On positive solution for a class of degenerate quasilinear elliptic positone/semipositone systems

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Abstract

This paper deals with the existence and nonexistence of positive weak solutions of degenerate quasilinear elliptic systems with subcritical and critical exponents. The nonlinearities involved have semipositone and positone structures and the existence results are obtained by applying the lower and upper-solution method and variational techniques.

Key words: Degenerate quasilinear equations, elliptic systems, positone, semipositone, perturbation.

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1 Introduction

Castro, Hassanpour, and Shivaji in [1], and then different authors, focused their attention on a class of problems, so called semipositione problems, of the form

$$-\Delta u = \lambda f(u) \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega,$$

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where Ω is a smooth bounded domain in $\mathbb{R}^N$, λ is a positive parameter, and $f : [0, \infty) \to \mathbb{R}$ is a monotone and continuous function satisfying the conditions

$$f(0) < 0, \tag{f_0}$$

$$\lim_{s\to\infty} f(s) = +\infty, \tag{f_1}$$

and also the sublinear condition at infinity, namely, $\lim_{s\to\infty} f(s)/s = 0$.

This kind of problem was motivated by the paper of Keller-Cohen in [2], where they studied a positone problem which means the Dirichlet problem involving as nonlinearity a positive and monotone function.

The semipositone problems are mathematically a challenging area in the study of positive solution of the Dirichlet problem. We refer the reader to the survey paper [3] (and [4] for $p \neq 2$) as well as to [5] and references therein. Actually, the authors in [5] were able to treat not only the sublinear case, but also the superlinear case by applying variational techniques.

Chhetri, Hai, and Shivaji [6], by using degree theory, studied the semipositone systems involving p-laplacian operator with $p > 1$, of the type

$$(P_p): \quad \begin{cases} -\Delta_p u = \lambda f_1(v) \ \text{ in } \Omega, \\ -\Delta_p v = \lambda f_2(u) \ \text{ in } \Omega, \\ u = v = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, λ is a positive parameter, and $f_1, f_2 : [0, \infty) \to \mathbb{R}$ are monotone and continuous functions satisfying conditions (f_0), (f_1), and

$$\lim_{s\to\infty} \frac{\max\{f_1(s), f_2(s)\}}{s^{p-1}} = 0. \tag{f_2}$$

While in [7], Hai and Shivaji proved an existence result for system (P_p) with the condition

$$\lim_{s\to\infty} \frac{f_1(M(f_2(s))^{1/(p-1)})}{s^{p-1}} = 0, \quad \text{for all } M > 0, \tag{f_3}$$

instead of condition (f_2) above mentioned. They applied the lower and upper solution method. We would like to mention that in both papers above mentioned were considered only the autonomous case.

On the other hand, in the scalar case, García and Peral in [8] proved that the problem involving p-laplacian operator with $1 < p < N$, of the type

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, \\ \quad u = 0 \qquad \text{on } \partial\Omega \end{cases}$$

does not have any nontrivial weak solution for all $\lambda > 0$ small enough. While the perturbed problem involving p-laplacian operator with $1 < p < N$

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + |u|^{q-2} u \text{ in } \Omega, \\ \quad u \geq 0 \qquad \text{in } \Omega, \\ \quad u = 0 \qquad \text{on } \partial\Omega \end{cases}$$

has a nontrivial weak solution for all $\lambda > 0$ sufficiently small and $1 < q < p^*$ (see also Ghoussoub and Yuan [9]). Here $p^* = Np/(N-p)$ denotes the critical Sobolev exponent. Still in the scalar case, an important situation happens when $q = p^*$; in this case the inclusion $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is continuous, but not compact any longer. However, in a pioneering paper of Brezis and Nirenberg [10] is treated the case $p = 2$. This type of problem involving the lack of compactness has been studied by many authors, and we would like to mention some of them, for instance, [8,9,11] and references therein. Recently, Adriouch and Hamidi in [12] studied the system

$$\begin{cases} -\Delta_p u = \lambda |u|^{p_1-2} u + (\alpha+1)|u|^{\alpha-1}|v|^{\beta+1} u \ \text{ in } \Omega, \\ -\Delta_q u = \lambda |v|^{q-2} v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1} v \ \text{ in } \Omega, \end{cases}$$

with Dirichlet or mixed boundary conditions, and supposing that the exponents verify $(\alpha+1)/p^* + (\beta+1)/q^* < 1$, and $1 < p_1 < p$. In [13], they handled the critical case $(\alpha+1)/p^* + (\beta+1)/q^* = 1$, with $p = q$. For the systems involving $p-$Laplacian or Laplacian operators, we would like to mention the papers, e.g., [14–16] and a survey paper [17].

The aim of this work is to extend or complement some of the above results for the quasilinear elliptic systems involving singularities of the type

$$\begin{cases} -div(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda g_1(x,u,v) \quad \text{in} \quad \Omega, \\ -div(|x|^{-bq}|\nabla v|^{q-2}\nabla v) = \lambda g_2(x,u,v) \quad \text{in} \quad \Omega, \\ \qquad u = v = 0 \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $g_1, g_2 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and monotone functions, Ω satisfies

$$\Omega \ \text{ is a smooth bounded domain of } \mathbb{R}^N \text{ with } 0 \in \Omega, \tag{H_Ω}$$

λ is a positve parameter, and the exponents verify the following conditions

$$1 < p, q < N,\ 0 \leq a < (N-p)/p,\ 0 \leq b < (N-q)/q. \qquad (H_{exp})$$

We point out that the systems of type (1) arise naturally as stationary states of certain models, for instance, in fluids mechanics.

Our three results below involve semipositone systems. We recall that the positone systems, that is, system (P_{pq}) with $f_i \geq 0, i = 1, 2$, even in the regular case have attracted much attention in recent years. We refer the reader to the papers [14,18–20] and references therein. Still in this situation, for the quasilinear elliptic equations/systems involving singularities, we would like to mention, e.g. [9,11,21–31] and references therein.

Theorem 1.1 *In addition to* (H_Ω) *and* (H_{exp})*, assume that* $g_1(x,u,v) = |x|^{-(a+1)p+c_1} f_1(v)$, $g_2(x,u,v) = |x|^{-(b+1)q+c_2} f_2(u)$*, with* $c_1, c_2 > 0$*, and* $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ *are nondecreasing and continuous functions such that*

$$\lim_{s\to\infty} \frac{f_1(M(f_2(s))^{\frac{1}{q-1}})}{s^{p-1}} = 0,\ \forall\ M > 0,\ \text{and}\ \lim_{s\to\infty} f_i(s) = \infty,\ \text{for } i = 1, 2. \quad (H_1)$$

Then, there exists λ_0 *a large enough positive parameter such that system* (1) *possesses a weak solution, where each component is positive, for each* $\lambda \geq \lambda_0$.

The next result treats the non-autonomous case.

Theorem 1.2 *In addition to* (H_Ω) *and* (H_{exp})*, assume that* $g_1(x,u,v) = |x|^{-(a+1)p+c_1} f_1(x,u,v)$, $g_2(x,u,v) = |x|^{-(b+1)q+c_2} f_2(x,u,v)$*, with* $c_1, c_2 > 0$, $f_1, f_2 : \overline{\Omega}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *are continuous functions, nondecreasing in the variables* s *and* t*, and they satisfy*

$$\lim_{s\to\infty} \frac{f_1(x,s,t)}{s^{p-1}} = 0 \text{ uniformly in } (x,t) \in \Omega \times \mathbb{R}, \qquad (H_2)$$

$$\lim_{t\to\infty} \frac{f_2(x,s,t)}{t^{q-1}} = 0 \text{ uniformly in } (x,s) \in \Omega \times \mathbb{R}, \qquad (H_3)$$

$$\lim_{(s,t)\to\infty} f_i(x,s,t) = \infty \text{ uniformly in } x \in \Omega,\ \text{for } i = 1, 2. \qquad (H_4)$$

Then, there exists λ_0 *a large enough positive parameter such that system* (1) *possesses a weak solution, where each component is positive, for each* $\lambda \geq \lambda_0$.

Also, we will establish a nonexistence result for our system.

Theorem 1.3 *In addition to* (H_Ω) *and* (H_{exp})*, assume that* $g_1(x,u,v) = |x|^{-(a+1)p+c_1} f_1(x,u,v)$, $g_2(x,u,v) = |x|^{-(b+1)q+c_2} f_2(x,u,v)$, $(a+1)p - c_1 = (b+1)q - c_2$, $c_1, c_2 > 0$*, and* $f_1, f_2 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions* *satisfying*

$$|f_1(x,s,t)s| \le k_1|s|^p + k_2|t|^q \text{ and } |f_2(x,s,t)t| \le k_3|s|^p + k_4|t|^q,$$

for all $s,t \in \mathbb{R}$ and every $x \in \Omega$, where k_1, k_2, k_3, k_4 are positive real numbers. Then, there exists λ_0 a positive parameter such that system (1) *does not possess any weak solution, where each component is nontrivial, for all* $0 < \lambda < \lambda_0$.

