THE GENERALIZED BBM-BURGER EQUATIONS WITH NONLINEAR DISSIPATIVE TERM: EXISTENCE AND CONVERGENCE RESULTS

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Abstract. We study the global existence of solutions for certain equations of the form $u_t + f(u)_x = \gamma B(u_x)_x + \delta u_{xxt} - \alpha u_{xxxx}$, as $\gamma > 0$, $\delta > 0$, and $\alpha > 0$ aproach zero, and f, and B are sufficiently smooth functions satisfying certain appropriate assumptions. We consider solutions of hyperbolic conservation laws regularized of this equations. Following a pioneering work by Schonbek and a work by LeFloch and Natalini, we establish the convergence of the regularized solutions toward discontinuous solutions of the hyperbolic conservation law.

1. Introduction

In this paper we study the existence and convergence of the smooth solutions $\{u(x,t;\gamma,\delta,\alpha)\}$ for partial differential equations of the form

$$u_t + f(u)_x = \gamma B(u_x)_x + \delta u_{xxt} - \alpha u_{xxxx}, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{1}$$

with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{2}$$

as $\gamma > 0$, $\delta > 0$, and $\alpha > 0$ aproach zero. Here f, $B(\lambda) = \beta(\lambda) + \lambda$ are sufficiently smooth functions satisfying certain assumptions to be listed in Section 2. The equations of type (1) and (2) are related to the well-known BBM equations which were advocated by Benjamin-Bona-Mahony [1] as a refinement of the KdV equation [1] and [3]. Since the viscous term $\gamma B(u_x)_x$ and the dissipative term αu_{xxxx} are of physical backgrounds [1] and [4] and, as pointed out in [4] and [5], the convergences of the solution sequences $\{u(x,t;\gamma,\delta,\alpha)\}$ as $\gamma \to 0$, $\delta \to 0$ and $\alpha \to 0$ correspond to some physical processes, such as vanishing viscosity, etc. It is of interest and importance to consider the existence (for each fixed $\gamma > 0$, $\delta > 0$ and $\alpha > 0$) and the convergence of the solution sequences $\{u(x,t;\gamma,\delta,\alpha)\}$ (as $\gamma \to 0$, $\delta \to 0$ and $\alpha \to 0$) to the Cauchy problems (1) and (2). When $f(u)$ satisfies certain growth conditions of infinity, $\alpha = 0$, and $B(\lambda) = \lambda$, f is convex, Schonbek [5] discussed the strong convergence of the solution sequence $\{u(x,t;\delta,\gamma)\}$ as $\delta \to 0$ and $\gamma \to 0$. When $B(\lambda) = \lambda$, then the problems reduces to te generalizaded BBM- Burgers equations proposed by Huijiang Zhao and Benjin Xuan [7], in the case $n = 1$. We show that if $\overline{\beta}(\lambda) =$

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$\frac{\epsilon}{\gamma}\beta(\lambda)$, $\delta = O(\gamma^{\frac{(4+2p)}{(2-p)}})$, $\alpha = O(\gamma^{\frac{(6+p)}{(2-p)}})$, and $\epsilon = O(\gamma^{\frac{2}{(2-p)}})$, then there exists a subsequence of the solutions $\{u(x,t;\epsilon,\gamma,\delta,\alpha)\}$ which converges pointwise almost everywhere to a solution of the limiting hiperbolic conservation law

$$u_t + f(u)_x = 0 \tag{3}$$

Following Schonbek [5], we state a representation theorem for the Young measures associated with a given sequence of uniformly bounded functions on L^q. The corresponding setting in L^∞ was first established by Tartar [8].

Lemma 1. *Let $\{u_j\}$ be a uniformly bounded sequence in $L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$. Then there exists a subsequence $\{u'_j\}$ and a weakly-$\star$ measurable mapping $\nu : \mathbb{R}\times\mathbb{R}_+ \to Prob(\mathbb{R})$ taking its values in the space of nonnegative measures with unit total mass (probability measures) such that, for all functions $g \in C(\mathbb{R})$ satisfying*

$$g(u) = o(|u|^r) \ \ as \ \ |u| \to \infty \tag{4}$$

for some $r \in [0,q)$, the following limit representation holds

$$\lim_{j'\to\infty} \int\int_{\mathbb{R}\times\mathbb{R}_+} g(u_{j'}(x,t))\phi(x,t)dxdt = \int\int_{\mathbb{R}\times\mathbb{R}_+}\int_{\mathbb{R}} g(\lambda)d\nu_{(x,t)}(\lambda)\phi(x,t)dxdt \tag{5}$$

for all $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}_+)$.

The measure-valued function $\nu_{(x,t)}$ is a Young measure associated with the sequence $\{u_j\}$.

Following DiPerna [10] and Szepessy [11], we now define the measure-valued (m.-v.) solutions to the first-order Cauchy problem (3) and (2).

Definition 2. Assume that f satisfies the growth condition (4) and $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$. A Young measure ν associated with a sequence $\{u_j\}$, which is assumed to be uniformly bounded in $L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$, is called an entropy m.-v. solution to the problem (3) and (2) if

$$\partial_t < \nu(.), |\lambda - k| > + \partial_x < \nu(.), sgn(\lambda - k)(f(\lambda) - f(k)) > \leq 0 \tag{6}$$

in the distributional sense for all $k \in \mathbb{R}$ and, for all intervals $I \subset \mathbb{R}$,

$$\lim_{T\to 0^+} \frac{1}{T}\int_0^T \int_I < \nu_{(x,t)}, |\lambda - u_0(x)| > dxdt = 0. \tag{7}$$

Following LeFloch and Natalini [9] we state a convergence result:

Lemma 3. *Assume that f satisfies Eq. (4) and $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$. Let $\{u_j\}$ be a sequence, uniformly bounded in $L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$ for $q \geq 1$, and let ν be a Young measure associated with this sequence. If ν is an entropy m.-v. solution to the problem (3) and (2), then*

$$\lim_{j\to\infty} u_j = u \ \ in \ \ L^\infty(\mathbb{R}_+; L^r_{loc}(\mathbb{R}))$$

for all $r \in [1,q)$, where $u \in L^\infty(\mathbb{R}_+; L^q(\mathbb{R}))$ is the unique entropy solution to (3) and (2).

The remainder of this paper is divided into three sections. After this introduction, which constitutes Section 1, we consider in Section 2 the global existence results, and the convergence results are stated in Section 3.

2. Global Existence of Solutions

In this section, first we study the existence of global smooth solutions to the Cauchy problem (1)-(2). Where γ, δ, and α are positive constants, f and β are sufficiently smooth functions. Furthermore, suppose that there exists a constant $M > 0$ such that

$$|\beta'(\lambda)| \leq M, \quad \forall \lambda \in \mathbb{R}.$$

The following result is a consequence from Theorem 4.7 (page 30) of [6]:

Theorem 4. *Suppose that $G = G(w)$ is sufficiently smooth. If functions $\overline{w} = \overline{w}(x)$ and $\overline{\overline{w}} = \overline{\overline{w}}(x)$ satisfy $\|w\|_\infty \leq N$ (N is a positive constant), and $\overline{w}$, $\overline{\overline{w}} \in H^s(\mathbb{R})$ with $s \geq 1$ then for*

$$w^* = \overline{w} - \overline{\overline{w}}$$

we have

$$\|G(\overline{w}) - G(\overline{\overline{w}})\|_{H^s(\mathbb{R})} \leq C_s \|w^*\|_{H^s(\mathbb{R})} (|G'(0)| + \|\overline{w}\|_{H^s(\mathbb{R})} + \|\overline{\overline{w}}\|_{H^s(\mathbb{R})})$$

where $w^* = \overline{w} - \overline{\overline{w}}$ and C_s is a positive constant depending on N and on s. □

