THE MILNOR NUMBER OF A FUNCTION ON A SPACE CURVE GERM

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ABSTRACT. Given a finite function germ $f : (X, 0) \to (\mathbb{C}, 0)$ on a reduced space curve singularity $(X, 0)$, we show that $\mu(f) = \mu(X, 0) + \deg(f) - 1$. Here, $\mu(f)$ and $\mu(X, 0)$ denote the Milnor numbers of the function and the curve respectively and $\deg(f)$ is the degree of f. We use this formula to obtain several consequences related to the topological triviality and Whitney equisingularity of families of curves and families of functions on curves.

1. Introduction

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a germ of reduced space curve and let $f : (X, 0) \to (\mathbb{C}, 0)$ be a finite function germ on it. The Milnor number $\mu(f)$ has been introduced in [7] for curves in $\mathbb{C}^3$ and later in [13] for the general case. It is an invariant with the following two properties:

(1) it is preserved under simultaneous deformation of f and $(X, 0)$;
(2) if $(X, 0)$ is smooth, it coincides with the usual Milnor number.

In fact, if $(X, 0)$ is smoothable, these two conditions determine in a unique way the Milnor number $\mu(f)$, although we remark that there are curve singularities which are not smoothable [15]. It is shown in [13] that under some conditions, $\mu(f) = \tau(f)$, where $\tau(f)$ is the Tjurina number of f (i.e., the dimension of the base of a semi-universal deformation). This gives a positive answer to a conjecture stated in [7].

On the other hand, it has been observed by the first author and Jorge-Pérez in [8] that if $(X, 0)$ is smoothable, then

$$\mu(f) = \mu(X, 0) + \deg(f) - 1, \tag{1}$$

where $\mu(X, 0)$ is the Milnor number of the curve itself as defined by Buchweitz and Greuel in [3] and $\deg(f)$ is the degree of f (that is, it is the number of preimages $f^{-1}(z)$ of a generic value $z \in \mathbb{C}$ near 0). The proof follows by taking a deformation $f_t : X_t \to \mathbb{C}$, with X_t smooth, applying the Riemann-Hurwitz formula, and using the fact that $\mu(X, 0) = 1 - \chi(X_t)$, where $\chi(X_t)$ is the Euler characteristic of X_t. Of course, this is a topological argument and in order to apply it, we only need the two above conditions on $\mu(f)$.

We also remark that if $(X, 0)$ is a 1-dimensional isolated complete intersection singularity (ICIS), the above equality can be obtained easily as a consequence of the Lê-Greuel formula [2], [10]. There are also similar formulas in [4], where they consider different versions of the Milnor numbers, $\nu(f)$ and $\nu(X, 0)$, which also coincide in the case of ICIS.

In this paper, we give a simple algebraic proof of formula (1) in the general case, that is, without the assumption that $(X, 0)$ is smoothable or ICIS. In fact, the proof only uses

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the definition of the invariants and the Milnor formula $\mu(X,0)=2\delta-r+1$, where δ, r are the delta invariant and the number of branches of the curve, respectively.

We apply our formula in order to study Whitney equisingularity of a family of space curves X_t. We show that X_t is Whitney equisingular if and only if the first polar multiplicity $m_1(X_t,0)$ is constant. For a space curve germ $(X,0)\subset(\mathbb{C}^n,0)$, this invariant is defined as $m_1(X,0)=\mu(p|_{X,0})$, where $p:\mathbb{C}^n\to\mathbb{C}$ is a generic linear projection. This definition is based on the notion of d^{th} polar multiplicity of a d-dimensional ICIS [6].

Finally, we also consider families of functions on space curves $f_t:X_t\to\mathbb{C}$. We prove that f_t is topologically trivial if and only if $\mu(f_t)$ is constant and that f_t is Whitney equisingular if and only if $\mu(f_t)$ and the multiplicity of the curve $m_0(X_t,0)$ are both constant.

2. Milnor number and degree of a function on a space curve

We first recall the definition and main properties of the Milnor number of a reduced space curve, which generalizes the notion of Milnor number of a plane curve. We refer to [3] for full details and proofs.