As we can see, a consequence of theorem 1.3 is that the system (1), with $g_1(x,u,v) = \theta|x|^{-\beta_1}|u|^{\theta-2}|v|^{\delta}u$, $g_2(x,u,v) = \delta|x|^{-\beta_1}|u|^{\theta}|v|^{\delta-2}v$, $\theta,\delta > 1$, $\theta/p + \delta/q = 1$, $\beta_1 = (a+1)p - c_1 = (b+1)q - c_2$, and $c_1, c_2 > 0$, does not have any weak solution for all $\lambda > 0$ small enough.

We will use a version of the mountain pass theorem due to Ambrosetti and Rabinowitz [32] and the Ekeland's variational principle to establish some conditions for the existence of a nontrivial weak solution for a nonlinear perturbation of the above system, namely,

$$\begin{cases} Lu_{ap} = \lambda\theta|x|^{-\beta_1}|u|^{\theta-2}|v|^{\delta}u + \mu\alpha|x|^{-\beta_2}|u|^{\alpha-2}|v|^{\gamma}u & \text{in } \Omega, \\ Lv_{bq} = \lambda\delta|x|^{-\beta_1}|u|^{\theta}|v|^{\delta-2}v + \mu\gamma|x|^{-\beta_2}|u|^{\alpha}|v|^{\gamma-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $Lw_{er} = -div(|x|^{-er}|\nabla w|^{r-2}\nabla w)$, λ, μ are nonnegative parameters, and, in addition to (H_Ω) and (H_{exp}), the exponents verify the following conditions:

$$\begin{gathered} a \le e_1 < a+1,\ b \le e_2 < b+1,\ d_1 = 1 + a - e_1,\ d_2 = 1 + b - e_2, \\ p^* = Np/(N - d_1 p),\ q^* = Nq/(N - d_2 q),\ \beta_1, \beta_2 \in \mathbb{R}, \\ \alpha, \gamma, \theta, \delta > 1, \text{ and } \theta/p + \delta/q = 1. \end{gathered} \tag{H^*_{exp}}$$

We will distinguish the following cases

$$\frac{\alpha}{p} + \frac{\gamma}{q} > 1 \text{ and } \frac{\alpha}{p^*} + \frac{\gamma}{q^*} < 1; \qquad p,q - \text{superlinear} \tag{3}$$

$$\frac{\alpha}{p} + \frac{\gamma}{q} < 1; \qquad p,q - \text{sublinear} \tag{4}$$

$$\frac{\alpha}{p^*} + \frac{\gamma}{q^*} = 1. \qquad \text{critical case} \tag{5}$$

Our main results related to the perturbed system are the following ones.

Theorem 1.4 *In addition to (H_Ω), (H_{exp}), and (H^*_{exp}), assume that $p_1 = p$, $q_1 = q$, $p_2, q_2 > 1$ are such that $\alpha/p_2 + \gamma/q_2 = 1$, and β_1, β_2 satisfy*

$$\beta_i < \min\left\{(a+1)p_i + N\left(1 - \frac{p_i}{p}\right), (b+1)q_i + N\left(1 - \frac{q_i}{q}\right)\right\},\ i = 1,2. \tag{6}$$

Suppose that $p_2 \in (p, p^*)$, $q_2 \in (q, q^*)$, *if* (3) *is held; and* $p_2 \in (1,p)$, $q_2 \in (1,q)$, *if* (4) *is holds. Then, there exists* $\lambda_0 > 0$ *such that system* (2) *possesses a weak solution, where each component is nontrivial and nonnegative, for all* $\mu > 0$ *and* $0 \leq \lambda < \lambda_0$.

We will establish our last result which involves the critical situation.

Theorem 1.5 *In addition to* (H_Ω), (H_{exp}), *and* (H^*_{exp}), *assume that* $p = q$, $a = b \geq 0$, $p^* = q^*$, $\beta_1 = (a+1)p - c$, $\beta_2 = e_1 p^*$, *and* $0 < c \leq (N - p - ap)/(p-1)$. *Suppose that* (5) *holds. Then, there exists* $\lambda_0 > 0$ *such that system* (2) *possesses a weak solution, where each component is nontrivial and nonnegative, for each* $\mu > 0$ *and* $0 < \lambda < \lambda_0$.

We observe that due to the singularity in the weights a careful analysis is required, for instance, to get some properties of the eigenfunction associated to the first eigenvalue related to the our operator.

2 Preliminaries

We will set some spaces and their norms. If $\alpha \in \mathbb{R}$ and $l \geq 1$, $L^l(\Omega, |x|^\alpha)$ denotes the subspace of $L^l(\Omega)$ of the Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$||u||_{L^l(\Omega,|x|^\alpha)} := \left(\int_\Omega |x|^\alpha |u|^l dx\right)^{\frac{1}{l}} < \infty.$$

If $1 < p < N$ and $-\infty < a < (N-p)/p$, we define $W^{1,p}(\Omega, |x|^{-ap})$ (resp. $W_0^{1,p}(\Omega, |x|^{-ap})$) as being the completion of $C^\infty(\Omega)$ (resp. $C_0^\infty(\Omega)$) with respect to the norm $||\cdot||$ defined by

$$||u|| := \left(\int_\Omega |x|^{-ap} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

The following Hardy-Sobolev inequality with weights was proved by Caffarelli, Kohn and Nirenberg in [33]. Suppose that $1 < p < N$, then there exists $C_{a,e} > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-ep^*} |u|^{p^*} dx\right)^{\frac{p}{p^*}} \leq C_{a,e} \left(\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx\right), \forall u \in C_0^\infty(\mathbb{R}^N),$$

where $-\infty < a < (N-p)/p$, $a \leq e \leq a+1$, $d = 1 + a - e$, and $p^* := p^*(a,e) = Np/(N - dp)$. From the boundedness of Ω and by standard approximation arguments, it is easy to see that there exists $C > 0$ such that

$$\left(\int_\Omega |x|^{-\delta} |u|^r dx\right)^{\frac{p}{r}} \leq C \left(\int_\Omega |x|^{-ap} |\nabla u|^p dx\right), \forall u \in W_0^{1,p}(\Omega, |x|^{-ap}),$$

where $1 \leq r \leq Np/(N-p)$ and $\delta \leq (a+1)r + N\left[1-(r/p)\right].$

The proof of the lemma below is completely similar to the proof of [14, Theorem 5] (see also [15, Lemma 3] for $p \neq 2$).

Lemma 2.1 *Suppose that Ω is a smooth bounded domain of $\mathbb{R}^N$, $0 \in \Omega$, $1 < p < N$, $-\infty < a < (N-p)/p$, $a \leq e_1 < a+1$, $d_1 = 1 + a - e_1$, $p^* = Np/(N - d_1 p)$, and $\alpha + \gamma = p^*$. Then,*

$$\tilde{S} := \inf_{(u,v)\in \tilde{W}} \left\{ \frac{\int_\Omega |x|^{-ap} \left(|\nabla u|^p + |\nabla v|^p\right) dx}{\left(\int_\Omega |x|^{-e_1 p^*} |u|^\alpha |v|^\gamma dx\right)^{\frac{p}{p^*}}} \right\},$$

where

$$\tilde{W} = \left\{(u,v) \in \left(W_0^{1,p}(\Omega, |x|^{-ap})\right)^2 : |u||v| \not\equiv 0\right\},$$

satisfies

$$\tilde{S} = \left[(\alpha/\gamma)^{\gamma/p^*} + (\alpha/\gamma)^{-\alpha/p^*}\right] C^*_{a,p}.$$

Furthermore, if $C^*_{a,p}$ is achieved by $w_0 \in W_0^{1,p}(\Omega, |x|^{-ap})$, then $\tilde{S}$ is achieved by (sw_0, tw_0), for all $s, t > 0$ satisfying $s/t = (\alpha/\gamma)^{1/p}$. $C^*_{a,p} = C^*_{a,p}(\Omega)$ is the best Hardy-Sobolev constant, which is characterized for

$$C^*_{a,p}(\Omega) := \inf_{u \in W_0^{1,p}(\Omega, |x|^{-ap})\setminus\{0\}} \left\{ \frac{\int_\Omega |x|^{-ap} |\nabla u|^p dx}{\left(\int_\Omega |x|^{-e_1 p^*} |u|^{p^*} dx\right)^{\frac{p}{p^*}}} \right\}.$$

