

ALMOST PERIODIC ORBITS AND STABILITY FOR QUANTUM TIME-DEPENDENT HAMILTONIANS

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ABSTRACT. We study almost periodic orbits of quantum systems and prove that for periodic time-dependent Hamiltonians an orbit is almost periodic if, and only if, it is precompact. In the case of quasiperiodic time-dependence we present an example of a precompact orbit that is not almost periodic. Finally we discuss some simple conditions assuring dynamical stability for nonautonomous quantum system.

1. INTRODUCTION

The time evolution of a quantum mechanical system with time-dependent Hamiltonians $H(t)$ is determined by the Schrödinger equation

$$i\frac{d\psi(t)}{dt} = H(t)\psi(t),$$

where $H(t)$ is a family of self-adjoint operators in the Hilbert space \mathcal{H} and $\psi(t) \in \mathcal{H}$ for all $t \in \mathbb{R}$. The initial value problem $\psi(0) = \psi$ has a unique solution

$$\psi(t) \doteq U(t, 0)\psi,$$

under suitable conditions on $H(t)$ (see [21, 18, 19, 15]) and the propagators, or time evolution operators $U(t, s)$, form a strongly continuous family of unitary operators acting on \mathcal{H} , such that

$$U(t, r)U(r, s) = U(t, s), \quad \forall r, s, t, \in \mathbb{R}$$

$$U(t, t) = \mathbb{I}_d, \quad \forall t.$$

\mathbb{I}_d denotes the identity operator. If the Hamiltonian is time-periodic with period T , then $U(t+T, r+T) = U(t, r)$ and the Floquet operator at s is defined by $U_F(s) \doteq U(s+T, s)$; $U_F(0)$ is simply called Floquet operator and denoted by U_F , and $U_F(s)$ is unitarily equivalent to $U_F(r)$, $\forall r, s$. Let

$$\mathcal{O}(\psi) \doteq \{U(t, 0)\psi : t \in \mathbb{R}\}$$

be the orbit of a vector $\psi \in \mathcal{H}$.

If $H(t) = H$ is independent of t the time evolution operators are $U(t, s) = e^{-iH(t-s)}$. In this case, it is a well-known fact that if ψ is in the point

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subspace of H then the quantum time evolution of the state ψ , $\psi(t)$, is almost periodic, since it can be expanded in terms of the eigenfunctions φ_n of H , with eigenvalues E_n ,

$$\psi(t) = \sum_n c_n e^{-iE_n t} \varphi_n.$$

Reciprocally, if $\psi(t)$ is almost periodic then using the results in [20] (Chapter VI) it holds true that $\mathcal{O}(\psi)$ is precompact and then ψ is in the point subspace of H (see Theorem 3 ahead). In this work, we prove that this fact remains true in the periodic case, that is, ψ is in the point subspace of U_F if, and only if, $\psi(t)$ is almost periodic (see Theorem 5).

In the studies of time-dependent systems it is common to consider the quasienergy operator, i.e., a self-adjoint operator formally given by

$$K = -i \frac{d}{dt} + H(t)$$

acting in some enlarged Hilbert space. The quasienergy operator K was previously defined for periodic Hamiltonians [22, 13] and then generalized for general time dependence in [14]. In the periodic case it was proved that

$$e^{-iKT} \simeq \text{Id} \otimes U_F,$$

where \simeq means unitary equivalence.

A natural framework for considering general time-dependent perturbations, which includes both periodic and the random potentials as special cases, is to write $H(t)$ in the form

$$H(t) = H(g_t(\theta)) = H_0 + V(g_t(\theta)),$$

where $g_t : \Omega \rightarrow \Omega$ is an invertible flow on a compact manifold Ω with a probability ergodic measure μ and H_0 is the Hamiltonian of the isolated system (see [16, 2]). Again, under suitable conditions on V there exists a unitary time evolution operator $U_\theta(t, s)$ and the generalized quasienergy operator is defined [16] on $L^2(\Omega, \mathcal{H}, d\mu)$ by

$$(e^{-i\tilde{K}t} f)_\theta = \mathcal{F}_{-t} U_\theta(t, 0) f_\theta = U_\theta(0, -t) \mathcal{F}_{-t} f_\theta,$$

where $\mathcal{F}_{-t} f_\theta = f_{g_{-t}(\theta)}$; we refer to this construction as *Jauslin-Lebowitz formulation*. The operator \tilde{K} acts as

$$(\tilde{K}f)_\theta = i \frac{d}{dt} f_{g_{-t}(\theta)} \Big|_{t=0} + H_\theta f_\theta.$$

In the case of a periodic potential one has $\Omega = S^1 \equiv [0, 2\pi)$, $g_t(\theta) = \theta + \omega t$ and $d\mu = \frac{d\theta}{2\pi}$.

For quasiperiodic potentials with two incommensurate frequencies $\omega_1/\omega_2 \notin \mathbb{Q}$ the manifold Ω is $S^1 \times S^1$, $g_t(\theta_1, \theta_2) = (\theta_1 + \omega_1 t, \theta_2 + \omega_2 t)$ and $d\mu = \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi}$. We denote the two periods by $T_j = \frac{2\pi}{\omega_j}$. In this case the generalized Floquet operator acting on $\mathcal{K}_1 \doteq L^2(S^1, \mathcal{H}, \frac{d\theta_1}{2\pi})$ is defined by

$$(1) \quad U_F = \mathcal{T}_{-T_2} u_1,$$

where $u_1(\theta_1) = U_{(\theta_1,0)}(T_2, 0)$ (\doteq monodromy operator) and $(\mathcal{T}_{-T_2}\phi)(\theta_1) = \phi(\theta_1 - \omega_1 T_2)$.

Let $A : \text{dom } A \subset \mathcal{H} \rightarrow \mathcal{H}$ be an unbounded positive self-adjoint operator with discrete spectrum which we call a *probe operator*. Assuming that if $\psi \in \text{dom } A$, then $U(t, 0)\psi \in \text{dom } A$ for all $t \geq 0$, a very interesting question is about the behavior of the expectation value of A , that is,

$$E_\psi^A(t) \equiv \langle U(t, 0)\psi, AU(t, 0)\psi \rangle.$$

We say the system is A -dynamically stable if $E_\psi^A(t)$ is a bounded function of time, and A -dynamically unstable otherwise. A particular case is when the Hamiltonian has the form $H(t) = H_0 + V(t)$ and $A = H_0$. In this work we discuss some simple conditions assuring dynamical stability, mainly when either the Floquet or quasienergy operator has purely point spectrum; recall that in the periodic case it is known that continuous spectrum of the Floquet operator implies dynamical instability (see Section 2).

Usually it is not a simple task to get results on dynamical (in)stability in the original Hilbert space H through properties of K or \tilde{K} acting in the corresponding enlarged space. We present some theoretical results about this point in Section 4. An important result in the periodic case was proved in [11], i.e., that the applicability of the KAM method for the quasienergy operator K , which is a technique to find out a unitary operator U such that $UKU^{-1} = D$, where D is pure point, gives a uniform bound at the expectation value of the energy for a class of time-periodic Hamiltonians of the form $H(t) = H_0 + V(t)$ considered in [10].

