A NOTE ON THE THEOREMS OF LUSTERNIK-SCHNIRELMANN AND BORSUK-ULAM

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ABSTRACT. Let p be a prime number and X a simply-connected Hausdorff space equipped with a free $\mathbb{Z}_p$-action generated by $f_p : X \to X$ and let $\alpha : S^{2n-1} \to S^{2n-1}$ be a homeomorphism generating a free $\mathbb{Z}_p$-action over the $(2n-1)$-sphere, whose orbit space is some Lens space. Then we prove that, under some homotopical conditions on X, there exists an equivariant map $F : (S^{2n-1}, \alpha) \to (X, f_p)$. As applications, we derive new versions of generalized Lusternik-Schnirelmann and Borsuk-Ulam theorems.

Keywords. Borsuk-Ulam Theorem, Lusternik-Schnirelmann Theorem, free actions, equivariant maps.

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1. INTRODUCTION

Let X be a simply connected Hausdorff space equipped with a free $\mathbb{Z}_p$ action (p is a prime number) generated by $f_p : X \to X$. Given $l = (l_1, l_2, \ldots, l_n) \in \mathbb{Z}^n$ such that for each $j = 1, 2, \ldots n$, p does not divide l_j, consider the free $\mathbb{Z}_p$ action on S^{2n-1} generated by $\alpha_{p,l} : S^{2n-1} \to S^{2n-1}$, $\alpha_{p,l}(z_1, z_2, \ldots, z_n) = (\mathrm{e}^{\frac{2\pi i l_1}{p}} \cdot z_1, \mathrm{e}^{\frac{2\pi i l_2}{p}} \cdot z_2, \ldots, \mathrm{e}^{\frac{2\pi i l_n}{p}} \cdot z_n)$. We remember that a path-connected space Y is j-simple (for $j \geq 1$) if the canonical action of the fundamental group of Y over the group $\pi_j(Y)$ is trivial. Our main result is the following

Theorem 1. *Suppose that for $2 \leq j < m = 2n-1$ the orbit space X/f_p is j-simple and*

i) $\pi_j(X) = p \cdot \pi_j(X)$, if j is odd,

ii) $\pi_j(X)$ does not have elements of order p, if j is even.

Then there exists an equivariant map $F : (S^m, \alpha_{p,l}) \to (X, f_p)$.

If $p = 2$, theorem 1 remains valid for any m odd or even ($\alpha_{2,l}$ is the antipodal map for any choice of l). This theorem provides the following versions of the Borsuk-Ulam and Lusternik-Schnirelmann theorems.

Theorem 2. *Let X, $f_p : X \to X$ and m satisfying the same hypotheses of theorem 1. Then for each family $\mathcal{F} = \{C_0, \ldots C_k\}$ of $k+1$ sets covering X, each of which is either open or closed, and such that*

(1) $p = 2$ and $k \leq m$ or

(2) $p = 3$, m is odd and $k \leq m+1$ or

(3) $p > 3$, m is odd and $\left(\frac{p-1}{2}\right)(k-2) + 2 \leq m$

there exists $C_{j_0} \in \mathcal{F}$ such that $f_p(C_{j_0}) \cap C_{j_0} \neq \emptyset$.

Theorem 3. *Let X, $f_p : X \to X$ and m satisfying the same hypotheses of theorem 1.*

i) If $m \geq k(p-1)$, then for each continuous map $f : X \to \mathbb{R}^k$ there exists $x \in X$ such that $f(x) = f \circ (f_p)^j(x)$, $\forall\, 1 \leq j \leq p-1$.

ii) If $m \geq (k-1)(p-1)+1$, then for each continuous map $f : X \to \mathbb{R}^k$ there exists $x \in X$ such that $f(x) = f \circ f_p(x)$.

Theorem 4. *Let X be a paracompact, Hausdorff and simply connected space and let $f_2 : X \to X$ be an involution without fixed points, both satisfying the hypotheses of theorem 1. If Y is a separable metric space with topological dimension* $\dim(Y) \leq \frac{m-1}{2}$, *then for any map $f : X \to Y$, there exists $x \in X$ such that $f(x) = f \circ f_2(x)$.*

In [6] M. Izydorek and J. Jaworowski constructed for each k and $n \leq 2k-1$ a map f from the n-sphere S^n into a specific contractible k-dimensional complex Y such that $f(x) \neq f(-x)$, for all $x \in S^n$. Thus the upper bound for dimension in theorem 4 is sharp for general cases.

Theorem 3 generalizes, in a certain sense, the result of Cohen-Connet [4]. Generalizations of the same nature of the Borsuk-Ulam Theorem with "nice" topological spaces X and Y satisfying some homological conditions can be found in [8] and [3].

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2. Proof of Theorem 1

Let us consider the free action of $\mathbb{Z}_p$ on the sphere S^{2n-1} generated by the map $\alpha_{p,l}$ as defined in the Introduction. If X is a simply connected Hausdorff space equipped with a free $\mathbb{Z}_p$-action, then the diagonal action $\mathbb{Z}_p \times (S^{2n-1} \times X) \to (S^{2n-1} \times X)$, given by $g \cdot (q, x) = (g \cdot q, g \cdot x)$ is a free $\mathbb{Z}_p$-action. The projection $\pi : S^{2n-1} \times X \to S^{2n-1}$ onto the first coordinate is an equivariant map and induces a map

$$\xi_{n,l} : S^{2n-1} \times_{\mathbb{Z}_p} X \to L_p(l)$$

on orbit spaces. Since the $\mathbb{Z}_p$-action over S^{2n-1} is a free action, the quotient map $\pi_S : S^{2n-1} \to L_p(l)$ is a covering map. If $\tilde{U} \subset L_p(l)$ and $U \subset S^{2n-1}$ are open sets such that $\pi_S^{-1}(\tilde{U}) = \bigcup_{g \in \mathbb{Z}_p} g \cdot U$ (disjoint union) and $\pi_S|_{g \cdot U}$ is a homeomorphism from $g \cdot U$ onto $\tilde{U}$, then the map $\phi_{\tilde{U}} : \tilde{U} \times X \to \xi_{n,l}^{-1}(\tilde{U})$ given by $\phi_{\tilde{U}}([q], x) = [(q, g \cdot x)]$, where $g \in \mathbb{Z}_p$ is such that $q \in g \cdot U$, provides a local trivialization of $\xi_{n,l}$. Thus $\xi_{n,l}$ is a locally trivial fibration over the Lens space $L_p(l)$ with X as typical fiber.

With these notations we have the following

Lemma 1. *There exists an equivariant map $F : S^{2n-1} \to X$ if and only if there exists a cross-section of the fibration $\xi_{n,l}$.*

Proof: If $F : S^{2n-1} \to X$ is equivariant, then $\sigma_F : L_p(l) \to S^{2n-1} \times_{\mathbb{Z}_p} X$ defined by $\sigma_F([q]) = [(q, F(q))]$ is a cross-section of $\xi_{n,l}$.

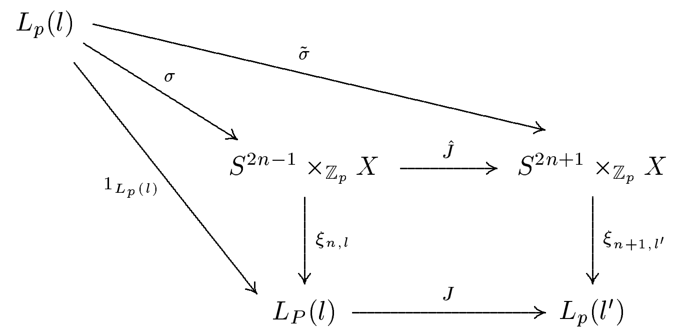
Conversely, given a cross-section $\sigma : L_p(l) \to S^{2n-1} \times_{\mathbb{Z}_p} X$, then $\sigma([q]) = [(q, F_\sigma(q))]$ for some continuous map $F_\sigma : S^{2n-1} \to X$. The continuity of F_σ follows from the fact that for $q \in g \cdot U$ we have $F_\sigma(q) = g \circ \pi_X \circ \phi_{\tilde{U}}^{-1} \circ \sigma \circ \pi_S(q)$, where $\pi_X : \tilde{U} \times X \to X$ is the projection onto X and we identify the element $g \in \mathbb{Z}_p$ with the homeomorphism $g : X \to X$ induced by the action over X.

Finally, to see that F_σ is equivariant, let $q' = g \cdot q$ for some $g \in \mathbb{Z}_p$ then for the element $[g \cdot q, F_\sigma(g \cdot q)] \in S^{2n-1} \times_{\mathbb{Z}_p} X$ we have

$$[g \cdot q, F_\sigma(g \cdot q)] = [q', F_\sigma(q')] = \sigma([q']) = \sigma([q]) = [q, F_\sigma(q)] = [g \cdot q, g \cdot F_\sigma(q)]$$

Thus there exists $h \in \mathbb{Z}_p$ such that $q' = h \cdot q'$ and $F_\sigma(g \cdot q) = h \cdot (g \cdot F_\sigma(q))$. But the action on S^{2n-1} is free, therefore $h = 1$ and $F_\sigma(g \cdot q) = g \cdot F_\sigma(q)$.

In the light of the above lemma, to prove theorem 1 it is enough to prove the existence of a cross-section of $\xi_{n,l} : S^{2n-1} \times_{\mathbb{Z}_p} X \to L_p(l)$. To do this, let us consider the fibration $\xi_{n+1,l'} : S^{2n+1} \times_{\mathbb{Z}_p} X \to L_p(l')$, where $l' = (l, 1) \in \mathbb{Z}^{n+1}$, then it can be easily checked that $\xi_{n,l}$ is isomorphic to the pull-back fibration of $\xi_{n+1,l'}$, induced by the inclusion $J : L_p(l) \to L_p(l')$, $J([z_1, z_2, \ldots, z_n]) = [z_1, z_2, \ldots, z_n, 0]$. A lift $\tilde{\sigma} : L_p(l) \to S^{2n+1} \times_{\mathbb{Z}_p} X$ of J is a partial cross-section of $\xi_{n+1,l'}$, defined on the $(2n-1)$-skeleton of $L_p(l')$. If we succeed in construct $\tilde{\sigma}$, then by the universal property of the pull-back, there exists a unique continuous map $\sigma : L_p(l) \to S^{2n-1} \times_{\mathbb{Z}_p} X$ such that $\xi_{n,l} \circ \sigma = 1_{L_p(l)}$ and $\hat{J} \circ \sigma = \tilde{\sigma}$



in particular, σ is a cross-section of $\xi_{n,l}$.

Let us prove now the existence of $\tilde{\sigma}$. Since the fiber X of the fibration $\xi_{n+1,l'}$ is path connected, there exists a cross-section over the 1-skeleton of $L_P(l')$. The hypothesis that X is simply-connected implies X is j-simple for all positive integer j, so the obstruction to the existence of a cross-section over the j-skeleton of $L_p(l')$ is an element of the cohomology with local coefficients, $H^j(L_p(l'), \pi_{j-1})$, where the system of local coefficients is formed by the groups $\pi_{j-1}(X_{[q]})$, where $X_{[q]}$ is the fiber over $[q]$, and $[q]$ runs over all points of the base space $L_p(l')$. The assumption that X/f_p is j-simple for all $2 \leq j < 2n-1$ (together with the 1-connectivity of X) guarantees that the local system of groups π_{j-1} is simple for $2 \leq j \leq 2n-1$, i.e., the cohomology with local coefficients $H^j(L_p(l'), \pi_{j-1})$ reduces to the ordinary singular cohomology $H^j(L_p(l'), \pi_{j-1}(X))$ for all $2 \leq j \leq 2n-1$ (cf. [10]).

Now to compute the cohomology groups we use the universal coefficient theorem to conclude that

$$H^j(L_p(l'); \pi_{j-1}(X)) \cong \begin{cases} \mathrm{H}om_{\mathbb{Z}}(\mathbb{Z}_p, \pi_{j-1}(X)) & \text{if } j \text{ is odd} \\ \mathrm{E}xt(\mathbb{Z}_p, \pi_{j-1}(X)) & \text{if } j \text{ is even} \end{cases}$$

but from the definitions of Hom and Ext we have

$$\mathrm{H}om_{\mathbb{Z}}(\mathbb{Z}_p, \pi_{j-1}(X)) \cong \{\alpha \in \pi_{j-1}(X) : p\alpha = 0\}$$

$$\mathrm{E}xt(\mathbb{Z}_p, \pi_{j-1}(X)) \cong \pi_{j-1}(X)/p\pi_{j-1}(X)$$

Hence, as X satisfies the hypotheses i) and ii) then $H^j(L_p(l'), \pi_{j-1}(X)) = 0$ for all $2 \leq j \leq 2n-1$. Therefore there exists a lift of J restrict to $L_p(l)$, the $(2n-1)$-skeleton of $L_p(l')$, and the theorem 1 follows.

Remark 1. *Given integers $m > n$, it is well known that there is no equivariant map $F : (S^m, a_m) \to (S^n, a_n)$, where a_m and a_n are the antipodal maps. We note that for n even (S^n, a_n) satisfies all hypotheses of theorem 1 except that $S^n/a_n = \mathbb{R}P^n$ is j-simple for each $2 \leq j < m$. Thus the assumption that the orbit space X/f_p is j-simple can not be dropped.*

Remark 2. *The procedure of extending the cross-section performed above does not apply directly over the fibration $\xi_{n,l}$, because in this case the obstruction to extend the cross-section σ to the $(2n-1)$-skeleton is an element of $H^{2n-1}(L_p(l); \pi_{2n}(X))$ but this is isomorphic to $Hom_{\mathbb{Z}}(H_{2n-1}(L_p(l)); \pi_{2n}(X))$, and since $H_{2n-1}(L_p(l)) = \mathbb{Z}$ we have $H^{2n-1}(L_p(l); \pi_{2n}(X)) = Hom_{\mathbb{Z}}(\mathbb{Z}; \pi_{2n}(X))$ which, upon our assumptions about X, is not zero in general.*

3. Proof of Theorem 2

The classical Lusternik-Schnirelmann Theorem says the following

Theorem LS: *Let $m \geq k$ and let $H_0, H_1, \ldots, H_k$ be closed subsets of the sphere S^m such that $S^m = \cup_{j=0}^{k} H_j$, then there exists $j_0 \in \{0, 1, \ldots, k\}$ with $H_{j_0} \cap -H_{j_0} \neq \emptyset$*

In 1979, Steinlein in [11] proved the following

Theorem S: *Let p be a prime number and* $\alpha_p : S^m \to S^m$ *a continuous map generating a free* $\mathbb{Z}_p$*-action on* S^m*. Let* $m, k \in \mathbb{N}$ *be such that m is odd and*

$$m \geq \begin{cases} k-1 & , \text{ if } p=3 \\ \left(\frac{p-1}{2}\right)(k-2)+2 & , \text{ if } p>3 \end{cases}$$

Then for each covering $S^m = \cup_{j=0}^{k} H_j$ *of the m-sphere* S^m *by* $k+1$ *closed sets, there exists some* H_{j_0} *such that* $H_{j_0} \cap \alpha_p(H_{j_0}) \neq \emptyset$

In [5], J. E. Greene proved that in theorem LS each set H_j can be either open or closed. With a similar reasoning theorem S can be improved to

Theorem SG *Let p be a prime number and* $\alpha_p : S^m \to S^m$ *a continuous map generating a free* $\mathbb{Z}_p$*-action on* S^m*. Let* $m, k \in \mathbb{N}$ *be such that m is odd and*

$$m \geq \begin{cases} k-1 & , \text{ if } p=3 \\ \left(\frac{p-1}{2}\right)(k-2)+2 & , \text{ if } p>3 \end{cases}$$

Then for each covering $S^m = \cup_{j=0}^{k} H_j$ *of the m-sphere* S^m *by* $k+1$ *sets, each of which is either open or closed, there exists some* H_{j_0} *such that* $H_{j_0} \cap \alpha_p(H_{j_0}) \neq \emptyset$

Proof: Following the reasoning of Greene we prove theorem SG by induction on the number t of closed sets in the cover of S^m. The case $t=0$ correspond to a cover of S^m by open sets $H_0, H_1, \ldots, H_k$. Select a Lebesgue number for this cover, that is, a positive number λ such that for all $x \in S^m$, the closed ball $\bar{B}(x, \lambda)$ is contained in some H_j. By compactness, there exists a finite collection of points $\{x_i\}$ such that the open balls $B(x_i, \lambda)$ cover S^m. For each j, let F_j denote the union of those $\bar{B}(x_i, \lambda)$ contained in H_j. Then F_j is closed, F_j is a subset of H_j for each j, and together the F_j cover S^m. Therefore, the theorem S implies that there exists some F_{j_0} such that $F_{j_0} \cap \alpha_p(F_{j_0}) \neq \emptyset$, and hence there exists some H_{j_0} such that $H_{j_0} \cap \alpha_p(H_{j_0}) \neq \emptyset$.

Thus we may assume that $0 < t < k+1$ and the theorem holds for fewer than t closed sets. We now show that it holds for t closed sets. Let $\mathcal{C}$ be a cover of S^m with $k+1$ sets, of which exactly t are closed and the remaining sets are open. Fix a closed set F in $\mathcal{C}$, and suppose that $F \cap \alpha_p(F) = \emptyset$. By normality, there exist open sets A and B such that $F \subset A$, $\alpha_p(F) \subset B$ and $A \cap B = \emptyset$. Let U denote the set $A \cap \alpha_p^{-1}(B)$. Then, U is open, $F \subset U$ and $U \cap \alpha_p(U) = \emptyset$. Therefore $\mathcal{C}' = (\mathcal{C} - \{F\}) \cup \{U\}$ is a cover of S^m with $k+1$ sets, of which exactly $t-1$ are closed and the remaining sets are open, so by the induction hypothesis some set H in the cover satisfies $H \cap \alpha_p(H) \neq \emptyset$ and by construction this H must be different from U, and hence some set H in the original cover must satisfy $H \cap \alpha_p(H) \neq \emptyset$. This complete the inductive step.

Now we are ready to prove theorem 2. Suppose that $X = \bigcup_{j=0}^{k} C_j$ is a covering of X by $k+1$ sets, each of which is either open or closed, and k satisfies the condition (1), (2) or (3) of theorem 2.

By theorem 1 for each $\alpha_{p,l} : S^m \to S^m$, generating a free $\mathbb{Z}_p$-action on S^m, and for each $f_p : X \to X$, generating a free $\mathbb{Z}_p$-action on X, there exists an equivariant continuous map from $(S^m, \alpha_{p,l})$ to (X, f_p). Note that, $(f_p)^{-1} = (f_p)^{p-1}$ also generates a free $\mathbb{Z}_p$-action on X, and analogously if $l = (l_1, \ldots, l_n)$ then $\alpha_{p,l}^{-1} = \alpha_{p,l'}$ $(l' = (p-l_1, p-l_2, \ldots, p-l_n))$ generates a free $\mathbb{Z}_p$-action on S^m. Then there exists an equivariant continuous map $F : (S^m, \alpha_{p,l}) \to (X, (f_p)^{-1})$. Thus we have that $S^m = \bigcup_{j=0}^{k} F^{-1}(C_j)$ is a covering of S^m by $k+1$ sets, each of which is either open or closed.

If $p \geq 3$ it follows from theorem SG that there exists some C_{j_0} such that

$$F^{-1}(C_{j_0}) \bigcap \alpha_{p,l'}(F^{-1}(C_{j_0})) \neq \emptyset$$

This together the facts that $\alpha_{p,l'} = \alpha_{p,l}^{-1}$ and $F : (S^m, \alpha_{p,l}) \to (X, f_p)$ is equivariant implies that $C_{j_0} \bigcap f_p(C_{j_0}) \neq \emptyset$.

If $p = 2$, m can be even or odd, and in any case α_2 is the antipodal map. The same reasoning applies to the Greene's version of theorem LS.