Let us consider Ω a open domain of $\mathbb{R}^N$ (not necessarily bounded), $0 \in \Omega$, $1 < p < N$, $0 \leq a \leq (N-p)/p$, $a \leq e_1 < a+1$, $d_1 = 1 + a - e_1$, and $p^* = Np/(N - d_1 p)$. We define the space

$$W^{1,p}_{a,e_1}(\Omega) = \left\{u \in L^{p^*}(\Omega, |x|^{-e_1 p^*}) : |\nabla u| \in L^p(\Omega, |x|^{-ap})\right\},$$

equipped with the norm

$$||u||_{W^{1,p}_{a,e_1}(\Omega)} = ||u||_{L^{p^*}(\Omega, |x|^{-e_1 p^*})} + ||\nabla u||_{L^p(\Omega, |x|^{-ap})}.$$

We consider the best Hardy-Sobolev constant given by

$$\tilde{S}_{a,p} = \inf_{W^{1,p}_{a,e_1}(\mathbb{R}^N)\setminus\{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |x|^{-e_1 p^*} |u|^{p^*} dx\right)^{\frac{p}{p^*}}} \right\}.$$

Also, we define the space

$$R^{1,p}_{a,e_1}(\mathbb{R}^N) = \left\{u \in W^{1,p}_{a,e_1}(\mathbb{R}^N) : u(x) = u(|x|)\right\},$$

endowed with the norm

$$||u||_{R^{1,p}_{a,e_1}(\mathbb{R}^N)} = ||u||_{W^{1,p}_{a,e_1}(\mathbb{R}^N)}.$$

Actually, Horiuchi in [34] proved that

$$\tilde{S}_{a,p,R} = \inf_{R^{1,p}_{a,e_1}(\mathbb{R}^N)\setminus\{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx}{(\int_{\mathbb{R}^N} |x|^{-e_1 p^*}|u|^{p^*}dx)^{\frac{p}{p^*}}} \right\} = \tilde{S}_{a,p},$$

and it is achieved by functions of the form

$$y_\epsilon(x) := k_{a,p}(\epsilon)U_{a,p,\epsilon}(x), \forall \epsilon > 0,$$

where

$$k_{a,p}(\epsilon) = c_0 \epsilon^{(N-d_1p)/d_1p^2} \text{ and } U_{a,p,\epsilon}(x) = \left(\epsilon + |x|^{\frac{d_1 p(N-p-ap)}{(p-1)(N-d_1p)}}\right)^{-\left(\frac{N-d_1p}{d_1p}\right)}.$$

Moreover, y_ϵ satisfies

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla y_\epsilon|^p dx = \int_{\mathbb{R}^N} |x|^{-e_1p^*}|y_\epsilon|^{p^*}dx. \tag{7}$$

See also Clément, Figueiredo and Mitidieri [26, Proposition 1.4].

The next lemma can be proved arguing as in [10], more exactly, see [11, Lemma 5.1].

Lemma 2.2 *Suppose that Ω is a smooth bounded domain of $\mathbb{R}^N$, $0 \in \Omega$, $1 < p < N$, $-\infty < a < (N-p)/p$, $a \le e_1 < a+1$, $d_1 = 1 + a - e_1$, $p^* = Np/(N-d_1p)$, and $c > 0$. Let $R_0 \in (0,1)$ be such that $B(0, 2R_0) \subset \Omega$ and $\psi \in C_0^\infty(B(0, 2R_0))$ with $\psi \ge 0$ and $\psi \equiv 1$ in $B(0, R_0)$, then the function defined by*

$$u_\epsilon(x) = \frac{\psi(x)U_{a,p,\epsilon}(x)}{||\psi U_{a,p,\epsilon}||_{L^{p^*}(\Omega,|x|^{-e_1p^*})}},$$

satisfies

$$||u_\epsilon||^{p^*}_{L^{p^*}(\Omega,|x|^{-e_1p^*})} = 1, \; ||\nabla u_\epsilon||^p_{L^p(\Omega,|x|^{-ap})} \le \tilde{S}_{a,p,R} + O(\epsilon^{(N-d_1p)/d_1p}),$$

and

$$||u_\epsilon||^p_{L^p(\Omega,|x|^{-(a+1)p+c})} \ge \begin{cases} O(\epsilon^{(N-d_1p)/d_1p}) \text{ if } c > \frac{N-p-ap}{p-1}, \\ O(\epsilon^{(N-d_1p)/d_1p}|ln(\epsilon)|) \text{ if } c = \frac{N-p-ap}{p-1}, \\ O(\epsilon^{(N-d_1p)(p-1)c/d_1p(N-p-ap)}) \text{ if } c < \frac{N-p-ap}{p-1}. \end{cases} \tag{8}$$

Definition 2.1 *We say that the pair $(\underline{u}, \underline{v})$, where $\underline{u} \in W_0^{1,p}(\Omega, |x|^{-ap}) \cap L^\infty(\Omega)$*

and $\underline{v} \in W_0^{1,q}(\Omega, |x|^{-bq}) \cap L^\infty(\Omega)$, *is a weak lower-solution of system* (1) *if*

$$\begin{cases} \displaystyle\int_\Omega |x|^{-ap}|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla\phi dx \leq \int_\Omega |x|^{-(a+1)p+c_1} f_1(x,\underline{u},\underline{v})\phi dx, \\ \displaystyle\int_\Omega |x|^{-bq}|\nabla \underline{v}|^{q-2}\nabla \underline{v}\nabla\psi dx \leq \int_\Omega |x|^{-(b+1)q+c_2} f_2(x,\underline{u},\underline{v})\psi dx, \\ \underline{u}, \underline{v} \leq 0 \ \text{on} \ \partial\Omega, \end{cases}$$

for all $\phi \in W_0^{1,p}(\Omega, |x|^{-ap})$ *and* $\psi \in W_0^{1,q}(\Omega, |x|^{-bq})$ *with* $\phi, \psi \geq 0$ *(A function* $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ *is said to be less than or equal to* $w \in W_0^{1,p}(\Omega, |x|^{-ap})$ *on* $\partial\Omega$ *when* $\max\{0, u - w\} \in W_0^{1,p}(\Omega, |x|^{-ap})$*).*

Similarly one defines a weak upper-solution $(\overline{u}, \overline{v})$ of system (1), by considering the reversed inequalities in the above definition.

3 Proof of Theorems 1.1, 1.2 and 1.3

In this section, we will prove theorems 1.1, 1.2 and 1.3. Our main tool will be a general method of lower and upper-solution. This method, in the scalar situation, has been used by many authors, for instance, [19] and [22]. The proof for the system case can be found in [28]. The proofs of our results above mentioned follow exploring some arguments used in [28], so for the sake of completeness we are only going to give the sketch of their proofs.

3.1 Proof of Theorem 1.1

First of all, we will prove that system (1) possesses a weak upper-solution, where each component is positive, for each $\lambda > 0$ fixed.

By combining [28, Lemma 2.1, Theorem 2.1] with [22, Theorem 2.1, Theorem 3.1] and a regularity result of [35], we can choose $e_i \in C^{0,\rho_i}(\overline{\Omega}) \cap C^{1,\alpha_i}(\Omega \setminus \{0\})$, where (e_1, e_2) is a weak solution of system (1) with $g_1(x,u,v) = \frac{1}{\lambda}|x|^{-(a+1)p+c_1}$ and $g_2(x,u,v) = \frac{1}{\lambda}|x|^{-(b+1)q+c_2}$, for some $\rho_i, \alpha_i > 0$ and $i = 1, 2$, and also each component is positive.

Define

$$(z_{1c}(x), z_{2c}(x)) := \left(c\,\mu^{-1}\lambda^{\frac{1}{p-1}} e_1(x), [\lambda f_2(c\lambda^{\frac{1}{p-1}})]^{\frac{1}{q-1}} e_2(x)\right),$$

where $\mu := \max\{||e_1||_\infty, ||e_2||_\infty\}$ and c is a positive constant that will be fixed later. Notice that $z_{ic} \in C^{0,\rho_i}(\overline{\Omega}) \cap C^{1,\alpha_i}(\overline{\Omega} \setminus \{0\})$ for $i = 1, 2$.

Let $\phi_1 \in W_0^{1,p}(\Omega, |x|^{-ap})$ with $\phi_1 \geq 0$. Then, we obtain

$$\int_\Omega |x|^{-ap}|\nabla z_{1c}|^{p-2}\nabla z_{1c}\nabla\phi_1 dx = \lambda\left(\tfrac{c}{\mu}\right)^{p-1}\int_\Omega |x|^{-(a+1)p+c_1}\phi_1 dx. \tag{9}$$

By monotonicity condition on f_1 and from (H_1), there exists c_0 a large enough positive constant such that

$$\lambda c^{p-1} \geq \lambda\mu^{p-1}f_1(\mu[\lambda f_2(c\lambda^{\frac{1}{p-1}})]^{\frac{1}{q-1}}) \geq \lambda\mu^{p-1}f_1(z_{2c}(x)),\ \forall x \in \Omega,\ c \geq c_0. \tag{10}$$

Consequently, from (9) and (10), we achieve

$$\int_\Omega |x|^{-ap}|\nabla z_{1c}|^{p-2}\nabla z_{1c}\nabla\phi_1 dx \geq \lambda\int_\Omega |x|^{-(a+1)p+c_1}f_1(z_{2c})\phi_1 dx,\ \forall c \geq c_0.$$

Since f_2 is monotone, for every $\phi_2 \in W_0^{1,q}(\Omega, |x|^{-bq})$ with $\phi_2 \geq 0$, we have

$$\begin{aligned}\int_\Omega |x|^{-bq}|\nabla z_{2c}|^{q-2}\nabla z_{2c}\nabla\phi_2 dx &= \lambda f_2\left(c\lambda^{\frac{1}{p-1}}\right)\int_\Omega |x|^{-(b+1)q+c_2}\phi_2 dx\\ &\geq \lambda\int_\Omega |x|^{-(b+1)q+c_2}f_2\left(c\lambda^{\frac{1}{p-1}}\mu^{-1}e_1\right)\phi_2 dx\\ &= \lambda\int_\Omega |x|^{-(b+1)q+c_2}f_2\left(z_{1c}\right)\phi_2 dx.\end{aligned}$$

Therefore, (z_{1c}, z_{2c}) is a weak upper-solution of system (1) for each $c \geq c_0$ and $\lambda > 0$ fixed. Moreover, since $\lim_{s\to\infty} f_2(s) = \infty$ we get $z_{ic} > 0$ in Ω, for $i = 1, 2$, by taking c sufficiently large.

Now, we will prove that system (1) possesses a weak lower-solution, where each component is positive, for each $\lambda \geq \lambda_0$ with $\lambda_0 > 0$ large enough.

Applying [28, Theorem 2.3], we can take $\lambda_1 > 0$ and $\varphi_1 \in C^0(\overline{\Omega})\cap C^1(\overline{\Omega}\setminus\{0\})$ the eigenvalue and eigenfunction, respectively, of the eigenvalue problem

$$Lu_{ap} = \lambda|x|^{-(a+1)p+c_1}|u|^{p-2}u \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \tag{11}$$

satisfying $\varphi_1 > 0$ in Ω and $|\nabla\varphi_1| \geq \sigma_1$ on $\partial\Omega$ for some $\sigma_1 > 0$. Changing p, a, and c_1 by q, b, and c_2, respectively, we get $\lambda_2 > 0$ and $\varphi_2 \in C^0(\overline{\Omega})\cap C^1(\overline{\Omega}\setminus\{0\})$ the eigenvalue and eigenfunction, respectively, of the above problem, satisfying $\varphi_2 > 0$ in Ω and $|\nabla\varphi_2| \geq \sigma_2$ on $\partial\Omega$ for some $\sigma_2 > 0$. Also, we can suppose that $||\varphi_i||_\infty = 1$, $i = 1, 2$. Furthermore, it is easy to prove that there exist $m, \eta > 0$ such that

$$\begin{aligned}&|x|^{-ap}|\nabla\varphi_1|^p - \lambda_1|x|^{-(a+1)p+c_1}\varphi_1^p \geq m,\\ &|x|^{-bq}|\nabla\varphi_2|^q - \lambda_2|x|^{-(b+1)q+c_2}\varphi_2^q \geq m,\end{aligned} \tag{12}$$

in $\Omega_\eta := \{x \in \Omega : dist(x, \partial\Omega) \leq \eta\}$.

Since f_i is continuous and nondecreasing, we have $f_i(s) \geq -k_0$ for all $s \geq 0$, $i = 1, 2$, and for some $k_0 > 0$. Choose $r > 0$ such that $r \leq |x|^{-(a+1)p+c_1}, |x|^{-(b+1)q+c_2}$ in Ω_η. Define

$$(\Psi_{1_\lambda}(x), \Psi_{2_\lambda}(x)) := \left((\tfrac{\lambda k_0 r}{m})^{\frac{1}{p-1}} (\tfrac{p-1}{p}) \varphi_1^{\frac{p}{p-1}}(x), (\tfrac{\lambda k_0 r}{m})^{\frac{1}{q-1}} (\tfrac{q-1}{q}) \varphi_2^{\frac{q}{q-1}}(x) \right),$$

where each component belongs to $C^0(\overline{\Omega})$.

Then, for $h_1 \in W_0^{1,p}(\Omega, |x|^{-ap})$ with $h_1 \geq 0$, we reach

$$\begin{aligned} &\int_\Omega |x|^{-ap} |\nabla \Psi_{1_\lambda}|^{p-2} \nabla \Psi_{1_\lambda} \nabla h_1 dx \\ &\quad = \left(\tfrac{\lambda k_0 r}{m}\right) \int_\Omega |x|^{-ap} |\nabla \varphi_1|^{p-2} \nabla \varphi_1 [\nabla(\varphi_1 h_1) - (\nabla \varphi_1) h_1] dx \\ &\quad = \left(\tfrac{\lambda k_0 r}{m}\right) \int_\Omega \left[\lambda_1 |x|^{-(a+1)p+c_1} \varphi_1^p - |x|^{-ap} |\nabla \varphi_1|^p \right] h_1 dx. \end{aligned} \tag{13}$$

Since $\Psi_{i_\lambda} \geq 0$ in Ω, for $i = 1, 2$, it follows that

$$-k_0 r \leq |x|^{-(a+1)p+c_1} f_1(\Psi_{2_\lambda}(x)), \ \forall x \in \Omega_\eta. \tag{14}$$

Hence, by (12) and (14) we get

$$\begin{aligned} &\left(\tfrac{\lambda k_0 r}{m}\right) \int_{\Omega_\eta} \left[\lambda_1 |x|^{-(a+1)p+c_1} \varphi_1^p - |x|^{-ap} |\nabla \varphi_1|^p \right] h_1 dx \\ &\quad \leq -\lambda k_0 r \int_{\Omega_\eta} h_1 dx \\ &\quad \leq \lambda \int_{\Omega_\eta} |x|^{-(a+1)p+c_1} f_1(\Psi_{2_\lambda}(x)) h_1 dx. \end{aligned} \tag{15}$$

There exists $\mu > 0$ such that $\varphi_i(x) \geq \mu$ in $\Omega \setminus \Omega_\eta$, for $i = 1, 2$. Therefore

$$\Psi_{2_\lambda}(x) \geq (\tfrac{\lambda k_0 r}{m})^{\frac{1}{q-1}} (\tfrac{q-1}{q}) \mu^{\frac{q}{q-1}} \longrightarrow \infty, \tag{16}$$

as $\lambda \to \infty$, uniformly in $x \in \Omega \setminus \Omega_\eta$.

By (16) and $\lim_{s \to \infty} f_1(s) = \infty$, we can find $\lambda_0 > 0$ sufficiently large such that

$$\tfrac{\lambda_1 k_0 r}{m} \varphi_1^p(x) \leq \tfrac{\lambda_1 k_0 r}{m} \leq f_1(\Psi_{2_\lambda}(x)), \forall x \in \Omega \setminus \Omega_\eta, \ \lambda \geq \lambda_0. \tag{17}$$

Hence, from (17), for each $\lambda \geq \lambda_0$ we have

$$\begin{aligned} &\left(\tfrac{\lambda k_0 r}{m}\right) \int_{\Omega\setminus\Omega_\eta} \left[\lambda_1 |x|^{-(a+1)p+c_1}\varphi_1^p - |x|^{-ap}|\nabla\varphi_1|^p\right] h_1 dx \\ &\quad \leq \lambda \int_{\Omega\setminus\Omega_\eta} |x|^{-(a+1)p+c_1} \left(\tfrac{\lambda_1 k_0 r}{m}\right) \varphi_1^p h_1 dx \\ &\quad \leq \lambda \int_{\Omega\setminus\Omega_\eta} |x|^{-(a+1)p+c_1} f_1(\Psi_{2_\lambda}) h_1 dx. \end{aligned} \tag{18}$$

Then, using (13), (15), and (18), we conclude

$$\int_\Omega |x|^{-ap}|\nabla\Psi_{1_\lambda}|^{p-2}\nabla\Psi_{1_\lambda}\nabla h_1 dx \leq \lambda \int_\Omega |x|^{-(a+1)p+c_1} f_1(\Psi_{2_\lambda}) h_1 dx$$

and similarly

$$\int_\Omega |x|^{-bq}|\nabla\Psi_{2_\lambda}|^{q-2}\nabla\Psi_{2_\lambda}\nabla h_2 dx \leq \lambda \int_\Omega |x|^{-(b+1)q+c_2} f_2(\Psi_{1_\lambda}) h_2 dx,$$

for all $h_1 \in W_0^{1,p}(\Omega, |x|^{-ap})$, $h_2 \in W_0^{1,q}(\Omega, |x|^{-bq})$, with $h_1, h_2 \geq 0$ and each $\lambda \geq \lambda_0$.

Consequently, $(\Psi_{1_\lambda}, \Psi_{2_\lambda}) \in C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ is a weak lower-solution of system (1), where each component is positive, for each $\lambda \geq \lambda_0$ with $\lambda_0 > 0$ large enough.

We observe that $\Psi_{i_\lambda} \leq z_{ic}$ in Ω for some $c > 0$ sufficiently large with $\lambda \geq \lambda_0$ fixed and $i = 1, 2$. Thus, by [28, Theorem 2.4] there exists a weak solution of system (1), where each component is positive a.e. in Ω. ■

3.2 Proof of Theorem 1.2

Firstly, we will prove that system (1) possesses a positive weak upper-solution for each $\lambda > 0$ fixed. By choosing (e_1, e_2) as in the beginning of the proof of theorem 1.1, we define

$$(z_{1_A}(x), z_{2_B}(x)) := (Ae_1(x), Be_2(x)),$$

where A and B are positive constants that will be fixed later. We observe that $z_{1_A} \in C^{0,\rho_1}(\overline{\Omega}) \cap C^{1,\alpha_1}(\Omega \setminus \{0\})$ and $z_{2_B} \in C^{0,\rho_2}(\overline{\Omega}) \cap C^{1,\alpha_2}(\Omega \setminus \{0\})$ for some $\alpha_i, \rho_i > 0$, for $i = 1, 2$.

Let $\phi_1 \in W_0^{1,p}(\Omega, |x|^{-ap})$ with $\phi_1 \geq 0$. Then, we infer that

$$\int_\Omega |x|^{-ap}|\nabla z_{1_A}|^{p-2}\nabla z_{1_A}\nabla\phi_1 dx = A^{p-1}\int_\Omega |x|^{-(a+1)p+c_1}\phi_1 dx. \tag{19}$$

Since f_1 is nondecreasing in the variables s, t and continuous in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$, we have

$$-k_0 \leq f_1(x, Ae_1(x), Be_2(x)) \leq f_1(x, A||e_1||_\infty, B||e_2||_\infty),$$

for some $k_0 > 0$ and every $x \in \Omega$. Using this inequality and the hypothesis (H_2), we obtain A_0 a positive real number such that

$$\lambda f_1(x, Ae_1(x), Be_2(x)) \leq A^{p-1}, \forall A \geq A_0, (x, B) \in \Omega \times \mathbb{R}^+. \tag{20}$$

Then, from (19) and (20), we obtain

$$\int_\Omega |x|^{-ap} |\nabla z_{1_A}|^{p-2} \nabla z_{1_A} \nabla \phi_1 dx \geq \lambda \int_\Omega |x|^{-(a+1)p+c_1} f_1(x, z_{1_A}, z_{2_B}) \phi_1 dx,$$

for all $\phi_1 \in W_0^{1,p}(\Omega, |x|^{-ap})$ with $\phi_1 \geq 0$, for each $A \geq A_0$, and all $B \in \mathbb{R}^+$.

Similarly, we have

$$\int_\Omega |x|^{-bq} |\nabla z_{2_B}|^{q-2} \nabla z_{2_B} \nabla \phi_2 dx \geq \lambda \int_\Omega |x|^{-(b+1)q+c_2} f_2(x, z_{1_A}, z_{2_B}) \phi_2 dx,$$

for all $\phi_2 \in W_0^{1,q}(\Omega, |x|^{-bq})$ with $\phi_2 \geq 0$, for all $B \geq B_0$, $A \in \mathbb{R}^+$, and some $B_0 > 0$. Therefore, (z_{1A}, z_{2B}) is a weak upper-solution of system (1), where each component is positive, for each $A \geq A_0$ and $B \geq B_0$.

The proof of the existence of a weak lower-solution for system (1), where each component is positive, is completely similar to the one made in theorem 1.1. That is, there exists a pair of the functions $(\Psi_{1_\lambda}, \Psi_{2_\lambda}) \in C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ which is a weak lower-solution of system (1), where each component is positive, for each $\lambda \geq \lambda_0$ with $\lambda_0 > 0$ large enough.

Now, observing that $\Psi_{1_\lambda} \leq z_{1_A}$ and $\Psi_{2_\lambda} \leq z_{2_B}$ in Ω for some $A \geq A_0$ and $B \geq B_0$ sufficiently large, and each $\lambda \geq \lambda_0$ fixed, we obtain by [28, Theorem 2.4] a weak solution of system (1), where each component is positive a.e. in Ω. ■

3.3 Proof of Theorem 1.3

We will prove this result by contradiction. As in the proof of theorem 1.1, let λ_1 and λ_2 be the eigenvalues of problem (11). We consider $\lambda_0 := \min\{\frac{\lambda_1}{k_1+k_3}, \frac{\lambda_2}{k_2+k_4}\}$. Supposing by contradiction that there exists a weak solution (u, v) of system (1), where each component is nontrivial, with $0 <$

$\lambda < \lambda_0$, then by characterization of λ_i, $i = 1, 2$, we achieve

$$\begin{aligned}\lambda_1 \int_\Omega |x|^{-(a+1)p+c_1} |u|^p dx &\le \int_\Omega |x|^{-ap} |\nabla u|^p dx \\ &\le \lambda \int_\Omega |x|^{-(a+1)p+c_1} \left(k_1 |u|^p + k_2 |v|^q\right) dx\end{aligned} \tag{21}$$

and similarly

$$\lambda_2 \int_\Omega |x|^{-(b+1)q+c_2} |v|^q dx \le \lambda \int_\Omega |x|^{-(b+1)q+c_2} \left(k_3 |u|^p + k_4 |v|^q\right) dx. \tag{22}$$

Since $(a+1)p - c_1 = (b+1)q - c_2$, from (21) and (22) we get

$$\begin{aligned}0 < [\lambda_1 - \lambda (k_1 + k_3)] &\int_\Omega |x|^{-(a+1)p+c_1} |u|^p dx \\ + [\lambda_2 - \lambda (k_2 + k_4)] &\int_\Omega |x|^{-(b+1)q+c_2} |v|^q dx \le 0,\end{aligned}$$

which is a contradiction. ■

4 Proof of Theorem 1.4

In this section and in the next, we will prove theorem 1.4 and theorem 1.5, respectively, and our approach will be the variational techniques. The critical points of the Euler-Lagrange functional $I : W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq}) \to \mathbb{R}$ given by

$$\begin{aligned}I(u, v) = &\frac{1}{p} \int_\Omega |x|^{-ap} |\nabla u|^p dx + \frac{1}{q} \int_\Omega |x|^{-bq} |\nabla v|^q dx \\ &- \lambda \int_\Omega |x|^{-\beta_1} u_+^\theta v_+^\delta dx - \mu \int_\Omega |x|^{-\beta_2} u_+^\alpha v_+^\gamma dx,\end{aligned}$$

are precisely the weak solutions of system (2). We recall that $I \in C^1$ with Gâteaux derivative given by

$$\begin{aligned}\langle I'(u, v), (w, z)\rangle = &\int_\Omega |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla w dx + \int_\Omega |x|^{-bq} |\nabla v|^{q-2} \nabla v \nabla z dx \\ &- \lambda\theta \int_\Omega |x|^{-\beta_1} u_+^{\theta-1} v_+^\delta w dx - \lambda\delta \int_\Omega |x|^{-\beta_1} u_+^\theta v_+^{\delta-1} z dx \\ &- \mu\alpha \int_\Omega |x|^{-\beta_2} u_+^{\alpha-1} v_+^\gamma w dx - \mu\gamma \int_\Omega |x|^{-\beta_2} u_+^\alpha v_+^{\gamma-1} z dx,\end{aligned}$$

where $u_\pm = \max\{0, \pm u\} \in W_0^{1,p}(\Omega, |x|^{-ap})$ (Similarly $v_\pm = \max\{0, \pm v\} \in W_0^{1,q}(\Omega, |x|^{-bq})$) (see [36]).

4.1 Case (3)

Consider $(u_0, v_0) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ with $u_{0+}.v_{0+} \not\equiv 0$, then it is easy to see that

$$I(t^{\frac{1}{p}}u_0, t^{\frac{1}{q}}v_0) \longrightarrow -\infty \text{ as } t \to \infty, \forall \lambda \geq 0,\ \mu > 0.$$

By choosing $p_2 \in (p, p^*)$ and $q_2 \in (q, q^*)$ such that $\alpha/p_2 + \gamma/q_2 = 1$ and applying the Young's inequality, we achieve for $(u, v) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ that

$$I(u,v) \geq (\tfrac{1}{p} - \lambda\tfrac{\theta C}{p})||u||^p + (\tfrac{1}{q} - \lambda\tfrac{\delta C}{q})||v||^q - \mu\tfrac{\alpha C^{p_2/p}}{p_2}||u||^{p_2} - \mu\tfrac{\gamma C^{q_2/q}}{q_2}||v||^{q_2}.$$

Taking $\lambda_0 > 0$ such that $\min\{1 - \lambda\theta C, 1 - \lambda\delta C\} > 0$ for all $0 \leq \lambda < \lambda_0$, it follows that for each $\mu > 0$ and $0 \leq \lambda < \lambda_0$, we can get $\rho, \sigma \in (0, 1)$ satisfying

$$I(u, v) \geq \sigma \text{ if } ||(u, v)|| = \rho.$$

Applying the mountain pass theorem [32], there exists a Palais Smale sequence ($(PS)_c$ sequence, in short) $\{(u_n, v_n)\}$ in $W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ for the operator I at the level c, that is, $I(u_n, v_n) \to c$ and $I^{'}(u_n, v_n) \to 0$, as $n \to \infty$, where

$$0 < \sigma \leq c = \inf_{h\in\Gamma} \max_{t\in[0,1]} I(h(t))$$

and

$$\Gamma = \{h \in C([0, 1], W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})) : h(0) = 0, h(1) = e\},$$

with $I(e) \equiv I(t_0u_0, t_0v_0) < 0$.

By taking p_2, q_2 as above and applying the Young's inequality, we obtain

$$\begin{aligned} c + ||(u_n, v_n)|| + O_n(1) &\geq I(u_n, v_n) - \langle I'(u_n, v_n), (u_n/p_2, v_n/q_2)\rangle \\ &\geq [(\tfrac{1}{p} - \tfrac{1}{p_2}) + \lambda(\tfrac{\theta}{p_2} + \tfrac{\delta}{q_2} - 1)\tfrac{\theta C}{p}]||u_n||^p \\ &\quad +[(\tfrac{1}{q} - \tfrac{1}{q_2}) + \lambda(\tfrac{\theta}{p_2} + \tfrac{\delta}{q_2} - 1)\tfrac{\delta C}{q}]||v_n||^q, \end{aligned}$$

where $O_n(1) \to 0$ as $n \to \infty$.

Therefore, changing λ_0 by another small constant, if necessary, we obtain that $\{(u_n, v_n)\}$ is bounded in $W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$. In particular, we have that $\{(u_{n_+}, v_{n_+})\}$ and $\{(u_{n_-}, v_{n_-})\}$ are bounded. We infer that $\{(u_{n_+}, v_{n_+})\}$ is also a $(PS)_c$ sequence. In fact, notice that

$$O_n(1) = \left\langle I^{'}(u_n, v_n), (u_{n_-}, 0)\right\rangle = -\int_\Omega |x|^{-ap}|\nabla u_{n_-}|^p dx$$

and

$$O_n(1) = -\int_\Omega |x|^{-bq}|\nabla v_{n_-}|^q dx,$$

where $O_n(1) \to 0$ as $n \to \infty$. Moreover, we get

$$I(u_{n+}, v_{n+}) = I(u_n, v_n) + O_n(1),$$

therefore, it follows that $I(u_{n+}, v_{n+}) \longrightarrow c$ as $n \to \infty$.

Similarly, we prove that if $(w, z) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$, then

$$\left\langle I^{'}(u_{n+}, v_{n+}), (w, z)\right\rangle = \left\langle I^{'}(u_n, v_n), (w, z)\right\rangle + O_n(1),$$

hence $I^{'}(u_{n+}, u_{n+}) \to 0$ as $n \to \infty$. Then, we can consider $u_n, v_n \geq 0$.

Due to the boundedness of this sequence, there exists (u, v) such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$ and $v_n \rightharpoonup v$ weakly in $W_0^{1,q}(\Omega, |x|^{-bq})$, as $n \to \infty$. We can assume that, by passing to a subsequence if necessary, $u_n \to u$ in $L^p(\Omega, |x|^{-\beta_1}) \cap L^{p_2}(\Omega, |x|^{-\beta_2})$ and $v_n \to v$ in $L^q(\Omega, |x|^{-\beta_1}) \cap L^{q_2}(\Omega, |x|^{-\beta_2})$, as $n \to \infty$ (see Xuan [11,31]). Thus, we have $u, v \geq 0$ and, using the Lebesgue's dominated convergence theorem, it is not hard to check that

$$\lim_{n\to\infty} \int_\Omega |x|^{-\beta_1} u_n^{\theta-1} v_n^\delta (u_n - u)dx = \lim_{n\to\infty} \int_\Omega |x|^{-\beta_2} u_n^{\alpha-1} v_n^\gamma (u_n - u)dx = 0.$$

Also, since $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla(u_n - u)dx = 0.$$

Consequently, we achieve

$$\begin{aligned}
&\limsup_{n\to\infty} \int_\Omega |x|^{-ap} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right) \nabla(u_n - u)dx \\
&\quad = \limsup_{n\to\infty} \Big\{ \left\langle I^{'}(u_n, v_n), (u_n - u, 0)\right\rangle + \lambda\theta \int_\Omega |x|^{-\beta_1} u_n^{\theta-1} v_n^\delta (u_n - u)dx \\
&\qquad + \mu\alpha \int_\Omega |x|^{-\beta_2} u_n^{\alpha-1} v_n^\gamma (u_n - u)dx - \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla(u_n - u)dx \Big\} \\
&\quad = 0,
\end{aligned}$$

therefore, using a well known lemma, for instance, [22, Lemma 3.1] or [9, Lemma 4.1], we get $u_n \to u$ in $W_0^{1,p}(\Omega, |x|^{-ap})$ as $n \to \infty$. Analogously, we have $v_n \to v$ in $W_0^{1,q}(\Omega, |x|^{-bq})$ as $n \to \infty$. Thus, we conclude $I(u_n, v_n) \to I(u, v) = c > 0$ and $I^{'}(u_n, v_n) \to I^{'}(u, v) \equiv 0$, as $n \to \infty$, that is, (u, v) is a weak solution of system (2), where each component is nonnegative. Moreover, it is easy check that $u, v \not\equiv 0$. ■

4.2 Case (4)

In this case, choose $p_2 \in (1,p)$ and $q_2 \in (1,q)$ such that $\alpha/p_2 + \gamma/q_2 = 1$. Arguing as in the case (3) we can see that $I(u,v) \longrightarrow \infty$ as $||(u,v)|| \longrightarrow \infty$, for all $\mu > 0$ and $0 \leq \lambda < \lambda_0$ with $\lambda_0 > 0$ sufficiently small.

Also, for each $\lambda \geq 0$ and $\mu > 0$, if $(u_0, v_0) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ with $u_{0+}.v_{0+} \not\equiv 0$, there exists $t_0 \in (0,1)$ such that $I(t_0^{1/p}u_0, t_0^{1/q}v_0) < 0$, then

$$-\infty < M := \inf\{I(u,v) : (u,v) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})\} < 0.$$

Applying the Ekeland's variational principle [37, Corollary 5.3], we obtain a sequence $\{(u_n, v_n)\} \subset W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$ satisfying $I(u_n, v_n) \rightarrow M < 0$ and $I^{'}(u_n, v_n) \rightarrow 0$, as $n \rightarrow \infty$.

Therefore, by using p_2, q_2 as before, we obtain

$$\begin{aligned} M + ||(u_n, v_n)|| + O_n(1) \geq\ & [(\tfrac{1}{p} - \tfrac{1}{p^*}) + \lambda(\tfrac{\theta}{p^*} + \tfrac{\delta}{q^*} - 1)\tfrac{\theta C}{p}]||u_n||^p \\ & +[(\tfrac{1}{q} - \tfrac{1}{q^*}) + \lambda(\tfrac{\theta}{p^*} + \tfrac{\delta}{q^*} - 1)\tfrac{\delta C}{q}]||v_n||^q \\ & +\mu(\tfrac{\alpha}{p^*} + \tfrac{\gamma}{q^*} - 1)(\tfrac{\alpha C^{p_2/p}}{p_2}||u_n||^{p_2} + \tfrac{\gamma C^{q_2/q}}{q_2}||v_n||^{q_2}). \end{aligned}$$

Then, changing λ_0 by another small constant, if necessary, we obtain that $\{(u_n, v_n)\}$ is bounded in $W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}(\Omega, |x|^{-bq})$. Now, proceeding as in the case (3) we conclude that system (2) possesses a weak solution, where each component is nontrivial and nonnegative. ■

5 Proof of Theorem 1.5

The geometric conditions of the mountain pass theorem [32] follow as in the case (3), that is, if $(u_0, v_0) \in (W_0^{1,p}(\Omega, |x|^{-ap}))^2$ with $u_{0+}.v_{0+} \not\equiv 0$, then

$$I(t^{\frac{1}{p}}u_0, t^{\frac{1}{p}}v_0) \longrightarrow -\infty \text{ as } t \rightarrow \infty,$$

for all $\lambda \geq 0$ and $\mu > 0$. Also, there exist $\rho, \sigma \in (0,1)$ such that

$$I(u,v) \geq \sigma \text{ if } ||(u,v)|| = \rho.$$

Claim. Let us consider $s_0 = s_1(s_1^{\alpha}t_1^{\gamma})^{\frac{-1}{p^*}}$ and $t_0 = t_1(s_1^{\alpha}t_1^{\gamma})^{\frac{-1}{p^*}}$, where $s_1, t_1 > 0$, and $s_1/t_1 = (\alpha/\gamma)^{1/p}$ as in lemma 2.1, and u_ϵ is the function defined in lemma