The study of precompactness (and related properties) of orbits of a time-dependent quantum system and their connection with spectral type and stability was carried out, e.g., in [12, 6, 5, 3, 16, 2]. In this work we prove that in the periodic case (including the autonomous case) the orbit $\mathcal{O}(\psi)$ is precompact if, and only if, $\psi(t)$ is an almost periodic function. Moreover, already in the quasiperiodic case we present an example with precompact orbits which are not almost periodic.

This paper is organized as follows. In Section 2 we recall some subspaces of \mathcal{H} that were studied in the literature and the results that connect this subspaces with dynamical (in)stability and spectral properties of the Floquet or quasienergy operators. In Section 3 we present our results about almost periodic orbits. In Section 4 we discuss some simple conditions assuring dynamical stability; we pay special attention to connection between enlarged spaces and the original quantum Hilbert space. A number of known results are recalled in the text in order to make it as readable as possible.

2. PRELIMINARIES

In this section we present a short account of suitable subspaces and relations among them, in order to put our results in context.

Consider a time-dependent Hamiltonian $H(t)$ acting in a separable Hilbert space \mathcal{H} , which may be nonperiodic, and let $U(t, 0)$ the corresponding propagators. Denote by $A : \text{dom } A \subset \mathcal{H} \rightarrow \mathcal{H}$ a probe operator, such that $\text{dom } A$ is invariant under time evolution $U(t, 0)$. Let $F(A > E)$ be the spectral projection onto the closed space spanned by the eigenvectors of A corresponding to the eigenvalues larger than $E \in \mathbb{R}$. The relevant definitions are as follows [12, 6, 5, 3].

- Definition 1.** (i) $\mathcal{H}_{\text{pc}} \doteq \{\xi \in \mathcal{H} : \mathcal{O}(\xi) \text{ is precompact in } \mathcal{H}\}$.
(ii) $\mathcal{H}_{\text{f}} \doteq \{\xi \in \mathcal{H} : \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \|CU(t, 0)\xi\| dt = 0 \text{ for any compact operator } C\}$.
(iii) $\mathcal{H}_{\text{be}} \doteq \{0 \neq \xi \in \mathcal{H} : \lim_{E \rightarrow \infty} \sup_{t \in \mathbb{R}} \|F(A > E)U(t, 0) \frac{\xi}{\|\xi\|}\| = 0\} \cup \{0\}$.
(iv) $\mathcal{H}_{\text{ue}} \doteq \{0 \neq \xi \in \mathcal{H} : \lim_{E \rightarrow \infty} \sup_{t \in \mathbb{R}} \|F(A > E)U(t, 0) \frac{\xi}{\|\xi\|}\| = 1\} \cup \{0\}$.
(v) $\mathcal{S}^{\text{bd}}(A) \doteq \{\xi \in \text{dom } A : \text{the function } t \mapsto E_\xi^A(t) \text{ is bounded}\}$.
(vi) $\mathcal{S}^{\text{un}}(A) \doteq \{\xi \in \text{dom } A : \text{the function } t \mapsto E_\xi^A(t) \text{ is unbounded}\}$.

Important compact operators are the projections onto finite subspaces of \mathcal{H} , so that the elements of \mathcal{H}_{f} are interpreted as the vectors that under time evolution leave, on average, any finite-dimensional subspace of \mathcal{H} .

Some basic properties of the sets that appeared in the above definition are summarized ahead. For proofs we refer the reader to [5, 6, 12, 3].

Theorem 1. *Let $H(t)$ be a time-dependent Hamiltonian and A as above; then:*

- (a) \mathcal{H}_{f} and \mathcal{H}_{pc} are closed subspaces of \mathcal{H} .
(b) $\mathcal{H}_{\text{pc}} \perp \mathcal{H}_{\text{f}}$.
(c) $\mathcal{H}_{\text{be}} = \mathcal{H}_{\text{pc}}$ and $\mathcal{H}_{\text{f}} \subset \mathcal{H}_{\text{ue}}$.
(d) If $\xi \in \text{dom } A$ and $\xi \notin \mathcal{H}_{\text{pc}}$ then $\xi \in \mathcal{S}^{\text{un}}(A)$, that is, $\mathcal{S}^{\text{bd}}(A) \subset \mathcal{H}_{\text{pc}}$. In particular, $(\text{dom } A \cap \mathcal{H}_{\text{f}}) \setminus \{0\} \subset \mathcal{S}^{\text{un}}(A)$.

Note that if the Hamiltonian $H(t)$ has the form $H(t) = H_0 + V(t)$ with H_0 an unbounded, positive, self-adjoint operator with discrete spectrum, then Theorem 1(d) holds true for $A = H_0$.

2.1. Periodic Case. If $H(t)$ is periodic of period T and $U_F = U(T, 0)$ is the corresponding Floquet operator, we denote by \mathcal{H}_{p} the point spectral subspace and by \mathcal{H}_{c} the continuous subspace of the Floquet operator U_F . Recall the important

Theorem 2 (RAGE). *Let $C : \mathcal{H} \rightarrow \mathcal{H}$ be a compact operator and $\xi \in \mathcal{H}_{\text{c}}$, then*

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \|CU(t, 0)\xi\| dt = 0.$$

A detailed proof of Theorem 2 can be found in [12]; this result was firstly proved for the autonomous case (see, e.g., [1]). As a consequence of this theorem it follows that if $\xi \in \mathcal{H}_c$ then $\xi \in \mathcal{H}_f$, so by Theorem 1(d) it follows that $\langle U(t, 0)\xi, AU(t, 0)\xi \rangle$ is unbounded. Thus, as it is well known, the presence of continuous spectrum for the Floquet operator is a signature of quantum instability. In principle, one would expect that a Floquet operator with purely point spectrum would imply quantum stability, however there are examples with purely point spectrum and dynamically unstable; see [9, 17, 7] for examples in the autonomous case and [8] for the time-periodic case.

Using the above theorem and a series of technical lemmas in [6], one gets

Theorem 3. *If the Hamiltonian operator is periodic in time, then*

- (a) $\mathcal{H}_p = \mathcal{H}_{be} = \mathcal{H}_{pc}$;
- (b) $\mathcal{H}_c = \mathcal{H}_{ue} = \mathcal{H}_f$.

We observe that Theorem 3 also holds in the autonomous case $H(t) = H$ and with \mathcal{H}_p and \mathcal{H}_c denoting, respectively, the point and continuous subspace of the Hamiltonian H .

According to the above-quoted results, for periodic systems we have

$$(2) \quad \mathcal{H} = \mathcal{H}_{pc} \oplus \mathcal{H}_f.$$

In [5] was presented an example for which relation (2) does not hold for nonperiodic time dependence. It was defined the “unusual” subspace \mathcal{H}_a by the relation

$$\mathcal{H} = \mathcal{H}_{pc} \oplus \mathcal{H}_f \oplus \mathcal{H}_a,$$

and constructed a nonperiodic Hamiltonian such that $\mathcal{H} = \mathcal{H}_a$. The example is given by the Floquet operator generated by the kicked Hamiltonian

$$H(t) = p^2 + x \sum_{n=1}^{\infty} \epsilon_n \delta(t - n), \quad x \in [0, 2\pi),$$

acting on $\mathcal{H} = L^2(\mathbb{T})$ and $\epsilon_n \in \{-1, 0, 1\}$ adequately chosen. This example illustrates some possible unusual properties of nonstationary quantum systems.