Let $(X,0)\subset(\mathbb{C}^n,0)$ be a reduced space curve germ. We denote its local ring by $\mathcal{O}_{X,0}=\mathcal{O}_n/I$, where $\mathcal{O}_n=\mathbb{C}\{x_1,\dots,x_n\}$ is the local ring of holomorphic function germs from $(\mathbb{C}^n,0)$ to $\mathbb{C}$ and $(X,0)=V(I)$. We also denote by $\Omega_{X,0}$ the module of holomorphic 1-forms on $(X,0)$ and by $\omega_{X,0}$ the dualizing module of Grothendieck, that is,

$$\omega_{X,0}:=\operatorname{Ext}^{n-1}_{\mathcal{O}_n}(\mathcal{O}_{X,0},\Omega^n_{\mathbb{C}^n,0}),$$

where $\Omega^n_{\mathbb{C}^n,0}$ is the module of holomorphic n-forms on $(\mathbb{C}^n,0)$. We consider a normalization of the curve $n:(\bar X,\bar 0)\to(X,0)$, which induces the so-called class map $c_{X,0}:\Omega_{X,0}\to\omega_{X,0}$ as the composition

$$\Omega_{X,0}\longrightarrow\Omega_{\bar X,\bar 0}\cong\omega_{\bar X,\bar 0}\longrightarrow\omega_{X,0}.$$

The composition of the differential operator $d:\mathcal{O}_{X,0}\to\Omega_{X,0}$ with the class map gives a map which we also denote by $d:\mathcal{O}_{X,0}\to\omega_{X,0}$.

Definition 2.1. We define the *Milnor number* of a reduced curve $(X,0)\subset(\mathbb{C}^n,0)$ as

$$\mu(X,0)=\dim_{\mathbb{C}}\frac{\omega_{X,0}}{d\mathcal{O}_{X,0}}.$$

The main property of the Milnor number is that it generalizes the Milnor formula for plane curves

$$\mu(X,0)=2\delta-r+1, \tag{2}$$

where δ is the delta invariant and r is the number of branches of $(X,0)$. We recall that the delta invariant is defined as

$$\delta=\dim_{\mathbb{C}}\frac{n_*\mathcal{O}_{\bar X,\bar 0}}{\mathcal{O}_{X,0}},$$

where $n:(\bar X,\bar 0)\to(X,0)$ is the normalization of $(X,0)$.

Some easy consequences of formula (2) are the following:

(1) $\mu(X,0)\geq 0$, with equality if and only if $(X,0)$ is smooth;
(2) $\mu(X,0)\geq\delta$, with equality if and only if $(X,0)$ is an ordinary r-tuple point;
(3) $\mu(X,0)\geq r-1$, with equality if and only if $(X,0)$ is an ordinary r-tuple point;

Another important property of the Milnor number is that it can be seen as the number which measures the vanishing cohomology of $(X, 0)$. Assume that $(X, 0)$ is smoothable and let $\pi : \mathcal{X} \to D$ be a good representative (in the sense of [3]) of a smoothing of $(X, 0)$, where $0 \in D \subseteq \mathbb{C}$. Then, for $t \in D \setminus \{0\}$,

$$\mu(X, 0) = 1 - \chi(X_t) = \dim_{\mathbb{C}} H^1(X_t; \mathbb{C}),$$

where $X_t = \pi^{-1}(t)$ and $\chi(X_t)$ is the Euler characteristic of X_t. Note that the second equality follows from the fact that X_t is connected.

We give now the definition of Milnor number of a finite function germ $f : (X, 0) \to (\mathbb{C}, 0)$ on a reduced curve $(X, 0)$ (see [13]).

Definition 2.2. Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a finite function germ, where $(X, 0) \subset (\mathbb{C}^n, 0)$ is a reduced curve. The *Milnor number* of f is defined as

$$\mu(f) = \dim_{\mathbb{C}} \frac{\omega_{X,0}}{df \wedge \mathcal{O}_{X,0}}.$$

Here, $df\wedge : \mathcal{O}_{X,0} \to \omega_{X,0}$ denotes the composition of $df\wedge : \mathcal{O}_{X,0} \to \Omega_{X,0}$ with the class map $c_{X,0} : \Omega_{X,0} \to \omega_{X,0}$.

The main properties of this Milnor number are (see [13]):

(1) It is preserved under simultaneous deformation of f and $(X, 0)$.
(2) If $(X, 0)$ is smooth, it coincides with the usual Milnor number.

Note that if $(X, 0)$ is smoothable, then these two properties are enough in order to determine the Milnor number of any function.

