LOCALLY SOLVABLE VECTOR FIELDS AND HARDY SPACES

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Abstract. We characterize the boundary value of homegeneous solutions of planar one-sided locally solvable vector fields with analytic coefficients with the property that the L^p norm of their traces is locally uniformly bounded, $0 < p \leq 1$. For $p \neq 1/n$, $n = 1, 2, \dots$, the boundary value must locally belong to the local Hardy space $h^p(\mathbb{R})$ of Goldberg while for $p = 1/n$, $n = 1, 2, \dots$, the answer calls for a new class of atomic Hardy spaces if the vector field is of infinite type at some boundary point.

0. Introduction

This paper studies the local boundary behavior of solutions of the equation $Lf = 0$ where

$$L = A(x,t)\frac{\partial}{\partial t} + B(x,t)\frac{\partial}{\partial x}$$

is a nonvanishing, complex vector field with real analytic coefficients defined on an open subset Ω of the plane. We approach questions of boundary regularity through the localized analogue of the Hardy space of holomorphic functions on the unit disc. As a motivation, suppose $h(z)$, $z = x+iy$, is a holomorphic function of one variable defined on some rectangle

$$Q = (-r, r) \times (0, T)$$

with a weak boundary value at $y = 0$. It is well known that if the boundary value $bh \in \mathcal{D}'(-a, a)$ is locally in the localizable Hardy space $h^p(\mathbb{R})$ [G], $0 < p < \infty$, then

($\sharp_p(h)$) *for any* $0 < c < a$, *the norms of the traces* $h(.,y)$ *are uniformly bounded in* $L^p[-c, c]$ *as* $y \mapsto 0^+$, *i.e.*,

$$\int_{-c}^{c} |h(x+iy)|^p \, dx \leq C, \quad y \searrow 0.$$

Conversely, if ($\sharp_p(h)$) holds, bh exists and belongs locally to $h^p(\mathbb{R})$. This is just a local version of a classical property of the Hardy space $H^p(\Delta)$ of holomorphic functions on the unit disc Δ whose theory started with the fundamental work of Hardy [Ha].

Holomorphic functions are solutions of a complex vector field and it is natural to study vector fields L for which the solutions of the homogeneous equation $Lf = 0$ show a similar behavior, i.e., ($\sharp_p(f)$) whenever bf belongs to an appropriate space.

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When this happens we will say for brevity that the vector field L itself possesses the (H^p) property. Assume that $A(x,t) \equiv 1$ and for every $0 < r' < r$ there exists $\varepsilon > 0$ such that L satisfies the Nirenberg-Treves condition $(\mathcal{P})$ [NT] on $(-r', r') \times (0, \varepsilon)$. It was proved in [BH1] and [BH2], assuming that L is locally integrable and has smooth coefficients and $1 \leq p \leq \infty$, that $(\sharp_p(f))$ holds for any continuous solution $Lf = 0$ with boundary value bf if and only if bf is locally in $L^p(\mathbb{R})$ (recall that for $1 < p \leq \infty$, $h^p(\mathbb{R}) = L^p(\mathbb{R})$). Conversely, if L satisfies the (H^p) property for some $1 \leq p \leq \infty$, L must satisfy $(\mathcal{P})$ on a family of rectangles $(-r', r') \times (0, \varepsilon)$, with $r' \nearrow r$, $\varepsilon \searrow 0$. Roughly speaking, for $p \geq 1$, L has the (H^p) property if and only if L is locally solvable on some neighborhood of $(-r, r) \times \{0\}$ in $(-r, r) \times [0, T)$. The implication $(H^p) \implies (P)$ was extended for vector fields with real analytic coefficients to the case $0 < p \leq 1$ in [H]. In this work we deal with vector fields with real analytic coefficients that are locally solvable on a neighborhood of $(-r, r) \times \{0\}$ in $(-r, r) \times [0, T)$, and study for $0 < p \leq 1$ the relationship between $(\sharp_p(f))$ and the nature of bf for continuous homogeneous solutions $Lf = 0$. At boundary points where L is of finite type, the situation is identical with that of the Cauchy-Riemann operator: $(\sharp_p(f))$ holds if an only if $bf \in h^p(\mathbb{R})$ locally. Due to the real analyticity assumptions, unless L is a real vector field, the set of boundary points $x_0 \in (-r, r)$ of infinite type form a discrete set F that we may assume to be finite after shrinking r. This leads naturally to a functional analysis question: given a finite set $F \subset \mathbb{R}$, can we find for any $g \in h^p(\mathbb{R})$ an atomic decomposition $g = \sum_j \lambda_j a_j$, where all atoms a_j with "big" L^∞ norm are supported in intervals that do not contain points of F in their interior? The answer depends on the value of $0 < p \leq 1$: if p avoids the discrete set of values $1, 1/2, 1/3, \ldots$ the answer is yes. If $p = 1/n$, $n = 1, 2, 3, \ldots$, those elements of $h^{1/n}(\mathbb{R})$ that admit such a decomposition constitute a proper dense subspace $h_F^{1/n}(\mathbb{R}) \subset h^{1/n}(\mathbb{R})$ carrying a natural complete translation invariant metric. It turns out that for $p \in (0,1) \setminus \{1/n : \ n = 1, 2, 3, \ldots\}$, $(\sharp_p(f))$ holds for a continuous solution $Lf = 0$ possessing a boundary value $bf \in \mathcal{D}'$ if and only if bf belongs (locally) to $h^p(\mathbb{R})$. However, in the presence of boundary points of infinite type, i.e., if $F \cap (-r, r) \neq \emptyset$, and $p = 1/n$, $n = 1, 2, 3, \ldots$, $(\sharp_p(f))$ holds for a continuous solution $Lf = 0$ possessing a boundary value $bf \in \mathcal{D}'$ if and only if bf belongs (locally) to the smaller space $h_F^p(\mathbb{R})$. These equivalences are the main result of this paper.

The organization of the paper is as follows: in Section 1 we introduce a special kind of atomic decompositions (called distinguished decompositions) associated to a finite set $F \subset \mathbb{R}$ and are led to the definition of the spaces $h_F^{1/n}(\mathbb{R})$. In Section 2, we prove the existence of distinguished decompositions in $h^p(\mathbb{R})$ when $p \neq 1, 1/2, 1/3, \ldots$. In Section 3 we discuss some functional properties of the spaces $h_F^{1/n}(\mathbb{R})$, including atomic decompositions with additional vanishing moments, invariance under multiplication by test functions, invariance under change of variables and compute their duals. In Section 4 we recall the one-sided version of condition $(\mathcal{P})$ and express L in a convenient local form that will be used throughout the rest of the paper. In Section 5 we state one of our main results, that if bf is locally in $h^p(\mathbb{R})$ (or in $h_F^p(\mathbb{R})$ if $1/p \in \mathbb{N}$) then the traces $x \mapsto f(x,t)$ have uniformly bounded L^p norm (Theorem 5.2). This theorem is proved in Section 6. The converse property, that if f has traces with uniformly bounded L^p norm the bf must belong locally to $h^p(\mathbb{R})$ (or to $h_F^p(\mathbb{R})$ if $1/p \in \mathbb{N}$) (Theorems 7.1 and 7.2) is stated in Section 7, where some key technical lemmas are also stated and proved. Section 8 discusses two a priori different types of complex Hardy spaces, $E^p(U)$ and $H^p(U)$, which however coincide for some classes of possibly rough domains and this identity is essential in our proof of Theorems 7.1 and 7.2 which is concluded in Section 9.

1. A class of Hardy spaces

We recall how the localizable Hardy spaces $h^p(\mathbb{R}^n)$, introduced by Goldberg in [G], are defined. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)dx \neq 0$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define the *small maximal function* $m_\phi f$ by

$$m_\phi f(x) = \sup_{0<t<1} |(f * \phi_t)(x)|.$$

Definition 1.1. *Let $0 < p < \infty$. A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $h^p(\mathbb{R}^n)$ if and only if $m_\phi f \in L^p(\mathbb{R}^n)$, i.e.,*

$$\|f\|_{h^p} \doteq \left(\int_{\mathbb{R}^n} \left(m_\phi f(x) \right)^p dx \right)^{1/p} < \infty.$$

From now on we restrict our attention to the case $n = 1$ and $0 < p \leq 1$. The space $h^p(\mathbb{R})$ is a complete metric space with the distance

$$d(f, g) = \|f - g\|_{h^p}^p, \quad f, g \in h^p(\mathbb{R}).$$

For $p = 1$, $\|f\|_{h^1}$ is a norm and $h^1(\mathbb{R})$ is a normed space densely contained in $L^1(\mathbb{R})$. In the sequel, $|I| = b - a$ will denote the length of an interval $I = [a, b]$ and $\operatorname{supp} f$ the support of a distribution f.

Definition 1.2. *A measurable function $a(x)$, $x \in \mathbb{R}$, is an h^p-atom if satisfies the following properties: there exist an interval $I \subset \mathbb{R}$ such that*

(i) $\operatorname{supp} a \subset I$;
(ii) $\|a\|_{L^\infty} \leq |I|^{-1/p}$;
(iii) *if* $|I| < 1$, $\int x^k a(x)\, dx = 0$, *for all* $0 \leq k \leq p^{-1} - 1$, $k \in \mathbb{Z}$.

Thus, we may distinguish between two types of atoms: those satisfying a small bound and allowed to be supported in large intervals for which no moment condition is required and those supported in small intervals which are allowed to assume relatively large values depending on the size of I and are required to have a number of vanishing moments. An interval I such that (i), (ii) and (iii) are satisfied is called a *carrier* of the h^p-atom $a(x)$. A standard argument shows that there exists $R > 0$ such that $\|a\|_{h^p} \leq R$ for all h^p-atoms. According to the atomic decomposition theorem ([G]), there exist two positive constants C_1, C_2, such that for every $f \in h^p(\mathbb{R})$, we may find a sequence of h^p-atoms $\{a_k\}$ and a sequence of complex numbers $\{\lambda_k\}$ in ℓ^p such that f is the limit of the series

$$f = \sum_{k=1}^{\infty} \lambda_k a_k$$

with convergence both in the distribution sense and in the topology of $h^p(\mathbb{R})$ and furthermore,

$$C_1 \|f\|_{h^p} \leq \left(\sum_k |\lambda_k|^p \right)^{1/p} \leq C_2 \|f\|_{h^p}.$$

In particular, the quantity

$$\inf \sum_j |\lambda_j|^p, \quad f = \sum_j \lambda_j a_j,$$

where the inf is taken over all atomic decompositions of f yields a quantity that is equivalent to $\|f\|_{h^p}^p$.

We now introduce a distinguished type of atomic decompositions in $h^p(\mathbb{R})$ associated to a finite set $F \subset \mathbb{R}$.

Definition 1.3. *Let $F \subset \mathbb{R}$ be a finite set and $0 < p \leq 1$. We say that $f \in h^p(\mathbb{R})$ admits a distinguished atomic decomposition with respect to F if and only if there exists an atomic decomposition $f = \sum \lambda_j a_j$, where each a_j is an h^p-atom, $\sum |\lambda_j|^p < \infty$ and*

($\natural$) if $\|a_j\|_{L^\infty} > 1$, a_j has a carrier $[\alpha_j, \beta_j]$ such that $(\alpha_j, \beta_j) \cap F = \emptyset$.

Such a decomposition will be called a distinguished atomic decomposition and an h^p-atom that satisfies ($\natural$) will be called a distinguished atom. In particular, an h^p-atom with $\|a_j\|_{L^\infty} \leq 1$ is always distinguished for any F.

Theorem 1.4. *Let $F \subset \mathbb{R}$ be a finite, nonempty set and assume that $p = 1/k$ for some $k = 1, 2, \ldots$. There exists $f \in h^p(\mathbb{R})$ that does not admit a distinguished atomic decomposition with respect to F.*

Proof. It will be enough to prove the theorem assuming $F = \{0\}$. Assume by contradiction that every $f \in h^p(\mathbb{R})$ admits a distinguished decomposition with respect to $F = \{0\}$. Then the quantity

$$\|g\|_{h_F^p}^p = \inf \sum_j |\lambda_j|^p, \quad g = \sum \lambda_j a_j \in h^p(\mathbb{R}),$$

where the inf is taken over all admissible atomic decompositions, defines a complete distance on $h^p(\mathbb{R})$, $d(f, g) = \|f - g\|_{h_F^p}^p$ so that $\big(h^p(\mathbb{R}), \| \, \|_{h_F^p}\big)$ is an F-space in the sense of Banach [B]. Notice that $\|a\|_{h_F^p} \leq 1$ for every distinguished h^p-atom $a(x)$. Clearly, we have

$$\|g\|_{h^p} \leq C \|g\|_{h_F^p}, \quad g \in h^p(\mathbb{R}),$$

for some positive constant independent of $g \in h^p(\mathbb{R})$. Thus, the open mapping theorem implies that $\|f\|_{h_F^p} \leq C\|f\|_{h^p}$, $f \in h^p(\mathbb{R})$. In particular, the finite linear combinations of distinguished atoms form a dense subspace of $h^p(\mathbb{R})$. We treat first the case $p = 1$. Set

$$\Phi(x) = \begin{cases} \ln |x|, & \text{for } -1 \leq x < 0, \\ -\ln x, & \text{for } 0 < x \leq 1, \\ 0, & \text{for } |x| > 1. \end{cases}$$

Then $\Phi(x)$ does not belong to bmo($\mathbb{R}$) while $\Psi(x) = |\Phi(x)|$ does (see, e.g., [St] for the non localizable case of BMO). For a positive integer N, define $\Phi_N(x) = \Phi(x)$ if $|\Phi(x)| \leq \ln N$, $\Phi_N(x) = \ln N$ if $0 < x \leq 1/N$, $\Phi_N(x) = -\ln N$ if $-1/N \leq x < 0$ and set $\Psi_N(x) = |\Phi_N(x)|$. Thus, $\|\Psi_N\|_{\text{bmo}} \leq C$ while $\|\Phi_N\|_{\text{bmo}} \to \infty$ as $N \to \infty$.

If $a(x)$ is an h^1-atom carried by an interval contained either in $[0,\infty)$ or in $(-\infty,0]$ we have $\langle\Phi_N,a\rangle=\pm\langle\Psi_N,a\rangle$. Hence,

$$|\langle\Phi_N,a\rangle|=|\langle\Psi_N,a\rangle|\le\|\Psi_N\|_{\text{bmo}}\,\|a\|_{h^1}\le C.$$

Let $f\in h^1(\mathbb{R})$ and consider a distinguished decomposition, $f=\sum_j\lambda_j a_j$, with $\sum_j|\lambda_j|\le C\|f\|_{h^1}$. We have

$$\begin{aligned}|\langle\Phi_N,f\rangle|&=\Big|\Big\langle\Phi_N,\sum_j\lambda_j a_j\Big\rangle\Big|\\ &=\Big|\sum_j\Big\langle\Phi_N,\lambda_j a_j\Big\rangle\Big|\le\sum_j|\lambda_j|\,|\langle\Phi_N,a_j\rangle|\le C\|f\|_{h^1},\end{aligned}\tag{1.1}$$

where we have used that the series $\sum_j\lambda_j a_j\to f$ in L^1 and $\Phi_N\in L^\infty$. On the other hand, we may find a sequence (f_N) in $h^1(\mathbb{R})$, $\|f_N\|_{h^1}\le C$, so that $|\langle\Phi_N,f_N\rangle|\to\infty$, because $\|\Phi_N\|_{\text{bmo}}\to\infty$ and $\text{bmo}(\mathbb{R})\simeq(h^1(\mathbb{R}))^*$ ([G]). For instance, if $f_N(x)=N$ for $0\le x\le 1/N$, $f_N(x)=0$ for $x>1/N$, and $f_N(x)=-f_N(-x)$, it is easy to check that $\|f_N\|_{h^1}\le C$ and $|\langle\Phi_N,f_N\rangle|=2\ln N$. This contradicts (1.1).