2.2. Quasiperiodic Case. In this case we have the generalized Floquet operator U_F as defined in (1), acting on the enlarged space $\mathcal{K}_1 = L^2(S^1, \mathcal{H}, \frac{d\theta_1}{2\pi})$, and the generalized quasienergy operator \tilde{K} acting in $L^2(S^1 \times S^1, \mathcal{H}, \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi})$. We denote, respectively, by $\mathcal{K}_{1,p}$ and $\mathcal{K}_{1,c}$ the point and continuous subspace of the generalized Floquet operator U_F .

For each fixed t let the unitary operator $U(t) : \mathcal{K}_1 \rightarrow \mathcal{K}_1$ be given by $(U(t)\psi)(\theta_1) = U_{(\theta_1,0)}(t,0)\psi(\theta_1)$, that is,

$$U(t) = \int_{S^1}^{\oplus} U_{(\theta_1,0)}(t,0) \frac{d\theta_1}{2\pi},$$

and given $\psi \in \mathcal{K}_1$ let $\tilde{\mathcal{O}}(\psi) = \{U(t)\psi : t \in \mathbb{R}\}$ be the orbit of ψ in the enlarged space \mathcal{K}_1 .

Let $A : \text{dom } A \subset \mathcal{K}_1 \rightarrow \mathcal{K}_1$ be a probe operator with $U(t)\text{dom } A \subset \text{dom } A$ and $F(A > E)$ as before. The relevant definitions are as follows [16, 2, 6]:

Definition 2. (a) $\mathcal{K}_{1,f} \doteq \{\psi \in \mathcal{K}_1 : \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \|CU(t)\psi\|_{\mathcal{K}_1} dt = 0 \text{ for any compact operator } C \text{ in } \mathcal{K}_1\}$.

(b) $\mathcal{K}_{1,pc} = \{\psi \in \mathcal{K}_1 : \tilde{\mathcal{O}}(\psi) \text{ is precompact in } \mathcal{K}_1\}$.

(c) $\mathcal{K}_{1,be} \doteq \{0 \neq \psi \in \mathcal{K}_1 : \lim_{E \rightarrow \infty} \sup_{t \in \mathbb{R}} \|F(A > E)U(t) \frac{\psi}{\|\psi\|}\| = 0\} \cup \{0\}$.

(d) $\mathcal{K}_{1,ue} \doteq \{0 \neq \psi \in \mathcal{K}_1 : \lim_{E \rightarrow \infty} \sup_{t \in \mathbb{R}} \|F(A > E)U(t) \frac{\psi}{\|\psi\|}\| = 1\} \cup \{0\}$.

In [16] it was proved the analog of the RAGE Theorem for the quasiperiodic case. The proof is an adaptation of the similar statement in the periodic case discussed in [12]. As in the periodic case one has:

Theorem 4. *If the Hamiltonian operator is quasiperiodic in time, then*

(a) $\mathcal{K}_{1,p} = \mathcal{K}_{1,pc} = \mathcal{K}_{1,be}$;

(b) $\mathcal{K}_{1,c} = \mathcal{K}_{1,ue} = \mathcal{K}_{1,f}$.

It is worth mentioning that the relation between the energy growth and the characterizations in Definition 2 is not as direct as in the case of periodic and autonomous potentials. The above theorem holds on the enlarged space \mathcal{K}_1 so that a generalized operator with continuous spectrum does not ensure unbounded energy growth in the original Hilbert space \mathcal{H} , although it does in \mathcal{K}_1 . See [16, 2] for interesting examples on systems with time-quasiperiodic dependence.

3. ALMOST PERIODIC ORBITS

Let \mathcal{B} be a Banach space. A continuous function $f : \mathbb{R} \rightarrow \mathcal{B}$ is called *almost periodic* if for any number $\epsilon > 0$, one can find a number $l(\epsilon) > 0$ such that any interval of the real line of length $l(\epsilon)$ contains at least one point τ with the property that

$$\|f(t + \tau) - f(t)\| < \epsilon, \quad \forall t \in \mathbb{R}.$$

For properties of almost periodic functions we refer the reader to [4, 20]. Now we introduce the following subset of \mathcal{H} :

$$\mathcal{H}_{\text{ap}} \doteq \{\xi \in \mathcal{H} : \text{the function } \mathbb{R} \ni t \mapsto \xi(t) = U(t,0)\xi \text{ is almost periodic}\}.$$

By abuse of language sometimes we say that the orbit $\mathcal{O}(\xi)$ is almost periodic.

For general time dependence one has

Proposition 1. \mathcal{H}_{ap} is a closed subspace of \mathcal{H} and $\mathcal{H}_{\text{ap}} \subset \mathcal{H}_{\text{pc}}$.

Proof. Clearly $0 \in \mathcal{H}_{\text{ap}}$. If $\xi, \psi \in \mathcal{H}_{\text{ap}}$ then $\xi(t) = U(t, 0)\xi$ and $\psi(t) = U(t, 0)\psi$ are almost periodic functions. Since the sum of two almost periodic functions with values in \mathcal{H} is an almost periodic function, it follows that $\xi(t) + \psi(t) = U(t, 0)\xi + U(t, 0)\psi = U(t, 0)(\xi + \psi) = (\xi + \psi)(t)$ is an almost periodic function. So $\xi + \psi \in \mathcal{H}_{\text{ap}}$. Now, let $\xi \in \mathcal{H}_{\text{ap}}$ and λ a complex number, then $\xi(t) = U(t, 0)\xi$ is an almost periodic function. Since $\lambda\xi(t) = \lambda U(t, 0)\xi = U(t, 0)(\lambda\xi)$ is an almost periodic function, it follows that $\lambda\xi \in \mathcal{H}_{\text{ap}}$. So \mathcal{H}_{ap} is a vector subspace of \mathcal{H} .

Suppose that $\{\xi_j\} \subset \mathcal{H}_{\text{ap}}$ and $\lim_{j \rightarrow \infty} \xi_j = \xi$. Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|\xi_j - \xi\| < \epsilon$ for all $j \geq N$; thus, there exists N as above such that $j \geq N$ implies that $\forall t \in \mathbb{R}$

$$\|\xi(t) - \xi_j(t)\| = \|U(t, 0)\xi - U(t, 0)\xi_j\| \leq \|\xi - \xi_j\| < \epsilon.$$

So $\xi_j(t) \rightarrow \xi(t)$ uniformly in \mathbb{R} in the sense of convergence in the norm. Since each $\xi_j(t)$ is an almost periodic function, it follows that $\xi(t)$ is an almost periodic function (Theorem 6.4 in [4]) and $\xi \in \mathcal{H}_{\text{ap}}$, which shows that \mathcal{H}_{ap} is a closed vector subspace of \mathcal{H} .

