On Pairs of Positive Solutions for a Class of Quasilinear Elliptic Problems *

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Abstract

We prove, by using bifurcation theory, the existence of at least two positive solutions for the quasilinear problem $-\Delta_p u = f(x, u)$ in Ω, $u = 0$ on $\partial\Omega$, where $p \geq 2$ and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$ and the nonlinearity f is a locally Lipschitz continuous function among others assumptions.

1 Introduction

This paper is concerned with the existence and multiplicity of positive solutions for the problem

$$\begin{cases} -\Delta_p u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $p \geq 2$ is the well known p-Laplacian operator. We will assume throughout the paper that Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 2$ and $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ is a locally Lipschitz continuous function and satisfies the following basic assumptions:

(f_1) $f(x,t) > 0$ for all $(x,t) \in \overline{\Omega} \times (0,+\infty)$;

(f_2) $f(x,0) > 0$ or if $f(x,0) = 0$ then $0 < a_0(x) := \lim_{t\to 0^+} \dfrac{f(x,t)}{t^{p-1}}$, uniformly in $\overline{\Omega}$, and $a_0 \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$;

(f_3) there exists a function $a_\infty \in C^\alpha(\overline{\Omega})$, for some $\alpha \in (0,1)$, such that $\lim_{t\to\infty} \dfrac{f(x,t)}{t^{p-1}} = a_\infty(x) > 0$, uniformly in $\overline{\Omega}$.

Let us denote by $\lambda_i(m), i = 1,2,...$, the eigenvalues of the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta_p u &= \lambda m(x)|u|^{p-2}u, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $m \in C^\alpha(\overline{\Omega})$ and $m(x) > 0$ in Ω. It is well known, cf. P. Drábek [8], that the first eigenvalue $\lambda_1(m) > 0$ is simple and its associated eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega)$ may be taken positive in Ω and it is the unique eigenfunction with this property. Actually, $\varphi_1 \in C^{1,\alpha}(\overline{\Omega})$ (see P. Tolksdorf [15]).

Hereafter we assume the following:

$$\lambda_1(a_0) < 1 \quad \text{and} \quad \lambda_1(a_\infty) < 1. \tag{3}$$

From now on, we also suppose that Ω lies between two parallel planes a distance d apart and we put $C = e^d - 1$.

Theorem 1.1 *In addition to conditions $(f_1) - (f_3)$ and (3), suppose that f satisfies the following condition*

(f_4) *there exist $m, t_0 > 0$ such that $C(f(x,t_0) + mt_0^{p-1}) < t_0$ for all $x \in \overline{\Omega}$ and $f(x,t) + mt^{p-1} \leq f(x,t_0) + mt_0^{p-1}$ for all $(x,t) \in \overline{\Omega} \times [0,t_0]$ with $p \geq 2$.*

Then problem (1) has at least two positive solutions u_1 and u_2 satisfying

$$0 < ||u_1||_\infty < t_0 < ||u_2||_\infty.$$

Remark 1.1 The existence of a positive solution for the problem has been studied by many authors. More recently, cf. De Paiva [13] considered the case where $\lambda_1(a_0) < 1 < \lambda_1(a_\infty)$, with λ_1 denoting the first eigenvalue of the weighted nonlinear problem (2) and $m(x)$ beeing a function in L^r space, $r > \frac{N}{p}$ if $1 < p \leq N$ and $r = 1$ if $p > N$, which can change sign in Ω. For problems using inequalities of type (3), see De Figueiredo & Lions [7], where the authors studied the semilinear case $p = 2$, and De Coster [6], for the one-dimensional case.