Denote $F(u)$ the Fourier transform of u with respect to the spatial variable x, F^{-1} is the inverse transform of F. Formally, from (1)

$$F(u_t - \delta u_{xxt} - \gamma u_{xx} + \alpha u_{xxxx}) = F(\gamma\beta(u_x)_x - f(u)_x)$$
$$(1 + \delta\xi^2)F(u)_t + (\gamma\xi^2 + \alpha\xi^4)F(u) = F(\gamma\beta(u_x)_x - f(u)_x).$$

Then

$$\left[\exp\left\{\frac{(\gamma\xi^2 + \alpha\xi^4)t}{(1 + \delta\xi^2)}\right\} F(u)\right]_t = \exp\left\{\frac{(\gamma\xi^2 + \alpha\xi^4)t}{(1 + \delta\xi^2)}\right\} \frac{F(\gamma\beta(u_x)_x - f(u)_x)}{(1 + \delta\xi^2)},$$

and integrate on $[0, t]$ we have

$$u(x,t) = F^{-1}\left(\exp\left\{\frac{-(\gamma\xi^2 + \alpha\xi^4)t}{(1 + \delta\xi^2)}\right\} F(u_0)\right) + \int_0^t F^{-1}\left(\frac{\exp\left\{\frac{-(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\} F(\gamma\beta(u_x)_x - f(u)_x)}{(1 + \delta\xi^2)}\right) ds.$$

Hence the integral equation of the solution is

$$u(x,t) = G(t)u_0 + \int_0^t G(t-s)F^{-1}\left(\frac{F(\gamma\beta(u_x)_x - f(u)_x)}{(1 + \delta\xi^2)}\right) ds \tag{8}$$

where $G(t)u = F^{-1}\left(\exp\left\{\frac{-(\gamma\xi^2 + \alpha\xi^4)t}{(1 + \delta\xi^2)}\right\} F(u)\right)$. The family of linear operators $\{G(t)\}_{t\geq 0}$ satisfies the properties of semigroup.

In the following lemma we give some estimates which will be used in this section:

Lemma 5. *For $\gamma > 0$, $\alpha > 0$, $\delta > 0$, and $\theta > 0$ we have the following inequalities:*
i) $\dfrac{\xi^2}{(1+\delta\xi^2)^2}\exp\left\{\dfrac{-2(\gamma\xi^2+\alpha\xi^4)\theta}{(1+\delta\xi^2)}\right\} \le \delta^{-2}$, *if* $\delta \le 4$;
ii) $\dfrac{\xi^2}{(1+\delta\xi^2)^2}\exp\left\{\dfrac{-2(\gamma\xi^2+\alpha\xi^4)\theta}{(1+\delta\xi^2)}\right\} \le (2\gamma e\theta)^{-1}$;
iii) $\dfrac{\xi^4}{(1+\delta\xi^2)^2}\exp\left\{\dfrac{-2(\gamma\xi^2+\alpha\xi^4)\theta}{(1+\delta\xi^2)}\right\} \le \delta^{-2}$;
iv) $\xi^2\exp\left\{\dfrac{-2(\gamma\xi^2+\alpha\xi^4)\theta}{(1+\delta\xi^2)}\right\} \le \dfrac{(\alpha+\gamma\delta)}{2\gamma\alpha\theta}$;
v) $\dfrac{\xi^6}{(1+\delta\xi^2)^2}\exp\left\{\dfrac{-2(\gamma\xi^2+\alpha\xi^4)\theta}{(1+\delta\xi^2)}\right\} \le (2\alpha\delta\theta)^{-1}$;
vi) $\dfrac{\xi^4}{(1+\delta\xi^2)^2}\exp\left\{\dfrac{-2(\gamma\xi^2+\alpha\xi^4)\theta}{(1+\delta\xi^2)}\right\} \le (2\alpha e\theta)^{-1}$.

Proof. The proof follows from simple computations. □

To begin, define the operator

$$\mathcal{L}u(t) = G(t)u_0 + \int_0^t G(t-s)F^{-1}\left(\frac{F(\gamma\beta(u_x)_x - f(u)_x)}{(1+\delta\xi^2)}\right)ds$$

on

$$\mathcal{A}_T = \{u \in C([0,T];H^1(\mathbb{R})); \|u(t)-G(t)u_0\|_{H^1(\mathbb{R})} \le \|u_0\|_{H^1(\mathbb{R})}, t\in[0,T]\}$$

and the norm in $\mathcal{A}_T$ by $\|u(x,t)\|_{\mathcal{A}_T} = \sup\limits_{0\le t\le T}\|u(t)\|_{H^1(\mathbb{R})}$.

Our local existence result will follow from the properties of $\mathcal{L}$ given in the following lemma:

Lemma 6. *Suppose that f and β are sufficiently smooth. Assume that $u(t), u_0 \in H^1(\mathbb{R})$ and that*

$$\|u(t)-G(t)u_0\|_{H^1(\mathbb{R})} \le \|u_0\|_{H^1(\mathbb{R})}, \qquad \forall t\in[0,T]. \tag{9}$$

If $T>0$ is sufficiently small, then the following hold:
(i) $\mathcal{L}u(t)\in H^1(\mathbb{R})$ with

$$\|\mathcal{L}u(t)-G(t)u_0\|_{H^1(\mathbb{R})} \le \|u_0\|_{H^1(\mathbb{R})}, \qquad \forall t\in[0,T]$$

and

$$\|\mathcal{L}u(t)\|_{H^1(\mathbb{R})} \le 2\|u_0\|_{H^1(\mathbb{R})}, \qquad \forall t\in[0,T];$$

(ii) $\|\mathcal{L}u(t)\|_\infty \le 2\sqrt{2}\|u_0\|_{H^1(\mathbb{R})}$;
(iii) $\mathcal{L}$ maps $\mathcal{A}_T$ into itself;
iv) $\mathcal{L}$ is a contraction on $\mathcal{A}_T$.

Proof. Let $E = \sup\limits_{|v|\le 2\sqrt{2}\|u_0\|_{H^1(\mathbb{R})}} |f'(v)|$. Without loss of generallity, we take $f(0)=0$ and $\beta(0)=0$.

(i) Let $u(t) \in H^1(\mathbb{R})$ satisfying (9), then using the properties of Fourier transform on $L^2(\mathbb{R})$, and the Parseval's equality we have

$$
\begin{aligned}
\|G(t)u_0\|_{H^1(\mathbb{R})} &= \left[\sum_{k=0}^{1}\int_{\mathbb{R}}\left|\frac{\partial^k G(t)u_0}{\partial x^k}\right|^2 dx\right]^{\frac{1}{2}}\\
&= \left[\sum_{k=0}^{1}\int_{\mathbb{R}}\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)t}{(1+\delta\xi^2)}\right\}\left|F\left(\frac{\partial^k u_0}{\partial x^k}\right)\right|^2 d\xi\right]^{\frac{1}{2}}\\
&\leq \|u_0\|_{H^1(\mathbb{R})}.
\end{aligned}
$$

Furthermore, using the Parseval's equality

$$
\begin{aligned}
&\|\mathcal{L}u(t)-G(t)u_0\|_{H^1(\mathbb{R})} \leq \\
&\int_0^t\left\{\sum_{k=0}^{1}\int_{\mathbb{R}}\frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\}\xi^{2k}|F(\gamma\beta(u_x)_x-f(u)_x)|^2}{(1+\delta\xi^2)^2}d\xi\right\}^{\frac{1}{2}} ds\\
&\leq \int_0^t\left\{\int_{\mathbb{R}}\frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\}(\xi^2+\xi^4)|F(\gamma\beta(u_x)-f(u))|^2}{(1+\delta\xi^2)^2}d\xi\right\}^{\frac{1}{2}} ds
\end{aligned}
$$

using the Lemma 5 (ii) and (vi) and the Parseval's equality

$$
\begin{aligned}
&\leq \int_0^t\frac{[(2\gamma e)^{-1}+(2\alpha e)^{-1}]^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}}\left\{\int_{\mathbb{R}}|\gamma\beta(u_x)-f(u)|^2 d\xi\right\}^{\frac{1}{2}} ds\\
&= \int_0^t\frac{[(2\gamma e)^{-1}+(2\alpha e)^{-1}]^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}}[\gamma M\|u_x(s)\|_{L^2(\mathbb{R})}+E\|u(s)\|_{L^2(\mathbb{R})}]ds\\
&\leq 4[(2\gamma e)^{-1}+(2\alpha e)^{-1}]^{\frac{1}{2}}[\gamma M+E]\|u_0\|_{H^1(\mathbb{R})}T^{\frac{1}{2}}\\
&\leq \|u_0\|_{H^1(\mathbb{R})}
\end{aligned}
$$

if $T \leq \dfrac{1}{8[((\gamma e)^{-1}+(\alpha e)^{-1})(\gamma M+E)^2]}$.