2.2. Then, there exists $\epsilon > 0$ such that

$$\sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) < (\tfrac{1}{p} - \tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}}.$$

Indeed. Notice that for each $\epsilon > 0$ there exists $t_\epsilon > 0$ such that

$$0 < \sigma \leq \sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) = I(t_\epsilon(s_0u_\epsilon), t_\epsilon(t_0u_\epsilon)).$$

Moreover, supposing by contradiction that there exist a subsequence $\{t_{\epsilon_n}\}$ with $t_{\epsilon_n} \to 0$ as $n \to \infty$ and by combining with lemma 2.2, we obtain

$$\begin{aligned} 0 < \sigma &\leq I(t_{\epsilon_n}(s_0u_{\epsilon_n}), t_{\epsilon_n}(t_0u_{\epsilon_n})) \\ &\leq \tfrac{t_{\epsilon_n}^p s_0^p}{p}||u_{\epsilon_n}||^p + \tfrac{t_{\epsilon_n}^p t_0^p}{p}||u_{\epsilon_n}||^p \\ &\leq \tfrac{t_{\epsilon_n}^p}{p}(s_0^p + t_0^p)(\tilde{S}_{a,p,R} + O(\epsilon_n^{(N-d_1p)/d_1p})) \longrightarrow 0, \end{aligned}$$

as $n \to \infty$, which is a contradiction. Then, we have $l > 0$ with $t_\epsilon \geq l$, for all $\epsilon > 0$. Consequently, we get

$$\begin{aligned} \sup_{t\geq 0} I(t(s_0u_\epsilon), t(t_0u_\epsilon)) &\leq \tfrac{(s_1^p+t_1^p)t_\epsilon^p}{(s_1^\alpha t_1^\gamma)^{p/p^*}p}||u_\epsilon||^p - \mu\tfrac{s_1^\alpha t_1^\gamma t_\epsilon^{p^*}}{(s_1^\alpha t_1^\gamma)^{(\alpha+\gamma)/p^*}}\int_\Omega |x|^{-e_1p^*}u_\epsilon^{p^*}dx \\ &\quad -\lambda l^{\theta+\delta}s_0^\theta t_0^\delta\int_\Omega |x|^{-\beta_1}u_\epsilon^{\theta+\delta}dx \\ &= \tfrac{(s_1^p+t_1^p)t_\epsilon^p}{(s_1^\alpha t_1^\gamma)^{p/p^*}p}||u_\epsilon||^p - \mu t_\epsilon^{p^*} - \lambda l^p s_0^\theta t_0^\delta\int_\Omega |x|^{-\beta_1}u_\epsilon^p dx. \end{aligned} \tag{23}$$

Let $f_\epsilon : (0,\infty) \to \mathbb{R}$ be given by

$$f_\epsilon(t) = \tfrac{(s_1^p+t_1^p)\,t^p}{(s_1^\alpha t_1^\gamma)^{p/p^*}p}||u_\epsilon||^p - \mu t^{p^*}.$$

Thus

$$t_{1_\epsilon} = (\mu p^*)^{\frac{-1}{p^*-p}}\left(\tfrac{s_1^p+t_1^p}{(s_1^\alpha t_1^\gamma)^{p/p^*}}\right)^{\frac{1}{p^*-p}}||u_\epsilon||^{\frac{p}{p^*-p}} \tag{24}$$

is the unique maximum point of f.

We know that

$$(A+B)^k \leq A^k + k(A+B)^{k-1}B, \tag{25}$$

for all $A, B \geq 0$ and $k \geq 1$ (See [38]). Also observe, that

$$\left[\tfrac{s_1^p+t_1^p}{(s_1^\alpha t_1^\gamma)^{p/p^*}}\right] = \left[(\alpha/\gamma)^{\gamma/p^*} + (\alpha/\gamma)^{-\alpha/p^*}\right]. \tag{26}$$

By the Caffarelli-Kohn-Nirenberg's inequality [33] we have $W_0^{1,p}(\Omega, |x|^{-ap}) \subset W_{a,e_1}^{1,p}(\mathbb{R}^N)$, then

$$\tilde{S}_{a,p} \leq C_{a,p}^*. \tag{27}$$

Substituting (24) in (23), from (25), (26), (27), and using lemma 2.2, we obtain

$$\begin{aligned}\sup_{t\geq 0} I(t(s_0u_\epsilon),t(t_0u_\epsilon)) &\leq (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\Big\{[(\tfrac{\alpha}{\gamma})^{\frac{\gamma}{p^*}}+(\tfrac{\alpha}{\gamma})^{\frac{-\alpha}{p^*}}]\tilde{S}_{a,p,R}\\ &\quad +O(\epsilon^{\frac{N-d_1p}{d_1p}})\Big\}^{\frac{p^*}{p^*-p}}-\lambda l^p s_0^\theta t_0^\delta\int_\Omega |x|^{-\beta_1}u_\epsilon^p dx\\ &\leq (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\Big\{[(\tfrac{\alpha}{\gamma})^{\frac{\gamma}{p^*}}+(\tfrac{\alpha}{\gamma})^{\frac{-\alpha}{p^*}}]\tilde{S}_{a,p}\Big\}^{\frac{p^*}{p^*-p}}\\ &\quad +O(\epsilon^{\frac{N-d_1p}{d_1p}})-\lambda l^p s_0^\theta t_0^\delta\int_\Omega |x|^{-\beta_1}u_\epsilon^p dx\\ &\leq (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\Big\{[(\tfrac{\alpha}{\gamma})^{\frac{\gamma}{p^*}}+(\tfrac{\alpha}{\gamma})^{\frac{-\alpha}{p^*}}]C^*_{a,p}\Big\}^{\frac{p^*}{p^*-p}}\\ &\quad +O(\epsilon^{\frac{N-d_1p}{d_1p}})-\lambda l^p s_0^\theta t_0^\delta\int_\Omega |x|^{-\beta_1}u_\epsilon^p dx.\end{aligned}$$

So from lemma 2.1, we get

$$\begin{aligned}\sup_{t\geq 0} I(t(s_0u_\epsilon),t(t_0u_\epsilon)) &\leq (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}}+O(\epsilon^{\frac{N-d_1p}{d_1p}})\\ &\quad -\lambda l^p s_0^\theta t_0^\delta\int_\Omega |x|^{-\beta_1}u_\epsilon^p dx.\end{aligned}\tag{28}$$

Assuming that $c=(N-p-ap)/(p-1)$, from lemma 2.2 and of (28), for each $0<\lambda<\lambda_0$ there exists $\epsilon>0$ sufficiently small such that

$$\begin{aligned}\sup_{t\geq 0} I(t(s_0u_\epsilon),t(t_0u_\epsilon)) &\leq (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}}+O(\epsilon^{\frac{N-d_1p}{d_1p}})-O(\epsilon^{\frac{N-d_1p}{d_1p}}|ln(\epsilon)|)\\ &< (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}}.\end{aligned}$$

Now, considering $c<(N-p-ap)/(p-1)$ we have

$$\tfrac{(N-d_1p)(p-1)c}{d_1p(N-p-ap)}<\tfrac{N-d_1p}{d_1p}.$$

Therefore, by lemma 2.2 and by (28), we can choose $\epsilon>0$ small enough such that

$$\begin{aligned}I(t(s_0u_\epsilon),t(t_0u_\epsilon)) &\leq (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}}+O(\epsilon^{\frac{N-d_1p}{d_1p}})-O(\epsilon^{\frac{(N-d_1p)(p-1)c}{d_1p(N-p-ap)}})\\ &< (\tfrac{1}{p}-\tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}}\tilde{S}^{\frac{p^*}{p^*-p}}.\end{aligned}$$

Thus, we conclude the proof of the claim.

Fix $\epsilon>0$ as in the above claim. Then, by the mountain pass theorem we get a sequence $\{(u_n,v_n)\}\subset(W_0^{1,p}(\Omega,|x|^{-ap}))^2$ satisfying $I(u_n,v_n)\to c$ and

$I'(u_n, v_n) \to 0$, as $n \to \infty$, where

$$0 < \sigma \le c = \inf_{h\in\Gamma} \max_{t\in[0,1]} I(h(t)) < (\tfrac{1}{p} - \tfrac{1}{p^*})(\mu p^*)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}}, \tag{29}$$

and

$$\Gamma = \{h \in C([0,1], (W_0^{1,p}(\Omega, |x|^{-ap}))^2) : h(0) = 0, h(1) = \tilde{e}\},$$

with $I(\tilde{e}) \equiv I(\tilde{t}(s_0 u_\epsilon), \tilde{t}(t_0 u_\epsilon)) < 0$.