We now discuss briefly the cases $p=1/2,1/3,\ldots$. We recall ([G]) that the dual of $h^p(\mathbb{R})$ for $p=1/k$, $k=2,3,\ldots$, is the Zygmund space $\Lambda_*^{k-1}(\mathbb{R})$. If $I\subset\mathbb{R}$ is an interval, $\Lambda^1_*(I)$ is defined as the space of continuous and bounded functions defined on I such that $\Delta_h^2 f(x)=|f(x+h)+f(x-h)-2f(x)|\le C_f|h|$ for all $x,x+h,x-h\in I$. Thus, a proof similar to that given for $p=1$ can be produced starting from a continuous function $f(x)\in\Lambda^1_*[0,\infty)\cup\Lambda^1_*(-\infty,0]\setminus\Lambda^1_*(\mathbb{R})$ since such a function will satisfy

$$|\langle f,a\rangle|\le C\,\|a\|_{h^p}\le C'$$

for all distinguished h^p-atoms but will not define a bounded linear functional on $h^{1/2}(\mathbb{R})$. For instance, we may take $f(x)=x\ln x$ for $x>0$ and $f(x)=0$ for $x\le 0$ —then it is a well known exercise that $\Delta_h^2 f(x)=|f(x+h)+f(x-h)-2f(x)|\le C_f|h|$ for all $x,x+h,x-h\ge 0$, [Du,p.91], and it is clear that $\Delta_h^2 f(0)=h\ln 1/h$, $h>0$— and cut if off with a smooth compactly supported function that is identically 1 for $|x|\le 1$ to make it bounded. Similarly, the case $p=1/k$, $k=2,3,\ldots$, can be dealt with by using a primitive of order $k-2$ of $x\ln x$. We leave details to the reader. □

Definition 1.5. *Let $F\subset\mathbb{R}$ be a finite set, $p=1/n$, $n\in\mathbb{N}$. We denote by $h^p_F(\mathbb{R})\subset h^p(\mathbb{R})$ the space of distributions $f\in h^p(\mathbb{R})$ such that $f=\sum\lambda_j a_j$, for some atomic distinguished decomposition with respect to F.*

Then

$$\|f\|^p_{h^p_F}=\inf\sum_j|\lambda_j|^p,\quad f=\sum_J\lambda_j a_j\in h^p_F,$$

where the inf is taken over all distinguished atomic decompositions, defines a translation invariant metric $d(f,g)=\|f-g\|^p_{h^p_F}$ and $(h^p_F(\mathbb{R}),\|\ \|_{h^p_F})$ is an F-space. The inclusion $h^p_F(\mathbb{R})\subset h^p(\mathbb{R})$ is continuous with dense range and the topology induced by $\|\ \|^p_{h^p_F}$ on $h^p_F(\mathbb{R})$ is strictly finer than that one inherited from $h^p(\mathbb{R})$.

We point out that if $f\in h^p(\mathbb{R})$ and $\operatorname{supp} f\cap F=\emptyset$, then f admits a distinguished decomposition with respect to F. Indeed, since the distance from $\operatorname{supp} f$ to F is positive, we may find an atomic decomposition of f with atoms supported in the complement of F.

2. Distinguished atomic decompositions in Hardy spaces

In contrast with Theorem 1.4, the situation for $p \in (0,1)\setminus\{1/2, 1/3, \dots\}$ is quite different.

Theorem 2.1. *Let $F \subset \mathbb{R}$ be a finite set, $0 < p < 1$, $p \neq 1/2, 1/3, \dots$ and K be a non negative integer. There exists a positive constant $C > 0$ such that every $f \in h^p(\mathbb{R})$ admits a distinguished atomic decomposition $f = \sum_j \lambda_j a_j$ with respect to F satisfying*

(1) *every a_j with $\|a_j\|_{L^\infty} > 1$ has vanishing moments up to order K;*

(2) $\sum_{j=1}^{\infty} |\lambda_j|^p \leq C\|f\|^p_{h^p}$.

Corollary 2.2. *If $n \in \mathbb{N}$, $1/n < p < \infty$ and $J \subset \mathbb{R}$ is a finite interval, then $h^p(\mathbb{R}) \cap \mathcal{E}'(J) \subset h_F^{1/n}(\mathbb{R}) \subset h^{1/n}(\mathbb{R})$ with a continuous inclusion.*

□

The proof of Theorem 2.1 will depend on two technical lemmas.

Lemma A. *Let $F \subset \mathbb{R}$ be a finite set, n a positive integer, $1/(n+1) < p < 1/n$. There exists a positive constant $C = C_p > 0$ such that every $f \in h^p(\mathbb{R})$ admits a decomposition in $\mathcal{S}'(\mathbb{R})$,*

$$f = U(x) + \sum_{j=1}^{\infty} \lambda_j a_j,$$

where U is a distribution supported in F with $\|U\|_{h^p} \leq C_p\|f\|_{h^p}$, the functions $a_j(x)$ are bounded, supported in intervals I_j whose interior does not intersect F and

(a) $\|a_j\|_{L^\infty} \leq |I_j|^{-1/p}$;

(b) $\sum_{j=1}^{\infty} |\lambda_j|^p \leq C\|f\|^p_{h^p}$;

(c) *every a_j with $\|a_j\|_{L^\infty} > 1$ has vanishing moments up to order $n-1$, i.e.,*

$$\int x^k a_j(x)\, dx = 0, \quad k = 0, 1, \dots, n-1.$$

In particular, the a_j's are distinguished h^p-atoms.

Proof. Since $h^p(\mathbb{R})$ is invariant under multiplication by $C_c^\infty(\mathbb{R})$, any $f \in h^p(\mathbb{R})$ may be written as a finite sum $f = f_1 + \cdots + f_N$, $f_j \in h^p(\mathbb{R})$, so that the support of each f_j meets at most one point of F. Thus, there is no restriction in proving the theorem under the assumption that $F = \{0\}$, which we make from now on. Since, whatever the value of $0 < p < 1$, any $f \in h^p(\mathbb{R})$ has a standard atomic decomposition, it will be enough to prove that any standard h^p-atom $a(x)$ has a decomposition $a = \sum_j \lambda_j a_j$ with $\sum_j |\lambda_j|^p \leq C$ for some universal $C > 0$ depending only on p and $\int x^k a(x)\, dx = 0$, $k = 0, 1, \dots, n-1$. Of course, we need only worry with h^p-atoms $a(x)$ such that: i) $\|a\|_{L^\infty} > 1$ and ii) any carrier $J = [\alpha, \beta]$ of $a(x)$ contains $\{0\}$ in its interior, i.e., $\alpha < 0 < \beta$. In fact, replacing $a(x)$ by a convenient multiple if necessary we may assume that $\alpha = -\beta$ and we will do so.

We assume that $J = [-\ell, \ell]$, $0 < \ell < 1$, is a carrier for $a(x)$ which, being an atom satisfies $\|a\|_{L^\infty} \leq 1/(2\ell)^{1/p}$. Denote by $H(x)$ the Heaviside function, $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x < 0$, and set $a^+(x) = H(x)a(x)$. Then a^+ is supported on $J^+ = [0, \ell]$ and $\|a^+\|_{L^\infty} \leq \|a\|_{L^\infty} \leq |J|^{-1/p} \leq |J^+|^{-1/p}$. We will initially write a^+ as a sum of functions with vanishing mean plus a multiple of Dirac's measure $\delta(x)$. For $j = 0, 1, 2, \ldots$, set

$$b_j(x) = 2^j \chi_{[0,\infty]}(x) a(2^j x) = 2^j a^+(2^j x)$$

and check that

$$\int_0^\infty b_j(x)\, dx = c, \quad j = 0, 1, 2, \ldots. \tag{2.1}$$

We have

$$\|b_j\|_{L^\infty} \leq \frac{2^j}{(2\ell)^{1/p}}, \quad \|b_{j-1} - b_j\|_{L^\infty} \leq \frac{2\, 2^j}{(2\ell)^{1/p}}, \quad j = 1, 2, \ldots,$$

and writing $I_j = [0, 2^{-j}\ell]$ we observe that

$$\operatorname{supp}(b_{j-1} - b_j) \subset I_{j-1}.$$

Suppose first that $n = 1$ so $1/2 < p < 1$. Define, for $j = 1, 2, \ldots$,

$$a_j(x) = \frac{1}{2} 2^{j(1/p-1)}(b_{j-1} - b_j) \quad \text{so} \quad \|a_j\|_{L^\infty} \leq \frac{1}{|I_{j-1}|^{1/p}}$$

and, in view of (2.1),

$$\int a_j(x)\, dx = 0,$$

showing that each a_j satisfies (a) and (c). Set $\lambda_j = 2\, 2^{(1-1/p)j}$. Then

$$a^+(x) = b_N + \sum_{j=1}^{N} b_{j-1}(x) - b_j(x) = b_N(x) + \sum_{j=1}^{N} \lambda_j a_j(x). \tag{2.2}$$

Since $p < 1$

$$\sum_{j=1}^{N} |\lambda_j|^p \leq c_p \sum_{j=1}^{\infty} 2^{(p-1)j} \leq C_p < \infty$$

with C_p independent of ℓ, so letting $N \to \infty$ in (2.2) gives

$$a^+(x) = c_0 \delta(x) + \sum_{j=1}^{\infty} \lambda_j a_j(x). \tag{2.3}$$

In fact, since $\sum_j |\lambda_j|^p < \infty$ and each $a_j(x)$ is an h^p-atom, the series $\sum_j \lambda_j a_j(x)$ converges in $h^p(\mathbb{R})$, while $b_N = 2^N a^+(2^N x)$ converges to $c_0\delta(x)$, $c_0 = \int a^+(x)\, dx$, in $\mathcal{S}'(\mathbb{R})$ as $N \to \infty$. A similar reasoning gives a decomposition for $a^-(x) = H(-x)a(x)$ and then $a(x) = a^+(x) + a^-(x)$ has also a similar decomposition. This proves the lemma for $n = 1$. If $n = 2$, $1/3 < p < 1/2$, we start by setting $b_j(x) = 2^j a^+(2^j x)$ as before, then choose $1/2 < q < 1$ and write $b_{j-1}(x) - b_j(x) = \mu_j \tilde{a}_j(x)$ so that $\sum_j |\mu_j|^q < \infty$, each $\tilde{a}_j(x)$ is an h^q-atom and the series $\sum_j \lambda_j a_j(x) = \sum_j \mu_j \tilde{a}_j(x)$ converges in $h^q(\mathbb{R})$. For instance, recalling the estimate $\|b_{j-1} - b_j\|_{L^\infty} \leq 2\,2^j/(2\ell)^{1/p}$, we define

$$\mu_j = c_{p,q} \ell^{(1/q-1/p)} 2^{j(1-1/q)}, \quad \tilde{a}_j(x) = \mu_j^{-1}(b_{j-1}(x) - b_j(x)).$$

We have already noticed that b_N converges to a multiple of the Dirac measure. Therefore we have

$$a^+(x) = c_0 \delta(x) + \sum_{j=1}^{\infty} \mu_j \tilde{a}_j(x)$$

and each $\tilde{a}_j$ satisfies (a) and (c) for $k = 0$. Now we will further decompose each $\tilde{a}_j$ with $\|a_j\|_{L^\infty} > 1$ in terms of a sum of functions with both mean and first moment equal to zero plus a distribution supported at the origin. Fix j and set

$$d_{m,j}(x) = 2^{2m} \tilde{a}_j(2^m x), \quad m = 0, 1, \dots.$$

By the properties of $\tilde{a}_j$ we see that

$$\int_0^\infty d_{m,j}(x)\, dx = 0, \quad \int_0^\infty x\, d_{m,j}(x)\, dx = \rho_j, \quad m = 0, 1, 2, \dots,$$
$$\|d_{m,j}\|_{L^\infty} \leq 2^{2m} \|\tilde{a}_j\|_{L^\infty} \leq 2^{2m} (2^{1-j}\ell)^{-1/q},$$
$$\|d_{m-1,j} - d_{m,j}\|_{L^\infty} \leq 2\, 2^{2m} (2^{1-j}\ell)^{-1/q}$$

and

$$\operatorname{supp}(d_{m-1,j} - d_{m,j}) \subset [0, 2^{2-j-m}\ell] \doteq J_{m-1,j}.$$

For $m = 1, 2, \dots,$ define

$$\nu_{m,j} = c_{p,q} 2^{j(1/q-1/p)} \ell^{(1/p-1/q)} 2^{m(2-1/p)}, \quad e_{m,j}(x) = \nu_{m,j}^{-1}(d_{m-1,j}(x) - d_{m,j}(x)).$$

Is easy to see that with this definition each $e_{m,j}$ is a distinguished h^p-atom with carrier $J_{m-1,j}$. Moreover,

$$\sum_{m=1}^{N} |\nu_{m,j}|^p \leq c_{p,q}^p 2^{j(p/q-1)} \ell^{(1-p/q)} \sum_{m=1}^{\infty} 2^{m(2p-1)}$$
$$\leq C(p,q) 2^{j(p/q-1)} \ell^{(1-p/q)}$$

because $p < 1/2$. Next we write

$$\tilde{a}_j(x) = d_{N,j}(x) + \sum_{m=1}^{N} d_{m-1,j}(x) - d_{m,j}(x) = d_{N,j}(x) + \sum_{m=1}^{N} \nu_{m,j} e_{m,j}(x),$$

Letting $N \to \infty$ we get

$$\tilde{a}_j(x) = c_{1,j} \delta'(x) + \sum_{m=1}^{\infty} \nu_{m,j} e_{m,j}(x).$$

where we have used that, since $\int \tilde{a}_j(x)\,dx = 0$, $2^{2N}\tilde{a}_j(2^N x) \to c_{1,j}\delta'(x)$ as $N \to \infty$, with $c_{1,j} = -\int x\tilde{a}_j(x)\,dx = 2^{-j}\mu_j^{-1}c_1$, $c_1 \doteq -\int x\, a^+(x)\,dx$. Using this representation for each $\tilde{a}_j$ with $\|a_j\|_{L^\infty} > 1$ we will obtain, after rearrangement and renaming, a decomposition

$$a^+(x) = c_0\delta(x) + c_1\delta'(x) + \sum_{j=1}^{\infty}\lambda_j a_j(x),$$

where the a_j's satisfy (a) and (c) with $k = 0, 1$ and $\sum|\lambda_j|^p \le C_p < \infty$. To see this we start by observing that $\sum_j \mu_j^p \sum_m \nu_{m,j}^p < \infty$. In fact,

$$\begin{aligned}\sum_j \mu_j^p \sum_m \nu_{m,j}^p &= c_{p,q}\sum_{j,m} \underbrace{\ell^{p/q-1}2^{j(p-p/q)}}_{\mu_j^p}\underbrace{2^{j(p/q-1)}\ell^{(1-p/q)}2^{m(2p-1)}}_{\nu_{m,j}^p} \\ &= c_{p,q}\sum_{j,m} 2^{j(p-1)}2^{m(2p-1)} = C_{p,q} < \infty.\end{aligned}$$

Consider the sum

$$\begin{aligned}c_0\delta(x) + \sum_{j=1}^{N}\mu_j\tilde{a}_j(x) &= c_0\delta(x) + \sum_{j=1}^{N} c_{1,j}\mu_j\,\delta'(x) + \sum_{j=1}^{N}\mu_j\sum_{m=1}^{\infty}\nu_{m,j}e_{m,j}(x) \\ &= c_0\delta(x) + \sum_{j=1}^{N} c_1 2^{-j}\,\delta'(x) + \sum_{j=1}^{N}\mu_j\sum_{m=1}^{\infty}\nu_{m,j}e_{m,j}(x).\end{aligned}$$

We already know that the left hand side converges to $a^+(x)$ in $h^p(\mathbb{R})$ as $N \to \infty$. Furthermore, the term $\sum_{j=1}^{N}\mu_j\sum_{m=1}^{\infty}\nu_{m,j}e_{m,j}(x)$ on the right hand side converges in $h^p(\mathbb{R})$ because $\sum_{m,j}\mu_j^p\nu_{m,j}^p < \infty$ and the middle term clearly tends to $c_1\delta'(x)$ proving the desired decomposition.

A similar reasoning gives a decomposition for $a^-(x) = H(-x)a(x)$ and then $a(x) = a^+(x) + a^-(x)$ has also an analogous decomposition. This proves the lemma for $n = 2$. It is clear that if $n > 2$ this procedure can be further continued to obtain the desired representation for any value of n. □

Remark. The proof of Lemma A gives an explicit expression for the non atomic term $U^+(x)$ in the representation of $a^+(x)$, namely,

$$U^+(x) = \sum_{j=0}^{n-1} c_j\delta^{(j)}(x), \qquad c_j = (-1)^j\int x^j a^+(x)\,dx.$$

The following lemma gives an atomic expansion for the non atomic term $U(x)$ in Lemma A.