The last ingredient in our formula is the degree of a map germ. It is a well known fact that a finite map germ $f : (X, 0) \to (\mathbb{C}^d, 0)$, from a (reduced) d-dimensional complex analytic set germ $(X, 0) \subset (\mathbb{C}^n, 0)$, has a well defined topological degree. See for instance [14] for a proof of the following result.

Lemma 2.3. *Let $f : (X, 0) \to (\mathbb{C}^d, 0)$ be a finite map germ, where $(X, 0)$ is a d-dimensional complex analytic set germ. Then, there is a representative $f : U \to V$, where U, V are open neighbourhoods of the origin in $X, \mathbb{C}^d$ respectively, and a closed analytic subset $B \subset V$ such that*

(1) *$V \setminus B$ is connected;*
(2) *f is proper and $f^{-1}(0) = \{0\}$;*
(3) *$U \setminus f^{-1}(B)$ is smooth of dimension d and $f|_{U\setminus f^{-1}(B)} : U \setminus f^{-1}(B) \to V \setminus B$ is regular.*

Definition 2.4. Let $f : (X, 0) \to (\mathbb{C}^d, 0)$ be a finite map germ, where $(X, 0)$ is a d-dimensional complex analytic set germ. Let us consider a representative $f : U \to V$ and a closed analytic subset $B \subset V$ which verify conditions (1), (2) and (3) of the above lemma. We define the *degree* of f as

$$\deg(f) = \# f^{-1}(z), \quad \forall z \in V \setminus B.$$

Since $f|_{U\setminus f^{-1}(B)}$ is a covering map and $V \setminus B$ is connected, we have that this number is well defined and does not depend on the value of z.

The degree of a finite map germ can be computed algebraically by means of the Samuel formula. We denote by $e(I; R)$ the multiplicity of an ideal I in a local Noetherian ring R. If $f : (X, 0) \to (\mathbb{C}^d, 0)$ is a finite map germ, then

$$\deg(f) = e(\langle f_1, \dots, f_d \rangle; \mathcal{O}_{X,0}),$$

where $f_1, \dots, f_d \in \mathcal{O}_{X,0}$ are the components of f (see for instance [14]). In fact, $f_1, \dots, f_d$ generate a system of parameters in $\mathcal{O}_{X,0}$. Thus, if $\mathcal{O}_{X,0}$ is Cohen-Macaulay (which will be our case), the multiplicity of the ideal is equal to the colength (see [12]),

$$\deg(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,0}}{\langle f_1, \dots, f_d \rangle}. \tag{3}$$

In the case that $(X, 0)$ is a smooth curve, its local ring is $\mathcal{O}_{X,0} = \mathbb{C}\{t\}$. Given a finite map germ $f : (X, 0) \to (\mathbb{C}, 0)$, we have

$$f(t) = a_k t^k + \text{ higher order terms},$$

for some $k \geq 1$ and $a_k \neq 0$. It follows that $\deg(f) = k = \mu(f) + 1$, since

$$df = (k a_k t^{k-1} + \dots) dt.$$

We will generalize this for singular curves $(X, 0)$ by means of the normalization. The first step is given in the following lemma.

Lemma 2.5. *Let $(X, 0)$ be a reduced curve and let $n : (\bar{X}, \bar{0}) \to (X, 0)$ be its normalization. Given a finite function germ $f : (X, 0) \to (\mathbb{C}, 0)$, we have*

$$\dim_{\mathbb{C}} \frac{\Omega_{\bar{X},\bar{0}}}{d\bar{f} \wedge \mathcal{O}_{\bar{X},\bar{0}}} = \deg(f) - r,$$

where $\bar{f} = f \circ n$ and r is the number of branches of $(X, 0)$.

Proof. We assume that $(X, 0)$ has r branches $X = X_1 \cup \dots \cup X_r$ and we also denote $\bar{X} = \bar{X}_1 \sqcup \dots \sqcup \bar{X}_r$ so that $n(\bar{X}_i) = X_i$. Note that each component $(\bar{X}_i, 0)$ is smooth and we denote its local ring by $\mathcal{O}_{\bar{X}_i,0} = \mathbb{C}\{t_i\}$.

For each $i = 1, \dots, r$, we can consider the restriction $\bar{f}_i = \bar{f}|_{\bar{X}_i,0}$. Since $\bar{f}_i \in \mathbb{C}\{t_i\}$, it can be written as

$$\bar{f}_i(t_i) = a_{k_i} t_i^{k_i} + \text{ higher order terms},$$

for some $k_i \geq 1$ such that $a_{k_i} \neq 0$. It follows from the definition of degree that

$$\deg(f) = k_1 + \dots + k_r.$$

On the other hand, it is now easy to compute the desired dimension,

$$\begin{aligned}
\dim_{\mathbb{C}} \frac{\Omega_{\bar{X},\bar{0}}}{d\bar{f} \wedge \mathcal{O}_{\bar{X},\bar{0}}} &= \dim_{\mathbb{C}} \frac{\mathbb{C}\{t_1\}dt_1 \oplus \dots \oplus \mathbb{C}\{t_r\}dt_r}{d\bar{f}_1 \wedge \mathbb{C}\{t_1\} \oplus \dots \oplus d\bar{f}_r \wedge \mathbb{C}\{t_r\}} \\
&= \dim_{\mathbb{C}} \frac{\mathbb{C}\{t_1\}dt_1}{d\bar{f}_1 \wedge \mathbb{C}\{t_1\}} + \dots + \dim_{\mathbb{C}} \frac{\mathbb{C}\{t_r\}dt_r}{d\bar{f}_r \wedge \mathbb{C}\{t_r\}} \\
&= \sum_{i=1}^{r} (k_i - 1) = \deg(f) - r.
\end{aligned}$$

□

We now state the main theorem.

Theorem 2.6. *Let $(X,0)$ be a reduced curve and let $f:(X,0)\to(\mathbb{C},0)$ be a finite function germ on X. Then,*

$$\mu(f)=\mu(X,0)+\deg(f)-1.$$

Proof. Let $n:(\bar{X},\bar{0})\to(X,0)$ be the normalization of $(X,0)$ and let us denote $\bar{f}=f\circ n$. We have a commutative diagram

$$\begin{array}{ccccc}\mathcal{O}_{X,0} & \xrightarrow{n^*} & \mathcal{O}_{\bar{X},\bar{0}} & & \\ {\scriptstyle df\wedge}\downarrow & & {\scriptstyle d\bar{f}\wedge}\downarrow & & \\ \Omega_{X,0} & \xrightarrow{n_1^*} & \Omega_{\bar{X},\bar{0}} & \xrightarrow{\bar{c}} & \omega_{X,0},\end{array}$$

from which we obtain

$$\mu(f)=\dim_{\mathbb{C}}\operatorname{coker}(\bar{c}\circ n_1^*\circ(df\wedge))=\dim_{\mathbb{C}}\operatorname{coker}(\bar{c}\circ(d\bar{f}\wedge)\circ n^*).$$

Now we use that the three morphisms $\bar{c}$, n^* and $d\bar{f}\wedge$ are in fact monomorphisms. Then we can split the dimension into three summands:

$$\mu(f)=\dim_{\mathbb{C}}\operatorname{coker}\bar{c}+\dim_{\mathbb{C}}\operatorname{coker}(d\bar{f}\wedge)+\dim_{\mathbb{C}}\operatorname{coker}n^*.$$

By definition, the last term $\dim_{\mathbb{C}}\operatorname{coker}n^*$ is equal to δ, the delta invariant of $(X,0)$. But we can see in [3] that the first term $\dim_{\mathbb{C}}\operatorname{coker}\bar{c}$ is also equal to δ. Finally, Lemma 2.5 says that $\dim_{\mathbb{C}}\operatorname{coker}(d\bar{f}\wedge)$ is equal to $\deg(f)-r$, where r is the number of branches of $(X,0)$. Hence,

$$\mu(f)=2\delta+\deg(f)-r=\mu(X,0)+\deg(f)-1,$$

where the last equality follows from the Milnor formula (2). □

Corollary 2.7. *Let $(X,0)$ be a reduced curve and let $f:(X,0)\to(\mathbb{C},0)$ be a finite function germ. We have:*

(1) $\mu(f)=0$ *if and only if* $(X,0)$ *is smooth and* f *is regular.*
(2) $\mu(f)\geq 2(r-1)$*, where* r *is the number of branches of* $(X,0)$*.*
(3) $\mu(f)\geq 2\delta$*, where* δ *is the delta invariant of* $(X,0)$*.*

Moreover, the equality holds in (2) *or* (3) *if and only if* $(X,0)$ *is an ordinary* r*-tuple point and the restriction of* f *to each branch is regular.*