Since the set of values of an almost periodic function with values in \mathcal{H} is precompact in \mathcal{H} (Theorem 6.5 in [4]), it follows that $\mathcal{H}_{\text{ap}} \subset \mathcal{H}_{\text{pc}}$. \square

3.1. Periodic Systems. If the Hamiltonian time dependence is periodic (or autonomous) more can be said.

Proposition 2. If the Hamiltonian operator is periodic in time and $\xi \in \mathcal{H}_{\text{p}}$ is an eigenvector of U_{F} , that is, $U_{\text{F}}\xi = e^{-i\alpha}\xi$, $\alpha \in \mathbb{R}$, then $\xi \in \mathcal{H}_{\text{ap}} \subset \mathcal{H}_{\text{pc}}$.

Proof. Since $U(t, 0)$ is strongly continuous the map $t \mapsto \xi(t)$ is continuous.

Any $t \in \mathbb{R}$ can be written in the form $t = nT + s$, with $n \in \mathbb{Z}$ and $0 \leq s < T$. We have $U_{\text{F}}\xi = e^{-i\alpha}\xi$ and $U_{\text{F}}^{-1}\xi = e^{i\alpha}\xi$. Since for $t \geq 0$ ($n \geq 0$)

$$\begin{aligned} U(t, 0)\xi &= U(s + nT, nT)U(nT, (n-1)T) \dots U(T, 0)\xi \\ &= U(s, 0) \underbrace{U(T, 0) \dots U(T, 0)}_{n \text{ factors}} \xi = U(s, 0)e^{-in\alpha}\xi, \end{aligned}$$

and for $t < 0$ ($n < 0$)

$$\begin{aligned} U(t, 0)\xi &= U(s + nT, nT)U(nT, (n+1)T) \dots U(-T, 0)\xi \\ &= U(s, 0) \underbrace{U(T, 0)^{-1} \dots U(T, 0)^{-1}}_{n \text{ factors}} \xi = U(s, 0)e^{-in\alpha}\xi, \end{aligned}$$

it follows that

$$U(t, 0)\xi = U(s, 0)e^{-in\alpha}\xi,$$

for $t = nT + s \in \mathbb{R}$, $n \in \mathbb{Z}$ and $0 \leq s < T$. So for each $t = nT + s \in \mathbb{R}$

$$\begin{aligned}\xi(t+T) &= U(t+T, 0)\xi = U(s, 0)e^{-i(n+1)\alpha}\xi \\ &= e^{-i\alpha}U(s, 0)e^{-in\alpha}\xi = e^{-i\alpha}\xi U(t, 0)\xi = e^{-i\alpha}\xi(t),\end{aligned}$$

so $t \rightarrow \xi(t)$ is an almost periodic function and the result is proved. \square

Summing up, we conclude:

Theorem 5. *If the Hamiltonian operator is periodic in time, then*

- (a) $\mathcal{H}_p = \mathcal{H}_{be} = \mathcal{H}_{pc} = \mathcal{H}_{ap}$;
- (b) $\mathcal{H}_c = \mathcal{H}_{ue} = \mathcal{H}_f$.

Proof. It is enough to prove that $\mathcal{H}_{pc} = \mathcal{H}_{ap}$. The inclusion $\mathcal{H}_{ap} \subset \mathcal{H}_{pc}$ was proved in Proposition 1. On the other hand, it is a consequence of Propositions 1 and 2 that $\mathcal{H}_p \subset \mathcal{H}_{ap}$. Since $\mathcal{H}_p = \mathcal{H}_{pc}$, it follows that $\mathcal{H}_{pc} \subset \mathcal{H}_{ap}$. \square

Theorem 5 holds also for autonomous Hamiltonians.

3.2. Quasiperiodic Systems. In the above theorem we proved that for time-periodic Hamiltonians an orbit $\mathcal{O}(\xi)$ is precompact if, and only if, $t \mapsto \xi(t)$ is almost periodic. Now we construct an example showing that already in the case of time-quasiperiodic Hamiltonians there are precompact orbits that are not almost periodic.

Example Given the matrix

$$u_1(\theta_1) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix},$$

it is known (see Lemma 5.1 in [2]) that there exists a quasiperiodic Hamiltonian $H_\theta(t)$, $\theta = (\theta_1, \theta_2)$, acting on $\mathcal{H} = \mathbb{C}^2$, of the form

$$(3) \quad H_\theta(t) = h_0(t)I_d + \sum_{j=1}^3 h_j(t)\sigma_j,$$

where σ_j are the Pauli matrices, and $h_j(t)$ are real quasiperiodic functions, i.e., $h_j(t) = \bar{h}_j(\omega_1 t + \theta_1, \omega_2 t + \theta_2)$, where $\bar{h}_j(\theta_1, \theta_2)$ are continuous and 2π -periodic in the two arguments $\theta_1, \theta_2 \in S^1$, and ω_1, ω_2 are positive real numbers so that $u_1(\theta_1) = U_{(\theta_1, 0)}(T_2, 0)$ is the corresponding monodromy operator. Moreover, the corresponding generalized Floquet operator $U_F = \mathcal{T}_{-T_2} u_1$ has absolutely continuous spectrum for any irrational $\alpha \doteq \frac{\omega_1}{\omega_2}$.

By the construction in the proof of Lemma 5.1 in [2], it is found that for $k \in \mathbb{Z}$, $k > 0$,

$$\begin{aligned}
U_{(\theta_1,0)}(kT_2, 0) &= u_1(\theta_1 + (k-1)2\pi\alpha) \dots u_1(\theta_1 + 2\pi\alpha)u_1(\theta_1) \\
&= \begin{pmatrix} e^{i(\theta_1+(k-1)2\pi\alpha)} & 0 \\ 0 & e^{-i(\theta_1+(k-1)2\pi\alpha)} \end{pmatrix} \cdots \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} \\
&= \begin{pmatrix} e^{i(\theta_1+(k-1)2\pi\alpha)} \dots e^{i\theta_1} & 0 \\ 0 & e^{-i(\theta_1+(k-1)2\pi\alpha)} \dots e^{-i\theta_1} \end{pmatrix} \\
&= \begin{pmatrix} e^{i(k\theta_1+(1+2+\dots+(k-1))2\pi\alpha)} & 0 \\ 0 & e^{-i(k\theta_1+(1+2+\dots+(k-1))2\pi\alpha)} \end{pmatrix} \\
&= \begin{pmatrix} e^{ik(\theta_1+(k-1)\pi\alpha)} & 0 \\ 0 & e^{-ik(\theta_1+(k-1)\pi\alpha)} \end{pmatrix};
\end{aligned}$$

for $k < 0$ the same expression is found. Therefore, for all $k \in \mathbb{Z}$

$$U_{(\theta_1,0)}(kT_2, 0) = \begin{pmatrix} e^{ik(\theta_1+(k-1)\pi\alpha)} & 0 \\ 0 & e^{-ik(\theta_1+(k-1)\pi\alpha)} \end{pmatrix}.$$

Moreover, for $\theta_1 \in S^1$, $0 \leq t \leq T_2$, define

$$v(t; \theta_1) = \begin{pmatrix} e^{i\frac{t}{T_2}(\theta_1+(\frac{t}{T_2}-1)\pi\alpha)} & 0 \\ 0 & e^{-i\frac{t}{T_2}(\theta_1+(\frac{t}{T_2}-1)\pi\alpha)} \end{pmatrix},$$

which is differentiable with respect to t and satisfies

$$v(0; \theta_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{Id}, \quad v(T_2; \theta_1) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} = u_1(\theta_1).$$