Lemma B. *Let n be a positive integer, $1/(n+1) < p < 1/n$, $U(x) \in h^p(\mathbb{R})$ a distribution supported at the origin. Then, for any non negative integer K, $U(x)$ has a representation*

$$U(x) = \sum_{j=1}^{\infty}\lambda_j a_j(x),$$

where each $a_j(x)$ is an h^p-atom supported in an interval whose interior does not contain the origin, all moments up to order K of $a_j(x)$ are null if $\|a_j\|_{L^\infty} > 1$ and $\sum_j |\lambda_j|^p \leq C(p,K)\|U\|_{h^p}$.

Proof. The hypotheses show that $U(x) = \sum_{\ell=0}^{n-1} c_\ell D^\ell \delta(x)$. It follows that $\|U\|_{h^p} \simeq \sum_{\ell=0}^{n-1} |c_\ell|$. Therefore, it will be enough to show that $D^\ell \delta$ possesses a distinguished atomic decomposition in h^p with vanishing moments up to order K. To see this, choose $\psi \in C_c^\infty(\mathbb{R})$ such that: i) $\operatorname{supp}\psi \subset [2,4]$, ii) $\int \psi(x)\,dx = 1$, iii) $\int x^d \psi(x)\,dx = 0$ for $d = 1,2,\ldots,M$, where $M = n-1+\max[(n-1),K]$ and $\|\psi\|_{L^\infty} \leq 1$. For $j = 0,1,2,\ldots$, define

$$\psi_j(x) = 2^j \psi(2^j x)$$

so ψ_j satisfies properties ii) and iii) above and i) must be replaced by $\operatorname{supp}\psi_j \subset [2^{1-j}, 2^{2-j}]$. We may write

$$\psi_k(x) = \psi_0(x) + \sum_{j=1}^{k} (\psi_j(x) - \psi_{j-1}(x)) = \psi(x) + \sum_{j=1}^{k} (\psi_j(x) - \psi_{j-1}(x)).$$

For $j \geq 1$, we have

$$|\psi_j(x) - \psi_{j-1}(x)| \leq 2^j, \quad \text{and} \quad \operatorname{supp}(\psi_j - \psi_{j-1}) \subset [0, 2^{3-j}].$$

Define, for $j = 0,1,2,\ldots$,

$$\alpha_0(x) = \psi(x), \quad \alpha_j(x) = 2^{-3/p} 2^{j(1/p-1)} (\psi_j(x) - \psi_{j-1}(x)),$$

and set $\lambda_0 = 1$ and $\lambda_j = 2^{3/p} 2^{j(1-1/p)}$, for $j \geq 1$. Hence,

$$\psi_k(x) = \sum_{j=0}^{k} \lambda_j \alpha_j(x).$$

The α_j's are distinguished h^p-atoms, $j = 1,2,\ldots$ while α_0 is an h^p-atom with $\|\alpha_0\|_{L^\infty} \leq 1$ carried by $[2,4]$. Furthermore, $\sum_{j=0}^\infty |\lambda_j|^p \leq C_p \sum_{j=0}^\infty 2^{j(p-1)} < \infty$, showing that $\sum_{j=0}^k \lambda_j \alpha_j(x)$ converges in h^p to a distribution $g \in h^p(\mathbb{R})$ as $k \to \infty$. However, $g = \delta$ because $\psi_k \to \delta$ in $\mathcal{E}'(\mathbb{R})$ when $k \to \infty$. Similarly, it is easy to see that

$$D^\ell \psi_k(x) = \sum_{j=0}^{k} \lambda_j D^\ell \alpha_j(x), \quad 0 \leq \ell \leq n-1,$$

converges in h^p to $D^\ell \delta$ as $k \to \infty$ since each $D^\ell \alpha_j$ is an h^p-atom with vanishing moments up to order $M - \ell \geq M - n + 1 \geq \max(n-1, K)$, so

$$D^\ell \delta(x) = \sum_{j=0}^{\infty} \lambda_k D^\ell \alpha_j(x)$$

gives the required decomposition. □

Remark. The proof of Lemma B shows in particular the well known fact that $D^\ell \delta$, the derivative of order ℓ of the Dirac measure δ, belongs to h^p if $\ell \leq n-1$ and $p < 1/n$.

Proof of Theorem 2.1. As in the proof of Lemma A, we may assume without restriction that $F = \{0\}$. Choose $n \in \mathbb{N}$ such that $1/(n+1) < p < 1/n$. By combining lemmas A and B we obtain a distinguished decomposition with respect to F, $f = \sum \lambda_j a_j$, so that (1) and (2) hold for $K = n-1$. Assume as an inductive hypothesis that the proposition is true for some $K \geq n-1$ and let us see that it also holds for $K+1$. We may start from a distinguished decomposition

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x) \tag{2.4}$$

where $\sum_j |\lambda_j|^p < \infty$ and the distinguished atoms $a_j(x)$ with $\|a_j\|_{L^\infty} > 1$ have vanishing moments up to order K. Pick up one of the a_j's with vanishing moments and call it $g(x)$. Assume that $g(x)$ is carried, say, by $[0, \ell]$. Set

$$b_k(x) = 2^{-(K+2)k} g(2^{-k}x)$$

and note that

$$\int_{\mathbb{R}} x^d b_k(x)\, dx = 0, \quad d = 0, 1, \ldots, K, \quad \int_0^\infty x^{K+1} b_k(x)\, dx = c. \tag{2.5}$$

It is easily verified that

$$\|b_k\|_{L^\infty} \leq \frac{2^{-(K+2)k}}{\ell^{1/p}}, \quad \|b_{k-1} - b_k\|_{L^\infty} \leq \frac{c2^{-(K+2)k}}{\ell^{1/p}}, \quad k = 1, 2, \ldots,$$

and, writing $I_k = [0, 2^k \ell]$

$$\operatorname{supp}(b_{k-1} - b_k) \subset I_k.$$

Define, for $k = 1, 2, \ldots,$

$$\alpha_k(x) = \frac{2^{k(K+2-1/p)}}{c}(b_{k-1} - b_k) \quad \text{so} \quad \|\alpha_k\|_{L^\infty} \leq \frac{1}{|I_k|^{1/p}}$$

and, taking account of (2.5), observe that

$$\int x^d \alpha_k(x)\, dx = 0, \quad d = 0, 1, 2, \ldots, K+1,$$

showing that each α_k is a distinguished h^p-atom with vanishing moments up to order $K+1$. Choose $N \in \mathbb{N}$ such that $2^{N-1} \leq \ell^{-1/p} < 2^N$ and set $\mu_k = c2^{-k(K+1-1/p)}$. Then

$$g(x) = b_N + \sum_{k=1}^{N} b_{k-1}(x) - b_k(x) = b_N(x) + \sum_{k=1}^{N} \mu_k \alpha_k(x).$$

Note that

$$\|b_N\|_{L^\infty} \leq \frac{2^{-N(k+2)}}{\ell^{1/p}} = 2^{-N(k+2-1/p)} \frac{2^{-N/p}}{\ell^{1/p}} = 2^{-N(k+2-1/p)} |I_N|^{-1/p} \leq \frac{1}{|I_N|^{1/p}}$$

because $k+2 \geq n+1$ and $1/(n+1) < p < 1/n$. By the choice of N

$$\|b_N\|_{L^\infty} \leq \frac{1}{|I_N|^{1/p}} = \frac{\ell^{-1/p}}{2^{N/p}} \leq \frac{2^N}{2^{N/p}} < 1$$

so b_N is a distinguished atom (without moment condition). Finally,

$$\sum_{k=1}^{N} |\mu_k|^p = c^p \sum_{k=1}^{N} 2^{-kp(K+2-1/p)} \leq c^p \sum_{k=1}^{\infty} 2^{-kp(k+2-1/p)} \leq C_p < \infty$$

with C_p independent of ℓ. We have proved that every a_j in (2.4) with $\|a_j\|_{L^\infty} > 1$ may be expanded as a finite sum of distinguished h^p-atoms with vanishing moments up to order $K+1$. Replacing each of those a_j's by its expansion, we obtain a new representation like (2.4), where any atom a_j with $\|a_j\|_{L^\infty} > 1$ has vanishing moments up to order $K+1$. □

3. Properties of the spaces $h^p_F(\mathbb{R})$

Atomic decompositions. Built in the definition of $h_F^{1/n}(\mathbb{R})$, $n \in \mathbb{N}$, is the fact that every element $f \in h_F^{1/n}(\mathbb{R})$ has a distinguished atomic decomposition whose atoms a_j with large $\|a_j\|_{L^\infty}$ have vanishing moments up to order $n-1$. However, it is useful to know the existence of atomic decompositions with additional vanishing moments.

Theorem 3.1. *Let $F \subset \mathbb{R}$ be a finite set, $n \in \mathbb{N}$ and K a non negative integer. There exists a positive constant $C > 0$ such that every $f \in h_F^{1/n}(\mathbb{R})$ admits a distinguished atomic decomposition $f = \sum_j \lambda_j a_j$ with respect to F satisfying*

(1) *every a_j with $\|a_j\|_{L^\infty} > 1$ has vanishing moments up to order K;*
(2) $\displaystyle\sum_{j=1}^{\infty} |\lambda_j|^{1/n} \leq C \|f\|_{h_F^{1/n}}^{1/n}$.

Proof. For $K = n-1$, (1) and (2) follow from the definition of $h_F^{1/n}(\mathbb{R})$. The general case may be obtained by induction on K adapting the proof of Theorem 2.1. We leave details to the reader. □

Multiplication invariance. Let us show that $h_F^{1/n}(\mathbb{R})$, $n = 1, 2, \ldots$ is invariant under multiplication by functions of $\mathcal{S}(\mathbb{R})$. It will be enough to show that if $a(x)$ is an $h_F^{1/n}$-atom and $0 \neq \psi \in \mathcal{S}(\mathbb{R})$ then $\psi(x)a(x) \in h_F^{1/n}(\mathbb{R})$ and $\|\psi a\|_{h_F^{1/n}} \leq C_\psi$. Let $a(x) \in h_F^{1/n}(\mathbb{R})$. If $\|a\|_{L^\infty} \leq 1$ and we write $a_1 = \psi a / \|\psi\|_{L^\infty}$ we see that a_1 is an $h_F^{1/n}$-atom because $\|a_1\|_{L^\infty} \leq 1$, so no moment condition needs to hold. Hence, $\psi a = \|\psi\|_{L^\infty} a_1$ is scalar multiple of a distinguished $h^{1/n}$-atom and $\|\psi a\|_{h_F^{1/n}} \leq \|\psi\|_{L^\infty}$.

Assume now that $\|a\|_{L^\infty} > 1$ thus, $a(x)$ has a carrier I whose interior does not meet F. Let us assume without loss of generality that $I = [0, L]$, $L < 1$ and let $n \geq 1$. In view of Theorem 3.1, we may assume without loss of generality that

$$\int x^d a(x)\, dx = 0, \quad d = 0, 1, 2, \ldots, 2(n-1). \tag{3.1}$$

By Taylor's expansion

$$\psi(x) = \sum_{j<n} \frac{\psi^{(j)}(0)}{j!} x^j + \psi_n(x)x^n,$$

so

$$\psi(x)a(x) = \sum_{j<n} \frac{\psi^{(j)}(0)}{j!} x^j a(x) + \psi_n(x)x^n a(x) = \sum_{j<n} b_j(x) + b_n(x).$$

Note that (3.1) implies that the terms $b_j(x)$, $0 \le j \le n-1$, are scalar multiples of distinguished $h^{1/n}$-atoms with vanishing moments up to order $n-1$ and $\|b_j\|_{h_F^{1/n}} \le C_j \|\psi^{(j)}\|_{L^\infty}$, $0 \le j \le n-1$. Since

$$\|b_n\|_{L^\infty} \le C_n \|\psi^{(n)} a\|_{L^\infty} L^n \le C_n \|\psi^{(n)}\|_{L^\infty}$$

we see that $b_n(x)$ is a scalar multiple of a distinguished $h^{1/n}$-atom (without vanishing moments), the interval $[0,1]$ carries $b_n(x)$ and $\|b_n\|_{h_F^{1/n}} \le C_n \|\psi^{(n)}\|_{L^\infty}$. This shows that $\psi a \in h_F^{1/n}$ and $\|\psi a\|_{h_F^{1/n}} \le C_n \sum_{j\le n} \|\psi^{(n)}\|_{L^\infty}$.

Duality. If $f(x)$ is a locally integrable function defined on the real line and $I \subset \mathbb{R}$ is an interval, we denote the mean of $f(x)$ over I by

$$f_I = \frac{1}{|I|} \int_I f(x)\, dx.$$

Definition 3.2. *Let $F \subset \mathbb{R}$ be a finite set. We say that a function $f(x) \in L^1_{\text{loc}}(\mathbb{R})$ is in* $\text{bmo}_F(\mathbb{R})$ *if there exists a constant $C > 0$ such that for any interval $I \subset \mathbb{R}$:*

i) $\displaystyle\int_I |f(x) - f_I|\, dx \le C|I|$ *if* $|I| < 1$ *and* $I \cap F = \emptyset$*;*

ii) $\displaystyle\int_I |f(x)|\, dx \le C|I|$ *if* $|I| \ge 1$ *or* $I \cap F \ne \emptyset$*.*

The infimum of the constants $C > 0$ such that i) and ii) hold, is a norm that will be denoted by $\|f\|_{\text{bmo}_F}$*.*

The norm $\|\ \|_{\text{bmo}_F}$ turns $\text{bmo}_F(\mathbb{R})$ into a Banach space that contains $L^\infty(\mathbb{R})$ as a dense subspace. Take $f(x) \in L^\infty(\mathbb{R})$. Let $a(x)$ be an h^1_F-atom with $\|a\|_{L^\infty} > 1$ and let I be a carrier that does not intersect F. Since $a(x)$ has vanishing mean

$$\left| \int f(x)a(x)\, dx \right| = \left| \int_I (f(x) - f_I)a(x)\, dx \right| \le \|f\|_{\text{bmo}_F}.$$

If $a(x)$ has a carrier I such that $|I| \ge 1$ or $I \cap F \ne \emptyset$ we have

$$\left| \int f(x)a(x)\, dx \right| \le \|a\|_{L^\infty} \int_I |f|\, dx \le \|f\|_{\text{bmo}_F}.$$

This shows that integration against f defines a continuous linear functional on $h^1_F(\mathbb{R})$ such that $|\langle f, g\rangle| \le \|f\|_{\text{bmo}_F} \|g\|_{h^1_F}$: indeed, we have

$$\langle f, \sum_j \lambda_j a_j \rangle = \sum_j \langle f, \lambda_j a_j \rangle, \quad g = \sum_j \lambda_j a_j \in h^1_F(\mathbb{R}),$$

because the series converges in L^1 whenever $\sum_j \lambda_j a_j$ is a distinguished decomposition of g. Therefore, after the appropriate identifications, we have a continuous inclusion $(L^\infty(\mathbb{R}), \| \, \|_{\mathrm{bmo}_F}) \subset (h^1_F(\mathbb{R}))^*$ and taking the closure we get $\mathrm{bmo}_F(\mathbb{R}) \subset (h^1_F(\mathbb{R}))^*$.

Conversely, if $\Lambda \in (h^1_F(\mathbb{R}))^*$, the restriction of Λ to the Schwartz space $\mathcal{S}(\mathbb{R})$ defines a tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ and we wish to prove that $f \in \mathrm{bmo}_F(\mathbb{R})$. Taking the restriction of Λ to $C^\infty_c(I)$ where I is an arbitrary interval of length 2 we get $|\langle \Lambda, \phi\rangle| \leq \|\Lambda\|\|\phi\|_{L^\infty}$, $\phi \in C^\infty_c(I)$, since $\phi/\|\phi\|_{L^\infty}$ is a distinguished atom. We thus conclude, by introducing a partition of unity subordinated to some cover of $\mathbb{R}$ by open intervals of length 2, that there is a bounded measure μ such that $\langle \Lambda, \phi\rangle = \int \phi \, d\mu$, $\phi \in C^\infty_c(\mathbb{R})$. If we restrict Λ to $C^\infty_c(\mathbb{R} \setminus F)$, using the fact that h^1 atoms supported in $\mathbb{R} \setminus F$ are h^1_F atoms and adapting the arguments in the proof of $H^1(\mathbb{R})^* \simeq BMO(\mathbb{R})$ ([GR],[St]) we conclude that the restriction of μ to $\mathbb{R} \setminus F$ is absolutely continuous with respect to Lebesgue measure, $d\mu = f \, dx$, where $f \in L^1$ satisfies i) and ii) of Definition 3.2. Hence, the proof of $\Lambda|_{\mathcal{S}} \in \mathrm{bmo}_F$ will be finished if we show that $\mu = f$. We know that $\mu - f$ is a measure concentrated in F, so if $x_0 \in F$, $\mu - f = c\delta(x - x_0)$ in a neighborhood of x_0, where c is a constant and δ the Dirac mass. Let $\phi(x) \in C^\infty_c([-1, 1])$ be an even function satisfying

(1) $\|\phi\|_{L^\infty} \leq 1$ and $\phi(0) = 1$;

(2) $\displaystyle\int_0^1 \phi(x) \, dx = \int_{-1}^0 \phi(x) \, dx = 0$;

and set $\phi_n(x) = \phi(n(x - x_0))$. Then we see that $n \, \chi_{[0,1/n]}(x - x_0)\phi_n(x)$ and $n \, \chi_{[-1/n,0]}(x - x_0)\phi_n(x)$ are distinguished atoms and this shows that $\phi_n \to 0$ in $h^1_F(\mathbb{R})$. Thus,

$$c = \langle \delta(x - x_0), \phi_n(x)\rangle = \langle \Lambda, \phi_n(x)\rangle - \int f(x)\phi_n(x) \, dx \longrightarrow 0,$$

proving that $c = 0$. Since $x_0 \in F$ was taken arbitrarily, we have proved that $\mu \equiv f \in \mathrm{bmo}_F$. Summing up, we have proved that

$$\mathrm{bmo}_F(\mathbb{R}) \simeq (h^1_F(\mathbb{R}))^*.$$

Let us now look at the duals of the spaces $h^{1/n}_F(\mathbb{R})$, $n \geq 2$. We recall ([G]) that the dual of $h^p(\mathbb{R})$ for $p = 1/k$, $k = 2, 3, \ldots$, is the Zygmund space $\Lambda^{k-1}_*(\mathbb{R})$. If $I \subset \mathbb{R}$ is an interval, $\Lambda^1_*(I)$ is defined as the space of continuous and bounded functions defined on I such that

$$\Delta^2_h f(x) = |f(x+h) + f(x-h) - 2f(x)| \leq C_f|h| \quad \text{for all} \quad x, x+h, x-h \in I. \tag{3.2}$$

If I is an open interval and $\bar{I}$ is its closure the restriction map $\Lambda^1_*(\bar{I}) \ni f \mapsto f|_I \in \Lambda^1_*(I)$ is an isometric bijection and both spaces may be identified. We denote by $|f|_{1,I}$ the smallest constant C that makes (3.2) valid. A norm in $\Lambda^1_*(I)$ is defined by $\|f\|_{\Lambda^1_*(I)} = \|f\|_{L^\infty(I)} + |f|_{1,I}$. Similarly, the space $\Lambda^d_*(I)$, $d \in \mathbb{N}$ is defined as the space of bounded functions of class C^{d-1} defined on I such that $f^{(d-1)} \in \Lambda^1_*(I)$. A norm in $\Lambda^d_*(I)$ is defined by $\|f\|_{\Lambda^d_*(I)} = \|f\|_{L^\infty(I)} + |f^{(d-1)}|_{1,I}$.

Definition 3.3. *Let $F \subset \mathbb{R}$ be a finite set. We say that a bounded function f of class C^{d-1} defined on $\mathbb{R}$ is in $\Lambda^d_{*,F}(\mathbb{R})$, $d = 1, 2, \ldots$ if and only if the restriction of f to any component I of $\mathbb{R} \setminus F$ is in $\Lambda^d_*(I)$.*

The norm in $\Lambda^d_{*,F}(\mathbb{R})$ is defined as $\|f\|_{\Lambda^d_{*,F}(\mathbb{R})} = \|f\|_{L^\infty(\mathbb{R})} + \sum_I |f^{(d-1)}|_{1,I}$, where the sum is taken over all components of $\mathbb{R} \setminus F$.

Theorem 3.4.

(1) $(h^1_F(\mathbb{R}))^* \simeq \mathrm{bmo}_F(\mathbb{R})$;

(2) $(h^{1/n}_F(\mathbb{R}))^* \simeq \Lambda^{n-1}_{*,F}(\mathbb{R})$, $n = 2, 3, \ldots$.

Proof. We have already proved that (1) holds so from now on we assume that $n \geq 2$. To prove (2), consider $f \in \Lambda^{n-1}_{*,F}(\mathbb{R})$ and let $a(x)$ be a distinguished $h^{1/n}$-atom with $\|a\|_{L^\infty} > 1$, so $a(x)$ has a carrier J whose interior does not intersect F, say, $J = [\alpha, \beta]$ and $(\alpha, \beta) \subset I$, where I is a connected component of $\mathbb{R} \setminus F$. Since the restriction of f to I is in the Zygmund space $\Lambda^{n-1}_*(I)$, a standard argument that exploits the vanishing moments of $a(x)$ shows that

$$\begin{aligned} |\langle f, a\rangle| &\leq C\, \|f\|_{\Lambda^{n-1}_*(I)} \\ &\leq C\, \|f\|_{\Lambda^{n-1}_{*,F}(\mathbb{R})} \end{aligned}$$

and the second inequality is independent of the particular component I. If $\|a\|_{L^\infty} \leq 1$, we may assume that a has a carrier J of length $|J| \geq 1$ and

$$|\langle f, a\rangle| = \left|\int f(x)a(x)\,dx\right| \leq \|f\|_{L^\infty}\|a\|_{L^\infty}|I| \leq \|f\|_{\Lambda^{n-1}_{*,F}(\mathbb{R})}|I|^{1-n} \leq \|f\|_{\Lambda^{n-1}_{*,F}(\mathbb{R})}.$$

Then f defines a continuous linear functional on $h^{1/n}_F(\mathbb{R})$. Conversely, let Λ be a continuous linear functional on $h^{1/n}_F(\mathbb{R})$. Choose p such that $n-1 < 1/p < n$, and $\psi(x) \in C^\infty_c(-1,1)$ such that $\{\psi_j\}$, $\psi_j(x) = \psi(x-j)$, $j \in \mathbb{Z}$, is a partition of unity, i.e., $\sum_j \psi_j(x) \equiv 1$, $x \in \mathbb{R}$. Then, Corollary 2.2 shows that the linear functional $h^p(\mathbb{R}) \ni \phi \mapsto \Lambda(\psi_j\phi)$ is bounded on $h^p(\mathbb{R})$ with bound independent of j and by the characterization $(h^p(\mathbb{R}))^* \simeq \Lambda^{1/p-1}(\mathbb{R})$ ([G]) we conclude, in particular, that there exists functions $f_j(x)$ in the Hölder space $\Lambda^{1/p-1}(\mathbb{R})$, supported in $[j-1, j+1]$ with norm $\|f_j\|_{\Lambda^{1/p-1}} \leq C$, such that

$$\begin{aligned} \Lambda(\psi_j\phi) &= \int_{\mathbb{R}} f_j(x)\phi(x)\,dx, \\ \Lambda(\phi) &= \int_{\mathbb{R}} \sum_{j=-\infty}^{\infty} f_j(x)\,\phi(x)\,dx, \quad \phi \in \mathcal{S}(\mathbb{R}). \end{aligned}$$

Thus, Λ is represented by a function $f = \sum_j f_j \in \Lambda^{1/p-1}(\mathbb{R})$, in particular, f is bounded and of class $n-2$ by the choice of p. Fix a component I of $\mathbb{R}\setminus F$ and observe that the restriction of Λ to $h^{1/n}(\mathbb{R}) \cap \mathcal{E}'(I)$ is a bounded linear functional. This implies, invoking the arguments in [GR, Ch. III, Sec. 5], that for any subinterval $J \subset I$, the estimate

$$\frac{1}{|J|}\int_J |f(x) - P_J(x)\chi_J(x)|\,dx \leq C\|\Lambda\|\,|J|^{n-1} \tag{3.3}$$

holds, where $\chi_J(x)$ is the characteristic function of J and $P_J(x)$ is the unique polynomial of degree $n-1$ such that $(f(x) - P_J(x))\chi_J(x)$ has vanishing moments up to order $n-1$. Let $\phi(x) \in C^\infty_c[-1,1]$ an even function with $\int \phi(x)\,dx = 1$ and set

$\phi_t(x) = t^{-1}\phi(x/t)$, $t > 0$, $v(x,t) = \phi_t * f^{(n-2)}(x)$. It follows that $v(x,t) \to f^{(n-2)}(x)$ uniformly in $x \in \mathbb{R}$ as $t \to 0$. For $t > 0$ small, consider the interval

$$J_t = \{x \in I : \operatorname{dist}(x, \mathbb{R} \setminus I) \geq t\}.$$

Then (*cf.* [GR, Ch. III, Sec. 5]) (3.3) implies

$$|\partial_t^2 v(x,t)| \leq C\|\Lambda\| \, \frac{1}{t}, \quad x \in J_t,$$

which can be used to estimate the second order difference for $v(x,t)$ on J_t, namely,

$$|v(x+h,t) + v(x-h,t) - 2v(x,t)| \leq C\|\Lambda\||h|, \quad x, x+h, x-h \in J_t.$$

Fixing $x, x+h, x-h \in I$, we may take t small enough to apply the above estimate and letting $t \searrow 0$ obtain

$$|f(x+h) + f(x-h) - 2f(x)| \leq C\|\Lambda\||h|, \quad x, x+h, x-h \in I,$$

which shows that $f \in \Lambda_*^{n-1}(I)$. □

Change of variables.

Throughout this subsection $\phi : \mathbb{R} \to \mathbb{R}$ will be a diffeomorphism such that

$$\|(\phi^{-1})'\|_{L^\infty} < \infty, \qquad \|\phi^{(k)}\|_{L^\infty} < \infty, \quad k = 1, 2 \ldots \tag{3.4}$$

It is known that the linear map $h^p(\mathbb{R}) \ni f \mapsto \tilde{f} \circ \phi$ is bounded in $h^p(\mathbb{R})$, $0 < p \leq \infty$.

Theorem 3.5. *Let $F \subset \mathbb{R}$ be a finite set, $\phi : \mathbb{R} \to \mathbb{R}$ a diffeomorphism satisfying* (3.4)*, $n \in \mathbb{N}$ and set $\tilde{F} = \phi^{-1}(F)$. The map*

$$h_F^{1/n}(\mathbb{R}) \ni f \to f \circ \phi \in h_{\tilde{F}}^{1/n}(\mathbb{R})$$

is bounded.

Proof. We will assume that $n \geq 2$, the case $n = 1$ is simpler. By the usual arguments, the case of a general finite set F will follow as soon as we prove the theorem when $F = \{0\}$, so from now on we assume we are in that situation. If the support of f lies at a distance, say, greater than or equal to $1/2$ from the origin, we will have that $\|f \circ \phi\|_{h_{\tilde{F}}^{1/n}} \simeq \|f \circ \phi\|_{h^{1/n}}$ and $\|f\|_{h_F^{1/n}} \simeq \|f\|_{h^{1/n}}$ so the known estimate $\|f \circ \phi\|_{h^{1/n}} \leq C\|f\|_{h^{1/n}}$ gives $\|f \circ \phi\|_{h_{\tilde{F}}^{1/n}} \leq C'\|f\|_{h_F^{1/n}}$. Thus, we may assume that f has a distinguished atomic decomposition $f = \sum_j \lambda_j a_j$ with $\sum_j |\lambda_j|^{1/n} = K$ and $\operatorname{supp} a_j \subset [-1, 1]$. We have $\|f \circ \phi\|_{h_{\tilde{F}}^{1/n}}^{1/n} \leq \sum_j |\lambda_j|^{1/n} \|a_j \circ \phi\|_{h_{\tilde{F}}^{1/n}}^{1/n}$ which will give $\|f \circ \phi\|_{h_{\tilde{F}}^{1/n}}^{1/n} \leq CK$ if we are able to show that, for some constant C depending only on n, F and ϕ, $\|a \circ \phi\|_{h_{\tilde{F}}^{1/n}} \leq C$ for all distinguished $h_F^{1/n}$ atoms supported in $[-1, 1]$. Let us assume without loss of generality that ϕ is increasing and that the

$h_F^{1/n}$ atom $a(x)$ is supported in the interval $I = [\lambda, \lambda + \ell]$, $0 \le \lambda < \lambda + \ell \le 1$. We have

$$\|a\|_{L^\infty} \le \frac{1}{\ell^n},$$

$$\int (y-\lambda)^k a(y)\, dy = 0, \quad k = 0, \dots n-1. \tag{3.5}$$

Set $\tilde{a} = a \circ \phi$ and $\tilde{I} = \phi^{-1}(I) = [\tilde{\lambda}, \tilde{\lambda} + \tilde{\ell}]$. Then (3.4) implies that $\tilde{\ell} \simeq \ell$ and $\|\tilde{a}\|_{L^\infty} \le K\tilde{\ell}^{-n} = K|\tilde{I}|^{-n}$. Set

$$\mu_j = \mu_j(\tilde{a}) = \int (x - \tilde{\lambda})^j \tilde{a}(x)\, dx, \quad j = 0, \dots, n-1.$$

We will first show that $|\mu_j| \le C\tilde{\ell}$, $k = 0, \dots, n-1$, for some C depending only on ϕ and n. The change of variables $y = \phi(x)$ in (3.5) gives

$$0 = \int_{\tilde{\lambda}}^{\tilde{\lambda}+\tilde{\ell}} \left(\phi(x) - \phi(\tilde{\lambda})\right)^k \tilde{a}(x)\phi'(x)\, dx. \tag{3.6}$$

Taking $k = n-1$ in (3.6) and writing $\phi(x) - \phi(\tilde{\lambda}) = \phi'(\tilde{\lambda})(x - \tilde{\lambda}) + O(|x - \tilde{\lambda}|^2)$ and $\phi'(x) = \phi'(\tilde{\lambda}) + O(|x - \tilde{\lambda}|)$ we get $\phi'(\tilde{\lambda})^n \mu_{n-1} = O(\tilde{\ell})$. Hence, $|\mu_{n-1}| \le C_{n-1}\tilde{\ell}$. Assume by descending induction that we have proved that $|\mu_j| \le C_j\tilde{\ell}$ for $k+1 \le j \le n-1$ and let us estimate μ_k. Using the Taylor expansions of $\phi(x)$ and $\phi'(x)$ at $x = \tilde{\lambda}$ in (3.6), we may write

$$\left(\phi'(\tilde{\lambda})\right)^{k+1} \int (x - \tilde{\lambda})^k \tilde{a}(x)\, dx = \sum_{j=k+1}^{n-1} c_{jk}\mu_j + \int_{\tilde{\lambda}}^{\tilde{\lambda}+\tilde{\ell}} \tilde{a}(x)\, r_n(x)\, dx,$$

where $|r_n(x)| \le c_n|x - \tilde{\lambda}|^n$ and the constants c_n, c_{jk} depend on the size of the derivatives of ϕ up to order $n+1$. Therefore, $|\mu_k| \le C_k\tilde{\ell}$ with

$$C_k = \frac{\sum_{j=k+1}^{n-1} |c_{jk}|C_j + Kc_n}{\inf |\phi'|^{k+1}}.$$

This gives $|\mu_k| \le C_k\tilde{\ell} \le C_k$ for all $0 \le k \le n-1$. Write

$$\tilde{a}(x) = \underbrace{\tilde{a}(x) - P_{\tilde{I}}(x)\chi_{\tilde{I}}(x)}_{\tilde{a}_1(x)} + \underbrace{P_{\tilde{I}}(x)\chi_{\tilde{I}}(x)}_{\tilde{a}_2(x)}$$

where $P_{\tilde{I}}(x)$ is the unique polynomial of degree $n-1$ such that $(\tilde{a}(x) - P_{\tilde{I}}(x))\chi_{\tilde{I}}(x)$ has vanishing moments up to order $n-1$, i.e.,

$$\mu_j(\tilde{a}) = \mu_j(P_{\tilde{I}}) \doteq \int_{\tilde{\lambda}}^{\tilde{\lambda}+\tilde{\ell}} (x - \tilde{\lambda})^j P_{\tilde{I}}(x)\, dx.$$

There is a constant M depending on n but independent of $\tilde{\lambda}$ and $0 \le \tilde{\ell} \le 1$ such that for all polynomials $P(x)$ of degree $\le n-1$ the estimate

$$\sup_{\tilde{I}} |P(x)| \le M\, \tilde{\ell}^{-n} \max_{0 \le j \le n-1} |\mu_j(P)|$$

holds. Applying it to $P = P_{\tilde{I}}$ we obtain

$$\|\tilde{a}_2\|_{L^\infty} \leq C\frac{1}{\tilde{\ell}^{n-1}} \quad \text{and} \quad \|\tilde{a}_1\|_{L^\infty} \leq \|\tilde{a}\|_{L^\infty} + \|\tilde{a}_2\|_{L^\infty} \leq C'\frac{1}{\tilde{\ell}^n}. \tag{3.7}$$

Since $\tilde{a}_1(x)$ has vanishing moments up to order $n-1$, $C'^{-1}\tilde{a}_1(x)$ is an $h^{1/n}_{\tilde{F}}$ atom and $\|\tilde{a}_1\|_{h^{1/n}_{\tilde{F}}} \leq C$. On the other hand, the term $\tilde{a}_2$ is in the smaller space $h^p(\mathbb{R})$ for any $1/n < p < 1/(n-1)$. Indeed, $\tilde{a}_2$ is supported in $\tilde{I}$ and $\|\tilde{a}_2\|_{L^\infty} \leq C\tilde{\ell}^{-1/p}$ by (3.7), so using the arguments in the proof of Lemma A we may obtain a decomposition of $\tilde{a}_2(x)$ similar to that of $a^+(x)$ in that lemma (see the remark at the end of Lemma A), namely

$$\tilde{a}_2(x) = \sum_{k=0}^{n-1}(-1)^k\mu_k\,\delta^{(k)}(x-\tilde{\lambda}) + \sum_{j=1}^{\infty}\nu_j\alpha_j(x)$$

where each $\alpha_j(x)$ is an h^p atom and $\sum_j |\nu_j|^p < \infty$ is independent of $\tilde{\lambda}$ and $\tilde{\ell}$. Thus, $\|\tilde{a}\|_{h^{1/n}_{\tilde{F}}} \leq \|\tilde{a}_1\|_{h^{1/n}_{\tilde{F}}} + \|\tilde{a}_2\|_{h^{1/n}_{\tilde{F}}} \leq \|\tilde{a}_1\|_{h^{1/n}_{\tilde{F}}} + c\|\tilde{a}_2\|_{h^p} \leq C.$ □

The preceding proof contains an argument that we state explicitly as a proposition

Proposition 3.6. *Let $F \subset \mathbb{R}$ be a finite set, $n \in \mathbb{N}$. There exists $C = C(n) > 0$ such that if $J = [\lambda, \lambda + \ell] \subset \mathbb{R}$ is an interval with interior disjoint from F, $|J| \leq 1$, and $a(x)$ is a bounded function supported in J with $\|a\|_{L^\infty} \leq \ell^{-n}$ such that*

$$|\mu_j(a)| \leq \ell, \quad 0 \leq j \leq n-1, \quad \textit{where} \quad \mu_j(a) = \int (x-\lambda)^j a(x)\,dx,$$

then $\|a\|_{h^{1/n}_F} \leq C$.

□

Remark. Theorem 3.5 makes it possible to define the spaces $h^{1/n}_F(\Gamma)$ when Γ is a smooth closed curve and $h^{1/n}_{F,\mathrm{loc}}(\Gamma)$ when Γ is a smooth open curve.

Sometimes it is useful to estimate the $h^{1/n}_F$ norm of a bounded function $a(x)$ that is supported in an interval J and satisfies the size condition $\|a\|_{L^\infty} \leq \ell^{-n}$ but the moment conditions $\mu_j(a) = 0$, $0 \leq j \leq n-1$, are replaced by similar conditions where the role of the powers x^j are replaced by powers of a perturbation of x.

Proposition 3.7. *Let $F \subset \mathbb{R}$ be a finite set, $\varphi : \mathbb{R} \to \mathbb{R}$ a smooth function with bounded derivatives of all orders, $n \in \mathbb{N}$. Set $Z(x) = x + i\varphi(x)$ and assume that $J = [\lambda, \lambda+\ell] \subset \mathbb{R}$ is an interval with interior disjoint from F, $|J| \leq 1$, and $a(x)$ is a bounded function supported in J with $\|a\|_{L^\infty} \leq \ell^{-n}$ such that*

$$\nu_j(a) = \int (Z(x) - Z(\lambda))^j a(x)\,dx = 0, \quad 0 \leq j \leq n-1. \tag{3.8}$$

Then $\|a\|_{h^{1/n}_F} \leq C$ with C depending only on F and n.

Proof. Taking the Taylor expansion of $\varphi(x)$ around $x = \lambda$ we may write

$$(Z(x) - Z(\lambda))^j = (x-\lambda)^j(1 + i\varphi'(\lambda))^j + \sum_{k=1}^{N} c_k(x-\lambda)^{j+k} + O\left(|x-\lambda|^{j+N+1}\right).$$

If μ_j is defined as in Proposition 3.6, we may obtain from (3.8) a recursive relation

$$\mu_j = \sum_{k=1}^{N} c_{jk}\mu_{j+k} + O(|\ell|^{j+N+2-n})$$

which allows us to prove by descending induction that $|\mu_j| \le C\ell$, $0 \le j \le n-1$, for some C independent of $a(x)$ and then apply Proposition 3.6. $\square$

4. The one-sided Nirenberg-Treves condition

In this section we abandon momentarily the analyticity hypothesis on the coefficients of the vector fields, which we assume to be just smooth. Let $\Omega \subset \mathbb{R}^2$ be an open subset that is divided into two components, Ω^+ and Ω^- by an embedded curve $\Sigma \subset \Omega$ and consider a non vanishing vector field with smooth, complex coefficients

$$L = A(x,t)\frac{\partial}{\partial t} + B(x,t)\frac{\partial}{\partial x}$$

defined in Ω. We recall that if we write $L = X + iY$ with X and Y real and $\omega \subset \Omega$ is any open subset, the orbits of L in ω are by definition the orbits in the sense of Sussmann [S] of the pair of vector fields $\{X, Y\}$ restricted to ω. Since X and Y are assumed to have no common zeros, the orbits of L in ω are immersed submanifolds of ω of dimension one or two; furthermore, the two-dimensional orbits are open subsets of ω. Let $\mathcal{O} \subset \omega$ be a two-dimensional orbit of L in ω and consider $X \wedge Y \in C^\infty(\Omega; \bigwedge^2(T(\Omega)))$. Since $\bigwedge^2(T(\Omega))$ has a global non vanishing section $e_1 \wedge e_2$, $X \wedge Y$ is a real multiple of $e_1 \wedge e_2$ and this gives a meaning to the requirement that $X \wedge Y$ does not change sign on any two-dimensional orbits $\mathcal{O}$ of $\{X, Y\}$ in ω. The following definition gives a one-sided version of the well known Nirenberg-Treves solvability condition $(\mathcal{P})$ ([NT]).

Definition 4.1. *We say that L satisfies condition $(\mathcal{P}^+)$ at $p \in \Sigma$ if there is a disc $U \subset \Omega$ centered at p such that $X \wedge Y$ does not change sign on any two-dimensional orbit of L in $U^+ = U \cap \Omega^+$.*

Definition 4.2. *We say that L is one-sided locally integrable at $p \in \Sigma$ if there is a disc $U \subset \Omega$ centered at p such that —after interchanging Ω^+ and Ω^- if necessary— there exists $Z \in C^\infty(U)$ such that*

(1) *LZ vanishes identically on $U^+ = U \cap \Omega^+$;*
(2) *$dZ(p) \neq 0$.*

Let us assume that L is one-sided locally integrable at $p \in \Sigma$ and let Z satisfy (1) and (2) of Definition 4.2. Substituting Z by iZ if necessary and decreasing U we may choose local coordinates (x, t) such $x(p) = t(p) = 0$,

$$Z(x,t) = x + i\varphi(x,t) \tag{4.1}$$

with φ real, U is the rectangle $U = (-a, a) \times (-T, T)$, $\Sigma \cap U = \{(x, 0) : \ |x| < a\}$ and $U^+ = (-a, a) \times (0, T)$. Thus, replacing L by a convenient non vanishing multiple we may assume that

$$L = \frac{\partial}{\partial t} - i\frac{\varphi_t(x,t)}{1 + i\varphi_x(x,t)}\frac{\partial}{\partial x} \tag{4.2}$$

$$X = \frac{\partial}{\partial t} + \frac{\varphi_t\varphi_x}{1+\varphi_x^2}\frac{\partial}{\partial x}, \qquad Y = -\frac{\varphi_t}{1+\varphi_x^2}\frac{\partial}{\partial x},$$

and so

$$X \wedge Y = \frac{\varphi_t(x,y)}{1+\varphi_x^2}\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial t}.$$

We now recall how condition $(\mathcal{P}^+)$ may be characterized in terms of the one-sided first integral (4.1).

Lemma ([BH2])**.** *Let $Z(x,t)$ and L be given by* (4.1) *and* (4.2) *respectively. Then, L satisfies $(\mathcal{P}^+)$ at the origin if and only if there exist $T, a > 0$ such that $(0,T) \ni t \mapsto \varphi(x,t)$ is monotone for every $x \in (-a,a)$.*

Example. The Mizohata vector field

$$M = \frac{\partial}{\partial t} + it\frac{\partial}{\partial x}$$

is not locally solvable at the origin but satisfies $(\mathcal{P}^+)$ at the origin since $t \mapsto t^2$ is monotone for $t > 0$.

5. Sufficient conditions for the (H^p) property, $0 < p \leq 1$

From now on, we will deal with a real analytic vector field

$$L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x}$$

defined on a neighborhood of the closed rectangle $[-A,A]\times[-B,B]$ that satisfies condition $(\mathcal{P}^+)$ at every point in $\Sigma = [-A,A]\times\{0\}$ as described in Section 4, with a first integral $Z(x,t) = x + i\varphi(x,t)$ on $Q = (-A,A)\times(-B,B)$. Shrinking $B > 0$ if necessary we may assume that $(0,B) \ni t \mapsto \varphi(x,t)$ is monotone for every x in a neighborhood of $[-A,A]$. We set

$$m(x) = \min_{0\leq y\leq B} \varphi(x,y), \qquad M(x) = \max_{0\leq y\leq B} \varphi(x,y), \quad -A \leq x \leq A.$$

Thus, the function $Z(x,y)$ takes the rectangle $Q = [-A,A]\times[0,B]$ onto

$$Z(Q) = \{\xi + i\eta : \quad -A \leq \xi \leq A, \quad m(\xi) \leq \eta \leq M(\xi)\}.$$

The interior of $Z(Q)$ is

$$\{\xi + i\eta : \quad -A < \xi < A, \quad m(\xi) < \eta < M(\xi)\},$$

in particular, this interior is not empty if and only if $M(x) > m(x)$ for some $x \in (-A,A)$. Since L is a real analytic vector field, $M(x) = m(x)$ for at most a finite number of values of $x \in [-A,A]$, unless L is a real vector field, a trivial case that we will exclude from our considerations. We will denote by $F = F(L)$ this finite set. Notice that F does not depend on the choice of the first integral $Z(x,t)$ because it is precisely the set of those x's such that the linear span of $\{\Re L, \Im L\}$ has dimension one at the point of coordinates (x,y) for all $y \in (0,B)$. It follows that the set $Z((-A,A)\times(0,B))$ has nonempty interior. Every component of the interior of $Z((-A,A)\times(0,B))$ has the form

$$\{\xi + i\eta : \alpha < \xi < \beta, m(\xi) < \eta < M(\xi)\}$$

where (α,β) is a component of the open set $\{x \in (-A,A) : M(x) > m(x)\}$. Let

$$\{x \in (-A,A) : M(x) > m(x)\} = \bigcup_k (\alpha_k,\beta_k)$$

be a decomposition into components. Fix k and consider one of these components (α_k,β_k). Note that $m(\alpha_k) = M(\alpha_k)$ and $m(\beta_k) = M(\beta_k)$, unless $\alpha_k = -A$ or $\beta_k = A$. Since for each x, the function

$$t \longmapsto \varphi(x,t) \quad \text{is monotonic,}$$

either $m(x) = \varphi(x,0)$ and $M(x) = \varphi(x,B)$ or $m(x) = \varphi(x,B)$ and $M(x) = \varphi(x,0)$ on (α_k,β_k). Without loss of generality, we may assume that $m(x) = \varphi(x,0)$ and $M(x) = \varphi(x,B)$ for every $x \in (\alpha_k,\beta_k)$. Let $U_k =$ the interior of $Z((\alpha_k,\beta_k)\times(0,B))$. Thus

$$U_k = \{x+iy : \alpha_k < x < \beta_k,\ \varphi(x,0) < y < \varphi(x,B)\}.$$

For the next definition we must keep in mind the Hardy spaces $h^p(\mathbb{R})$ mentioned in Section 1.

Definition 5.1. *We denote by $h^p_{\mathrm{loc}}(-A,A)$ the subspace of the distributions $f \in \mathcal{S}'(\mathbb{R})$ such that $\psi f \in h^p(\mathbb{R})$ for any $\psi \in C^\infty_c(-A,A)$. If $p = 1/n$, $n \in \mathbb{N}$, we write $h^{1/n}_{F,\mathrm{loc}}(-A,A)$ for the subspace of the distributions $f \in \mathcal{S}'(\mathbb{R})$ such that $\psi f \in h^{1/n}_F(\mathbb{R})$ for any $\psi \in C^\infty_c(-A,A)$.*

Since $h^p(\mathbb{R})$ is invariant under multiplication by functions of $C^\infty_c(\mathbb{R})$ the space $h^p_{\mathrm{loc}}(-A,A)$ is a local space in the sense that $f \in h^p_{\mathrm{loc}}(-A,A)$ if and only if for every $x \in (-A,A)$ there is a function $\psi \in C^\infty_c(-A,A)$ with $\psi(x) \neq 0$ such that $\psi f \in h^p_{\mathrm{loc}}(-A,A)$ or, equivalently, if and only if for every $x \in (-A,A)$ there is an open interval J, $x \in J \subset (-A,A)$, such that $f \in h^p_{\mathrm{loc}}(J)$. A similar observation holds for $h^{1/n}_{F,\mathrm{loc}}(-A,A)$. The main result of this section is

Theorem 5.2. *Let L be real analytic and satisfy condition $(\mathcal{P}^+)$ as above. Assume that f is continuous on $[-A,A]\times(0,B]$, satisfies $Lf = 0$ on a neighborhood of $[-A,A]\times(0,B]$ in the sense of distributions and has a weak boundary value*

$$\langle bf,\psi\rangle = \lim_{t\searrow 0}\int_{\mathbb{R}} f(x,t)\psi(x)\,dx, \quad \psi \in C^\infty_c(-A,A). \tag{5.1}$$

Assume that

(a) *$bf \in h^p_{\mathrm{loc}}(-A,A)$ for some $0 < p \le 1$,*

(b) *if $p = 1/n$ for some $n \in \mathbb{N}$, assume further that $bf \in h^{1/n}_{F,\mathrm{loc}}(-A,A)$.*

Then for every $0 < a < A$ sufficiently small, $f(\cdot,t) \in L^p(-a,a)$, $0 < t < B$, and there is a constant C such that

$$\int_{-a}^{a} |f(x,t)|^p dx \le C, \quad 0 < t < B. \tag{5.2}$$

It is known ([BH3]) that the weak boundary value (5.1) exists if an only if for each K compact subset of $(-A,A)$ there is a positive integer N such that

$$\int_0^B \int_K |f(x,t)|\,|\varphi(x,t)-\varphi(x,0)|^N dxdt \le C, \tag{5.3}$$

for some $C = C(K)$. It turns out that (5.3) is in fact equivalent to the estimates

$$\sup_{x\in K} |f(x,t)|\, |\varphi(x,t) - \varphi(x,0)|^{N'} \leq C', \tag{5.3'}$$

valid on compact subsets $K \subset (-A, A)$ for some constants $C' = C'(K)$ and $N' = N'(K) \in \mathbb{N}$.

6. Proof of Theorem 5.2

For $p = 1$ the theorem follows from the results in [BH1], so we will assume in the sequel that $p < 1$. The set $F(L)$ introduced in the previous section will play a substantial role in the proof.

If $0 \notin F(L)$, the function $(0, B) \ni t \mapsto \varphi(0, t)$ is not identically zero and we may assume that it is strictly increasing for $0 < t < b$ for some $0 < b < B$. Thus, the function $Z(x, t) = x + i\varphi(x, t)$ maps a small open rectangle $Q = (-a, a) \times (0, b)$ into an open subset $Z(Q) = \{\xi + i\eta : |\xi| < a, 0 < \eta < \varphi(\xi, b)\}$ so that by the Baouendi-Treves approximation theorem we may write $f = \tilde{f} \circ Z$ with $\tilde{f}$ holomorphic in $Z(Q)$ and it is easy to check that $bf = b\tilde{f}$. By the classical local result for holomorphic functions, we conclude after shrinking $a > 0$ if necessary, that $K = \int_{-a}^{a} |\tilde{f}^{\perp}(\xi)|^p \, d\xi < \infty$, where $\tilde{f}^{\perp}(\xi)$ denotes the maximal function

$$\tilde{f}^{\perp}(\xi) = \sup_{0<\eta<\varphi(\xi,b)} |\tilde{f}(\xi + i\eta)|.$$

This implies that $\int_{-a}^{a} |\tilde{f}(\xi + i\varphi(\xi, t))|^p \, d\xi \leq K$, $0 < t < b$, which gives the uniform bound of the integrals of $|\tilde{f}|^p$ on the curves $\xi \mapsto \xi + i\varphi(\xi, t)$, that is, the images under $Z(x, t)$ of the segments $\gamma_t = (-a, a) \times \{t\}$. Then we may, by a change of parametrization, write $\int_{-a}^{a} |f(x, t)|^p \, dx = \int_{\gamma_t} |\tilde{f} \circ Z|^p \, ds$ as $\int_{Z(\gamma_t)} |\tilde{f}|^p \, ds$ and the latter is easily estimated by K.

Thus, it is enough to consider the case $\{0\} \subset F$. We may choose a local first integral $Z(x, t) = x + i\varphi(x, t)$ such that $\varphi(x, 0) \equiv 0$. In other words, we only need to prove the theorem assuming that, for some $0 < a < A$ the following holds:

$$L = \frac{\partial}{\partial t} - i \frac{\varphi_t(x,t)}{1 + i\varphi_x(x,t)} \frac{\partial}{\partial x}, \quad |x| \leq A, \quad |t| \leq B;$$
$$\varphi(x, 0) = 0, \quad |x| \leq A; \quad \varphi_t(0, t) = 0, \quad |t| \leq B.$$

Case 1. Assume $a \in F$, i.e., $\varphi_t(a, t) = 0$, $|t| \leq B$. We first focus our attention on the interval $[0, A]$ and assume momentarily that f is continuous up to $t = 0$ (this hypothesis will later be removed). If we set

$$U \doteq Z((0, a) \times (0, B)) = \{x + iy, \quad 0 < x < a, \quad 0 < y < \varphi(x, B)\},$$

by a well known consequence of the Baouendi-Treves theorem [BT] there is a holomorphic function $\hat{f} \in \mathcal{H}(U)$ such that $f(x, t) = \hat{f}(Z(x, t))$ for $0 < t \leq B$ and $0 < x < a$. Furthermore, $\hat{f}$ is continuous up to the boundary of U which is made up by the curves $y = \varphi(x, B)$ and $y = \varphi(x, 0)$. Thus, $\hat{f}$ has a boundary value at $0 < x < a$ (given by the restriction of $\hat{f}$ to $t = 0$) and since $Z(x, 0) \equiv x$ it is clear that $b\hat{f} = bf$ on $(0,a)$. Given $x_0 \in (0,a)$, $t_0 \in (0,B)$, set $\zeta_0 = x_0 + i\varphi(x_0,t_0)$ and let us write Cauchy's formula

$$\begin{aligned} 2\pi i \hat{f}(\zeta_0) &= \int_0^a \frac{b\hat{f}(\xi)}{\xi-\zeta_0}\,d\xi - \int_{\Gamma_B} \frac{\hat{f}(\zeta)}{\zeta-\zeta_0}\,d\zeta, \quad \zeta_0 \in U, \\ &= g(\zeta_0) + h(\zeta_0). \end{aligned} \tag{6.1}$$

where Γ_B denotes the graph $\{(\xi + i\varphi(\xi,B) : 0 \le \xi \le a\}$. To prove (5.2) we must study the integrals of $\int |\hat{f}|^p\,ds$ on the curves $\Gamma_t = \{(\xi + i\varphi(\xi,t) : 0 \le \xi \le a\}$, $0 < t < B$. We first focus on the contribution of the first term in (6.1):

$$g(\zeta_0) = \int_0^a \frac{b\hat{f}(\xi)}{\xi-\zeta_0}\,d\xi.$$

Choose $\psi(x) \in C_c^\infty(-A,A)$ such that $\psi(x) \equiv 1$ on a neighborhood of $[0,a]$ and consider a distinguished atomic decomposition of $\psi b\hat{f}$, so in a neighborhood of $[0,a]$ we have $bf = \sum_j \lambda_j a_j$, $\sum_j |\lambda_j|^p < \infty$ we have

$$|g(\zeta_0)|^p \le \sum_j |\lambda_j|^p \left| \int_0^a \frac{a_j(\xi)}{\xi-\zeta_0}\,d\xi \right|^p.$$

Then, it will be enough to prove that there is a constant C such that for any h^p-atom $a(x)$ that is distinguished with respect to $\{0,a\}$

$$\int_0^a \left| \int_0^a \frac{a_j(\xi)}{\xi - x - i\varphi(x,t)}\,d\xi \right|^p dx \le C, \quad 0 < t < B. \tag{6.2}$$

If a_j is an h^p-atom carried by an interval I_j with interior disjoint from $[0,a]$ the estimate above is trivial because the left hand side is zero. On the other hand, if a_j is an h^p-atom carried by an interval $I_j \subset [0,a]$ the Cauchy integral of a_j is dominated by the maximal function

$$C^\perp a_j(x) = \sup_{\eta>0} \left| \int_{\mathbb{R}} \frac{a_j(\xi)}{\xi - x - i\eta}\,d\xi \right|$$

and it is well known that $C^\perp : h^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$ is a bounded operator with norm K. Therefore, the left hand side in (6.2) is bounded by $K^p \|a_j\|_{h^p}^p \le C$ in this case. Thus, we need only worry with atoms a_j with carrier I_j that intersect both $(0,a)$ and $\mathbb{R} \setminus [0,a]$. However, these atoms must satisfy $\|a_j\|_{L^\infty} \le 1$ in addition to the standard estimate $\|a_j\|_{L^\infty} \le 1/|I_j|^{1/p}$ because their carriers intersect F and this implies that $\|a_j\|_{L^2} \le 1$. By the boundedness of $C^\perp : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ we obtain $\|C^\perp a_j\|_{L^2} \le C$ which implies $\|C^\perp a_j\|_{L^p[0,a]} \le C_a$ by Hölder inequality.

We will use a simpler estimate for the contribution of the second term in (6.1), $\int |h|^p\,ds$, that by Hölder's inequality is dominated by $(\int |h|^2\,ds)^{p/2}$. To control the latter expression it is enough to consider the maximal L^2 inequality

$$\int_0^a \left| \sup_{\eta<0} \int_0^a \frac{\hat{f}(\xi)}{\xi - x + i(\varphi(\xi,B) + \eta - \varphi(x,t))}\,d\xi \right|^2 dx \le C \|\hat{f}|_{\Gamma_B}\|_{L^2}^2 \le C' \|\hat{f}|_{\Gamma_B}\|_{L^\infty}^2.$$

Summing up, we have proved

$$\int_0^a |f(x,t)|^p dx \le C(\|\psi b f\|_{h^p} + \sup_{0\le x\le a} |f(x,B)|), \quad 0<t<B.$$

This proves the uniform $L^p[0,a]$ estimates of the traces under the additional hypothesis of continuity up to the boundary f. To complete the proof of case 1 we must remove the assumption that f is continuous up to $t=0$. In the general case, using the one-sided Baouendi-Treves approximation scheme, we may find a sequence of polynomials $p_n(\zeta)$ such that $f_n(x,t)=p_n(Z(x,t))$ converges uniformly to $f(x,t)$ on each rectangle of the form $[0,a]\times[\varepsilon,B]$, $\varepsilon>0$, and $f_n(x,0)=p_n(x)$ converges in $h^p(\mathbb{R})$ to $\psi(x)bf(x)$. By the case already considered, we are able to control the integrals of $[0,a]\ni x\mapsto |f_n(x,t)|^p$ by a constant that only depends on the L^∞ norm of $f_n(x,B)$ and the h^p-norm of $\psi(x)f_n(x,0)$. However, for n large, $\|f_n(\cdot,B)\|_{L^\infty} \le C\|f(\cdot,B)\|_{L^\infty}$ and $\|\psi(\cdot)f_n(\cdot,0)\|_{h^p} \le C\|\psi(\cdot)f(\cdot,0)\|_{h^p}$, showing that

$$\int_0^a |f_n(x,t)|^p dx \le C, \quad 0<t<B,$$

with C independent of n and t. Letting $n\to\infty$ we end the proof of this case.

Case 2. Assume $F=\{0\}$. We may easily find a subdomain $\tilde U\subset U$ of

$$U \doteq Z((0,a)\times(0,B)) = \{x+iy, \quad 0<x<a, \quad 0<y<\varphi(x,B)\},$$

such that $\partial\tilde U$ is smooth except at the point $\{(0,0)\}$ and $\tilde U\cap\{0\le x\le 3a/4\} = U\cap\{0\le x\le 3a/4\}$. Notice that the segment $[0,3a/4]\times\{0\}$ is a portion of the boundary $\partial\tilde U$. As in case 1 we may work under the assumption that f is continuous up to the boundary. We use Cauchy's formula to write

$$2\pi i\hat f(\zeta_0) = \int_{\partial\tilde U} \frac{\hat f(\zeta)}{\zeta-\zeta_0}\, d\zeta, \quad \zeta_0\in\tilde U. \tag{6.3}$$

Using a partition of unity of two elements, we may write $\hat f|_{\partial\tilde U} = b_1+b_2$ with b_2 vanishing identically on $([0,a/2]\times\mathbb{R})\cap\partial\tilde U$ and b_1 vanishing identically on $([3a/4,\infty)\times\mathbb{R})\cap\partial\tilde U$. This allows to write the Cauchy integral as the sum of two terms. The contribution to (6.3) stemming from b_1 may by treated, using a distinguished decomposition of $\hat f(x,0)$, with the methods used in the proof of case 1, while the contribution of b_2 will be a bounded function for $0\le\Re\zeta_0\le a/3$ because the factor $\zeta\mapsto(\zeta-\zeta_0)^{-1}$ in (6.3) belongs to a bounded set of smooth functions for $0\le\Re\zeta_0\le a/3$. We may then obtain

$$\int_0^a |f(x,t)|^p dx \le C, \quad 0<t<B, \tag{6.4}$$

when f is continuous up to $t=0$ and this implies the general case. We leave details to the reader.

End of the proof. We have already seen how to prove (6.4). An analogous reasoning leads to the similar estimate

$$\int_{-a}^{0} |f(x,t)|^p dx \leq C, \quad 0 < t < B, \tag{6.5}$$

so (6.4) and (6.5) give (5.2). □

Remark. Notice that because of the analyticity of the coefficients of the vector field, by choosing a small value of a, we may always assume that we are in case 2. Thus, the discussion of case 1 is, strictly speaking, superfluous. However, we have included it because it seems interesting in the context of bell-shaped domains [BH1].

7. Necessary conditions for the (H^p) property, $0 < p \leq 1$

Let L be a one-sided locally integrable vector field. It was proved in [BH2,Thm.3] for $1 \leq p \leq \infty$ that if all for homogeneous one-sided solutions $Lf = 0$, defined in a neighborhood of a boundary point p_0 with boundary value $bf \in L^p$, have traces uniformly bounded in L^p, then L must satisfy condition $(\mathcal{P}^+)$ at p_0. The argument may be extended to consider all values $0 < p \leq \infty$ ([H]) in the sense that if $(\mathcal{P}^+)$ does not hold at p_0 we can find local smooth solutions f to the equation $Lf = 0$ having small $\|bf\|_{L^p}$ and possessing traces of arbitrarily big L^p norm. Thus, we only consider vector fields that satisfy $(\mathcal{P}^+)$, in addition to the assumption of real analytic coefficients, and focus on the following question: if the L^p traces are uniformly bounded, what can we say about bf? For $p = 1$ we already know ([BH2]) that if f has traces that are uniformly bounded in L^1 norm, then $bf \in L^1$. When the coefficients are analytic this can be improved as follows. As in section 4, we assume that the vector field L is given by (4.2) and satisfies condition $(\mathcal{P}^+)$ at the origin in $\Sigma = (-A, A) \times \{0\}$, with a one-sided first integral $Z(x,t) = x + i\varphi(x,t)$ on $Q = (-A, A) \times (-B, B)$. We also consider the finite set $F = F(L) \subset [-A, A]$ associated to L in section 5.

Theorem 7.1. *Suppose that L is real analytic, f is continuous in $(-a, a) \times (0, b)$, for some $a > A$ and $b > B$, and satisfies $Lf = 0$ on $(-A, A) \times (0, B)$ in the weak sense. Assume that there is a positive integer N such that for each K compact in $(-A, A)$, we have*

$$\int_0^b \int_K |f(x,t)||\varphi(x,t) - \varphi(x,0)|^N dxdt \leq C, \tag{7.1}$$

for some $C = C(K)$. If the traces of f at $t > 0$ are uniformly bounded in L^1, that is,

$$\forall a' < A, \quad \int_{-a'}^{a'} |f(x,t)| dx \leq C(a'), \quad t \to 0^+, \tag{7.2}$$

then $bf \in h^1_{F,\mathrm{loc}}(-a', a')$, for every $a' < A$.

For $0 < p < 1$ we have

Theorem 7.2. *Suppose that L is real analytic, f is continuous in $(-a,a)\times(0,b)$, for some $a>A$ and $b>B$, and satisfies $Lf=0$ on $(-A,A)\times(0,B)$ in the weak sense. Assume that* (7.1) *holds. If the traces of f at $t>0$ are uniformly bounded in L^p for some $0<p<1$, that is,*

$$\forall a'<A,\quad \int_{-a'}^{a'}|f(x,t)|^p dx \le C(a'),\quad t\to 0^+, \tag{7.3}$$

then $bf\in h^p_{\mathrm{loc}}(-a',a')$, for every $a'<A$. Furthermore, if $p=1/n$, $n=2,3\ldots$, $bf\in h^p_{F,\mathrm{loc}}(-a',a')$.