Remark 1.2 In [12], a-priori estimates for positive solutions of the problem

$$\begin{cases} -\Delta_p u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

are obtained if $1 < p < 2$, f satisfies the conditions (f_5), (f_6), (f_7) and (f_8) below and the domain is strictly convex. Here, we may show a multiplicity result slightly similar to Theorem 2 in [2] for a class of sub-superlinear elliptic problems, by using a-priori bounds for positive solutions of problem (4) with $p \geq 2$ (see Proposition 2.2 below), and supposing in addition to this that $f : [0, +\infty) \to \mathbb{R}$ is strictly positive in $(0, +\infty)$ satisfying the assumptions (f_1), (f_2), (f_4), and $(f_5) - (f_8)$ below.

Before enunciating our next result we establish some conditions on f. We will suppose that $f : [0, +\infty) \to \mathbb{R}$ is strictly positive and continuous in $(0, +\infty)$ satisfying:

(f_5) $f(t) \geq c_1 t^q$ for all $t \geq 0$ for $c_1 > 0$ and $q > p - 1$;

(f_6) $\exists k > 0$ such that $f(t) \leq c_2 t^{p-1}$ for $0 \leq t \leq k$ where $c_2 > 0$,

(f_7) $\displaystyle\lim_{t\to\infty} \frac{f(t)}{t^{p^*-1}} = 0$ where $p^* = \frac{pN}{N-p}$ and $p < N$;

(f_8) $\displaystyle\limsup_{t\to\infty} \frac{tf(t) - \tau F(t)}{t^{\frac{p(N-p+1)}{N}} f(t)^{\frac{p}{N}}} \leq 0$ for some $0 \leq \tau < p^*$.

Now we may establish the following theorem, which is only stated for functions $f(x, r) = f(r)$.

Theorem 1.2 *Suppose that $f : [0, +\infty) \to \mathbb{R}$ is a strictly positive in the interval $(0, +\infty)$ and a C^1 function satisfying $(f_1), (f_2), (f_4)$ and $\lambda_1(a_0) < 1$. If, in addition, f satisfies $(f_5), (f_6)$, (f_7) and (f_8) with Ω strictly convex and $p \geq 2$, then problem (4) has at least two positive solutions u_1 and u_2 satisfying*

$$0 < ||u_1||_\infty < t_0 < ||u_2||_\infty.$$

2 Proofs.

Proof of Theorem 1.1. Consider the one parameter family of problems, namely

$$\left\{ \begin{array}{rcll} -\Delta_p u & = & \lambda f(x,u), & \text{in } \Omega, \\ u & = & 0, & \text{on } \partial\Omega, \\ \lambda & \geq & 0. & \end{array} \right. \tag{5}$$

Observe that for $\lambda = 1$, we have problem (1).

Using results on bifurcation theory (see [14]) we construct a component $\Sigma \subset \mathbb{R}^+ \times C^0(\overline{\Omega})$ of positive solutions of (5) which will be suitably conducted in order to meet the fiber $\{1\} \times C^0(\overline{\Omega}) \subset \mathbb{R}^+ \times C^0(\overline{\Omega})$ two times and so we produce two positive solutions for (1). More precisely, we proceed as follows. First we consider the auxiliary nonlinear eigenvalue problem

$$\left\{ \begin{array}{rcll} -\Delta_p u & = & \lambda \tilde{f}(x,u), & \text{in } \Omega, \\ u & = & 0, & \text{on } \partial\Omega, \\ \lambda & \geq & 0, & \end{array} \right. \tag{6}$$

where $\tilde{f} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is the extension of f given by

$$\tilde{f}(x,t) = \left\{ \begin{array}{ll} f(x,0) & \text{if } t \leq 0 \\ f(x,t) & \text{if } t > 0. \end{array} \right.$$

Let F be the Nemytskii operator $F : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ associated to $\tilde{f}$, that is, $F(u)(x) = \tilde{f}(x, u(x))$ for all $u \in W_0^{1,p}(\Omega)$, where $p' = \frac{p}{p-1}$. We may see, via elliptic regularity, that problem (6) is equivalent to the functional equation

$$u = \lambda L F(u),\ u \in W_0^{1,p}(\Omega) \text{ and } \lambda \geq 0, \tag{7}$$

where $L := (-\Delta_p)^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$. Using the maximum principle [16], we have that if (λ, u) is a solution of (6) with $\lambda > 0$ then $u > 0$ in Ω

and so (λ, u) satisfies (5). Since $F : C^0(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \subset W_0^{1,p}(\Omega) \to C^0(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \subset W^{-1,p'}(\Omega)$ is continuous and $L = (-\Delta_p)^{-1} : C^0(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \subset W^{-1,p'}(\Omega) \to C^0(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \subset W_0^{1,p}(\Omega) \hookrightarrow L^q$, $q \in (1, p^*)$ is compact, $LF : C^0(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \to C^0(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ is compact.

We now distinguish between two cases.

Case 1. $f(x, 0) > 0$. For this case, Rabinowitz theorem [14] implies that there is an unbounded component Σ of positive solutions of (5) meeting $(0, 0) \in \mathbb{R}^+ \times C^0(\overline{\Omega})$. In view of conditions $(f_1) - (f_3)$, there is a $\beta > 0$ so that $f(x, t) \geq \beta t^{p-1}$ for all $(x, t) \in \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$. By Picone's identity [1] and (5) we obtain

$$\lambda_1 \int_\Omega \varphi_1^p \geq \int_\Omega \frac{-\Delta_p u \varphi_1^p}{u^{p-1}} = \int_\Omega \frac{\lambda f(x, u)\varphi_1^p}{u^{p-1}} \geq \lambda \int_\Omega \frac{\beta u^{p-1}}{u^{p-1}} \varphi_1^p = \lambda\beta \int_\Omega \varphi_1^p$$

and so $\lambda \leq \frac{\lambda_1}{\beta}$. Consequently, the component Σ must be unbounded in u. So there is a sequence $(\lambda_n, u_n) \subset \Sigma$ in the strip $[0, \frac{\lambda_1}{\beta}] \times C^0(\overline{\Omega})$ such that $\lambda_n \to \lambda_0$ and $||u_n||_\infty \to \infty$. Thus

$$\begin{cases} -\Delta_p u_n &= \lambda_n f(x, u_n), & \text{in } \Omega, \\ u_n &= 0, & \text{on } \partial\Omega. \end{cases}$$

Setting $v_n = \frac{u_n}{||u_n||_\infty}$ we have

$$-\Delta_p v_n = \lambda_n \frac{f(x, u_n)}{||u_n||_\infty^{p-1}} \ \text{ in } \Omega, \quad v_n = 0, \ \text{on } \partial\Omega. \tag{8}$$

Since f satisfies (f_2) and (f_3), there is a constant $C > 0$ depending only on Ω and on the behavior of f in the limits arising in (f_2) and (f_3) such that $||v_n||_{W^{1,p}} \leq C$. In fact, multiplying both sides of equation (8) by v_n and integrating once we obtain

$$||v_n||^p_{W_0^{1,p}} = \int_\Omega |\nabla v_n|^p dx \leq \lambda_n \sup_{x \in \overline{\Omega}} \frac{f(x, u_n(x))}{||u_n||_\infty^{p-1}} ||v_n||_\infty |\Omega| \leq C \ \ \forall \ \ n.$$

From which, it follows that $v_n \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$, $v_n \to v_0$ in $L^q(\Omega)$, $q \in (1, p^*)$ and $v_n(x) \to v_0(x)$ a.e. in Ω. Hence, from the Dominated Convergence Theorem we have

$$\int_\Omega |\nabla v_0|^{p-2} \nabla v_0 \nabla \varphi = \int_\Omega \lambda_0 a_\infty(x) v_0^{p-1} \varphi,$$

$\forall\varphi \in W_0^{1,p}(\Omega)$. Because $v_0 \geq 0$, it follows, by the maximum principle [16], that $v_0 > 0$ in Ω and $\frac{\partial v_0}{\partial \eta} < 0$ on $\partial\Omega$. Then $\lambda_0 = \lambda_1(a_\infty)$ which implies that $(\lambda_1(a_\infty), \infty)$ is the only bifurcation point at infinity for positive solutions of (5). Note that we can not conclude, up to now, that Σ crosses $\lambda = 1$ because $\lambda_1(a_\infty) < 1$. For this we need the following result for quasilinear equations whose proof is similar to that found in [11], page 35.

Proposition 2.1 *If Ω lies between two parallel planes a distance d apart and $u \in C^{1,\alpha}(\overline{\Omega})$ satisfies*

$$-\Delta_p u + a|u|^{p-2}u = g(x) \in C(\overline{\Omega}) \text{ in } \Omega, u = 0 \text{ on } \partial\Omega,$$

where $a \geq 0$ is a constant, then

$$||u||_\infty \leq (e^d - 1)||g||_\infty. \tag{9}$$

Sketch of proof of Proposition 2.1. Let Ω lies in the slab $0 < x_1 < d$ and set $L_0 := \Delta_p$. We have

$$L_0 e^{x_1} = (p-1)e^{(p-1)x_1} \geq 1 \quad \text{for} \quad p \geq 2.$$

Let

$$v = \sup_{\partial\Omega} u^+ + (e^d - e^{x_1}) \sup_{\Omega} |(-g)^-|.$$

Then, since

$$L_p v := L_0(v) - a|v|^{p-2}v \leq -(p-1) \sup_{\Omega} |(-g)^-|,$$

$$L_p v - L_p u \leq -\big((p-1)\sup_{\Omega} |(-g)^-| - g(x)\big) \leq 0$$

in Ω, and $v - u \geq 0$ on $\partial\Omega$. Hence, for $C = e^d - 1$, we obtain the desired result from the strong comparison principle [3]. ∎

We claim that Σ meets $\lambda = 1$. Suppose by contradiction, that there is $(\lambda_0, u_0) \in \Sigma$ with $0 < \lambda_0 \leq 1$ and $||u_0||_\infty = t_0$. Thus,

$$-\Delta_p u_0 = \lambda_0 f(x, u_0) \text{ in } \Omega, u_0 = 0 \text{ on } \partial\Omega. \tag{10}$$

Let v_0 be the unique solution of

$$\begin{cases} -\Delta_p v_0 + m|v_0|^{p-2}v_0 = f(x, u_0) + m|u_0|^{p-2}u_0, & \text{in } \Omega, \\ v_0 = 0, & \text{on } \partial\Omega, \end{cases} \tag{11}$$

(see [10]). Applying the strong comparison principle (see Theorem 1 in [3]) for the weak solutions u_0 and v_0 of the partial differential equations (10) and (11), respectively, and using that $0 < \lambda_0 \leq 1$, it follows that $v_0 \geq u_0$ and so $t_0 = ||u_0||_\infty \leq ||v_0||_\infty$. By Proposition 2.1, we have

$$||v_0||_\infty \leq (e^d - 1)||f(x, u_0(x)) + mu_0^{p-1}(x)||_\infty.$$

Using (f_4), we obtain

$$0 \leq f(x, u_0) + mu_0^{p-1} \leq f(x, t_0) + mt_0^{p-1},$$

and so

$$t_0 = ||u_0||_\infty \leq ||v_0||_\infty \leq (e^d - 1)f(x, t_0) + mt_0^{p-1} < t_0,$$

which is a contradiction. Hence, there is no $(\lambda, u) \in \Sigma$ with $0 \leq \lambda \leq 1$ and $||u||_\infty = t_0$. Since $(0, 0) \in \Sigma$ and Σ is a continuum of positive solutions, then crosses $\{1\} \times C^0(\overline{\Omega})$ below t_0. This crossing produces the first solution u_1 of (1) with $0 < ||u_1||_\infty < t_0$. The assumption $\lambda_1(a_\infty) < 1$ implies that Σ must bend back on itself and a second crossing of Σ with $\{1\} \times C^0(\overline{\Omega})$ is produced, generating a second solution u_2 above t_0, that is, $||u_2||_\infty > t_0$.