So for $t \in \mathbb{R}$, $t = kT_2 + \delta_t$, $0 \leq \delta_t \leq T_2$, one has

$$U_{(\theta_1,0)}(t, 0) = v(\delta_t; \theta_1 + k2\pi\alpha)U_{(\theta_1,0)}(kT_2, 0).$$

Therefore, for $\xi \in \mathcal{H} = \mathbb{C}^2$, $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, we have

$$U_{(\theta_1,0)}(t, 0)\xi = \begin{pmatrix} e^{i\frac{t}{T_2}(\theta_1+(\frac{t}{T_2}-1)\pi\alpha)}\xi_1 \\ e^{-i\frac{t}{T_2}(\theta_1+(\frac{t}{T_2}-1)\pi\alpha)}\xi_2 \end{pmatrix}.$$

Since the map, for $0 \neq a \in \mathbb{R}$, $t \mapsto \sin at^2$ is not almost periodic, because it is not uniformly continuous, we conclude that the map $t \mapsto e^{iat^2}$ is not almost periodic. Thus,

$$t \mapsto g(t) = e^{i\frac{t}{T_2}(\theta_1+(\frac{t}{T_2}-1)\pi\alpha)} = e^{i\frac{t}{T_2}\theta_1} e^{it^2\frac{\omega_1\omega_2}{4\pi}} e^{-it\frac{\omega_1}{2}}$$

is not almost periodic, because on the contrary the map

$$e^{-i\frac{t}{T_2}\theta_1}g(t)e^{it\frac{\omega_1}{2}} = e^{it^2\frac{\omega_1\omega_2}{4\pi}}$$

would be almost periodic.

Therefore, if $\xi \neq 0$ then the map $t \mapsto U_{(\theta_1,0)}(t, 0)\xi$ is not almost periodic for all $\theta_1 \in S^1$. Hence we have got an example of precompact orbit (a closed

and bounded set on \mathbb{C}^2 is compact) which is not almost periodic. This finishes the example.

The above example can be extended to the infinite dimensional Hilbert space $\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathbb{C}^2$ of the elements $\xi = (\xi_n)_{n \in \mathbb{N}}$ with $\xi_n \in \mathbb{C}^2$ and $\sum_n |\xi_n|^2 < \infty$. Denote

$$\tilde{u}_1(\theta_1) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix};$$

we know that there exists a quasiperiodic $\tilde{H}_\theta(t)$ such that $\tilde{u}_1(\theta_1)$ is the corresponding monodromy operator. Moreover, $\sigma(\tilde{U}_F)$ is absolutely continuous for all irrational α . Let

$$u_1(\theta_1) = \begin{pmatrix} \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} & & & & \\ & \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} & & & \\ & & \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} & & \\ & & & \ddots & \end{pmatrix}$$

or, writing in the another way, $u_1(\theta_1) = \bigoplus \tilde{u}_1(\theta_1)$. For $\xi \in \mathcal{H}$ one has $u_1(\theta_1)\xi = \bigoplus \tilde{u}_1(\theta_1)\xi_n$. The Floquet operator corresponding to $u_1(\theta_1)$, $U_F = \mathcal{T}_{-T_2} u_1 : L^2(S^1, \mathcal{H}, \frac{d\theta_1}{2\pi}) \rightarrow L^2(S^1, \mathcal{H}, \frac{d\theta_1}{2\pi})$ has absolutely continuous spectrum for all irrational α .

If $H_\theta(t) = \bigoplus_{n \in \mathbb{N}} \tilde{H}_\theta(t)$ then the propagator of $H_\theta(t)$ is $U_\theta(t, 0) = \bigoplus \tilde{U}_\theta(t, 0)$. Thus, $H_\theta(t)$ has $u_1(\theta_1)$ as the corresponding monodromy operator, and given $0 \neq \xi \in \mathcal{H}$ and $\theta = (\theta_1, 0) \in S^1 \times S^1$ one has

$$\begin{aligned} U_{(\theta_1, 0)}(t, 0)\xi &= \bigoplus_n \begin{pmatrix} e^{i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2} - 1)\pi\alpha)} & 0 \\ 0 & e^{-i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2} - 1)\pi\alpha)} \end{pmatrix} \xi_n \\ &= \bigoplus_n \begin{pmatrix} e^{i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2} - 1)\pi\alpha)} \xi_n^1 \\ e^{-i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2} - 1)\pi\alpha)} \xi_n^2 \end{pmatrix}. \end{aligned}$$

So $t \mapsto U_{(\theta_1, 0)}(t, 0)\xi$ is not almost periodic. If ξ satisfies $\xi = \bigoplus \xi_n$ with $\xi_n \neq 0$ if, and only if, $n = l$, then

$$U_{(\theta_1, 0)}(t, 0)\xi = \begin{pmatrix} e^{i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2} - 1)\pi\alpha)} \xi_l^1 \\ e^{-i\frac{t}{T_2}(\theta_1 + (\frac{t}{T_2} - 1)\pi\alpha)} \xi_l^2 \end{pmatrix},$$

and the orbit is precompact since it lives in a finite dimension subspace. In the same way, if ξ is of the form $\xi = \bigoplus \xi_n$ with $\xi_n \neq 0$ only for finitely many indices n , we have an example of a theoretical quantum model with precompact orbits which are not almost periodic.

3.3. Quasienergy Operator and Almost Periodic Orbits. Let $H(t)$ be a general time-dependent Hamiltonian in a Hilbert space \mathcal{H} such that the propagator $U(t, s)$ is well defined. In this case we have defined the quasienergy operator $K = -i\frac{d}{dt} + H(t)$ acting in the extended Hilbert space $\mathcal{K} = L^2(\mathbb{R}, \mathcal{H}, dt)$. It is known [13, 14] that the quasienergy operator and the propagator are connected by the relation

$$(4) \quad (e^{-iK\sigma} f)(t) = U(t, t - \sigma)f(t - \sigma).$$

Let $\mathcal{K}_p(K)$ and $\mathcal{K}_c(K)$ denote, respectively, the point and continuous subspaces of K . We get the following result:

Proposition 3. *Let $\xi \in \mathcal{H}$ be such that $1 \otimes \xi \in \mathcal{K}_p(K)$. Then:*

- i) *The map $t \mapsto U(t, 0)^{-1}\xi$ is almost periodic.*
- ii) *If the eigenvectors of K have the form $\psi_m = 1 \otimes \xi_m$, with $\xi_m \in \mathcal{H}$, then $\xi \in \mathcal{H}_{\text{ap}}$.*

Proof. If $1 \otimes \xi \in \mathcal{K}_p(K)$ then $1 \otimes \xi = \sum_m c_m \psi_m$, with $K\psi_m = \lambda_m \psi_m$. So

$$e^{iK\sigma}(1 \otimes \xi) = \sum_m c_m e^{i\lambda_m \sigma} \psi_m,$$

therefore by (4) for each $t \in \mathbb{R}$,

$$U(t, t + \sigma)\xi = (e^{iK\sigma}(1 \otimes \xi))(t) = \sum_m c_m e^{-i\lambda_m \sigma} \psi_m(t)$$

and we conclude that, for each fixed t , the map $\sigma \mapsto U(t, t + \sigma)\xi$ is almost periodic. In particular taking $t = 0$ we obtain that $\sigma \mapsto U(0, \sigma)\xi$ is almost periodic and i) is proved.