The proofs of theorems 7.1 and 7.2 run parallel, the main point will be to show that for any $0<p\le 1$, estimate (7.3) implies, roughly speaking, the existence of a distinguished atomic decomposition in h^p and this will give the result. Since the conclusion is local, it is enough to analyze just two situations: i) $F=\emptyset$ or ii) $F=\{0\}$. If i) holds, the problem is reduced to study the case of a holomorphic function $\tilde f$ defined on $Z(Q)$, $Q=(-\beta,\beta)\times(0,b)$ such that $f=\tilde f\circ Z$ and $bf=b\tilde f$ as in the first part of the proof of Theorem 7.3. By the classical local result for holomorphic functions we will get that $bf=b\tilde f\in h^p_{\mathrm{loc}}(-\beta,\beta)$ (notice that for $p=1$ we have $h^1_{\mathrm{loc}}(-\beta,\beta)=h^1_{F,\mathrm{loc}}(-\beta,\beta)$ because $F=\emptyset$). Thus, we may assume without loss of generality that $F=\{0\}$ and we will do so. Most of the time, we will also be able to assume in the proof that $bf\in L^1_{\mathrm{loc}}(-A,A)$: when $p=1$ this is so because we already know that bf is locally integrable by the results in [BH1] while for $p<1$ we will find a sequence of homogeneous solutions f_j continuous up to the boundary such that $bf_j\to bf$ weakly and it will be enough to show that $\{bf_j\}$ is bounded in h^p, so we will work with the f_j's rather than with f. Initially, we will carry out our analysis on the interval $(0,\beta)$ with the following working assumptions:

$$\begin{aligned}
&L=\frac{\partial}{\partial t}-i\frac{\varphi_t(x,t)}{1+i\varphi_x(x,t)}\frac{\partial}{\partial x},\quad |x|\le A,\quad |t|\le B;\\
&\varphi(x,0)=0,\quad |x|\le A;\\
&f(x,t)=\tilde f\circ Z(x,t),\quad 0<t\le B,\ 0<x<\beta,
\end{aligned}$$

where $\tilde f(\xi+i\eta)$ is holomorphic on $\tilde U=\{\xi+i\eta:\ 0<\xi<A,\ 0<\eta<\varphi(\xi,B)\}$, continuous on $\overline{\tilde U}\cap\{\xi+i\eta:\ \eta>0\}$, has a boundary value $b\tilde f\in h^p_{\mathrm{loc}}(0,A)$ and is bounded on the vertical segment $\{\beta\}\times(0,B]$. The latter assumption depends on the choice of β and is granted by the fact that $b\tilde f\in h^p_{\mathrm{loc}}(0,A)$ implies that the limit $\lim_{\eta\to 0}\tilde f(\beta+i\eta)$ exists for a.e. $\beta\in(0,A)$. Furthermore, we assume that $\tilde f\in L^1(0,\beta)$. Given $0<x<\beta$ we consider, for a large $\gamma>0$, the approach region

$$\Gamma_x=\{\,\zeta=\xi+i\eta\in\overline{\tilde U}:\quad \eta>\gamma|\xi-x|\,\}$$

and the associated maximal function

$$\tilde f^*(x)=\sup_{\zeta\in\Gamma_x}|\tilde f(\zeta)|.$$

Set

$$\tilde\lambda=\inf_{\zeta\in\partial\tilde U\cap\{\eta>0\}}\tilde f^*(\zeta),\quad \text{and}\quad \mathcal{O}_\lambda=\{x\in\mathbb{R}:\tilde f^*(x)>\lambda\}.$$

Notice that $\tilde\lambda>0$ unless $\tilde f\equiv 0$ and that $\mathcal{O}_\lambda$ is an open subset of $(0,\beta)$, proper only if $\lambda>\tilde\lambda$. The next lemma yields a Calderón-Zygmund decomposition of $b\tilde f$ that is one of the key points in the proofs of Theorems 7.1 and 7.2.

Lemma 7.3. *With the hypotheses and notations above, let $\lambda > \tilde{\lambda}$, fix $k \in \mathbb{N}$ and set*

$$M = \max\left(\sup_{0<x<\beta} |\tilde{f}(x + i\varphi(x, B))|, \sup_{0<y<\varphi(\beta,B)} |\tilde{f}(\beta + iy)| \right).$$

There exists a constant $C > 0$ depending on γ and M but not on $\tilde{f}$, such that for every $\lambda > \tilde{\lambda}$ there exists a decomposition

$$b\tilde{f}(x) = G_\lambda(x) + B_\lambda(x), \quad 0 < x < \beta,$$

where $|G_\lambda(x)| \le C\lambda$ and $B_\lambda(x)$ satisfies

1. *B_λ vanishes identically on $(0, \beta) \setminus \mathcal{O}_\lambda$ and*
2. $\displaystyle\int_0^\beta x^\ell B_\lambda(x)\, dx = 0, \quad 0 \le \ell \le k.$

Proof. Notice that (2) becomes less restrictive as k decreases so if the lemma holds for some value of k it will also hold for smaller values of k. Here we will give the proof just for $k = 1$ (which implies the case $k = 0$) where all the ingredients of the general case are already present. Let $\mathcal{O}_\lambda = \bigcup_j I_j$ be the decomposition into connected components. Let the interval $I_j = (a, b)$ be one of the connected components and assume first that $b < \beta$. Consider two straight lines: one passing through the point of coordinates $(a, 0)$ and slope γ, the other one passing through the point of coordinates $(b, 0)$ and slope $-\gamma$. These two lines intersect at a point P that belongs to the strip $a < x < b$ and has a positive imaginary part. Then we call T the intersection of $\tilde{U}$ with the sector opening downwards of vertex P and sides of slopes $\pm\gamma$. If $P \in \tilde{U}$, T is a triangle bounded by two sides with slopes $\pm\gamma$ and a horizontal side equal to $I_j \times \{0\}$, while if P lies above the graph of $\varphi(\cdot, B)$ and γ is large enough we see that T has in addition a fourth upper side formed by a portion of the graph of $\varphi(\cdot, B)$ (unless $a = 0$, in which case the side of slope γ collapses to a point). At any rate, we have

$$\int_T \tilde{f}(z)\, dz = 0$$

by Cauchy's theorem (it is at this point that we use the assumption $bf \in L^1(0, \beta)$). This implies

$$\left| \int_{I_j} b\tilde{f}(x)\, dx \right| \le \int_L |\tilde{f}(z)|\, |dz|$$

where L denotes the path formed by the sides of T other than I_j. By the definition of I_j, $\tilde{f}(z)$ is bounded by λ on the straight sides of T with slopes $\pm\gamma$ and bounded by M on the side of T contained in the graph of $\varphi(\cdot, B)$ if it exists. Thus, for $\lambda > M$, we get

$$\left| \int_{I_j} b\tilde{f}(x)\, dx \right| \le \lambda |L|.$$

Since the length of L is bounded by a constant (that depends only on $\tilde{U}$ and γ) times the length of I_j, we easily get

$$\left| \int_{I_j} b\tilde{f}(x)\, dx \right| \le C|I_j|\, \lambda, \quad \lambda > \tilde{\lambda}, \tag{7.4}$$

where C depends on $\tilde{U}$, γ and M.

If $b = \beta$, consider the straight line passing through the point of coordinates $(a, 0)$ and slope γ. This line either intersects both the graph of $\varphi(\cdot, B)$ and the vertical line $x = \beta$ or only the vertical line $x = \beta$. In the first case, we call T the four sided region bounded by this line, the x axis, the vertical line $x = \beta$ and the graph of $\varphi(\cdot, B)$. In the second case, T will be the triangle bounded by this line, the x axis and the vertical line $x = \beta$. We also obtain estimate (7.4) in this case.

Denote by c_j the center of I_j. A similar reasoning with the function $(z - c_j)\tilde{f}(z)$ in the place of $\tilde{f}(z)$ yields

$$\left| \int_{I_j} (x - c_j)\, b\tilde{f}(x)\, dx \right| \leq C|I_j|^2 \lambda, \quad \lambda > \tilde{\lambda}, \tag{7.5}$$

where we have used that $\sup_{z \in T} |z - c_j| \leq C|I_j|$. Let us write $I_j = (c_j - \delta_j, c_j + \delta_j)$ and define

$$G_\lambda(x) = \begin{cases} \alpha_j(x - c_j) + \beta_j, & \text{for } x \in I_j, \\ b\tilde{f}(x), & \text{for } x \in (0, \beta) \setminus \mathcal{O}_\lambda. \end{cases}$$

Here, α_j, β_j are constants that we determine by the conditions

$$m_0^j \doteq \int_{I_j} b\tilde{f}(x)\, dx = \int_{I_j} (\alpha_j(x - c_j) + \beta_j)\, dx,$$
$$m_1^j \doteq \int_{I_j} b\tilde{f}(x)(x - c_j)\, dx = \int_{I_j} (\alpha_j(x - c_j)^2 + \beta_j(x - c_j))\, dx.$$

Since $\int_{I_j} (x - c_j)\, dx = 0$, we obtain

$$\alpha_j = \frac{m_1^j}{\rho_2^j}, \qquad \beta_j = \frac{m_0^j}{\rho_0^j},$$

with

$$\rho_\ell^j = \int_{I_j} (x - c_j)^\ell\, dx, \quad \ell = 0, 2.$$

In view of (7.4) and (7.5) $|m_0^j| \leq C\lambda\delta_j$ and $|m_1^j| \leq C\lambda\delta_j^2$, while $\rho_0^j = 2\delta_j$ and $\rho_2^j = (2/3)\delta_j^3$. Thus, $|\alpha_j| \leq C\dfrac{\lambda}{\delta_j}$ and $|\beta_j| \leq C\lambda$ which implies that $|G_\lambda(x)| \leq C\lambda$ on I_j. Since $|b\tilde{f}(x)| \leq \tilde{f}^*(x) \leq \lambda$ off $\mathcal{O}_\lambda$ we conclude that $|G_\lambda(x)| \leq C\lambda$ on $(0, \beta)$. Set now

$$B_\lambda(x) = \begin{cases} b\tilde{f}(x) - (\alpha_j(x - c_j) + \beta_j), & \text{for } x \in I_j, \\ 0, & \text{for } x \in (0, \beta) \setminus \mathcal{O}_\lambda. \end{cases}$$

It is clear that $b\tilde{f}(x) = G_\lambda(x) + B_\lambda(x)$. Concerning the properties that $B_\lambda(x)$ must fulfill, (1) holds by the very definition of $B_\lambda(x)$ while, to show (2), it is enough to check that $\displaystyle\int_{I_j} B_\lambda(x)\, dx = \int_{I_j} xB_\lambda(x)\, dx = 0$ for any j and this follows from the choice of α_j and β_j. $\square$

Remark. The proof of Lemma 7.3 exploits the fact that $\varphi(x,0) \equiv 0$ in order to obtain vanishing moments for the "good" function $B_\lambda(x)$. Otherwise, we would just obtain

$$\int_{I_j} [x - c_j + i(\varphi(x,0) - \varphi(c_j,0)]^\ell B_\lambda(x)(1 + i\varphi_x(x,0))\, dx = 0, \quad 0 \le \ell \le k.$$

On the other hand, the vanishing of those "generalized moments" is sufficient to obtain good estimates for the standard moments, which in turn is enough to control h^p norms (see, e.g., the proof of Proposition 3.7).

Proposition 7.4. *Let $0 < p \le 1$. With the above hypothesis and notations, there exist*

(a) *a sequence (a_j) of h^p-atoms supported in $[0,\beta]$ and*
(b) *a sequence (λ_j) of the complex numbers satisfying*

$$\sum_j |\lambda_j|^p \le C \int_0^\beta |\tilde{f}^*(x)|^p dx,$$

such that $b\tilde{f}(x) = \sum_j \lambda_j a_j(x)$ for almost every $x \in (0,\beta)$ and the series converges in $h^p(\mathbb{R})$ to a distribution supported in $[0,\beta]$ that coincides with $b\tilde{f}$ on $(0,\beta)$.

Proof. Fix $k \in \mathbb{N}$ such that

$$\frac{1}{k+2} < p \le \frac{1}{k+1}.$$

Hence, any measurable function $a(x)$ supported in an interval I and satisfying

$$\|a\|_{L^\infty} \le |I|^{-1/p} \quad \text{and} \quad \int x^j a(x) dx = 0, \quad j = 0, 1, \dots, k,$$

will be an h^p-atom. Choose $n_0 \in \mathbb{Z}$ such that $2^{n_0} \le \tilde{\lambda} < 2^{n_0+1}$. For every $n > n_0$, let

$$b\tilde{f}(x) = g_n(x) + b_n(x) \tag{7.6}$$

be the decomposition given by Lemma 7.3 for the choice $\lambda = 2^n$, with the notation $g_n = G_{2^n}$, $b_n = B_{2^n}$ and k as above. Define $g_n = b_n \equiv 0$ if $n < n_0$ and $g_{n_0}(x) = p(x)\, \chi_{(0,\beta)}(x)$, $b_{n_0}(x) = b\tilde{f}(x) - g_{n_0}(x)$, where $p(x)$ is the polynomial of degree k determined by

$$\int_0^\beta b\tilde{f}(x)\, x^\ell\, dx = \int_0^\beta p(x)\, x^\ell\, dx, \quad 0 \le \ell \le k.$$

With this definition, we have that

$$|g_{n_0}(x)| \le C 2^{n_0} \quad \text{and} \quad \int_0^\beta b_{n_0}(x)\, x^\ell\, dx = 0, \quad 0 \le \ell \le k.$$

We will consider simultaneously all decompositions (7.6) for different values of n. Observe that $g_n(x) \to b\tilde{f}(x)$ a.e. when $n \to \infty$, because the function $b\tilde{f}(x) - g_n(x)$ is supported in the set $\{x : \tilde{f}_k^*(x) > 2^n\}$ (if $n \geq n_0$) that decreases to a set of null measure when $n \to \infty$. Therefore, from the construction of g_n, is easy to see that $|g_n(x)| \leq C\tilde{f}^*(x)$ for $n > n_0$. Then we can write $b\tilde{f}$ as a sum of a telescopic series that converge a.e. plus a function that is a multiple of an h^p-atom:

$$\begin{aligned} b\tilde{f}(x) &= g_{n_0}(x) + \sum_{n=n_0}^{\infty} (g_{n+1}(x) - g_n(x)) \\ &= p(x)\, \chi_{(0,\beta)}(x) + \sum_{n=n_0}^{\infty} (b_n(x) - b_{n+1}(x)). \end{aligned}$$

The point here is that, although each term $b_n(x) - b_{n+1}(x)$, $n \geq n_0$, has no longer in general vanishing moments, it can be expanded in a series of multiples of h^p-atoms with coefficients in ℓ^p. Denote, for each n,

$$O_{2^n} = \bigcup_l I_n(l).$$

For fixed n, all the intervals $I_{n+1}(j)$ are included in the disjoint union of the $I_n(l)$ (for different values of l). For each pair (n, l), let

$$\psi_{n,l}(x) = \begin{cases} b_n(x) - b_{n+1}(x), & \text{for } x \in I_n(l) \\ 0, & \text{for } x \in \mathbb{R} \setminus I_n(l). \end{cases}$$

Then, for $\nu = 0, 1, \ldots, k$,

$$\begin{aligned} \int_{I_n(l)} x^\nu \psi_{n,l}(x)dx &= \int_{I_n(l)} x^\nu b_n(x)dx - \int_{I_n(l)} x^\nu b_{n+1}(x)dx \\ &= \int_{I_n(l)} x^\nu b_n(x)dx - \sum_{j \in S(n,l)} \int_{I_{n+1}(j)} x^\nu b_{n+1}(x)dx = 0, \end{aligned}$$

where $S(n, l)$ is the set of indices j for which $I_{n+1}(j) \subset I_n(l)$. We have also

$$\begin{aligned} |b_n(x) - b_{n+1}(x)| &= |g_{n+1}(x) - g_n(x)| \leq |g_{n+1}(x)| + |g_n(x)| \\ &= C2^{n+1} + C2^n = 3C2^n. \end{aligned}$$

Define

$$\phi_{n,l}(x) = \frac{2^{-n}}{3C} |I_n(l)|^{-1/p} \psi_{n,l}(x).$$

With this definition, all $\phi_{n,l}$ are H^p-atoms (i.e. h^p-atoms with vanishing moments) and

$$\sum_{n=n_0+1}^{\infty} (b_n(x) - b_{n+1}(x)) = \sum_{n=n_0+1}^{\infty} \sum_{l=1}^{\infty} 3C2^n |I_n(l)|^{1/p} \phi_{n,l}(x).$$

Moreover,

$$\begin{aligned}\sum_{n=n_0+1}^{\infty}\sum_{l=1}^{\infty}(3C2^n|I_n(l)|^{1/p})^p &= 3^pC^p\sum_{n=n_0+1}^{\infty}\sum_{l=1}^{\infty}2^{np}|I_n(l)|\\ &= C\sum_{n=n_0+1}^{\infty}2^{np}|\{x:\tilde{f}^*(x)>2^n\}|\\ &\leq C\int_0^{\infty}\lambda^{p-1}|\{x:\tilde{f}^*(x)>\lambda\}|\,d\lambda\\ &= C\int_0^{\beta}[\tilde{f}^*(x)]^p\,dx<\infty.\end{aligned}$$

Then, the series $\sum_{n=n_0+1}^{\infty}\sum_{l=1}^{\infty}3C2^n|I_n(l)|^{1/p}\phi_{n,l}(x)$ converges a.e. and in $h^p(\mathbb{R})$ to some distribution in $h^p(\mathbb{R})$, because $\|\phi_{n,l}\|_{H^p}\leq C$. Since

$$b_{n_0}-b_{n_0+1}+p(x)\,\chi_{(0,\beta)}$$

is a multiple of an h^p-atom supported in $[0,\beta]$, after rearranging the double series, we get the desired expression

$$b\tilde{f}(x)=\sum_j\lambda_ja_j(x),\quad \text{both a.e. and in the sense of } \mathcal{D}'(0,\beta),$$

where all the a_j's are h^p-atoms supported in $[0,\beta]$, $(\lambda_j)\in\ell^p$ and (a) and (b) hold. □

The size in ℓ^p of the sequence (λ_j) of the coefficients in the atomic decomposition of Proposition 7.4 is controlled by the L^p norm of the maximal function $\tilde{f}^*$. In the next section we translate the information we have on f (bounded L^p norm of traces) to boundedness of $\int_0^{\beta}(\tilde{f}^*)^p\,dx$.