(ii) The estimate (ii) is the consequence from (i) and $\|\mathcal{L}u(t)\|_\infty \leq (2\|\mathcal{L}u(t)\|_{L^2(\mathbb{R})}\|(\mathcal{L}u(t))_x\|_{L^2(\mathbb{R})})^{\frac{1}{2}}$.

(iii) To proof (iii), we only need to show if $u \in C([0,T];H^1(\mathbb{R}))$ then $\mathcal{L}u \in C([0,T];H^1(\mathbb{R}))$. Let $t_0 \in (0,T]$. Let $t \in (0,T]$. Without loss of generallity, we take $t_0 < t$. We have

$$
\begin{aligned}
\|\mathcal{L}u(t)-\mathcal{L}u(t_0)\|_{H^1(\mathbb{R})} &\leq \|G(t)u_0-G(t_0)u_0\|_{H^1(\mathbb{R})}\\
&+\left\|\int_0^t G(t-s)F^{-1}\left(\frac{F(\gamma\beta(u_x)_x-f(u)_x)}{(1+\delta\xi^2)}\right)ds\right.\\
&\left.-\int_0^{t_0} G(t_0-s)F^{-1}\left(\frac{F(\gamma\beta(u_x)_x-f(u)_x)}{(1+\delta\xi^2)}\right)ds\right\|_{H^1(\mathbb{R})}\\
&\leq \|G(t)u_0-G(t_0)u_0\|_{H^1(\mathbb{R})}\\
&+\int_{t_0}^t\left\|G(r)F^{-1}\left(\frac{F(\gamma\beta(u_x(t-r))_x-f(u(t-r))_x)}{(1+\delta\xi^2)}\right)\right\|_{H^1(\mathbb{R})} dr
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{t_0}\left\|G(r)F^{-1}\left(\frac{1}{(1+\delta\xi^2)}F[\gamma\beta(u_x(t-r))_x f(u(t-r))_x\right.\right.\\
&\qquad\left.\left. -(\gamma\beta(u_x(t_0-r))_x - f(u(t_0-r))_x)]\right)\right\|_{H^1(\mathbb{R})}dr\\
&= A+B+C.
\end{aligned}
$$

For A,

$$
\begin{aligned}
A &\le \left\{\sum_{k=0}^{1}\int_{\mathbb{R}}\frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)\theta}{1+\delta\xi^2}\right\}(\gamma\xi^2+\alpha\xi^4)^2|t-t_0|^2}{(1+\delta\xi^2)^2}\left|F\left(\frac{\partial^k u_0}{\partial x^k}\right)\right|^2 d\xi\right\}^{\frac{1}{2}}\\
&\le \frac{|t-t_0|\|u_0\|_{H^1(\mathbb{R})}}{\sqrt{2}t_0}.
\end{aligned}
$$

For B, we have

$$
\begin{aligned}
B &\le \int_{t_0}^{t}\frac{[(2\gamma e)^{-1}+(2\alpha e)^{-1}]^{\frac{1}{2}}[\gamma M\|u_x(t-r)\|_{L^2(\mathbb{R})}E\|u(t-r)\|_{L^2(\mathbb{R})}]}{r^{\frac{1}{2}}}dr\\
&\le 4[(2\gamma e)^{-1}+(2\alpha e)^{-1}]^{\frac{1}{2}}[\gamma M+E]\|u_0\|_{H^1(\mathbb{R})}|t-t_0|^{\frac{1}{2}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
C = \int_0^{t_0}&\left\{\int_{\mathbb{R}}\frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)r}{1+\delta\xi^2}\right\}(\xi^2+\xi^4)}{(1+\delta\xi^2)^2}|F[\gamma\beta(u_x(t-r)) - f(u(t-r))\right.\\
&\left. -(\gamma\beta(u_x(t_0-r)) - f(u(t_0-r)))]|^2 d\xi\right\}^{\frac{1}{2}}dr
\end{aligned}
$$

we use the Lemma 5 (i) and (iii)

$$
\le \int_0^{t_0}\sqrt{2}\delta^{-1}[\gamma M+E]\|u(t-r)-u(t_0-r)\|_{H^1(\mathbb{R})}dr \le \overline{C}
$$

where $\overline{C}\to 0$ when $|t-t_0|\to 0$ because $u\in C([0,T];H^1(\mathbb{R}))$ and $t_0-r\in[0,T]$.

iv) Let u, $v\in\mathcal{A}_T$,

$$
\begin{aligned}
\|\mathcal{L}u(t)-\mathcal{L}v(t)\|_{H^1(\mathbb{R})} &\le \int_0^t\left\{\int_{\mathbb{R}}\frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\}(\xi^2+\xi^4)}{(1+\delta\xi^2)^2}.\right.\\
&\qquad\left.|F[\gamma(\beta(u_x)-\beta(v_x)) - (f(u)-f(v))]|^2 d\xi\right\}^{\frac{1}{2}}ds\\
&\le \int_0^t\frac{[(2\gamma e)^{-1}+(2\alpha e)^{-1}]^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}}\left\{\int_{\mathbb{R}}|\gamma(\beta(u_x)-\beta(v_x))|^2dx\right.\\
&\qquad\left.+\int_{\mathbb{R}}|f(v)-f(u)|^2dx\right\}^{\frac{1}{2}}ds\\
&\le \int_0^t\frac{[(2\gamma e)^{-1}+(2\alpha e)^{-1}]^{\frac{1}{2}}[\gamma M+E]\|u(s)-v(s)\|_{H^1(\mathbb{R})}}{(t-s)^{\frac{1}{2}}}ds\\
&\le 2[(2\gamma e)^{-1}+(2\alpha e)^{-1}]^{\frac{1}{2}}[\gamma M+E]\sup_{[0,T]}\|u(t)-v(t)\|_{H^1(\mathbb{R})}T^{\frac{1}{2}}\\
&\le \frac{1}{2}\|u(x,t)-v(x,t)\|_{\mathcal{A}_T}
\end{aligned}
$$

if $T \leq \dfrac{1}{8[((\gamma e)^{-1} + (\alpha e)^{-1})(\gamma M + E)^2]}$. □

Remark 7. *We can to replace the space $C([0,T];H^1(\mathbb{R}))$ by $L^\infty(0,T;H^1(\mathbb{R}))$ and the properties above also hold.*

We can now obtain the local existence of solutions of (1)-(2):

Theorem 8. *Suppose that u_0, f, and β satisfy the same assumptions as in Lemma 6 then the Cauchy problem (1)-(2) admits a unique local smooth solution*

$$u \in C([0,T];H^1(\mathbb{R})) \cap C^1([0,T];L^2(\mathbb{R})).$$

Furthermore, for each integer $k \geq 1$, we have

$$u \in C((0,T];H^k(\mathbb{R})) \cap C^1((0,T];H^{k-1}(\mathbb{R})),$$

where T depends on γ, α, E, and $\|u_0\|_{H^1(\mathbb{R})}$.

