A NOTE ON S-ASYMPTOTICALLY PERIODIC FUNCTIONS

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Abstract. In [1, Lemma 2.1] is established that an scalar S-asymptotically ω-periodic function (that is, a continuous and bounded function $f : [0, \infty) \to \mathbb{R}$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$) is asymptotically ω-periodic. In this note we give two examples to show that this assertion is false.

1. Introduction

In [1, Lemma 2.1] is established that every scalar continuous and bounded function $f : [0, \infty) \to \mathbb{R}$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$, for some real number $\omega > 0$, is asymptotically ω-periodic. In this note, we provide two examples which shows that this assertion is false. For completeness, we recall some concepts and definitions.

In this note, $C_b([0, \infty), \mathbb{R})$ is the space of all continuous and bounded functions from $[0, \infty)$ into $\mathbb{R}$ endowed with the norm of the uniform convergence norm denoted by $| \cdot |_\infty$. Its subspaces, $C_0([0, \infty), \mathbb{R})$ and $C_\omega([0, \infty), \mathbb{R})$, $\omega > 0$, are defined by

$$\begin{aligned} C_0([0, \infty), \mathbb{R}) &= \left\{ x \in C_b([0, \infty), \mathbb{R}) : \lim_{t\to\infty} | x(t) |= 0 \right\}, \\ C_\omega([0, \infty), \mathbb{R}) &= \{ x \in C_b([0, \infty), \mathbb{R}) : x \text{ is } \omega\text{-periodic } \}. \end{aligned}$$

Definition 1.1. *A function $f \in C(\mathbb{R}, \mathbb{R})$ is called almost periodic if for every $\varepsilon > 0$ there exists a relatively dense subset $\mathcal{H}(\varepsilon, f)$ of $\mathbb{R}$ such that $| f(t+\xi) - f(t) |< \varepsilon$, for every $t \in \mathbb{R}$ and all $\xi \in \mathcal{H}(\varepsilon, f)$.*

Definition 1.2. *A function $f \in C_b([0, \infty), \mathbb{R})$ is called asymptotically almost periodic if there exists an almost periodic function g and $\varphi \in C_0([0, \infty), \mathbb{R})$ such that $f = g + \varphi$. If g is periodic (resp. ω-periodic) f is said asymptotically periodic (resp. asymptotically ω-periodic).*

For additional facts on almost periodic and asymptotically almost periodic functions, we refer the reader to [2, 3].

Definition 1.3. *A function $f \in C_b([0, \infty), \mathbb{R})$ is called S-asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. In this case, we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic.*

2. Examples

The following examples are contrary to [1, Lemma 2.1].

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† Corresponding author. This author thanks the financial support by CNPq, Brasil.

Example 2.1. Let $(b_n)_n$ be a sequence of real numbers such that $b_n \neq 0$, $n = 0, 1, 2, \ldots$, $b_n \to 0$ as $n \to \infty$, and the sequence $(a_n)_n = (\sum_{i=0}^{n} b_i)_n$ is bounded and non-convergent. We note that under these conditions, $a_n - a_{n-1} \to 0$ as $n \to \infty$.

Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by $f(n) = a_n$ and

$$f(t) = a_{n+1} + (a_{n+1} - a_n)(t - n - 1), \quad t \in [n, n+1], \quad n = 0, 1, 2, \ldots$$

That is, the graph of f consists of the line segments joining the points (n, a_n), $n = 0, 1, 2, \ldots$ Therefore, f is bounded. Moreover, since $|f(t)-f(s)| \leq \max_{k \geq n} |a_{k+1}-a_k||t-s|$, for every $t, s \in [n, \infty)$, it following that f is uniformly continuous and

$$\lim_{t \to \infty} |f(t + \omega) - f(t)| = 0,$$

for every $\omega > 0$. Therefore, f is an S-asymptotically ω-periodic function, for any $\omega > 0$. In particular, f is S-asymptotically 1-periodic.

