribaucour transform of the unique element of  $\mathcal{C}_0$  and such that, for every  $\hat{f} \in \mathcal{C}_{s+1}$ ,  $1 \leq s \leq k-1$ , there exist unique elements  $\hat{f}_1, \dots, \hat{f}_{s+1} \in \mathcal{C}_s$  satisfying the following conditions: