On Gevrey solvability and regularity

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Abstract

In this paper we study global C^∞ and Gevrey solvability for a class of sublaplacian defined on the torus $\mathbf{T}^3$. We also prove Gevrey regularity for a class of solutions of certain operators that are globally C^∞ hypoelliptic in the N-dimensional torus.

1 Introduction

Our first goal in this paper is to study global solvability. We recall that there is an intimate connection between the concepts of global solvability and global hypoellipticity (a standard functional analysis argument applies: the global hypoellipticity of ${}^tP$ implies the global solvability of P). In order to make clear the above statement we start by recalling the definition of the Gevrey spaces.

We denote by $\mathbf{T}^N$ the N dimensional torus, i.e., $\mathbf{T}^N = \frac{\mathbf{R}^N}{2\pi \mathbf{Z}^N}$. For $s \in \mathbf{R}$ and $s \geq 1$ we say that a function $f(x) \in C^\infty(\mathbf{T}^N)$ is in the Gevrey class $G^s(\mathbf{T}^N)$ if there exists a constant $C > 0$ such that $|\partial_x^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s$, for all $\alpha \in \mathbf{Z}_+^N$, $x \in \mathbf{T}^N$. In particular, $G^1(\mathbf{T}^N)$ is the space of all periodic analytic functions, denoted by $C^\omega(\mathbf{T}^N)$. We denote by $D'_s(\mathbf{T}^N)$ the dual space of $G^s(\mathbf{T}^N)$ and its elements are called ultradistributions of order s. As usually, we also denote by $D'(\mathbf{T}^N)$ the dual space of $C^\infty(\mathbf{T}^N)$. If either $u \in D'(\mathbf{T}^N)$ or $u \in D'_s(\mathbf{T}^N)$ then one can prove that u is in $G^s(\mathbf{T}^N)$, if and only if, there exist positive constants ε and C such that

$$|\widehat{u}(\xi)| \leq Ce^{-\varepsilon|\xi|^{1/s}}, \ \ \forall \ \xi \in \mathbf{Z}^N \setminus \{0\}.$$

We set $G^\infty(\mathbf{T}^N) \doteq C^\infty(\mathbf{T}^N)$ and $D'_\infty(\mathbf{T}^N) \doteq D'(\mathbf{T}^N)$.

A linear partial differential operator P defined on $\mathbf{T}^N$ with coefficients in $C^\omega(\mathbf{T}^N)$ is said to be globally G^s, $1 \leq s \leq \infty$, hypoelliptic in $\mathbf{T}^N$ if for any $u \in D'(\mathbf{T}^N)$ the condition $Pu \in G^s(\mathbf{T}^N)$ implies that $u \in G^s(\mathbf{T}^N)$. When $s = 1$ we say that P is globally analytic hypoelliptic in $\mathbf{T}^N$. The local version of this definition is given by: if P is defined on an open set U of $\mathbf{R}^N$, then P is said to be locally G^s, $1 \leq s \leq \infty$, hypoelliptic if for any V open set of U and any $u \in D'(V)$ the condition $Pu \in G^s(V)$ implies that $u \in G^s(V)$, where we are using

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the following notation: $G^{\infty}(V) = C^{\infty}(V)$. Note that local G^s hypoellipticity implies global G^s hypoellipticity. The opposite problem is not true in general, see for example, Greenfield and Wallach [24], Gramchev, Popivanov and Yoshino [22] and Himonas and Petronilho [33].

Local and global G^s, $1 \leq s \leq \infty$, hypoellipticity has been studied by many authors, including Albanese, Corli and Rodino [1], Amano [3], Baouendi and Goulaouic [4], Bell and Mohammed [5], Bergamasco, Cordaro, and Malagutti [6], Bove and Tartakoff [8], [9], Christ [12], [13], [14], Cordaro and Himonas [15], [16], Dickinson, Gramchev and Yoshino [17], Fedii [18], Fujiwara and Omori [19], Gramchev, Popivanov and Yoshino [20], [21], [22], Greenfield and Wallach [24], Hanges and Himonas [25], [26], Helffer [27], Himonas [28], [29], Himonas and Petronilho [30], [31], [32], [33], Himonas, Petronilho and dos Santos [34], Hörmander [35], [36], Kohn [38], Metivier [39], Oleinik and Radkevic [40], Omori and Kobayashi [41], Pham The Lai and Robert [46], Rodino [47], Rothschild and Stein [48], Tartakoff [49].

In order to understand the next results we shall recall the following definitions:

Definition 1.1 A number $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ is said to be a non-Liouville number if there exist $C > 0$ and $K > 0$ such that

$$|\alpha - \frac{p}{q}| \geq \frac{C}{|q|^K}, \ \ \forall \ (p,q) \in \mathbf{Z} \times \mathbf{Z}\backslash\{0\}.$$

Definition 1.2 $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ is said to be an exponentially non-Liouville number with exponent $s \geq 1$ if for any $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$|\eta - \alpha\xi| \geq C_\epsilon e^{-\epsilon|\xi|^{1/s}}, \ \ \forall \ (\eta,\xi) \in \mathbf{Z} \times \mathbf{Z}\backslash\{0\}.$$

Definition 1.3 Let $\mathcal{M}$ be a smooth manifold of dimension n. A point $p \in \mathcal{M}$ is said to be of finite type for the vector fields $X_1, \cdots, X_m$ if among the vector fields $X_{j_1}, [X_{j_1}, X_{j_2}]$, $[X_{j_1}, [X_{j_2}, X_{j_3}]], \cdots [X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots, X_{j_k}]]], \cdots$ where $j_i = 1, \cdots, m$, there exist n which are linearly independent at p. Otherwise p is said to be of infinite type.

Now we recall the definition of global solvability.

Definition 1.4 Let $1 \leq s \leq \infty$. Let $\mathbf{E}^s(P)$ be the set

$$\begin{aligned} \mathbf{E}^s(P) &= \{f \in G^s(\mathbf{T}^N) : \int_{\mathbf{T}^N} vf = 0 \\ &\quad \text{for all } v \in G^s(\mathbf{T}^N) \text{ such that } {}^tPv = 0\}. \end{aligned}$$

We say that P is *globally G^s solvable* if for every $f \in \mathbf{E}^s(P)$ there exists $u \in D'_s(\mathbf{T}^N)$ such that $Pu = f$.

We will study global G^s solvability for a special class of operators. We consider the class of sublaplacians for that Himonas, Petronilho and dos Santos [34] have proved global $G^s, 1 \leq s \leq \infty$, hypoellipticity. More precisely, they proved the following results:

Theorem 1.5. *Let P be given by*

$$P = -\partial_{t_1}^2 - (\partial_{t_2} + a(t_1, t_2)\partial_x)^2 \tag{1.1}$$

where $(t_1, t_2, x) \in \mathbf{T}^3$ and $a \in C^\infty(\mathbf{T}^2)$ is real-valued. We set

$$a_0(t_1) = \frac{1}{2\pi} \int_{\mathbf{T}} a(t_1, s) ds.$$

Then, P is globally C^∞ hypoelliptic in $\mathbf{T}^3$ if and only if either the range of $a_0(t_1)$ contains an irrational non-Liouville number or there exists a point $p \in \mathbf{T}^3$ of finite type for the vector fields X_1 and X_2 defined by $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t_1, t_2)\partial_x$

and

Theorem 1.6. *Let $1 \leq s < \infty$ and P be given by*

$$P = -\partial_{t_1}^2 - (\partial_{t_2} + a(t_1, t_2)\partial_x)^2,$$

where $(t_1, t_2, x) \in \mathbf{T}^3$ with $a \in C^\omega(\mathbf{T}^2)$ is real-valued. We set

$$a_0(t_1) = \frac{1}{2\pi} \int_{\mathbf{T}} a(t_1, s) ds.$$

Then, P is globally G^s hypoelliptic in $\mathbf{T}^3$ if and only if either the range of $a_0(t_1)$ contains an exponentially non-Liouville number with exponent s or there exists a point $p \in \mathbf{T}^3$ of finite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t)\partial_x$.

Now we are ready to state our results.

Theorem 1.7. *Let P be given by*

$$P = -\partial_{t_1}^2 - (\partial_{t_2} + a(t_1, t_2)\partial_x)^2$$

where $(t_1, t_2, x) \in \mathbf{T}^3$ and $a \in C^\infty(\mathbf{T}^2)$ is real-valued. We set

$$a_0(t_1) = \frac{1}{2\pi} \int_{\mathbf{T}} a(t_1, s) ds.$$

Then, P is globally C^∞ solvable in $\mathbf{T}^3$ if and only if one of the following conditions holds true
I) P is globally C^∞ hypoelliptic in $\mathbf{T}^3$.
II) All points in $\mathbf{T}^3$ are of infinite type for the vector fields X_1 and X_2 and $a_0(t_1) \equiv \alpha$ with α being a rational number.

and

Theorem 1.8. *Let $1 \leq s < \infty$ and let P be given by*

$$P = -\partial_{t_1}^2 - (\partial_{t_2} + a(t_1, t_2)\partial_x)^2$$

where $(t_1, t_2, x) \in \mathbf{T}^3$ and $a \in C^\omega(\mathbf{T}^2)$ is real-valued. We set

$$a_0(t_1) = \frac{1}{2\pi} \int_{\mathbf{T}} a(t_1, s) ds.$$

Then, P is globally G^s solvable in $\mathbf{T}^3$ if and only if one of the following conditions holds true
I') P is globally G^s hypoelliptic in $\mathbf{T}^3$.
II) All points in $\mathbf{T}^3$ are of infinite type for the vector fields X_1 and X_2 and $a_0(t_1) \equiv \alpha$ with α being a rational number.

We would like to point out that in the papers [42-45] we have studied global G^s solvability for sublaplacians in the torus $\mathbf{T}^{m+n}$ given by $-\Delta_t - \left(\sum_{j=1}^n a_j(t)\partial_{x_j}\right)^2$ assuming that the coefficients of the real vector field $L = \sum_{j=1}^n a_j(t)\partial_{x_j}$ depend only on $t \in \mathbf{T}^m$. In this paper we deal with sublaplacians in $\mathbf{T}^3$ in the form $-\partial_{t_1}^2 - (\partial_{t_2} + a(t_1, t_2)\partial_x)^2$. The novelty here is that the coefficient of the real vector field $\partial_{t_2} + a(t_1, t_2)\partial_x$ depend on t_1 and t_2.

For results on interesting open problems of local and global $G^s, 1 \leq s \leq \infty$, solvability we refer the reader to the following papers as well as the references therein: Albanese, Corli and Rodino [1], Bergamasco, Cordaro and Petronilho [7], Cardoso [10], Cardoso and Hounie [11], Gramchev, Popivanov and Yoshino [20], [21] and [22], Gramchev and Yoshino [23], Hounie [37], Petronilho [42], [43], [44], [45], Rodino [47].

We are also interested in studying Gevrey regularity for certain classes of operators.

In the paper [33] Himonas and Petronilho have analyzed the following question: Let P be a linear partial differential operator with coefficients in $C^\omega(\mathbf{T}^N)$ and suppose that P is globally C^∞ hypoelliptic in $\mathbf{T}^N$. Is P globally $G^s, s \geq 1$, hypoelliptic in $\mathbf{T}^N$? First of all they proved the following result about Gevrey regularity:

Theorem 1.9. *For $(t, x) \in \mathbf{T}^{m+n}$ let $P = P(t, D_t, D_x)$ be a linear partial differential operator with coefficients in $C^\omega(\mathbf{T}^m)$ and suppose that P is globally C^∞ hypoelliptic in $\mathbf{T}^{m+n}$. If $u \in D'(\mathbf{T}^{m+n}), Pu \in G^s(\mathbf{T}^{m+n})$, and $(t, x, \tau, 0) \notin WF_s(u)$, where $(t, x) \in \mathbf{T}^{m+n}, \tau \in \mathbf{R}^m \backslash \{0\}$, and if Ker $P \subset G^s(\mathbf{T}^{m+n})$, then $u \in G^s(\mathbf{T}^{m+n})$.*

After, they used this Gevrey regularity result in order to present a large class of operators for which their question has a positive answer.

Our next goal is improve Himonas and Petronilho's Theorem 1.9 by proving that the hypothesis that Ker $P \subset G^s(\mathbf{T}^{m+n})$, in Theorem 1.9, is superflous, i.e., we will prove the following result:

Theorem 1.10. *For $(t, x) \in \mathbf{T}^{m+n}$ let $P = P(t, D_t, D_x)$ be a linear partial differential operator with coefficients in $C^\omega(\mathbf{T}^m)$ and suppose that P is globally C^∞ hypoelliptic in $\mathbf{T}^{m+n}$. If $u \in D'(\mathbf{T}^{m+n}), Pu \in G^s(\mathbf{T}^{m+n})$, and $(t, x, \tau, 0) \notin WF_s(u)$, where $(t, x) \in \mathbf{T}^{m+n}, \tau \in \mathbf{R}^m \backslash \{0\}$, then $u \in G^s(\mathbf{T}^{m+n})$.*

In order to clarify the above statements we shall recall the definition of the Gevrey wave front.

Definition 1.11 If $X \subset \mathbf{R}^M$ and $u \in D'(X)$ we denote by $WF_s(u)$ the complement in $X \times (\mathbf{R}^M \backslash \{0\})$ of the set of (x_0, ξ_0) such that there is a neighborhood $U \subset X$ of x_0, a conic neighborhood Γ of ξ_0 and a bounded sequence $u_N \in E'(X)$ which is equal to u in U and satisfies the condition

$$|\widehat{u_N}(\xi)| \leq C(CN/|\xi|^{1/s})^N, \quad N = 1, 2, \cdots, \ \xi \in \Gamma$$

for some constant $C > 0$ independent of N.

Assume $s > 1$, $x_0 \in X$, $\xi_0 \in \mathbf{R}^M \backslash \{0\}$, and $u \in D'(X)$. Then $(x_0, \xi_0) \notin WF_s(u)$ if and only if there exists $\varphi \in G_0^s(X)$, $\varphi(x) = 1$ in a neighborhood of x_0, and a conic neighborhood Γ of ξ_0, such that for some $C, \epsilon > 0$

$$|\widehat{(\varphi u)}(\xi)| \leq Ce^{-\epsilon|\xi|^{1/s}} \text{ for all } \xi \in \Gamma.$$

Definition 1.12 If X is a real analytic manifold and $u \in D'(X)$, the Gevrey wave front set of u, $WF_s(u)$, is defined as a subset of $T^*X \backslash \{0\}$ so that its restriction to a coordinate patch X_κ is equal to $\kappa^* WF_s(u \circ \kappa^{-1})$.

Finally we present an application of Theorem 1.10 which generalizes Theorem 1.9.

2 Proof of Theorem 1.7 and Theorem 1.8

We recall the following general result

Proposition 2.1. *Let $1 \leq s \leq \infty$ and let P be a linear partial differential operator with coefficients in $G^s(\mathbf{T}^N)$. If ${}^tP$ is globally G^s hypoelliptic in $\mathbf{T}^N$ then P is globally G^s solvable in $\mathbf{T}^N$.*

We would like to point out that in the C^∞ case the proof of proposition 2.1 follows from a variation of [36, Theorem 26.1.7] while Albanese and Zanghirati [2] have proved it in the Gevrey case, $s \geq 1$.

Since for our operator P we have ${}^tP = P$ it follows from proposition 2.1 that the condition I), of Theorem 1.7, implies that P is globally C^∞ solvable in $\mathbf{T}^3$ as well as the condition I'), of Theorem 1.8, also implies that P is globally $G^s, s \geq 1$, solvable in $\mathbf{T}^3$.

We now assume that the condition II) holds, i.e., all points in $\mathbf{T}^3$ are of infinite type for the vector fields X_1 and X_2 and $a_0(t_1) \equiv \alpha$ with α being a rational number.

Since $[X_1, X_2] = \partial_{t_1} a(t)\partial_x$ and all points in $\mathbf{T}^3$ are of infinite type for the vector fields X_1 and X_2 we can conclude that $\partial_{t_1} a(t) \equiv 0$. Therefore, the coefficient a depends only on the variable t_2, i.e., $a(t_1, t_2) = a(t_2)$.

Thus, under this hypothesis we may write our operator as

$$P = -\partial_{t_1}^2 - (\partial_{t_2} + a(t_2)\partial_x)^2$$

and

$$a_0(t_1) = \frac{1}{2\pi}\int_0^{2\pi} a(t_1, s)ds = \frac{1}{2\pi}\int_0^{2\pi} a(s)ds \equiv \alpha$$

which we are assuming to be a rational number.

We introduce the following new variables in $\mathbf{T}^3$:

$$s_j = t_j,\ j = 1, 2;\ y = x - \int_0^{t_2} a(s)ds + \alpha t_2. \tag{2.1}$$

In these new variables the operator P becomes

$$Q = -\partial_{s_1}^2 - (\partial_{s_2} + \alpha\partial_y)^2$$

and we have that P is globally $G^s, 1 \le s \le \infty$, solvable in $\mathbf{T}^3$ if and only if Q is globally $G^s, 1 \le s \le \infty$, solvable in $\mathbf{T}^3$.

We set $\alpha = \frac{p}{q}$ where $p, q \in \mathbf{Z}$ and p and q do not have a common factor which is larger than one. Let $f \in \mathbf{E}^s(Q)$ be given. In order to solve the equation $Qu = f$, by using Fourier series, we must solve the equations

$$[\tau_1^2 + (\tau_2 + \frac{p}{q}\eta)^2]\hat{u}(\tau_1, \tau_2, \eta) = \hat{f}(\tau_1, \tau_2, \eta), \quad (\tau_1, \tau_2, \eta) \in \mathbf{Z}^3.$$

We note that $\tau_1^2 + (\tau_2 + \frac{p}{q}\eta)^2 = 0$ if and only if $\tau_1 = 0$ and $\tau_2 = -\ell p$ and $\eta = \ell q$ where $\ell \in \mathbf{Z}$.

We also notice that $Q(e^{i\ell(ps_2 - qy)}) = 0,\ \forall\ \ell \in \mathbf{Z}$. Thanks to the fact that $f \in \mathbf{E}^s(Q)$ we have

$$\hat{f}(0, -\ell p, \ell q) = \frac{1}{(2\pi)^3}\langle f, e^{i\ell(ps_2 - qy)}\rangle = 0,\ \forall\ \ell \in \mathbf{Z}.$$

Thus we can take

$$\widehat{u}(\tau_1, \tau_2, \eta) = \begin{cases} \frac{\hat{f}(\tau_1, \tau_2, \eta)}{\tau_1^2 + (\tau_2 + \frac{p}{q}\eta)^2}, & \text{if } \tau_1 \neq 0 \text{ or } (\tau_2 + \frac{p}{q}\eta) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Now it is easy to see that for $C = \max\{1, q^2\}$ we have

$$|\widehat{u}(\tau_1, \tau_2, \eta)| \le C|\hat{f}(\tau_1, \tau_2, \eta)|,\ \forall\ (\tau_1, \tau_2, \eta) \in \mathbf{Z}^3$$

in turns implies that $u(s_1, s_2, y) = \sum_{(\tau_1, \tau_2, \eta) \in \mathbf{Z}^3} \widehat{u}(\tau_1, \tau_2, \eta)e^{i(s_1\tau_1 + s_2\tau_2 + y\eta)} \in G^s(\mathbf{T}^3)$. Futhermore, $Qu = f$.

Summing up we have proved the sufficiency of the condition I) and II) in Theorem 1.7 as well as the sufficiency of the condition I') and II) in Theorem 1.8.

We also point out that we have actually found solutions in $G^s(\mathbf{T}^3)$, and hence the sufficiency theorems are global G^s solvability theorems.

Now we suppose that neither I) nor II) holds. Then it follows from Theorem 1.5 that all points in $\mathbf{T}^3$ are of infinite type for the vector fields X_1 and X_2 and $a_0(t_1) \equiv \alpha$ with α being a Liouville number. If now we suppose that neither I') nor II) holds, then by using Theorem

1.6 we can conclude that all points in $\mathbf{T}^3$ are of infinite type for the vector fields X_1 and X_2 and $a_0(t_1) \equiv \alpha$ with α being an exponentially Liouville number with exponent $s \geq 1$.

Thus we may use the variables s_1, s_2, y as we have introduced in (2.1) and recall that in these variables the operator P becomes $Q = -\partial_{s_1}^2 - (\partial_{s_2} + \alpha\partial_y)^2$.

We set $L = \partial_{s_2} + \alpha\partial_y$. Assuming that the vector field L is not globally G^s solvable, i.e., there exists $g \in \mathbf{E}^s(L)$ such that $Lv \neq g$ for all $v \in D'_s(\mathbf{T}^2_{s_2,y})$, we will show that the operator Q is not globally G^s solvable and therefore the operator P will not globally G^s solvable.

For this we define

$$f(s_1, s_2, y) = \sum_{(\tau_1,\tau_2,\eta)\in\mathbf{Z}^3} \widehat{f}(\tau_1, \tau_2, \eta)e^{i(s_1\tau_1+s_2\tau_2+y\eta)}$$

where

$$\widehat{f}(\tau_1, \tau_2, \eta) = \begin{cases} \widehat{g}(\tau_2, \eta), & \text{if } \tau_1 = 0 \\ 0, & \text{otherwise.} \end{cases}$$

We now show that $f \in \mathbf{E}^s(Q)$. We have $f \in G^s(\mathbf{T}^3)$ since $g \in G^s(\mathbf{T}^2_{s_2,y})$.

Let $w \in \text{Ker}\ {}^tQ$. By using the Fourier series we see that ${}^tQw = 0$ if and only if

$$[\tau_1^2 + (\tau_2 + \alpha\eta)^2]\widehat{w}(\tau_1, \tau_2, \eta) = 0, \ \forall\ (\tau_1, \tau_2, \eta) \in \mathbf{Z}^3.$$

Since α is irrational it follows from the last equality that

$$\widehat{w}(\tau_1, \tau_2, \eta) = 0, \ \forall\ (\tau_1, \tau_2, \eta) \in \mathbf{Z}^3\backslash\{0\}.$$

It gives that

$$\langle w, f\rangle = \widehat{w}(0, 0, 0)\widehat{f}(0, 0, 0) = \widehat{w}(0, 0, 0)\widehat{g}(0, 0), \tag{2.2}$$

since

$$\langle w, f\rangle = (2\pi)^3 \sum_{(\tau_1,\tau_2,\eta)\in\mathbf{Z}^3} \widehat{w}(\tau_1, \tau_2, \eta)\widehat{f}(-\tau_1, -\tau_2, -\eta).$$

Since $1 \in \text{Ker}\ {}^tL$ and $g \in \mathbf{E}^s(L)$ we have $\widehat{g}(0, 0) = 0$ and it follows from this and (2.2) that $\langle w, f\rangle = 0$. Hence $f \in \mathbf{E}^s(Q)$.

Now we will show that there is no solution to the equation $Qu = f$. Suppose, for a moment, that there exists $u \in D'_s(\mathbf{T}^3)$ such that $Qu = f$.

We define

$$v(s_2, y) = -\sum_{(\tau_2,\eta)\in\mathbf{Z}^2} i(\tau_2 + \alpha\eta)\widehat{u}(0, \tau_2, \eta)e^{i(s_2\tau_2+y\eta)},$$

which belongs to $D'_s(\mathbf{T}^2_{s_2,y})$.

Since $Qu = f$ we obtain that

$$[\tau_1^2 + (\tau_2 + \alpha\eta)^2]\widehat{u}(\tau_1, \tau_2, \eta) = \widehat{f}(\tau_1, \tau_2, \eta), \ \forall\ (\tau_1, \tau_2, \eta) \in \mathbf{Z}^3.$$

In particular we have

$$(\tau_2 + \alpha\eta)^2\widehat{u}(0, \tau_2, \eta) = \widehat{f}(0, \tau_2, \eta) = \widehat{g}(\tau_2, \eta), \ \forall\ (\tau_2, \eta) \in \mathbf{Z}^2.$$

It follows from the last equality that

$$\begin{aligned} Lv &= \sum_{(\tau_2,\eta)\in\mathbf{Z}^2} (\tau_2+\alpha\eta)^2\widehat{u}(0,\tau_2,\eta)e^{i(s_2\tau_2+y\eta)} \\ &= \sum_{(\tau_2,\eta)\in\mathbf{Z}^2} \widehat{g}(\tau_2,\eta)e^{i(s_2\tau_2+y\eta)} \\ &= g(s_2,y), \end{aligned}$$

which is a contradiction with our hypothesis that $Lv \neq g$ for all $v \in D'_s(\mathbf{T}^2_{s_2,y})$. Therefore there is no $u \in D'_s(\mathbf{T}^3)$ such that $Qu = f$.

In order to finish the proof it is enough to guarantee that the vector field L is not globally G^s solvable, provided α is a Liouville number (in the C^∞ case) or an exponentially Liouville number with exponent s (in the case $s \geq 1$). But it holds true thanks to Greenfield and Wallach [24] and Gramchev, Popivanov and Yoshino [22], respectively.

Summing up we have proved the necessity of conditions I) and II) in Theorem 1.7 as well as the necessity of conditions I') and II) in Theorem 1.8. The proof of Theorem 1.7 and Theorem 1.8 are now complete.

Remark 2.2 It is standard to prove that an operator is not locally or globally solvable by violating certain inequalities but here we prefer to do this by a constructive method.

3 Proof of Theorem 1.10

In this section we consider $1 \leq s < \infty$ and let P be as in Theorem 1.10. We shall need the following

Lemma 3.1. *Suppose that P is globally C^∞ hypoelliptic in $\mathbf{T}^{m+n}$. Then there exist $\ell \in \mathbf{Z}_+$ and $C > 0$ such that*

$$||\varphi||_0 \leq C(||P\varphi||_\ell + ||\varphi||_{-1}),\ \forall\ \varphi \in C^\infty(\mathbf{T}^{m+n}). \tag{3.1}$$

Proof. Himonas and Petronilho [33, Lemma 2.5] have proved that there exist $\ell \in \mathbf{Z}_+$ and $C > 0$ such that

$$||\varphi||_1 \leq C(||P\varphi||_\ell + ||\varphi||_0),\ \forall\ \varphi \in C^\infty(\mathbf{T}^{m+n}), \tag{3.2}$$

provided P is globally C^∞ hypoelliptic in $\mathbf{T}^{m+n}$.