However, f is not asymptotically 1-periodic. In fact, let us assume $f = g + \alpha$, where $g \in C_1([0, \infty), \mathbb{R})$ and $\alpha \in C_0([0, \infty), \mathbb{R})$. In such case, $a_n = f(n) = g(n) + \alpha(n) = g(0) + \alpha(n) \to g(0)$, as $n \to \infty$, which is contrary to the construction of $(a_n)_n$.

From the above remarks, we have that f is S-asymptotically 1-periodic but not asymptotically 1-periodic.

The following elementary lemma is immediate and plays a role in the example below.

Lemma 2.1. *If $g \in C_\omega([0, \infty), \mathbb{R})$, then $g([t, t + \omega]) = \mathcal{R}(g)$, the range of g, for any $t \in [0, \infty)$*

Example 2.2. Define $f : [0, \infty) \to \mathbb{R}$ by $f(t) = \sin \ln(t + 1)$, $t \in [0, \infty)$. Since $f'(t) = \big(\cos \ln(t + 1)\big)/(t + 1)$ we have $\lim_{t \to \infty} f'(t) = 0$.

For any $\omega > 0$, $f(t + \omega) - f(t) = f'(t + \sigma\omega)\omega$, where $0 < \sigma < 1$, which implies that $\lim_{t \to \infty} |f(t + \omega) - f(t)| = 0$. That is, f is S-asymptotically ω-periodic, for any $\omega > 0$.

But f is not asymptotically ω-periodic. In fact, given $\omega > 0$, let k be sufficiently large such that $e^{2k\pi} + \omega < e^{2k\pi + \frac{\pi}{2}}$. Note that f is increasing in $[e^{2k\pi} - 1, e^{2k\pi + \frac{\pi}{2}} - 1]$, with let $0 < \varepsilon < 1/2$, and take k larger if necessary in such a way that $t \geqslant e^{2k\pi} - 1$ implies $|\varphi(t)| < \varepsilon$. Therefore, if $g = f - \varphi$, one sees that $g([e^{2k\pi} - 1, e^{2k\pi + \frac{\pi}{2}} - 1]) \subset (-\varepsilon, 1 + \varepsilon)$. Since $g(e^{(2k + \frac{3}{2})\pi} - 1) < -1 + \varepsilon < -\varepsilon$ we have $g(e^{(2k + \frac{3}{2})\pi} - 1) < -1 + \varepsilon < -\varepsilon$ we have

$$g(e^{(2k+\frac{3}{2})\pi} - 1) \notin (-\varepsilon, 1 + \varepsilon) \supset g([e^{2k\pi} - 1, e^{2k\pi + \frac{\pi}{2}} - 1]) \supset g([e^{2k\pi} - 1, e^{2k\pi} - 1 + \omega]).$$

This means that for $t = e^{2k\pi} - 1$, $\mathcal{R}(g) \setminus g([t, t + \omega]) \neq \varnothing$ and, according to Lemma 2.1, there is not $\omega > 0$ such that g can be ω-periodic. Consequently, f cannot be asymptotically ω-periodic.

Remark 2.1. The examples above exploit the fact that the functions f under consideration are Lipschitz continuous, with the possibility of choosing the Lipschitz constant arbitrarily small by restricting f to an interval $[T, \infty)$, with $T > 0$ sufficiently large. By consequence, the S-asymptotic period $\omega > 0$ of f is arbitrary. This fact suggests the following question: *Does lemma 2.1 of [1] hold if restricted to scalar S-asymptotically ω-periodic functions that have a minimum S-asymptotic period $\omega > 0$?* The answer is negative, as one can see by taking for instance the function F given by $F(t) = f(t) + \cos t$, $t \in \mathbb{R}$, where f is the function given in example 2.2. Straightforward arguments show that F is S-asymptotically 2π-periodic, being 2π its minimum S-asymptotic period, but it is not asymptotically 2π-periodic.

References

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