Proof. Let $u^0 \equiv 0$ and $u^n \equiv \mathcal{L}(u^{n-1})$. Then by induction the estimates of the Lemma 6 (i) and (ii) hold for each u^n. Furthermore, by Lemma 6 (iii) and (iv), $\mathcal{L}$ maps $\mathcal{A}_T$ onto itself and is contractive. From Banach's fixed point theorem, the integral equation (8) possesses a unique solution $u \in C([0,T];H^1(\mathbb{R}))$. To prove the regularity results, we only need to show if $u \in C((0,T];H^l(\mathbb{R}))$, $l \geq 1$, then $u \in C((0,T];H^{l+1}(\mathbb{R})) \cap C^1((0,T];H^l(\mathbb{R}))$. Let $t_1 \in (0,T)$, we only need to show that $u \in C([t_1,T];H^{l+1}(\mathbb{R}))$ and $u_t \in C([t_1,T];H^l(\mathbb{R}))$. We take $t_2 = \frac{t_1}{2}$. Then the semigroup property of G implies that, for $t > t_2$

$$u(x,t) = G(t-t_2)u(t_2) + \int_{t_2}^{t} G(t-s)F^{-1}\left[\frac{F(\gamma\beta(u_x)_x - f(u)_x)}{1+\delta\xi^2}\right] ds.$$

To begin, using the Parseval's equality and the Lemma 5 (iv), (i), (iii), and (v), we have

$$\begin{aligned}
\|u(t)\|_{H^{l+1}(\mathbb{R})} &\leq \|G(t-t_2)u(t_2)\|_{H^{l+1}(\mathbb{R})} \\
&\quad + \int_{t_2}^{t} \left\| G(t-s)F^{-1}\left[\frac{F(\gamma\beta(u_x)_x - f(u)_x)}{1+\delta\xi^2}\right]\right\|_{H^{l+1}(\mathbb{R})} ds \\
&\leq \left\{ \int_{\mathbb{R}} \exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-t_2)}{(1+\delta\xi^2)}\right\} |F(u(t_2))|^2 d\xi \right. \\
&\quad \left. + \sum_{k=0}^{l} \int_{\mathbb{R}} \exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-t_2)}{(1+\delta\xi^2)}\right\} \xi^2 \left| F\left(\frac{\partial^k u(t_2)}{\partial x^k}\right)\right|^2 d\xi \right\}^{\frac{1}{2}} \\
&\quad + \int_{t_2}^{t} \sqrt{2} \left\{ \left[\sum_{k=0}^{l+1} \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\}\xi^{2k}}{(1+\delta\xi^2)^2} |F(\gamma[\beta(u_x)_x)|^2 d\xi \right]^{\frac{1}{2}} \right. \\
&\quad \left. + \left[\sum_{k=0}^{l+1} \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\}\xi^{2k}}{(1+\delta\xi^2)^2} |F(f(u)_x)|^2 d\xi \right]^{\frac{1}{2}} \right\} ds
\end{aligned}$$

$$\leq \left\{ \int_{\mathbb{R}} |F(u(t_2))|^2 d\xi + \frac{(\alpha+\gamma\delta)}{2\gamma\alpha(t-t_2)} \sum_{k=0}^{l} \left| F\left(\frac{\partial^k u(t_2)}{\partial x^k}\right)\right|^2 d\xi \right\}^{\frac{1}{2}}$$

$$+ \int_{t_2}^{t} \sqrt{2} \left\{ \left[\int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\} (\xi^2+\xi^4)|F(\gamma\beta(u_x))|^2}{(1+\delta\xi^2)^2} d\xi \right.\right.$$

$$\left. + \sum_{k=0}^{l-1} \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\} \xi^{2k+6}|F(\gamma\beta(u_x))|^2}{(1+\delta\xi^2)^2} d\xi \right]^{\frac{1}{2}}$$

$$+ \left[\int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\} \xi^2|F(f(u))|^2}{(1+\delta\xi^2)^2} d\xi \right.$$

$$\left.\left. + \sum_{k=0}^{l} \int_{\mathbb{R}} \frac{\exp\left\{\frac{-2(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\} \xi^{2k+4}|F(f(u))|^2}{(1+\delta\xi^2)^2} d\xi \right]^{\frac{1}{2}} \right\} ds$$

$$\leq \left[1 + \frac{(\alpha+\gamma\delta)}{2\gamma\alpha(t_1-t_2)}\right]^{\frac{1}{2}} \|u(t_2)\|_{H^l(\mathbb{R})} + \int_{t_2}^{t} \sqrt{2}\left\{\gamma[2\delta^{-2} + (2\alpha\delta(t-s))^{-1}]^{\frac{1}{2}}.\right.$$

$$\left. \|\beta(u_x(s))\|_{H^{l-1}(\mathbb{R})} + \sqrt{2}\delta^{-1}\|f(u(s))\|_{H^l(\mathbb{R})}\right\} ds$$

using the Theorem 4.3 on [6] (page 22),

$$\leq \left[1 + \frac{(\alpha+\gamma\delta)}{2\gamma\alpha(t_1-t_2)}\right]^{\frac{1}{2}} \|u(t_2)\|_{H^l(\mathbb{R})}$$

$$+ \int_{t_2}^{t} \left[\frac{2}{\delta^2} + \frac{1}{2\alpha\delta(t-s)}\right]^{\frac{1}{2}} C[\gamma\|u_x(s)\|_{H^{l-1}(\mathbb{R})} + \|u(s)\|_{H^{l-1}(\mathbb{R})}] ds$$

where C is a positive constant depending on l and $\|u_0\|_{H^1(\mathbb{R})}$. Thus

$$\sup_{t_1 \leq t \leq T} \|u(t)\|_{H^{l+1}(\mathbb{R})} < \infty.$$

On the other hand, note

$$F(u)_t = -\frac{(\gamma\xi^2+\alpha\xi^4)}{(1+\delta\xi^2)} F(u) + \frac{F(\gamma\beta(u_x)_x - f(u)_x)}{(1+\delta\xi^2)}$$

and for $t_2 = \frac{t_1}{2} < t_1 \leq t$

$$u_t(t) = -F^{-1}\left[\frac{(\gamma\xi^2+\alpha\xi^4)}{(1+\delta\xi^2)} \exp\left\{\frac{-(\gamma\xi^2+\alpha\xi^4)(t-t_2)}{(1+\delta\xi^2)}\right\} F(u(t_2))\right]$$

$$- \int_{t_2}^{t} F^{-1}\left[\frac{(\gamma\xi^2+\alpha\xi^4)\exp\left\{\frac{-(\gamma\xi^2+\alpha\xi^4)(t-s)}{(1+\delta\xi^2)}\right\} F(\gamma\beta(u_x)_x - f(u)_x)}{(1+\delta\xi^2)^2}\right] ds$$

$$+ F^{-1}\left[\frac{F(\gamma\beta(u_x)_x - f(u)_x)}{(1+\delta\xi^2)}\right].$$

To finish, the proof is quite similar to the Lemma 6 (iii), using the Theorem 4.3 of [6] and the Theorem 4, and its proof will be omitted. □

In the order to extend these solutions globally, that is, to all of $t > 0$, we first give the following lemma.