8. Complex Hardy spaces on a class of rough domains

We need to recall two definitions of Hardy spaces:

Definition 8.1. ([Du]) *For $0<p<\infty$, a holomorphic function g on a bounded domain D with rectifiable boundary is said to be in $E^p(D)$ if there exists a sequence of rectifiable curves C_j in D tending to bD in the sense that the C_j eventually surround each compact subdomain of D, such that*

$$\int_{C_n}|g(z)|^p|dz|\leq M<\infty$$

The norm of $g\in E^p(D)$ is defined as

$$||g||^p_{E^p(D)}=\inf\sup_j\int_{C_j}|g(z)|^p|dz|$$

where the inf is taken over all sequences of rectifiable curves C_j in D tending to ∂D.

Definition 8.2. ([JK]) *Let $\Omega \subset \mathbb{C}$ be a bounded region with rectifiable boundary and for $x \in \partial\Omega$ consider the non tangential approach subregion*

$$\Gamma_\alpha(x) = \{z \in \Omega : |z - x| \leq (1 + \alpha) dist(z, \partial\Omega)\},$$

which, for a.e. x, is open and contains $\{x\}$ in its closure. For $0 < p < \infty$, the Hardy space $H^p(\Omega)$ is defined by

$$H^p(\Omega) = \{G \in \mathcal{H}(\Omega) : G^* \in L^p(\partial\Omega)\}$$

where $\mathcal{H}(\Omega)$ denotes the holomorphic functions on Ω and G^* denotes the non tangential maximal function defined using the $\Gamma_\alpha(p)$.

For our purposes the class of regions considered in the next lemma will suffice.

Lemma 8.3 (**Canonic Factorization**). *Let U be a region that is bounded by a finite number of smooth curves $\mathcal{C}_j$, $j = 1, \dots, N$ that cross each other at N points A_j forming inner angles $0 \leq \theta_j < \pi$.*[1] *Let $f \in E^p(U)$, $0 < p < \infty$, not be identically zero, then $f = gB$ with $g \in E^p(U)$, $|B| \leq 1$, g has no zeros in U and $\|f\|_{E^p} = \|g\|_{E^p}$.*

Proof. This was first proved by F. Riesz in the case of the unit disk Δ and the general case follows from the classical result. Indeed, if $\omega : \Delta \to U$ is a conformal map, it follows that $\tilde{f}(z) = f(\omega(z))(\omega'(z))^{1/p} \in H^p(\Delta)$ by the corollary of Theorem 10.1 in [Du]. Denote by $\tilde{B}(z)$ the Blaschke product associated to the zeros of $\tilde{f}$, counted with multiplicity. Then $|\tilde{B}(z)| \leq 1$, has the same zeros as $f_1 = f \circ \omega$ with the same multiplicity and if $0 < r_j \nearrow 1$, it follows that

$$\sup_j \int_0^{2\pi} \frac{|f_1(r_j e^{i\theta})|^p}{|\tilde{B}(r_j e^{i\theta})|^p} |\omega'(r_j e^{i\theta})| \, d\theta$$

$$= \sup_j \int_0^{2\pi} |f_1(r_j e^{i\theta})|^p |\omega'(r_j e^{i\theta})| \, d\theta \leq C. \tag{8.1}$$

Indeed, it is clear that the supremum on the right-hand side of (8.1) is bounded by the supremum of the left-hand side, because $|\tilde{B}| \leq 1$. To prove the reverse inequality one considers the finite product $\tilde{B}_N$ of the first N factors of the $\tilde{B}$. These partial products $\tilde{B}_N \to \tilde{B}$ normally in Δ as $N \to \infty$, $|\tilde{B}_N| = 1$ for $|z| = 1$ and $\tilde{B}_N$ is continuous on $|z| \leq 1$, so

$$\sup_j \int_0^{2\pi} \frac{|f_1(r_j e^{i\theta})|^p}{|\tilde{B_N}(r_j e^{i\theta})|^p} |\omega'(r_j e^{i\theta})| d\theta = \sup_j \int_0^{2\pi} |f_1(r_j e^{i\theta})|^p |\omega'(r_j e^{i\theta})| d\theta.$$

Then, using Fatou's lemma,

$$\begin{aligned} \sup_j \int_0^{2\pi} & \lim_{N\to\infty} \frac{|f_1(r_j e^{i\theta})|^p}{|\tilde{B_N}(r_j e^{i\theta})|^p} |\omega'(r_j e^{i\theta})| d\theta \\ &\leq \sup_j \liminf_{N\to\infty} \int_0^{2\pi} \frac{|f_1(r_j e^{i\theta})|^p}{|\tilde{B}_N(r_j e^{i\theta})|^p} |\omega'(r_j e^{i\theta})| d\theta \\ &\leq \liminf_{N\to\infty} \sup_j \int_0^{2\pi} \frac{|f_1(r_j e^{i\theta})|^p}{|\tilde{B}_N(r_j e^{i\theta})|^p} |\omega'(r_j e^{i\theta})| d\theta \\ &= \sup_j \int_0^{2\pi} |f_1(r_j e^{i\theta})|^p |\omega'(r_j e^{i\theta})| d\theta. \end{aligned}$$

[1] Hence, outward-pointing cusps are allowed but not inward-pointing cusps.

Thus, if we define $B(\zeta) = \tilde{B}(\omega^{-1}(\zeta))$, we see that $|B| \leq 1$ in U, $g = f/B$ is well defined, does not vanish in U and (8.1) implies that $g \in E^p(U)$ with $\|f\|_{E^p} = \|g\|_{E^p}$. $\square$

Remark. If $f \in E^p(U)$, by Lemma 8.1, we may write $f = gB$ with $g \in E^p(U)$ and g has no zeros, so if U is simply connected, we can define any power of g. Let $h = g^r$, $r \in \mathbb{R}$, we have that $f = h^{1/r}B$, $h \in E^{p/r}(U)$ has no zeros, $|bf| = |bh|^{1/r}$ and $\|f\|_{E^p} = \|h\|^{1/r}_{E^{p/r}}$.

For the unit disc, it is classical that the Hardy spaces of definitions 8.1 and 8.2 are the same, i.e., $E^p(\Delta) = H^p(\Delta)$ ([Du]). By the Riemann mapping theorem, this is also true for any bounded, simply connected domain with smooth boundary. In the work [JK] it is shown that $E^p(\Omega) = H^p(\Omega)$, $0 < p < \infty$, if Ω is a chord-arc domain. The arguments in [BH1] show that $E^p(U) = H^p(U)$ also holds for $1 \leq p < \infty$ when U is a region of the kind considered in Lemma 8.3. Here we will need to extend the identity $E^p(U) = H^p(U)$ for $0 < p < \infty$ for this class.

Lemma 8.4. *$E^p(U) = H^p(U)$, $0 < p < \infty$.*

Proof. Suppose that $g \in H^p(U)$. Let C_j a sequence of rectifiable curves tending to the boundary of U. So,

$$\begin{aligned}\int_{C_j} |g(\zeta_j)|^p |d\zeta_j| &= \int_{\partial U} |g(\Lambda_j(\zeta))|^p m_j(\zeta)|d\zeta| \\ &\leq \limsup_j \int_{\partial U} |g(\Lambda_j(\zeta))|^p m_j(\zeta)|d\zeta| \\ &\leq \int_{\partial U} |g^*(\zeta)|^p |d\zeta| = C < \infty.\end{aligned}$$

In the first equality we use the formula of change of variables (formula (2.3) in [L],[V]). See the construction of C_j in [BH1,p.475-476].

On the other hand, if $f \in E^p(U)$ we can write, using Lemma 8.3, $f = gB$ where $g \in E^p(U)$ does not vanishes and $\|g\|_{E^p} = \|f\|_{E^p}$. Let n such that $np > 1$. By remark after Lemma 8.3 we can define $h = g^{1/n}$ that belongs to $E^{np}(U) = H^{np}(U)$, this latter equality is a consequence of the case already known [BH1]. So $h^* \in L^{np}(\partial U)$ and $(g)^* = [h^n]^* = (h^*)^n \in L^p(\partial U)$. Since $|B| \leq 1$ we have that $f^* = (gB)^* \in L^p(\partial U)$. Then $f \in H^p(U)$. $\square$

We now return to the holomorphic function $\tilde{f}(x+iy)$ defined on the region

$$\tilde{U} = \{x + y : \ 0 < x < A, \ 0 < y < \varphi(x, B)\}$$

which is of the type considered in Lemma 8.4, so $E^p(\tilde{U}) = H^p(\tilde{U})$, $0 < p < \infty$. Using that $|\tilde{f}|^p$ has uniformly bounded integrals on the graphs $(0, \beta) \ni x \mapsto (x, \varphi(x, t))$, $0 < t < B$, and that $\tilde{f}$ is bounded on the graph $(0, \beta) \ni x \mapsto (x, \varphi(x, B))$ and on the vertical segment $\{\beta\} \times (0, B)$, it is easy to conclude that $\tilde{f} \in H^p(\tilde{U})$. Thus, $\tilde{f} \in H^p(\tilde{U})$, which implies that

$$\int_0^\beta \left(\tilde{f}^*(x)\right)^p dx < \infty. \tag{8.2}$$

9. End of the proof of Theorems 7.1 and 7.2

End of the proof of Theorem 7.1. It is enough to prove that $bf \in h^1_{F,\mathrm{loc}}(-\beta,\beta)$ with $F = \{0\}$. By the results in [BH1] we already know that $bf \in L^1(-\beta,\beta)$ and $\tilde f \in E^1(\tilde U) = H^1(\tilde U)$, in particular (8.2) holds with $p = 1$. By Proposition 7.4 and (8.2) with $p = 1$, we may find a distribution $f_0^+ \in h^1_F(\mathbb{R})$, $F = \{0\}$, supported in $[0,\beta]$ such that $b\tilde f - f_0^+ = bf - f_0^+ = 0$ on $(0,\beta)$. Similarly, there is $f_0^- \in h^1_F(\mathbb{R})$ supported in $[-\beta,0]$ such that $bf - f_0^- = 0$ on $(-\beta,0)$. It follows that $b(f - f^+ - f^-)$ is supported in $\{0\}$ so $bf = f_0^+ + f_0^- + \sum_{0\le k\le m} C_k\delta^{(k)}$ on $(-\beta,\beta)$. However, $bf \in L^1(-\beta,\beta)$ and $f_0^+ + f_0^- \in h^1_F(\mathbb{R}) \subset L^1(-\beta,\beta)$ so $C_k = 0$, $0 \le k \le m$ and bf is the restriction to $(-\beta,\beta)$ of an element of $h^1_F(\mathbb{R})$, in particular, $bf \in h^1_{F,\mathrm{loc}}(-\beta,\beta)$. □

Since we cannot apply Proposition 7.4 directly to $\tilde f$ when $b\tilde f = bf$ is not locally integrable, we will assume initially that $f \in C^0[-\beta,\beta]\times[0,B]$ and the general case will be derived by an approximation argument. It is clear that $\tilde f \in E^p(\tilde U) = H^p(\tilde U)$ and since the norms are comparable $\|\tilde f\|^p_{H^p} \le C\|\tilde f\|^p_{E^p}$. If M is a bound for

$$\max\left(\sup_{0<x<\beta}\frac{|\tilde f(x+i\varphi(x,B))|^p}{\beta},\ \sup_{0<y<\varphi(\beta,B)}\frac{|\tilde f(x+iy)|^p}{\varphi(\beta,B)},\ \sup_{0\le t\le B}\int_0^\beta |f(x,t)|^p\,dx\right)$$

we have $\|\tilde f\|^p_{H^p} \le CM$ which implies

$$\int_0^\beta \left(\tilde f^*(x)\right)^p dx \le CM. \tag{9.1}$$

By Proposition 7.4 and (9.1), we may find a distribution $f_0^+ \in h^p(\mathbb{R})$, (and $f_0^+ \in h^p_F(\mathbb{R})$, $F = \{0\}$, in case $1/p$ is an integer) supported in $[0,\beta]$ such that $bf = f_0^+$ on $(0,\beta)$ and $\|f_0^+\|_{h^p} \le CM$ ($\|f_0^+\|_{h^p_F} \le CM$ in case $1/p$ is an integer). From now on, to be specific, assume $1/p$ is not an integer. Notice that, by the continuity of f, f_0^+ is a bounded function. Similarly, there is $f_0^- \in h^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ supported in $[-\beta,0]$ such that $bf = f_0^-$ on $(-\beta,0)$ and $\|f_0^-\|_{h^p} \le CM$. It follows that $b(f - f^+ - f^-)$ is supported in $\{0\}$ so $bf = f_0^+ + f_0^- + U$ on $(-\beta,\beta)$, with U supported in $\{0\}$. However, $bf, f_0^+, f_0^- \in L^\infty(-\beta,\beta)$ so $U = 0$ and bf is the restriction to $(-\beta,\beta)$ of an element of $h^p(\mathbb{R})$ with norm bounded by CM, in particular, for every $\psi \in C_c^\infty(-\beta,\beta)$, $\psi bf \in h^p_c(-\beta,\beta)$ and $\|\psi bf\|_{h^p} \le C_\psi M$.

Let us now consider the general case, with $f \in C^0(Q)$, $Q = (-A,A)\times(0,B]$, $bf \in h^p_{\mathrm{loc}}(-A,A)$ and $|f(\beta,t)| + |f(-\beta,t)| \le C$, $0 < t \le B$. For a large integer j, $Q_j = [-\beta,\beta]\times[1/j,B]$, $f_j = f|_{Q_j}$. Then f_j satisfies $Lf_j = 0$ in the interior of Q_j and is continuous up to the bottom of the cube Q_j, so we are essentially in the restricted case already considered with a small difference. Indeed, while in the previous case the restriction of $Z(x,t)$ to the bottom of the cube where the solution is defined (i.e., Q) is the identity, we see that now $Z(x,t)$ maps the bottom $[-\beta,\beta]\times\{1/j\}$ of Q_j to the smooth graph $\{(x,\varphi(x,1/j)\ -\beta \le x \le \beta\}$ rather than to an interval. However, inspection of the proofs of Lemma 7.3 and Proposition 7.4 shows that they remain valid in this case, with bounds independent of j. If $f(x,t) = \tilde f \circ Z(x,t)$ for $0 \le x \le \beta$, $0 < t \le B$, it follows that $f_j(x,t) = \tilde f_j \circ Z(x,t)$ with $\tilde f_j(x+iy) = \tilde f(x+iy)$ for $0 \le x \le \beta$, $\varphi(x,1/j) \le y \le \varphi(x,B)$. Thus, we conclude that $\|\tilde{f}_j\|^p_{H^p(\tilde{Q}_j)} = \|\tilde{f}\|^p_{H^p(\tilde{Q}_j)} \leq CM$, where $\tilde{Q}_j$ is the region comprised between the graphs $y = \varphi(x, 1/j)$ and $y = \varphi(x, B)$ for $0 \leq x \leq \beta$, which implies

$$\int_0^\beta \left(\tilde{f}_j^*(x + i\varphi(x, 1/j))\right)^p dx \leq CM. \tag{9.2}$$

Since $\tilde{f}_j \in C^0(\tilde{Q}_j)$ we obtain that, for every $\psi \in C_c^\infty(-\beta, \beta)$, $\|\psi(x)f(x, 1/j)\|_{h^p} \leq C_\psi M$. Since $\psi(x)bf(x) = \lim_{j\to\infty} \psi(x)f(x, 1/j)$ in $\mathcal{D}'(-A, A)$, this shows that $bf \in h^p_{\text{loc}}(-A, A)$ as we wished to prove. When $1/p$ is an integer the proof is analogous. □

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