Now we recall that for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$||\varphi||_0 \leq \delta||\varphi||_1 + C_\delta||\varphi||_{-1},\ \forall\ \varphi \in C^\infty(\mathbf{T}^{m+n}). \tag{3.3}$$

It follows from (3.2) and (3.3) that for any $\delta > 0$ there exist $C_\delta > 0$, $C > 0$ and $\ell \in \mathbf{Z}_+$ such that

$$||\varphi||_1 \leq C(||P\varphi||_\ell + \delta||\varphi||_1 + C_\delta||\varphi||_{-1}),\ \forall\ \varphi \in C^\infty(\mathbf{T}^{m+n}). \tag{3.4}$$

By choosing $\delta > 0$ such that $C\delta = 1/2$ and using the letter C to represent a constant which may change a finite number of times we obtain

$$||\varphi||_1 \leq C(||P\varphi||_\ell + ||\varphi||_{-1}),\ \forall\ \varphi \in C^\infty(\mathbf{T}^{m+n}).$$

Since $||\varphi||_0 \le ||\varphi||_1$ the last inequality implies (3.1). The proof of Lemma 3.1 is complete.

Now we will use inequality (3.1) in order to prove Theorem 1.10.

Let $u \in D'(\mathbf{T}^{m+n})$ be such that

$$Pu = f \in G^s(\mathbf{T}^{m+n}) \tag{3.5}$$

and $(t, x, \tau, 0) \notin WF_s(u)$, where $(t, x) \in \mathbf{T}^{m+n}, \tau \in \mathbf{R}^m \backslash \{0\}$.

Since P is globally C^∞ hypoelliptic in $\mathbf{T}^{m+n}$ it follows from (3.5) that

$$u(t, x) \in C^\infty(\mathbf{T}^{m+n}). \tag{3.6}$$

It follows from (3.6) and (3.1), by replacing φ by $\partial_x^\alpha u$, that there exists $\ell \in \mathbf{Z}$ such that

$$||\partial_x^\alpha u||_0 \le C(||P(\partial_x^\alpha u)||_\ell + ||\partial_x^\alpha u||_{-1}),\ \forall\ \alpha \in \mathbf{Z}_+^n. \tag{3.7}$$

Also, since the coefficients of P depend only on t we have

$$[\partial_x^\alpha, P] = 0,\ \forall\ \alpha \in \mathbf{Z}_+^n \backslash \{0\}.$$

It follows from the last equality and from (3.7) that for $\alpha \in \mathbf{Z}_+^n \backslash \{0\}$ we have

$$||\partial_x^\alpha u||_0 \le C(||\partial_x^\alpha Pu||_\ell + ||\partial_x^\alpha u||_{-1}). \tag{3.8}$$

By recalling that

$$||\partial_x^\alpha u||_{-1} \le ||\partial_x^{\alpha - e_k} u||_0,$$

where e_k is an element of the orthonormal basis of $\mathbf{R}^n$ such that the correponding $\alpha_k \ge 1$, it follows from (3.8) and (3.5) that we have

$$||\partial_x^\alpha u||_0 \le C(||\partial_x^\alpha f||_\ell + ||\partial_x^{\alpha - e_k} u||_0), \tag{3.9}$$

for $\alpha \in \mathbf{Z}_+^n \backslash \{0\}$.

Since $f \in G^s(\mathbf{T}^{m+n})$ there exists $B > 1$ such that

$$||\partial_x^\delta \partial_t^\gamma f||_0 \le B^{|\delta|+|\gamma|+1}(\delta!)^s(\gamma!)^s,\ \forall\ \delta \in \mathbf{Z}_+^n \text{ and } \gamma \in \mathbf{Z}_+^m. \tag{3.10}$$

It follows from (3.10) that for $\alpha \in \mathbf{Z}_+^n$ we have

$$\begin{aligned}
||\partial_x^\alpha f||_\ell^2 &= \sum_{|(\beta,\gamma)| \le \ell} ||\partial_x^\beta \partial_t^\gamma (\partial_x^\alpha f)||_0^2 \\
&\le \sum_{|(\beta,\gamma)| \le \ell} \left(B^{|\beta|+|\gamma|+|\alpha|+1}((\alpha+\beta)!)^s(\gamma!)^s\right)^2 \\
&\le \sum_{|(\beta,\gamma)| \le \ell} \left(B^{|\beta|+|\gamma|+|\alpha|+1}(2^s)^{|\alpha|+|\beta|}(\alpha!)^s(\beta!)^s(\gamma!)^s\right)^2 \\
&\le \left(\sum_{|(\beta,\gamma)| \le \ell} (2^s)^{2|\beta|}(\beta!)^{2s}(\gamma!)^{2s}\right)((2^s)^{|\alpha|}B^{\ell+1}B^{|\alpha|})^2(\alpha!)^{2s} \\
&= (D(2^s)^{|\alpha|}B^{\ell+1}B^{|\alpha|})^2(\alpha!)^{2s} \\
&\le C_1^{2(|\alpha|+1)}(\alpha!)^{2s}
\end{aligned}$$

where $C_1 = \max\{DB^{\ell+1}, (2^s)B, ||f||_\ell\}$ and $D^2 = \sum_{|(\beta,\gamma)|\leq \ell}(2^s)^{2|\beta|}(\beta!)^{2s}(\gamma!)^{2s}$ and therefore the constants C_1 and D denpend only on ℓ, which is fixed.