Lemma 9. *Suppose $u(x,t) = u(x,t;\gamma,\delta,\alpha)$ a solution of (1)-(2) on $\mathbb{R} \times [0,t_2]$. If β is nondecreasing and satisfies $\beta(0) = 0$ then we have the following estimate for $0 \le t_1 \le t_2$:*

$$\int_{\mathbb{R}} u^2(x,t_1)\,dx + \delta \int_{\mathbb{R}} u_x^2(x,t_1)\,dx \le \int_{\mathbb{R}} u_0^2(x)\,dx + \delta \int_{\mathbb{R}} u_{0x}^2(x)\,dx.$$

Proof. We multiply (1) by $2u$ and integrate in $\mathbb{R}$ and in $[0,t_1]$, we have

$$\begin{aligned}\int_{\mathbb{R}} u^2(x,t_1)\,dx + 2\gamma \int_0^{t_1}\int_{\mathbb{R}} u_x^2(x,t)\,dxdt + \delta \int_{\mathbb{R}} u_x^2(x,t_1)\,dx \\ +2\gamma \int_0^{t_1}\int_{\mathbb{R}} \beta(u_x(x,t))u_x(x,t)\,dxdt + 2\alpha \int_0^{t_1}\int_{\mathbb{R}} u_{xx}^2(x,t)\,dxdt \\ = \int_{\mathbb{R}} u_0^2(x)\,dx + \delta \int_{\mathbb{R}} u_{0x}^2(x)\,dx.\end{aligned} \tag{10}$$

□

We can now state our global existence result:

Theorem 10. *Suppose f, β, and u_0 satisfy the same assumptions of the Theorem 8. If β is nondecreasing and satisfies $\beta(0) = 0$ then the problem (1)-(2) has a global smooth solution*

$$u \in C([0,\infty); H^1(\mathbb{R})) \cap C^1([0,\infty); L^2(\mathbb{R})).$$

Furthermore, for each integer $k \ge 1$, we have

$$u \in C((0,\infty); H^k(\mathbb{R})) \cap C^1((0,\infty); H^{k-1}(\mathbb{R})).$$

Proof. From Theorem 8, there is a unique solution $u(x,t;\gamma,\delta,\alpha) = u(x,t)$ defined up to time T and satisfies $u \in C([0,T]; H^1(\mathbb{R})) \cap C^1([0,T]; L^2(\mathbb{R}))$, and for each integer $l \ge 1$, we have $u \in C((0,T]; H^l(\mathbb{R})) \cap C^1((0,T]; H^{l-1}(\mathbb{R}))$. Furthermore, from Lemma 9, we have for $0 \le t \le T$

$$\|u(t)\|^2_{L^2(\mathbb{R})} + \delta\|u_x(t)\|^2_{L^2(\mathbb{R})} \le \|u_0\|^2_{L^2(\mathbb{R})} + \delta\|u_{0x}\|^2_{L^2(\mathbb{R})}. \tag{11}$$

We consider the problem (1) with initial data $u(x,T) = u_T(x)$. Then $u_T \in H^1(\mathbb{R})$, so that, by Theorem 8, $u(x,t)$ can be extended up to time $2T$. Now suppose that $u(x,t)$ been defined up to time kT for some integer k, and that for each integer $l \ge 1$, we have

$$u \in C((0,kT]; H^l(\mathbb{R})) \cap C^1((0,kT]; H^{l-1}(\mathbb{R})) \tag{12}$$

and (11) hold for $0 \le t \le kT$. Then by Theorem 8, $u(x,t)$ can be extended up to time $(k+1)T$ and (12) hold for $(k+1)T$. But then, from Lemma 9 (for $kT \le t \le (k+1)T$) and (11) (for kT) , we have

$$\begin{aligned}\|u(t)\|^2_{L^2(\mathbb{R})} + \delta\|u_x(t)\|^2_{L^2(\mathbb{R})} &\le \|u(kT)\|^2_{L^2(\mathbb{R})} + \delta\|u_x(kT)\|^2_{L^2(\mathbb{R})} \\ &\le \|u_0\|^2_{L^2(\mathbb{R})} + \delta\|u_{0x}\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

and $u_{(k+1)T} \in H^1(\mathbb{R})$. Proceeding inductively, we thus establish the existence of the solution $u(x,t)$ in all of $t \geq 0$ and, for each integer $l \geq 1$, we have $u \in C((0,\infty); H^l(\mathbb{R})) \cap C^1((0,\infty); H^{l-1}(\mathbb{R}))$.

□

Remark 11. *We consider the equation*

$$u_t + f(u)_x = \epsilon\beta(u_x)_x + \gamma u_{xx} + \delta u_{xxt} - \alpha u_{xxxx}, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{13}$$

We take $\overline{\beta}(\lambda) = \dfrac{\epsilon}{\gamma}\beta(\lambda)$, *then we have (1) with* $B(\lambda) = \overline{\beta}(\lambda) + \lambda$. *If* f, β, *and* u_0 *satisfy the same assumptions of the Theorem 10 then we have the result from Theorem 10 for the problem (13) and (2).*

3. Convergence Results

Let $u = u(x,t;\epsilon,\gamma,\delta,\alpha)$ be a sequence of solutions of (13) and (2) obtained previously, for ϵ, δ, γ, and α sufficiently smalls ($\epsilon + \gamma + \delta + \alpha \to 0$) and we take the smooth initial data $u_0 = u(x,0;\epsilon,\gamma,\delta,\alpha)$ which has compact support and satisfies

$$\|u_0\|_{H^2(\mathbb{R})} + \|u_0\|_{L^{2(p+1)}(\mathbb{R})} \leq C_0, \tag{14}$$

for some $p \in [0,2)$. Observe that u depends on ϵ, γ, δ and α, but in theorem below we take ϵ, δ, and α depending on γ and then we denote $u = u^\gamma$.

Theorem 12. *Assume f sufficiently smooth satisfy the growth condition* $|f'(u)| \leq C(1+|u|^p)$, $0 \leq p < 2$. *If* $\delta = O(\gamma^{\frac{(4+2p)}{(2-p)}})$, $\alpha = O(\gamma^{\frac{(6+p)}{(2-p)}})$, *and* $\epsilon = O(\gamma^{\frac{2}{(2-p)}})$ *there exist a subsequence* $\{u^{\gamma_k}\}$ *such that* $u^{\gamma_k} \rightharpoonup \overline{u}$, $f(u^{\gamma_k}) \rightharpoonup f(\overline{u})$ *in the sense of distributions and* $\overline{u}$ *is a weak solution of (3). Furthermore, if* $f'' > 0$ *then* $u^{\gamma_k} \to \overline{u}$ *strongly in* $L^q_{loc}(\mathbb{R} \times \mathbb{R}_+)$, $1 < q \leq 2(p+1)$.

Proof. Let $\Omega = \mathbb{R} \times (0,T)$ for some $T > 0$. Following Schonbek [5](we have $2(p+1) > 1$, $f(u) = o(|u|^{(p+\frac{3}{2})})$ as $|u| \to \infty$, and $p + \frac{3}{2} \in [0, 2(p+1))$ we only need to show
(i) $\{u^\gamma\}$ lies in a bounded set of $L^{2(p+1)}(\Omega)$;
(ii) $\frac{\partial}{\partial t}\eta(u^\gamma) + \frac{\partial}{\partial x}\psi(u^\gamma) \in$ {compact set of $H^{-1}(\Omega)$}+ {bounded set of $M(\Omega)$}
(a consequence from Murat's lemma [8]),
where $M(\Omega)$ denotes the space of measures and $\eta(u)$ is a smooth function with linear growth at infinity and, more precicely, such that η' and η'' are uniformly bounded in $\mathbb{R}$ and $\psi'(u) = \eta'(u)f'(u)$. Throughout the calculation and for simplicity, we omit the upper-index γ. We multiply (13) by $-u_{xx}$ and integrate in $\mathbb{R}$ and in $(0,T)$, we obtain

$$\alpha \int_0^T \int_{\mathbb{R}} u_{xxx}^2 dxdt + \gamma \int_0^T \int_{\mathbb{R}} u_{xx}^2 dxdt + \frac{\delta}{2}\int_{\mathbb{R}} u_{xx}^2 dx + \epsilon \int_0^T \int_{\mathbb{R}} \beta'(u_x) u_{xx}^2 dxdt$$

$$+ \frac{1}{2}\int_{\mathbb{R}} u_x^2 dx = \frac{1}{2}\int_{\mathbb{R}} {u_0}_x^2 dx + \frac{\delta}{2}\int_{\mathbb{R}} {u_0}_{xx}^2 dx + \int_0^T \int_{\mathbb{R}} f'(u) u_x u_{xx} dxdt.$$