Now, if the eigenvectors of K have the form $\psi_m = 1 \otimes \xi_m$, then

$$\begin{aligned} \xi(t) &= U(t, 0)\xi = (e^{-iKt}(1 \otimes \xi))(t) \\ &= \sum_m c_m e^{-i\lambda_m t} \psi_m(t) \\ &= \sum_m c_m e^{-i\lambda_m t} \xi_m. \end{aligned}$$

If the sum is finite the map $t \mapsto \xi(t)$ is almost periodic since it is a trigonometric polynomial. If the sum is infinite then $\sum_{m=1}^k c_m e^{-i\lambda_m t} \xi_m \rightarrow \sum_{m=1}^{\infty} c_m e^{-i\lambda_m t} \xi_m$ uniformly as $k \rightarrow \infty$ and so the map $t \mapsto \xi(t)$ is almost periodic, that is, $\xi \in \mathcal{H}_{\text{ap}}$, which is ii). \square

4. BOUNDED ENERGY

In this section we consider time-dependent Hamiltonians $H(t) = H_0 + V(t)$ for which H_0 is a probe operator.

If $\psi_0 \in \text{dom } H_0$ and $\psi(t) = U(t, 0)\psi_0$ is the solution of the Schrödinger equation, under which conditions

$$E_{\psi_0}^0(t) = \langle \psi(t), H_0\psi(t) \rangle$$

is a bounded function on t ? Also, when

$$E_{\psi_0}(t) = \langle \psi(t), H(t)\psi(t) \rangle$$

is a bounded function? Next we present a set of simple general conditions related to the boundedness of such energy functions.

4.1. General Systems.

Proposition 4. *If $V(t)$ is an uniformly bounded family of operators, that is, $\sup_t \|V(t)\| < \infty$, then $E_{\psi_0}^0(t)$ is bounded if, and only if, $E_{\psi_0}(t)$ is bounded.*

Proof. It is sufficient to note that

$$E_{\psi_0}(t) = \langle \psi(t), H(t)\psi(t) \rangle = E_{\psi_0}^0(t) + \langle \psi(t), V(t)\psi(t) \rangle$$

and

$$\sup_t |\langle \psi(t), V(t)\psi(t) \rangle| \leq \sup_t \|\psi(t)\|^2 \|V(t)\| = \sup_t \|\psi_0\|^2 \|V(t)\| < \infty.$$

□

Proposition 5. *If $\psi(t) \in C^1(\mathbb{R}; \mathcal{H})$ is almost periodic and $\psi'(t)$ is uniformly continuous, then $E_{\psi_0}(t)$ is bounded.*

Proof. For each $n \in \mathbb{N}^*$ define

$$f_n(t) = n \left[\psi \left(t + \frac{1}{n} \right) - \psi(t) \right] = n \int_t^{t+\frac{1}{n}} \psi'(s) ds.$$

Since ψ is almost periodic it follows that f_n is almost periodic for each n . As $\psi'(t)$ is uniformly continuous, for each $\epsilon > 0$ there exists $\delta > 0$ such that $|s - t| < \delta$ implies $\|\psi'(t) - \psi'(s)\| < \epsilon$. Given $\epsilon > 0$ let $N(\epsilon)$ the smallest integer larger or equal to $\frac{1}{\delta}$; then for all $n > N(\epsilon)$ and $t \in \mathbb{R}$

$$\begin{aligned} \|f_n(t) - \psi'(t)\| &= \left\| n \int_t^{t+\frac{1}{n}} (\psi'(s) - \psi'(t)) ds \right\| \\ &\leq n \int_t^{t+\frac{1}{n}} \|\psi'(s) - \psi'(t)\| ds < \epsilon. \end{aligned}$$

So $f_n \rightarrow \psi'$ uniformly and therefore $\psi'(t)$ is almost periodic. Hence $i\psi'(t)$ and $\psi(t)$ are bounded maps. Since

$$E_{\psi_0}(t) = \langle \psi(t), H(t)\psi(t) \rangle = \langle \psi(t), i \frac{d\psi}{dt}(t) \rangle$$

the result follows. □

Note that the boundedness of energy follows if $t \mapsto \psi(t)$ and $t \mapsto \psi'(t)$ are bounded maps. Though well known, it is worth mentioning Proposition 6 in this set of conditions.

Proposition 6. *If $t \mapsto V(t)$ is strongly C^1 and $\psi'(t) \in \text{dom } H(t)$ for all t , then:*

(a) *The map $t \mapsto E_\psi(t)$ is differentiable and*

$$\frac{d}{dt}E_\psi(t) = \langle \psi(t), V'(t)\psi(t) \rangle.$$

(b) $|E_\psi(t) - E_\psi(0)| \leq t \times \sup_s \|V'(s)\|$.

(c) *If there are $C > 0, a > 1$ so that $\|V'(t)\| \leq \frac{C}{(1+|t|)^a}$, then $E_\psi(t)$ and $E_\psi^0(t)$ are bounded functions.*

Proof. (a) $E_\psi(t) = \langle \psi(t), (H_0 + V(t))\psi(t) \rangle$ and so

$$\begin{aligned} \frac{d}{dt}E_\psi(t) &= \langle \psi'(t), H(t)\psi(t) \rangle + \langle \psi(t), H(t)\psi'(t) \rangle + \langle \psi(t), V'(t)\psi(t) \rangle \\ &= \langle \psi'(t), i\psi'(t) \rangle + \langle i\psi'(t), \psi'(t) \rangle + \langle \psi(t), V'(t)\psi(t) \rangle \\ &= \langle \psi(t), V'(t)\psi(t) \rangle. \end{aligned}$$

(b) Since

$$E_\psi(t) - E_\psi(0) = \int_0^t \frac{d}{ds}E_\psi(s)ds = \int_0^t \langle \psi(s), V'(s)\psi(s) \rangle ds$$

the result follows.

(c) Similar to (b). □

A possibility for the proposition above is $V(t) = B_1 \sin t + \frac{B_2}{(1+|t|)^2}$ with $B_1, B_2 \in B(\mathcal{H})$ and self-adjoint. From this we see that certainly the choices of ψ depend on B_1, B_2 , since $B_1\psi$ and $B_2\psi$ must be kept in suitable domains so that $E_\psi(t)$ is meaningful.

4.2. Purely Point Systems. The next result is restricted to periodic time dependence and Floquet operators with nonempty point spectrum (see [11]).