Thus we have proved that there exists a constant $C_1 > 0$ such that

$$||\partial_x^\alpha f||_\ell \leq C_1^{|\alpha|+1}(\alpha!)^s, \ \forall\ \alpha \in \mathbf{Z}_+^n. \tag{3.11}$$

We also shall need the following result

Lemma 3.2. *There exists a positive constant $M > 1$ such that*

$$||\partial_x^\alpha u||_0 \leq (C_1 M)^{|\alpha|+1}(\alpha!)^s, \ \forall\ \alpha \in \mathbf{Z}_+^n \tag{3.12}$$

where C_1 is given in (3.11).

Proof. Since $u \in C^\infty(\mathbf{T}^{m+n})$ there exists a positive constant A such that

$$||u||_0 \leq A.$$

Let $M > 1$ be such that

$$C\left(\frac{1}{M} + \frac{1}{C_1 M}\right) < 1, \tag{3.13}$$

where C is given in (3.9), C_1 is given in (3.11) and

$$A \leq C_1 M. \tag{3.14}$$

We will prove (3.12) by induction on α. If $|\alpha| = 0$ then it follows from (3.14) that

$$||u||_0 \leq A \leq C_1 M$$

and therefore (3.12) holds true for $|\alpha| = 0$.

Next we assume that given $\alpha \in \mathbf{Z}_+^n\backslash\{0\}$ then (3.12) holds for any $\beta \in \mathbf{Z}_+^n$ such that $|\beta| \leq |\alpha| - 1$ and we will prove that (3.12) holds true for α.

It follows from (3.9), (3.11) and from the induction hypothesis that

$$\begin{aligned}
||\partial_x^\alpha u||_0 &\leq C\left(C_1^{|\alpha|+1}(\alpha!)^s + (C_1 M)^{|\alpha|}((\alpha - e_k)!)^s\right) \\
&\leq (C_1 M)^{|\alpha|+1}(\alpha!)^s C\left(\frac{1}{M^{|\alpha|+1}} + \frac{1}{C_1 M}\right) \\
&\leq (C_1 M)^{|\alpha|+1}(\alpha!)^s C\left(\frac{1}{M} + \frac{1}{C_1 M}\right) \\
&\leq (C_1 M)^{|\alpha|+1}(\alpha!)^s,
\end{aligned}$$

where in the last inequality we have used (3.13). The proof of Lemma 3.2 is complete. □

To complete the proof of Theorem 1.10, we shall need the following

Lemma 3.3. *There exist $\epsilon > 0$ and $C > 0$ such that*

$$|\hat{u}(t,\xi)| \leq Ce^{-\epsilon|\xi|^{1/s}}, \ \ \xi \in \mathbf{Z}^n\backslash\{0\}, \ \ t \in \mathbf{T}^m. \tag{3.15}$$

Proof. It follows from Cauchy-Schwarz inequality that

$$\begin{aligned} |\xi^\beta \hat{u}(t,\xi)| &= \left| \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} e^{-ix\cdot\xi} \xi^\beta u(t,x)dx \right| \\ &= \left| \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} (-1)^{|\beta|} D_x^\beta (e^{-ix\cdot\xi}) u(t,x)dx \right| \\ &= \left| \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} e^{-ix\cdot\xi} D_x^\beta u(t,x)dx \right| \\ &\leq \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |\partial_x^\beta u(t,x)|dx \\ &\leq C||\partial_x^\beta u(t,x)||_0. \end{aligned} \tag{3.16}$$

Let $N \in \mathbf{Z}_+$, $L = [N/s]$, i.e., L is the least integer such that $L \geq N/s$, and $\xi \in \mathbf{Z}^n \backslash \{0\}$. We recall that the following formula holds true:

$$(t_1 + \cdots + t_n)^J = \sum_{|\beta|=J} \frac{J!}{\beta_1! \cdots \beta_n!} t_1^{\beta_1} \cdots t_n^{\beta_n}$$

where $J \geq 1$ is an integer and $t_1, \cdots, t_n$ are n real numbers (see [47-formula (1.2.1),p.9]).

Thus we have

$$|\xi|^L \leq (|\xi_1| + \cdots + |\xi_n|)^L = \sum_{|\beta|=L} \frac{L!}{\beta!} |\xi^\beta|.$$

By using (3.16), Lemma 3.2 and the last inequality we obtain

$$\begin{aligned} |\xi|^{N/s} |\hat{u}(t,\xi)| &\leq |\xi|^L |\hat{u}(t,\xi)| \leq \sum_{|\beta|=L} \frac{L!}{\beta!} |\xi^\beta| |\hat{u}(t,\xi)| \\ &= \sum_{|\beta|=L} \frac{L!}{\beta!} |\xi^\beta \hat{u}(t,\xi)| \\ &\leq C \sum_{|\beta|=L} \frac{L!}{\beta!} ||\partial_x^\beta u(t,x)||_0 \\ &\leq C \sum_{|\beta|=L} \frac{L!}{\beta!} (C_1 M)^{|\beta|+1} (\beta!)^s \end{aligned}$$

By using the inequality $(\beta!)^s \leq |\beta|^{s|\beta|}$ it follows from the last inequality that

$$\begin{aligned} |\xi|^{N/s} |\hat{u}(t,\xi)| &\leq C \sum_{|\beta|=L} \frac{L!}{\beta!} (C_1 M)^{L+1} L^{sL} \\ &= C n^L (C_1 M)^{L+1} L^{sL} \\ &\leq C_2^{L+1} L^{sL} \end{aligned}$$

where C_2 is independent of N.