The last integral can be estimated using (10)

$$\int_0^T\int_{\mathbb{R}} f'(u)u_xu_{xx}dxdt \leq \frac{\gamma}{2}\int_0^T\int_{\mathbb{R}} u_{xx}^2dxdt + \frac{\gamma^{-1}}{2}\int_0^T\int_{\mathbb{R}} |f'(u)|^2u_x^2dxdt$$
$$\leq \frac{\gamma}{2}\int_0^T\int_{\mathbb{R}} u_{xx}^2dxdt + C\gamma^{-2}(1+\|u\|_\infty^{2p})$$

Thus

$$\alpha\int_0^T\int_{\mathbb{R}} u_{xxx}^2dxdt + \frac{\gamma}{2}\int_0^T\int_{\mathbb{R}} u_{xx}^2dxdt + \frac{\delta}{2}\int_{\mathbb{R}} u_{xx}^2dx$$
$$+\epsilon\int_0^T\int_{\mathbb{R}}\beta'(u_x)u_{xx}^2dxdt + \frac{1}{2}\int_{\mathbb{R}} u_x^2dx \leq C\gamma^{-2}(1+\|u\|_\infty^{2p}). \tag{15}$$

We use the Cauchy-Schwarz inequality, (15) and (10) and we obtain

$$|u(x,t)|^2 \leq 2\int_{-\infty}^x |uu_x|\,dx$$
$$\leq 2\|u(t)\|_{L^2(\mathbb{R})}\|u_x(t)\|_{L^2(\mathbb{R})}$$
$$\leq C(1+\|u\|_\infty^{2p})^{\frac{1}{2}}\gamma^{-1},$$

hence

$$\|u\|_\infty \leq C\gamma^{-\frac{1}{(2-p)}} \tag{16}$$

if $0 \leq p < 2$. Thus

$$\alpha\int_0^T\int_{\mathbb{R}} u_{xxx}^2dxdt + \frac{\gamma}{2}\int_0^T\int_{\mathbb{R}} u_{xx}^2dxdt + \frac{\delta}{2}\int_{\mathbb{R}} u_{xx}^2dx$$
$$+\epsilon\int_0^T\int_{\mathbb{R}}\beta'(u_x)u_{xx}^2dxdt + \frac{1}{2}\int_{\mathbb{R}} u_x^2dx \leq C\gamma^{-\frac{4}{(2-p)}}. \tag{17}$$

From (17) and (10) , we have

$$\alpha^2\int_0^T\int_{\mathbb{R}} u_{xxx}^2dxdt \leq C\alpha\gamma^{-\frac{4}{(2-p)}} \tag{18}$$

and

$$\alpha\int_0^T\int_{\mathbb{R}} |u_xu_{xxx}|dxdt \leq \alpha\left[\int_0^T\int_{\mathbb{R}} u_x^2dxdt\right]^{\frac{1}{2}}\left[\int_0^T\int_{\mathbb{R}} u_{xxx}^2dxdt\right]^{\frac{1}{2}} \tag{19}$$
$$\leq C\alpha\gamma^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}\gamma^{-\frac{2}{(2-p)}} = C\alpha^{\frac{1}{2}}\gamma^{-\frac{(6-p)}{2(2-p)}}.$$

We multiply (13) by $du^{2p+1} + 2\gamma u_t$ and integrate in $\mathbb{R}$ and in $(0,T)$

$$d\int_{\mathbb{R}}\frac{u^{2p+2}}{2p+2}dx + d(2p+1)\epsilon\int_0^T\int_{\mathbb{R}} u^{2p}\beta(u_x)u_xdxdt + d(2p+1)\gamma\int_0^T\int_{\mathbb{R}} u^{2p}u_x^2dxdt$$
$$+2\gamma\int_0^T\int_{\mathbb{R}} u_t^2dxdt + \alpha\gamma\int_{\mathbb{R}} u_{xx}^2dx + 2\gamma\delta\int_0^T\int_{\mathbb{R}} u_{xt}^2dxdt + \gamma^2\int_{\mathbb{R}} u_x^2dx$$
$$= d\int_{\mathbb{R}}\frac{u_0^{2p+2}}{2p+2}dx + \alpha\gamma\int_{\mathbb{R}} u_{0xx}^2dx + \gamma^2\int_{\mathbb{R}} u_{0x}^2dx - d(2p+1)\delta\int_0^T\int_{\mathbb{R}} u^{2p}u_xu_{xt}dxdt$$
$$+d(2p+1)\alpha\int_0^T\int_{\mathbb{R}} u^{2p}u_xu_{xxx}dxdt - 2\gamma\int_0^T\int_{\mathbb{R}} f'(u)u_x + u_tdxdt$$

$$+2\epsilon\gamma\int_0^T\int_{\mathbb{R}}\beta(u_x)_x u_t dxdt.$$

The last four integrals can be estimed the following. Using (10)

$$\begin{aligned}-d(2p+1)\delta\int_0^T\int_{\mathbb{R}}u^{2p}u_xu_{xt}dxdt &\le d(2p+1)\delta\|u\|_\infty^{2p}\int_0^T\int_{\mathbb{R}}|u_xu_{xt}|dxdt\\ &\le C\delta\gamma^{-1}\|u\|_\infty^{4p}\int_0^T\int_{\mathbb{R}}u_x^2dxdt+\delta\gamma\int_0^T\int_{\mathbb{R}}u_{xt}^2dxdt\\ &\le C\delta\gamma^{-\frac{(4+2p)}{(2-p)}}+\delta\gamma\int_0^T\int_{\mathbb{R}}u_{xt}^2dxdt.\end{aligned}$$

Using (10) and (18)

$$d(2p+1)\alpha\int_0^T\int_{\mathbb{R}}u^{2p}u_xu_{xxx}dxdt\le\frac{\gamma}{2}\int_0^T\int_{\mathbb{R}}u^{2p}u_x^2dxdt$$

$$+C\gamma^{-1}\|u\|_\infty^{2p}\alpha^2\int_0^T\int_{\mathbb{R}}u_{xxx}^2dxdt\le\frac{\gamma}{2}\int_0^T\int_{\mathbb{R}}u^{2p}u_x^2dxdt+C\alpha\gamma^{-\frac{(6+p)}{(2-p)}}.$$

Using (10)

$$2\gamma\int_0^T\int_{\mathbb{R}}f'(u)u_xu_tdxdt\le C\gamma\int_0^T\int_{\mathbb{R}}u_x^2dxdt+C^2\gamma\int_0^T\int_{\mathbb{R}}u^{2p}u_x^2dxdtdt$$

$$+\frac{\gamma}{2}\int_0^T\int_{\mathbb{R}}u_t^2dx\le C+C^2\gamma\int_0^T\int_{\mathbb{R}}u^{2p}u_x^2dxdt+\frac{\gamma}{2}\int_0^T\int_{\mathbb{R}}u_t^2dxdt$$