Proposition 7. *Let V be periodic with period T . If the subset $\{\xi_1, \dots, \xi_n\}$ of eigenvectors of U_F is in $\text{dom } H_0$ and $t \mapsto \xi_j(t)$ are C^1 maps, then for $\psi = \sum_{j=1}^n a_j \xi_j$, where $a_j \in \mathbb{C}$, $j = 1, \dots, n$, the map $E_\psi(t)$ is bounded. If, moreover, $V(t)$ are bounded operators and $\sup_t \|V(t)\| < \infty$, then $E_\psi^0(t)$ is also bounded.*

Proof. Suppose $U_F \xi_j = e^{i\lambda_j} \xi_j$ with $\lambda_j \in \mathbb{R}$, $1 \leq j \leq n$. We have

$$E_{\xi_j, \xi_k}(t) \doteq \langle \xi_j(t), H(t)\xi_k(t) \rangle = \langle \xi_j(t), i \frac{d}{dt} \xi_k(t) \rangle$$

and so $t \mapsto E_{\xi_j, \xi_k}(t)$ is continuous. Now

$$\begin{aligned} E_{\xi_j, \xi_k}(t+T) &= \langle U(t+T, 0)\xi_j, H(t+T)U(t+T, 0)\xi_k \rangle \\ &= \langle U(t+T, T)U_{\mathbb{F}}\xi_j, H(t)U(t+T, T)U_{\mathbb{F}}\xi_k \rangle \\ &= e^{-i\lambda_j} e^{i\lambda_k} \langle U(t, 0)\xi_j, H(t)U(t, 0)\xi_k \rangle \\ &= e^{i(\lambda_k - \lambda_j)} E_{\xi_j, \xi_k}(t) \end{aligned}$$

and then $t \mapsto E_{\xi_j, \xi_k}(t)$ is an almost periodic function. Since for $\psi = \sum_{j=1}^n a_j \xi_j$ we have $E_{\psi}(t) = \sum_{j,k=1}^n \bar{a}_j a_k E_{\xi_j, \xi_k}(t)$ it follows that $E_{\psi}(t)$ is almost periodic and so bounded. The second statement follows by Proposition 4. \square

According to Proposition 7, in order to get dynamical stability in the periodic case we need conditions assuring the eigenvectors of $U_{\mathbb{F}}$ are in $\text{dom } H_0$ and $t \mapsto \xi_j(t)$ to be C^1 functions. We present some sufficient conditions in terms of the quasienergy operator K .

Lemma 1. *Let $\xi \in \mathcal{H}$ be such that $H(t)U(t, s)\xi$ is well defined. Then the map $t \mapsto H(t)U(t, s)\xi$ is a C^r function if, and only if, $t \mapsto e^{i\lambda(t-s)}U(t, s)\xi$ is a C^{r+1} function for fixed $\lambda, s \in \mathbb{R}$.*

Proof. Note that

$$\frac{d}{dt}(e^{i\lambda(t-s)}U(t, s)\xi) = i\lambda e^{i\lambda(t-s)}U(t, s)\xi - ie^{i\lambda(t-s)}H(t)U(t, s)\xi.$$

Thus, if $t \mapsto H(t)U(t, s)\xi$ is C^r then $t \mapsto e^{i\lambda(t-s)}U(t, s)\xi$ is C^{r+1} and reciprocally if $t \mapsto e^{i\lambda(t-s)}U(t, s)\xi$ is C^{r+1} then $t \mapsto H(t)U(t, s)\xi$ is C^r . \square

Corollary 1. *If $f^{(\lambda)}$ is an eigenvector of K , $Kf^{(\lambda)} = \lambda f^{(\lambda)}$, then the map $t \mapsto f^{(\lambda)}(t)$ is C^r if, and only if, there exists $s \in \mathbb{R}$ so that $t \mapsto H(t)U(t, s)f^{(\lambda)}(s)$ is C^{r-1} .*

Proof. If $Kf^{(\lambda)} = \lambda f^{(\lambda)}$, then by relation (4),

$$e^{-i\lambda\sigma} f^{(\lambda)}(t) = U(t, t - \sigma) f^{(\lambda)}(t - \sigma);$$

so $f^{(\lambda)}(t) = e^{i\lambda\sigma} U(t, t - \sigma) f^{(\lambda)}(t - \sigma)$ for all $\sigma \in \mathbb{R}$. Denoting $t - \sigma = s$ it follows that $f^{(\lambda)}(t) = e^{i\lambda(t-s)} U(t, s) f^{(\lambda)}(s)$ and the result follows by Lemma 1. \square

By using relation (4) one can easily show

Lemma 2. *For periodic systems with period T , one has:*

- (a) *If $Kf = \lambda f$ then $U_{\mathbb{F}}(s)f(s) = e^{-i\lambda T} f(s)$, $\forall s \in \mathbb{R}$.*
- (b) *If $U_{\mathbb{F}}(s)\xi_s = e^{-i\lambda T}\xi_s$, $\xi_s \in \mathcal{H}$, $\forall s$, then*

$$f_{\xi}(t) \doteq e^{i\lambda(t-s)} U(t, s)\xi_s \in \text{dom } K$$

and $Kf_{\xi} = \lambda f_{\xi}$.

Corollary 2. (a) If $H(t+T) = H(t)$, and $\xi^{(\lambda)}$ is an eigenvector of $U_{\mathbb{F}}(s)$, $U_{\mathbb{F}}(s)\xi^{(\lambda)} = e^{-i\lambda T}\xi^{(\lambda)}$, then $\xi^{(\lambda)} \in \text{dom } H(s)$ if, and only if, there exists an eigenvector $f_{\xi^{(\lambda)}}$ of K , $Kf_{\xi^{(\lambda)}} = \lambda f_{\xi^{(\lambda)}}$, with $t \mapsto f_{\xi^{(\lambda)}}(t)$ continuous and differentiable.

(b) In particular, $U_{\mathbb{F}}(s)$ has a basis of eigenvectors in $\text{dom } H(s)$ if, and only if, K has a basis of eigenvectors $\{f_j\}$ such that $t \mapsto f_j(t)$ is continuous and differentiable for each j .

Proof. (a) Suppose that $\xi^{(\lambda)} \in \text{dom } H(s)$. By Lemma 2 $f_{\xi^{(\lambda)}}(t) = e^{i\lambda(t-s)}U(t,s)\xi^{(\lambda)} \in \text{dom } K$ and $Kf_{\xi^{(\lambda)}} = \lambda f_{\xi^{(\lambda)}}$. Since $\xi^{(\lambda)} \in \text{dom } H(s)$ it follows that $U(t,s)\xi^{(\lambda)} \in \text{dom } H(t)$ and $i\partial_t U(t,s)\xi^{(\lambda)} = H(t)U(t,s)\xi^{(\lambda)}$. Thus, $t \mapsto f_{\xi^{(\lambda)}}(t)$ is continuous and differentiable.

Reciprocally, if there exists an eigenvector $f_{\xi^{(\lambda)}}$ of K with $t \mapsto f_{\xi^{(\lambda)}}(t)$ continuous and differentiable, then $f_{\xi^{(\lambda)}}(t) = e^{i\lambda(t-s)}U(t,s)\xi^{(\lambda)}$ and $Kf_{\xi^{(\lambda)}} = \lambda f_{\xi^{(\lambda)}}$ implies $-i\partial_t f_{\xi^{(\lambda)}}(t) + H(t)f_{\xi^{(\lambda)}}(t) = \lambda f_{\xi^{(\lambda)}}(t)$; therefore, $\xi^{(\lambda)} \in \text{dom } H(s)$.