As in the proof of the case (3), we have that the sequence $\{(u_n, v_n)\}$ is bounded in $(W_0^{1,p}(\Omega, |x|^{-ap}))^2$ for all $0 \le \lambda < \lambda_0$ and $\mu > 0$. Also, we can suppose $u_n, v_n \ge 0$. Moreover, there exists $(u, v) \in (W_0^{1,p}(\Omega, |x|^{-ap}))^2$ with $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$, as $n \to \infty$. Still, we can assume that $u_n \to u$ and $v_n \to v$ in $L^p(\Omega, |x|^{-\beta_1})$, as $n \to \infty$. Then, by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega |x|^{-\beta_1} u_n^{\theta-1} v_n^{\delta} w dx = \int_\Omega |x|^{-\beta_1} u^{\theta-1} v^{\delta} w dx, \ \forall w \in W_0^{1,p}(\Omega, |x|^{-ap}) \tag{30}$$

and

$$\lim_{n\to\infty} \int_\Omega |x|^{-\beta_1} u_n^{\theta} v_n^{\delta-1} z dx = \int_\Omega |x|^{-\beta_1} u^{\theta} v^{\delta-1} z dx, \ \forall z \in W_0^{1,p}(\Omega, |x|^{-ap}). \tag{31}$$

Since $\{(u_n, v_n)\}$ is bounded in $(W_0^{1,p}(\Omega, |x|^{-ap}))^2$, we have that $\{|\nabla u_n|^{p-2}\nabla u_n\}$ and $\{|\nabla v_n|^{p-2}\nabla v_n\}$ are bounded in $(L^{\frac{p}{p-1}}(\Omega, |x|^{-ap}))^N$. On the other hand, since $\alpha + \gamma = p^*$, by the Hölder's inequality again, we infer that $\{{u_n}^{\alpha-1}{v_n}^{\gamma}\}$ and $\{{u_n}^{\alpha}{v_n}^{\gamma-1}\}$ are bounded in $L^{\frac{p^*}{p^*-1}}(\Omega, |x|^{-e_1 p^*})$.

Thus, as $\nabla u_n(x) \to \nabla u(x)$ and $\nabla v_n(x) \to \nabla v(x)$ for a.e. $x \in \Omega$, these facts can be proved arguing as in [39] (see also, [9,21,40]), by using a result in [41, Lemma 4.8], we get, as $n \to \infty$,

$$\nabla u_n \rightharpoonup \nabla u \text{ and } \nabla v_n \rightharpoonup \nabla v \text{ weakly in } (L^{\frac{p}{p-1}}(\Omega, |x|^{-ap}))^N \tag{32}$$

and

$$u_n^{\alpha} v_n^{\gamma-1} \rightharpoonup u^{\alpha} v^{\gamma-1} \text{ and } u_n^{\alpha-1} v_n^{\gamma} \rightharpoonup u^{\alpha-1} v^{\gamma} \text{ weakly in } L^{\frac{p^*}{p^*-1}}(\Omega, |x|^{-e_1 p^*}). \tag{33}$$

Consequently, using $(30) - (33)$ we achieve

$$\langle I'(u, v), (w, z)\rangle = \lim_{n\to\infty} \langle I'(u_n, v_n), (w, z)\rangle = 0, \forall (w, z) \in (W_0^{1,p}(\Omega, |x|^{-ap}))^2_,$$

that is, (u, v) is a weak solution of system (2).

We will conclude the proof of this theorem proving that each component of the weak solution (u, v) is nontrivial. Suppose by contradiction that $u(x) = 0$

for a.e. $x \in \Omega$. Using $u_n \to u$ and $v_n \to v$ in $L^p(\Omega, |x|^{-\beta_1})$, as $n \to \infty$ and Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_\Omega |x|^{-\beta_1} u_n^\theta v_n^\delta dx = 0.$$

Then, we obtain

$$0 = \lim_{n\to\infty}\left\langle I'(u_n, v_n), (u_n, 0)\right\rangle = \lim_{n\to\infty}\left(||u_n||^p - \mu\alpha\int_\Omega |x|^{-e_1 p^*} u_n^\alpha v_n^\gamma dx\right)$$

and

$$0 = \lim_{n\to\infty}\left\langle I'(u_n, v_n), (0, v_n)\right\rangle = \lim_{n\to\infty}\left(||v_n||^p - \mu\gamma\int_\Omega |x|^{-e_1 p^*} u_n^\alpha v_n^\gamma dx\right).$$

We can take $l \geq 0$ such that

$$l = \lim_{n\to\infty}\tfrac{||u_n||^p}{\alpha} = \lim_{n\to\infty}\tfrac{||v_n||^p}{\gamma} = \mu\lim_{n\to\infty}\int_\Omega |x|^{-e_1 p^*} u_n^\alpha v_n^\gamma dx.$$

Therefore, we obtain

$$c = \lim_{n\to\infty} I(u_n, v_n) = (\tfrac{\alpha}{p} + \tfrac{\gamma}{p} - 1)l \geq 0. \tag{34}$$

If $l = 0$, then $c = 0$, that is an absurd. Thus, we can suppose that $l > 0$, and by definition of $\tilde{S}$ we have

$$\left(\int_\Omega |x|^{-e_1 p^*} u_n^\alpha v_n^\gamma dx\right)^{\frac{p}{p^*}} \tilde{S} \leq ||u_n||^p + ||v_n||^p, \ \forall n.$$

Hence, taking the limit in the above inequality we get

$$\left(\tfrac{l}{\mu}\right)^{\frac{p}{p^*}} \tilde{S} \leq (\alpha + \gamma)l = p^* l,$$

then

$$l \geq (\mu)^{\frac{-p}{p^*-p}} (p^*)^{\frac{-p^*}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}}. \tag{35}$$

We obtain substituting the equation (35) in (34) that

$$c \geq (\tfrac{\alpha}{p} + \tfrac{\gamma}{p} - 1)(\mu)^{\frac{-p}{p^*-p}} (p^*)^{\frac{-p^*}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}} = (\tfrac{1}{p} - \tfrac{1}{p^*})(p^*\mu)^{\frac{-p}{p^*-p}} \tilde{S}^{\frac{p^*}{p^*-p}},$$

that contradicts the equation (29), therefore $u, v \not\equiv 0$. ■

6 Examples and Remark

Examples: Consider

$$f_1(x,s,t) = \begin{cases} c_1 t^\delta - k_1 & \text{if } t \geq 0, \\ -k_1 & \text{if } t < 0, \end{cases} \quad \text{and} \quad f_2(x,s,t) = \begin{cases} c_2 s^\gamma - k_2 & \text{if } s \geq 0, \\ -k_2 & \text{if } s < 0, \end{cases}$$

where c_1, c_2, k_1, k_2 are positive constants. In this case, system (1) is semipositone, and if $\delta, \gamma > 0$ are such that $\delta\gamma < (p-1)(q-1)$ we can apply theorem 1.1.

Let

$$f_1(x,s,t) = \begin{cases} c_1^\gamma c(x) s^\alpha - k_1 & \text{if } \ s \geq 0, t > c_1, \\ c(x) s^\alpha t^\gamma - k_1 & \text{if } \ s \geq 0, t \in [0, c_1], \\ -k_1 & \text{otherwise }, \end{cases}$$

and

$$f_2(x,s,t) = \begin{cases} c_2^\delta d(x) t^\beta - k_2 & \text{if } \ t \geq 0, s > c_2, \\ d(x) s^\delta t^\beta - k_2 & \text{if } \ t \geq 0, s \in [0, c_2], \\ -k_2 & \text{otherwise }, \end{cases}$$

where $c, d \in C(\bar{\Omega})$ are positive functions and c_1, c_2, k_1, k_2 are positive contants. If $\gamma, \delta \geq 0$, $0 < \alpha < p-1$, and $0 < \beta < q-1$ we have a example in which can be applied theorem 1.2.

Remark We can improve theorem 1.1 by changing the global monotonicity assumptions on nonlinearities by the monotonicity conditions in infinity; in other words, there exists $M_0 > 0$ such that $f_i(s)$ is nondecreasing for every $s \geq M_0$ and $i = 1, 2$; and also requiring that the nonlinearities satisfy the hypotheses

$$\begin{gathered} \text{There exists a function } g : \mathbb{R} \to \mathbb{R} \text{ continuous and nondecreasing with} \\ g(0) = 0, 0 \leq g(s) \leq C(1 + |s|^{r-1}), \ \forall s \in \mathbb{R}, \\ \text{where } r = \min\{p, q\}, \text{ for some } C > 0, \text{ and the applications} \\ (s,t) \longmapsto f_1(x,s,t) + g(s), \ (s,t) \longmapsto f_2(x,s,t) + g(t) \\ \text{are nondecreasing, for a.e. } x \in \Omega. \end{gathered}$$

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