Thus it follows from [47, p. 32] that there exists a new constant $C' > 0$ such that

$$|\xi|^{N/s}|\hat{u}(t,\xi)| \le C'(C'N)^N,$$

where C' is independent of N.

Also it follows from a variation of [47, Lemma 1.6.2] that there exist $C > 0$ and $\epsilon > 0$ such that

$$|\hat{u}(t,\xi)| \le Ce^{-\epsilon|\xi|^{1/s}}, \;\; \xi \in \mathbf{Z}^n\backslash\{0\}, \;\; t \in \mathbf{T}^m. \tag{3.17}$$

The proof of Lemma 3.3 is now complete. □

Now, we are going to complete the proof of Theorem 1.10 .

Thanks to the Lemma 3.3 we can conclude that there exist $C > 0$ and $\epsilon > 0$ such that

$$\begin{aligned} |\hat{u}(\tau,\xi)| &= \frac{1}{(2\pi)^m}\left|\int_{\mathbf{T}^m} e^{-it\cdot\tau}\hat{u}(t,\xi)dt\right| \\ &\le Ce^{-\epsilon|\xi|^{1/s}}, \;\; (\tau,\xi) \in \mathbf{Z}^m \times \mathbf{Z}^n\backslash\{0\}. \end{aligned} \tag{3.18}$$

Let $\tau_0 \in \mathbf{R}^m\backslash\{0\}$. Since, by hypothesis, $(t,x,\tau_0,0) \notin WF_s(u)$ for any $(t,x) \in \mathbf{T}^{m+n}$ it follows from the microlocal theory that there exist positive constants ϵ_1, C, δ and a cone $\Gamma = \{(\tau,\xi) \in \mathbf{Z}^{m+n} : |\xi| < \delta|\tau|\}$ contaning $(\tau_0, 0)$ such that

$$|\hat{u}(\tau,\xi)| \le Ce^{-\epsilon_1|(\tau,\xi)|^{1/s}}, \;\; \forall \;\; (\tau,\xi) \in \Gamma \cap \mathbf{Z}^{m+n}. \tag{3.19}$$

Now we set $\Gamma_1 \doteq \{(\tau,\xi) \in \mathbf{Z}^{m+n} : |\xi| > \frac{\delta}{2}|\tau|\}$. Thus, $(0,0) \notin \Gamma_1$, $(\tau_0,0) \notin \Gamma_1$, and if $(\tau,\xi) \in \Gamma_1$ then $\xi \neq 0$. It follows from (3.18) that

$$\begin{aligned} |\hat{u}(\tau,\xi)| &\le Ce^{-\epsilon(\frac{1}{2}|\xi|+\frac{1}{2}|\xi|)^{1/s}} \\ &\le Ce^{-\epsilon(\frac{\delta}{4}|\tau|+\frac{1}{2}|\xi|)^{1/s}} \\ &\le Ce^{-\epsilon'|(\tau,\xi)|^{1/s}}, \quad (\tau,\xi) \in \Gamma_1 \cap \mathbf{Z}^{m+n}. \end{aligned} \tag{3.20}$$

Therefore, (3.19) and (3.20) imply that $u \in G^s(\mathbf{T}^{m+n})$. This completes the proof of Theorem 1.10. □

4 Application of Theorem 1.10

As application of Theorem 1.9 Himonas and Petronilho [33-Theorem 4.1] studied global G^s hypoellipticity for a class of sublaplacians. We will present a generalization of their resul by considering perturbations of lower order of this class of sublaplacians. More precisely we have the following

Theorem 4.1. *Let*

$$X_j = \sum_{q=1}^{n} a_{jq}(t)\partial_{t_q} + \sum_{k=1}^{m} b_{jk}(t)\partial_{x_k}, \;\; j = 0, 1, \cdots, \nu,$$

with $a_{jk}, b_{jk} \in C^\omega(\mathbf{T}^n)$ *and real-valued. Let* $c \in C^\omega(\mathbf{T}^n)$*. Assume that:*

(i) every point in $\mathbf{T}^{n+m}$ *is of finite type for* $X_0, X_1, \cdots, X_\nu$*;*

(ii) $\sum a_{jq}(t)\partial_q,\ j = 1, \cdots, \nu$ *span* $T(\mathbf{T}^n)_t$*, for every* t*.*

Then $P = -\sum_{j=1}^{\nu} X_j^2 + X_0 + c$ *is globally* G^s *hypoelliptic in* $\mathbf{T}^{n+m}$*.*

Proof. It is easy to see that our hypotheses imply that the assumptions of Theorem 1.10 are fulfilled and therefore P is globally G^s hypoelliptic in $\mathbf{T}^{n+m}$. □

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