4. Proof of Theorem 3

Here we need the following theorem, which follows from the works of H. J. Munkholm [9] and E. L. Lusk [7].

Theorem ML: *Let p be a prime number, $k, m \in \mathbb{N}$ and $\alpha : S^m \to S^m$ a continuous map generating a free $\mathbb{Z}_p$-action on S^m.*

a) If $m \geq k(p-1)$, then for each continuous map $h : S^m \to \mathbb{R}^k$, there exists an $x \in S^m$ with $h(x) = h(\alpha^j(x))$ for all j, $1 \leq j \leq p-1$.

b) If $m \geq (k-1)(p-1)+1$, then for each continuous map $h : S^m \to \mathbb{R}^k$, there exists an $x \in S^m$ with $h(x) = h(\alpha(x))$.

Now, to prove theorem 3 let (X, f_p) be a pair satisfying the hypotheses of theorem 3 and let $f : X \to \mathbb{R}^k$ be a continuous map, then by theorem 1 there exists a continuous equivariant map $F : (S^m, \alpha_{p,l}) \to (X, f_p)$. Thus $h = f \circ F : S^m \to \mathbb{R}^k$ is a continuous map.

If $m \geq (k-1)(p-1)+1$, it follows from item b) of theorem ML that there exists $y \in S^m$ such that $h(y) = h(\alpha_{p,l}(y))$. Then if $x = F(y) \in X$ we have

$$f(x) = h(y) = h(\alpha_{p,l}(y)) = f(F(\alpha_{p,l}(y))) = f(f_p(F(y))) = f(f_p(x))$$

If $m \geq k(p-1)$, it follows from item a) of theorem ML that there exists $y \in S^m$ such that $h(y) = h((\alpha_{p,l})^j(y))$, for all $j = 0, 1, \dots p-1$. Then if $x = F(y) \in X$ we have in a similar way as above that $f(x) = f((f_p)^j(x))$ for all $j = 1, \dots p-1$.

5. Proof of Theorem 4

Here we use the following theorem due to Aarts, Fokkink and Vermeer [1]

Theorem AFV: *Let W be a paracompact, Hausdorff space such that $\dim(W) \leq m$. Suppose that α is a fixed point free involution of W. Then there exists a closed cover $\mathcal{C} = \{C_0, C_1, \dots, C_k\}$ of W with $k \leq m+1$ sets such that $C_j \cap \alpha(C_j) = \emptyset$ for each $j = 0, 1, \dots m+1$.*

To prove theorem 4, let us suppose by contradiction that $f(x) \neq f(f_2(x))$ for all $x \in X$. Let (W, τ) be a pair such that $W = Y \times Y - \triangle$ where $\triangle$ is the diagonal, and τ is the involution $\tau(x, y) = (y, x)$, then W is paracompact, Hausdorff, $\dim(W) \leq m-1$ and τ is a free continuous involution of W. By theorem AFV it follows that there exists a covering $W = \bigcup_{j=0}^{k} H_j$ of W by $k+1$ closed sets such that $H_j \cap \tau(H_j) = \emptyset$ for all $j = 0, 1, \dots k$ and $k \leq m$. By theorem 1 there exists an equivariant map $F : (S^m, \alpha_2) \to (X, f_2)$, and the map $g : (X, f_2) \to (W, \tau)$ given by $g(x) = (f(x), f(f_2(x)))$ is also an equivariant map, so $h = g \circ F : (S^m, \alpha_2) \to (W, \tau)$ is equivariant, therefore $S^m = \bigcup_{j=0}^{k} h^{-1}(H_j)$ is a covering of S^m by $k+1$ closed sets and since $H_j \cap \tau(H_j) = \emptyset$ for all $j = 0, 1, \dots k$, it follows that $h^{-1}(H_j) \cap \alpha_2(h^{-1}(H_j)) = \emptyset$ for all $j = 0, 1, \dots k$ and this contradicts the classical Lusternik-Schnirelmann theorem (theorem LS). Thus we conclude that there exists $x \in X$ such that $f(x) \neq f(f_2(x))$.

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