Case 2. $f(x, 0) = 0$ and $\lambda(a_0) < 1$. In this case (see P. Drábek, A. Elkhalil & A. Touzani [9]), there exists an unbounded component Σ of positive solutions of $(1)_\lambda$ bifurcating from $(\lambda_1(a_0), 0)$, this being its unique bifurcation point for positive solutions in the line of trivial solutions. ■

Before proving Theorem 1.2, we present the following result on a-priori estimates for positive solutions of problem (4).

Proposition 2.2 *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a positive solution of (4). Assume Ω is strictly convex, $1 < p < \infty$ and $f : [0, +\infty) \to \mathbb{R}$ is strictly positive and locally Lipschitz continuous function in $(0, +\infty)$ satisfying the assumptions (f_1), (f_2), (f_4), $(f_5) - (f_8)$. Then $||u||_{C^0(\overline{\Omega})}$ is a-priori bounded, that is, there is a constant C_∞ depending only on Ω and on the behavior of f such that $||u||_{C^0(\overline{\Omega})} \leq C_\infty$.*

Notice that for $1 < p < 2$, the Proposition 2.2 is Theorem 2.1 in [12]. The restriction on p appears, in Theorem 2.1, because the main tool used by the authors in the proof is the moving plane method applied to their problem with $1 < p < 2$ as in L. Damascelli & F. Pacella [4]. The Proposition 2.2 follows exactly as in Theorem 2.1 but here using a recent result proved by L.

Damascelli & B. Sciunzi [5], where they apply the moving plane method to a problem involving the p-Laplace operator with $p \geq 2$.

Proof of Theorem 1.2. The key to the proof is the construction of an truncation of f in order that we may use Theorem 1.1. In view of condition (f_5), there exists $K > \lambda_1$ and $t_\infty > 0$ such that $f(t) > Kt^{p-1}$ if $t \geq t_\infty$. We now choose C_∞ as in Proposition 2.2 so that $C_\infty > max\{t_0, t_\infty\}$. Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a function defined as follows:

If $0 \leq t \leq C_\infty$, define h by $h(t) = f(t) - Kt^{p-1}$. For $t > C_\infty$ we proceed in the following way. Note that there exists a straight line $\widetilde{h}$ through $(C_\infty, h(C_\infty))$, with negative slope, such that $\widetilde{h}(t) \leq f(t) - Kt^{p-1}$ if $t \geq C_\infty$ because f is C^1. Thus, there exists $C_1 > C_\infty$ so that $\widetilde{h}(C_1) = 0$. Now we define $h(t) = \widetilde{h}(t)$ if $C_\infty \leq t \leq C_1$ and $h(t) = 0$ if $t \geq C_1$. Thus h is bounded and locally Lipschitz continuous function.

Let $\widetilde{f} : \mathbb{R}^+ \to \mathbb{R}^+$ be a function given by $\widetilde{f}(t) = Kt^{p-1} + h(t)$. In this case $a_\infty = K$ and so $\lambda_1(a_\infty) < 1$ because $K > \lambda_1$. From the Proposition 2.2 we have that if u is a positive solution of

$$-\Delta u = \widetilde{f}(u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{12}$$

then $||u||_{C^0(\overline{\Omega})} \leq C_\infty$. In view of Theorem 1.1, problem (12) possesses two positive solutions u_1 and u_2 satisfying $0 < ||u_1||_\infty < t_0 < ||u_2||_\infty$. Furthermore, $0 < u_1(x), u_2(x) \leq C_\infty$ and, because $f(t) = \widetilde{f}(t)$ for all $t \in [0, C_\infty]$, we have that u_1 and u_2 are solutions of (3). ■

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