(b) It is a directly consequence of (a). \square

4.3. Jauslin-Lebowitz Formulation. We want to study an analogue of the expectation value of probe operators $A : \text{dom } A \subset \mathcal{H} \rightarrow \mathcal{H}$ on the formulation presented by Jauslin and Lebowitz [16, 2] briefly recalled in the Introduction. If the generalized quasienergy operator \tilde{K} has pure point spectrum, there exists an orthonormal basis $B \doteq \{f_n\}_{n=1}^{\infty}$ of $\tilde{\mathcal{K}}$ with $\tilde{K}f_n = \lambda_n f_n$. By Theorem 4.2 in [16], if $f = 1 \otimes \xi$ is in the point subspace of \tilde{K} the function $t \mapsto U_{\theta}(t,0)\xi$ is almost periodic a.e. θ with respect to the ergodic measure μ on the compact manifold Ω (see Section 1).

Denote

$$B_{n,m}(A) \doteq \int_{\Omega} \langle f_n(\theta), Af_m(\theta) \rangle_{\mathcal{H}} d\mu(\theta) = \langle f_n, (1 \otimes A)f_m \rangle_{\tilde{\mathcal{K}}}.$$

If $f \in \tilde{\mathcal{K}}$ then $f = \sum_n a_n f_n$, with $\sum_n |a_n|^2 = \|f\|_{\tilde{\mathcal{K}}}^2$. For each time t , consider the average over Ω of the expectation value of A , that is,

$$\begin{aligned} A_f(t) &\doteq \int_{\Omega} \langle U_{\theta}(t,0)f(\theta), AU_{\theta}(t,0)f(\theta) \rangle_{\mathcal{H}} d\mu(\theta) \\ &= \int_{\Omega} \langle (\mathcal{F}_t e^{-i\tilde{K}t} f)(\theta), A(\mathcal{F}_t e^{-i\tilde{K}t} f)(\theta) \rangle_{\mathcal{H}} d\mu(\theta) \end{aligned}$$

$$\begin{aligned}
&= \left\langle \mathcal{F}_t e^{-i\tilde{K}t} f, (1 \otimes A) \mathcal{F}_t e^{-i\tilde{K}t} f \right\rangle_{\tilde{\mathcal{K}}} \\
&= \left\langle e^{-i\tilde{K}t} f, (1 \otimes A) e^{-i\tilde{K}t} f \right\rangle_{\tilde{\mathcal{K}}} \\
&= \sum_{n,m} \bar{a}_n a_m e^{-it(\lambda_m - \lambda_n)} \langle f_n, (1 \otimes A) f_m \rangle_{\tilde{\mathcal{K}}} \\
&= \sum_{n,m} \bar{a}_n a_m e^{-it(\lambda_m - \lambda_n)} B_{n,m}(A).
\end{aligned}$$

Note that if this sum is absolutely convergent then $A_f(t)$ is a bounded and almost periodic function of t , and

$$t \mapsto \langle U_\theta(t, 0) f(\theta), AU_\theta(t, 0) f(\theta) \rangle_{\mathcal{H}}$$

is bounded a.e. θ . We conclude

Proposition 8. *If $f = \sum_{j=1}^m a_j f_j$, where f_j are eigenvectors of \tilde{K} and $f_j(\theta) \in \text{dom } A$, for all θ , then $t \mapsto A_f(t)$ is a bounded and almost periodic function. Moreover,*

$$t \mapsto \langle U_\theta(t, 0) f(\theta), AU_\theta(t, 0) f(\theta) \rangle_{\mathcal{H}}$$

is bounded for almost every θ .

More generally we obtain the following result:

Theorem 6. *Suppose that Ω is a compact manifold, $g_t : \Omega \rightarrow \Omega$ a C^1 flow with $\sup_{t,\theta} \|\partial_t g_t(\theta)\| < \infty$, and $\tilde{K} f^{(\lambda)} = \lambda f^{(\lambda)}$ with $\theta \mapsto f^{(\lambda)}(\theta)$ a C^1 map. Then for μ almost every θ one has $U_\theta(t, 0) f^{(\lambda)}(\theta) \in \text{dom } H_\theta(t)$ and*

$$\left\langle U_\theta(t, 0) f^{(\lambda)}(\theta), H_\theta(t) U_\theta(t, 0) f^{(\lambda)}(\theta) \right\rangle$$

is a bounded function of t . Moreover, if $H_\theta(t) = H_0 + V(g_t(\theta))$ with $V(g_t(\theta))$ bounded and $\sup_{t,\theta} \|V(g_t(\theta))\| < \infty$, then the energy expectation

$$\left\langle U_\theta(t, 0) f^{(\lambda)}(\theta), H_0 U_\theta(t, 0) f^{(\lambda)}(\theta) \right\rangle$$

is also bounded.

Proof. Since $\tilde{K} f^{(\lambda)} = \lambda f^{(\lambda)}$ then $f^{(\lambda)}(\theta) \in \text{dom } H_\theta(0)$ a.e. θ and therefore $U_\theta(t, 0) f^{(\lambda)}(\theta) \in \text{dom } H_\theta(t)$ a.e. θ . On the other hand

$$U_\theta(t, 0) f^{(\lambda)}(\theta) = \mathcal{F}_t e^{-i\tilde{K}t} f^{(\lambda)}(\theta) = \mathcal{F}_t e^{-i\lambda t} f^{(\lambda)}(\theta) = e^{-i\lambda t} f^{(\lambda)}(g_t(\theta))$$

and from the differentiability hypothesis it follows that

$$i \frac{\partial}{\partial t} U_\theta(t, 0) f^{(\lambda)}(\theta) = \lambda e^{-i\lambda t} f^{(\lambda)}(g_t(\theta)) + i e^{-i\lambda t} \frac{d}{d\theta} f^{(\lambda)}(g_t(\theta)) \frac{d}{dt} g_t(\theta),$$

which implies that

$$i \frac{\partial}{\partial t} U_\theta(t, 0) f^{(\lambda)}(\theta) = H_\theta(t) U_\theta(t, 0) f^{(\lambda)}(\theta)$$

is bounded and the first part of the result is proved. The second one follows as in Proposition 4. \square

Corollary 3. *Suppose the hypotheses of the above theorem hold and that for each eigenvector $f^{(\lambda_n)} \in \tilde{\mathcal{K}}$ the function $\theta \mapsto f^{(\lambda_n)}(\theta)$ is C^1 . Then for μ almost every θ and for all vectors $\xi \in \mathcal{H}$ of the form*

$$\xi = a_1 f^{(\lambda_1)}(\theta) + \dots + a_k f^{(\lambda_k)}(\theta),$$

the expectation value of the energy

$$\langle U_\theta(t, 0)\xi, H_\theta(t)U_\theta(t, 0)\xi \rangle$$

is a bounded function.

In case $\xi = \sum_{n=1}^{\infty} a_n f^{(\lambda_n)}(\theta)$ with $\sum |a_n|^2 < \infty$, a sufficient condition for $U_\theta(t, 0)\xi \in \text{dom } H_\theta(t)$ and bounded energy is

$$\sum_{j=1}^{\infty} |a_j| \left(|\lambda_j| + \sup_{\theta} \|\partial_\theta f^{\lambda_j}(\theta)\| \right) < \infty,$$

since this implies that

$$t \mapsto U_\theta(t, 0)\xi = \sum_{j=1}^{\infty} a_j e^{-i\lambda_j t} f^{\lambda_j}(g_t(\theta))$$

is a C^1 function and $i\partial_t U_\theta(t, 0)$ is bounded.

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