Finally, we use (17)

$$\begin{aligned}2\epsilon\gamma\int_0^T\int_{\mathbb{R}}\beta(u_x)_xu_tdxdt &\le C\epsilon^2\gamma\int_0^T\int_{\mathbb{R}}[\beta(u_x)_x]^2dxdt+\frac{\gamma}{2}\int_0^T\int_{\mathbb{R}}u_t^2dxdt\\ &\le C\epsilon^2\gamma\int_0^T\int_{\mathbb{R}}u_{xx}^2dxdt+\frac{\gamma}{2}\int_0^T\int_{\mathbb{R}}u_t^2dxdt\\ &\le C\epsilon^2\gamma^{-\frac{4}{(2-p)}}+\frac{\gamma}{2}\int_0^T\int_{\mathbb{R}}u_t^2dxdt.\end{aligned}$$

Thus

$$(20)\quad d\int_{\mathbb{R}}\frac{u^{2p+2}}{2p+2}dx+d(2p+1)\epsilon\int_0^T\int_{\mathbb{R}}u^{2p}\beta(u_x)u_xdxdt+d_1\gamma\int_0^T\int_{\mathbb{R}}u^{2p}u_x^2dxdt$$

$$+\gamma\int_0^T\int_{\mathbb{R}}u_t^2dxdt+\alpha\gamma\int_{\mathbb{R}}u_{xx}^2dx+\gamma\delta\int_0^T\int_{\mathbb{R}}u_{xt}^2dxdt+\gamma^2\int_{\mathbb{R}}u_x^2dx\le C,$$

when $\delta=O(\gamma^{\frac{(4+2p)}{(2-p)}})$, $\alpha=O(\gamma^{\frac{(6+p)}{(2-p)}})$, $\epsilon=O(\gamma^{\frac{2}{(2-p)}})$, and $d_1=(2p+1)d-C^2-\frac{1}{2}>0$. We replace T by $t*\in(0,T)$ in (20). We integrate once more in $(0,T)$. Hence condition (i) follows. We multiply (13) by $\eta'(u)$ and replacing $\eta'(u)f'(u)$ by $\psi'(u)$

$$\frac{\partial}{\partial t}\eta(u)+\frac{\partial}{\partial x}\psi(u)=\epsilon(\eta'(u)\beta(u_x))_x-\epsilon\eta''(u)u_x\beta(u_x)+\delta(\eta'(u)u_{xt})_x-\delta\eta''(u)u_xu_{xt}$$

$$+\gamma(\eta'(u)u_x)_x-\gamma\eta''(u)u_x^2-\alpha(\eta'(u)u_{xxx})_x+\alpha\eta''(u)u_xu_{xxx}=\sum_{j=1}^8\Gamma_j.$$

Let $\theta \in C_0^\infty(\Omega)$. To estimate Γ_1, Γ_2, Γ_5, and Γ_6 we use (10):

$$\begin{aligned}
|<\Gamma_1,\theta>| &\le \epsilon \int_0^T \int_{\mathbb{R}} |\eta'(u)\beta(u_x)\theta_x| dxdt \\
&\le C\epsilon \|\theta_x\|_{L^2(\Omega)} \left[\int_0^T \int_{\mathbb{R}} |\beta(u_x)|^2 dxdt\right]^{\frac{1}{2}} \\
&\le CM\epsilon \left[\int_0^T \int_{\mathbb{R}} u_x^2 dxdt\right]^{\frac{1}{2}} \\
&\le C\epsilon\gamma^{-\frac{1}{2}} \le C\gamma^{\frac{(2+p)}{2(2-p)}};
\end{aligned}$$

$$\begin{aligned}
|<\Gamma_2,\theta>| &\le \epsilon \int_0^T \int_{\mathbb{R}} |\eta''(u)\beta(u_x)u_x\theta| dxdt \\
&\le C\epsilon \int_0^T \int_{\mathbb{R}} \beta(u_x)u_x dxdt \le C;
\end{aligned}$$

$$\begin{aligned}
|<\Gamma_5,\theta>| &\le \int_0^T \int_{\mathbb{R}} |\gamma\eta'(u)u_x\theta_x| dxdt \\
&\le C\gamma^{\frac{1}{2}} \|\theta_x\|_{L^2(\Omega)} \left[\gamma \int_0^T \int_{\mathbb{R}} u_x^2 dxdt\right]^{\frac{1}{2}} \\
&\le C\gamma^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
|<\Gamma_6,\theta>| &\le \gamma \int_0^T \int_{\mathbb{R}} |\eta''(u)u_x^2\theta| dxdt \\
&\le C\gamma \int_0^T \int_{\mathbb{R}} u_x^2 dxdt \le C.
\end{aligned}$$

For Γ_3 we use (20)

$$\begin{aligned}
|<\Gamma_3,\theta>| &\le \delta \int_0^T \int_{\mathbb{R}} |\eta'(u)u_{xt}\theta_x| dxdt \\
&\le C\|\theta_x\|_{L^2(\Omega)} \delta^{\frac{1}{2}}\gamma^{-\frac{1}{2}} \left[\delta\gamma \int_0^T \int_{\mathbb{R}} u_{xt}^2 dxdt\right]^{\frac{1}{2}} \\
&\le C\gamma^{\frac{(2+3p)}{2(2-p)}}.
\end{aligned}$$

To estimate Γ_4 we use (20) and (10)

$$\begin{aligned}
|<\Gamma_4,\theta>| &\le \int_0^T \int_{\mathbb{R}} |\delta\eta''(u)u_x u_{xt}\theta| dxdt \\
&\le C\delta^{\frac{1}{2}}\gamma^{-1} \left[\gamma \int_0^T \int_{\mathbb{R}} u_x^2 dxdt\right]^{\frac{1}{2}} \left[\delta\gamma \int_0^T \int_{\mathbb{R}} u_{xt}^2 dxdt\right]^{\frac{1}{2}} \\
&\le C\gamma^{\frac{2p}{(2-p)}}.
\end{aligned}$$

For Γ_7, we use (18)

$$\begin{aligned}|<\Gamma_7,\theta>| &\le \alpha\int_0^T\int_{\mathbb{R}}|\eta'(u)u_{xxx}\theta_x|dxdt\\ &\le C\left[\alpha^2\int_0^T\int_{\mathbb{R}}u_{xxx}^2dxdt\right]^{\frac{1}{2}}\\ &\le C\gamma^{\frac{(2+p)}{2(2-p)}}.\end{aligned}$$

Finally, we use (19)

$$\begin{aligned}|<\Gamma_8,\theta>| &\le \alpha\int_0^T\int_{\mathbb{R}}|\eta''(u)u_xu_{xxx}\theta|dxdt\\ &\le C\alpha\int_0^T\int_{\mathbb{R}}|u_xu_{xxx}|dxdt\\ &\le C\gamma^{\frac{p}{(2-p)}}.\end{aligned}$$

We see that $\dfrac{\partial}{\partial t}\eta(u)+\dfrac{\partial}{\partial x}\psi(u)$ decomposed in the form

$$\frac{\partial}{\partial t}\eta(u)+\frac{\partial}{\partial x}\psi(u)=\widetilde{\Gamma}_1+\widetilde{\Gamma}_2,$$

where $\widetilde{\Gamma}_1\in\{\text{compact set in } H^{-1}(\Omega)\}$ and $\widetilde{\Gamma}_2\in\{\text{bounded set of } M(\Omega)\}$. When $p>0$ we have $\widetilde{\Gamma}_1=\Gamma_1+\Gamma_3+\Gamma_4+\Gamma_5+\Gamma_7+\Gamma_8$ and $\widetilde{\Gamma}_2=\Gamma_2+\Gamma_6$. When $p=0$ then $\widetilde{\Gamma}_1=\sum_{i=1}^{4}\Gamma_{2i-1}$ and $\widetilde{\Gamma}_2=\sum_{i=1}^{4}\Gamma_{2i}$ and the proof is completed. □

Assume again that there exists a limiting function $u_0\in L^1(\mathbb{R})\cap L^{2(p+1)}(\mathbb{R})$ such that

$$\lim_{\gamma\to 0}u_0^\gamma=u_0 \quad \in L^1(\mathbb{R})\cap L^{2(p+1)}(\mathbb{R}). \tag{21}$$

Theorem 13. *Suppose that f, ϵ, δ, and α satisfy the same assumptions as in Theorem 12. (When $p=0$ we consider $\epsilon=O(\gamma)$, $\delta=o(\gamma^2)$, and $\alpha=o(\gamma^3)$). Then*

$$\lim_{\gamma\to 0}u^\gamma=u \ \ in \ \ L^\infty(\mathbb{R}_+;L^r_{loc}(\mathbb{R}))$$

for all $r\in[1,2(p+1))$, where $u\in L^\infty(\mathbb{R}_+;L^{2(p+1)}(\mathbb{R}))$ is the unique entropy solution to (3) and (2).

Proof. First of all we establish that, for any convex function $\eta(u)$ such that η' and η'' are uniformly bounded on $\mathbb{R}$,

$$\Lambda^\gamma=\sum_{i=1}^{8}\Gamma_i \ \textit{converges to a nonpositive measure in } \mathfrak{D}'(\mathbb{R}\times\mathbb{R}_+), \tag{22}$$

conformes with the proof of Theorem 12. Furthermore, from this proof, for any given $\theta\in C_c^\infty(\mathbb{R}\times(0,T))$, $\theta\ge 0$, $<\Gamma_1+\Gamma_3+\Gamma_4+\Gamma_5+\Gamma_7+\Gamma_8,\theta>\to 0$ when

$\gamma \to 0$. The Γ_2 and Γ_6 are nonpositives:

$$< \Gamma_2, \theta >= -\epsilon \int_0^T \int_{\mathbb{R}} \eta''(u)\beta(u_x)u_x\theta \, dxdt \le 0;$$

$$< \Gamma_6, \theta >= -\gamma \int_0^T \int_{\mathbb{R}} \eta''(u)u_x^2\theta \, dxdt \le 0.$$

To apply Lemma 3 we show that (6) and (7) are satisfied for a Young measure ν associated with the sequence $\{u^\gamma\}$. It is a standard matter to deduce, for all convex entropy pairs,

$$\partial_t < \nu(.), \eta(\lambda) > +\partial_x < \nu(.), \psi(\lambda) > \le 0$$

from the convergence property (22). Namely it follows from the definition of the Young measure that the terms in (22) converge (in the sense of distributions) to their "natural" limits,

$$\eta(u^\gamma) \rightharpoonup < \nu, \eta >, \ \ \psi(u^\gamma) \rightharpoonup < \nu, \psi > .$$

Inequality (6) (for all $k \in \mathbb{R}$) then follows by a standard regularization of the function $|u-k|$. To show that (7) is satisfied, we combine the entropy inequalities and the weak consistency property, as was suggested by DiPerna [10]. We follow the detailed argument in Szepessy [11] and in LeFloch /Natalini [9]. Consider the function $g(\lambda) = \lambda^2$, and set

$$G(\lambda, \lambda_0) := g(\lambda) - g(\lambda_0) - g'(\lambda_0)(\lambda - \lambda_0) = \frac{g''(\theta)}{2}(\lambda - \lambda_0)^2 = (\lambda - \lambda_0)^2 \ge 0. \tag{23}$$

Let $I \subset \mathbb{R}$ be a closed and bounded interval. Using the Jensen inequality and (23), it is easily checked that

$$\frac{1}{T}\int_0^T \int_I < \nu_{(x,t)}, \lambda - u_0(x) > dxdt \tag{24}$$

$$\le C_I \left(\frac{1}{T}\int_0^T \int_I < \nu_{(x,t)}, G(\lambda, u_0(x)) > dxdt \right)^{\frac{1}{2}}.$$

Let $\{\psi_n\} \subset C_c^\infty(\mathbb{R})$ be a sequence of test functions such that

$$\lim_{n\to\infty} \psi_n = g'(u_0) \ \ in \ \ L^2(\mathbb{R}).$$

Using (5) and (10), we get

$$\int_0^T \int_I < \nu_{(x,t)}, G(\lambda, u_0(x)) > dxdt \le \int_0^T \int_I < \nu_{(x,t)}, u_0(x) - \lambda > dxdt \tag{25}$$

$$+T \int_{\mathbb{R}-I} |u_0|^2 dx + 2T\|u_0\|_{L^2(\mathbb{R})}\|g'(u_0) - \psi_n\|_{L^2(\mathbb{R})}.$$

Taking an increasing sequence of compact sets K_i covering $\mathbb{R}$, i.e. such that $I \subset K_1 \subset K_2 \subset ...$ and $\cup_{i=1}^\infty K_i = \mathbb{R}$, we have

$$\int_0^T \int_I < \nu_{(x,t)}, G(\lambda, u_0(x)) > dxdt \le \int_0^T \int_{K_i} < \nu_{(x,t)}, G(\lambda, u_0(x)) > dxdt.$$

This, together with (25) and I replaced by K_i, yields

$$\frac{1}{T}\int_0^T\int_I <\nu_{(x,t)}, G(\lambda, u_0(x))> dxdt \tag{26}$$

$$\leq \frac{1}{T}\int_0^T\int_{\mathbb{R}} <\nu_{(x,t)}, u_0(x)-\lambda> \psi_n(x)dxdt + 2T\|u_0\|_{L^2(\mathbb{R})}\|g'(u_0)-\psi_n\|_{L^2(\mathbb{R})}$$

since

$$\lim_{i\to\infty}\int_{\mathbb{R}-K_i}|u_0|^2dx = 0.$$

Therefore, in view of (24) and (26), the strong consistency property (7) will be established if we show

$$\lim_{T\to 0^+}\frac{1}{T}\int_0^T\int_{\mathbb{R}} <\nu_{(x,t)}, u_0(x)-\lambda> \psi_n dxdt \leq 0 \tag{27}$$

for all n. By definition of the Young measure (Equation (5)), we have

$$\frac{1}{T}\int_0^T\int_{\mathbb{R}} <\nu_{(x,t)}, u_0(x)-\lambda> \psi_n dxdt = \lim_{\gamma\to 0}\frac{1}{T}\int_0^T\int_{\mathbb{R}}[u_0(x)-u^\gamma(x,t)]\psi_n(x)dxdt$$

$$= \lim_{\gamma\to 0}\frac{1}{T}\left[\int_0^T\int_{\mathbb{R}}[u_0(x)-u_0^\gamma(x)]\psi_n(x)\,dx\,dt - \int_0^T\int_{\mathbb{R}}\int_0^t u_s^\gamma(x,s)ds\psi_n(x)\,dx\,dt\right]$$

$$:= \lim_{\gamma\to 0}(A+B).$$

In view of the property (21), the term A tends to zero as $\gamma\to 0$. Using (13) we have

$$B = \frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t u_s^\gamma(x,s)ds\psi_n(x)dxdt$$

$$= \frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^t[-f(u^\gamma)_x + \epsilon\beta(u_x^\gamma)_x + \gamma u_{xx}^\gamma + \delta u_{xxt}^\gamma - \alpha u_{xxxx}^\gamma]ds\psi_n(x)dxdt$$

$$\leq C_nT.$$

This shows inequality (27). The proof of Theorem 13 is completed. □

Remark 14. *The same convergence result is also established in the case* $p>0$ *and* $|f'(u)|\leq C$ *if* $\epsilon = O(\gamma)$ *(or* $\epsilon=\gamma$*),* $\delta = O(\gamma^{2p+2})$*, and* $\alpha = O(\gamma^{p+3})$*.*

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