Proof. We have that $\mu(f)=0$ if and only if $\mu(X,0)=0$ and $\deg(f)=1$, if and only if $(X,0)$ is smooth and f is regular. To see (2), we note that $\mu(X,0)\geq r-1$ and $\deg(f)\geq r$ with equality if and only if $(X,0)$ is an ordinary r-tuple point and the restriction of f to each branch is regular. Finally, (3) follows from the Milnor formula,

$$\mu(f)=2\delta-r+1+\deg(f)-1\geq 2\delta,$$

with equality if and only if $(X,0)$ is an ordinary r-tuple point and the restriction of f to each branch is regular. □

Corollary 2.8. *Let $(X,0)$ be a 1-dimensional ICIS, defined by $(X,0)=V(g_1,\dots,g_{n-1})$ in $(\mathbb{C}^n,0)$ and let $f:(X,0)\to(\mathbb{C},0)$ be a finite function germ. Then,*

$$\mu(f)=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{\langle g_1,\dots,g_{n-1},J\rangle},$$

where J is the Jacobian determinant of $(g_1,\dots,g_{n-1},f)$.

Proof. If $(X,0) = V(g_1,\dots,g_{n-1})$ is a 1-dimensional ICIS and $f:(X,0)\to(\mathbb{C},0)$ is finite, then $(X_1,0)=V(g_1,\dots,g_{n-1},f)$ is a 0-dimensional ICIS. Moreover, following [11] its Milnor number is equal to

$$\mu(X_1,0)=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{\langle g_1,\dots,g_{n-1},f\rangle}-1=\deg(f)-1,$$

where the last equality follows from equation (3). The result is now a consequence of the Lê-Greuel formula (see [2] and [10]),

$$\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{\langle g_1,\dots,g_{n-1},J\rangle}=\mu(X,0)+\mu(X_1,0)=\mu(X,0)+\deg(f)-1=\mu(f).$$

□

Example 2.9. Let $(X,0)\subset(\mathbb{C}^3,0)$ the space curve germ parametrized by $t\mapsto(t^3,t^4,t^5)$. Since $\mathcal{O}_{X,0}=\mathbb{C}\{t^3,t^4,t^5\}$ and $\mathcal{O}_{\bar{X},\bar{0}}=\mathbb{C}\{t\}$, we see that $\delta(X,0)=2$ and $\mu(X,0)=4$ ($(X,0)$ is irreducible, $r=1$). Let us consider the function germ $f:(X,0)\to(\mathbb{C},0)$ defined by $f(x,y,z)=x$. We have $\bar{f}(t)=t^3$, $\deg(f)=3$ and obviously $\mu(f)=6$.

We now take a smoothing X_s of $(X,0)$. To do this, we compute the equations of the curve $(X,0)=V\left(xz-y^2,x^2y-z^2,x^3-yz\right)\subset(\mathbb{C}^3,0)$. Then we can define $X_s=V(xz-y^2,x^2y-z^2+sy,x^3-yz+sx)$, which is a smooth deformation of $(X,0)$, when $s\neq0$. Moreover, we also consider $f_s:X_s\to\mathbb{C}$, $f_s(x,y,z)=x$, the corresponding deformation of f.

It is not difficult to see that f_s has three critical points $p_1=(0,0,0)$, $p_2=(\sqrt{-s},0,0)$ and $p_3=(-\sqrt{-s},0,0)\}$. Moreover, we also compute the corresponding parametrization of X_s near p_i and deduce that $\bar{f}_s(t)$ is equivalent to t^3 at p_i, $i=1,2,3$. This implies $\mu(f_s,p_i)=2$ and thus,

$$\mu(f_s,p_1)+\mu(f_s,p_2)+\mu(f_s,p_3)=6=\mu(f).$$

3. Equisingularity of families of space curves

In [6], Gaffney defines the d^{th} polar multiplicity of a d-dimensional ICIS $(X,0)$ as

$$m_d(X,0)=\dim_{\mathbb{C}}\frac{\mathcal{O}_{X,0}}{J(f_1,\dots,f_{n-d},p)},$$

where $(X,0)=V(f_1,\dots,f_{n-d})$ in $(\mathbb{C}^n,0)$, $p:\mathbb{C}^n\to\mathbb{C}$ is a generic linear projection and $J(h)$ denotes the ideal generated by the maximal minors of the Jacobian matrix of a map germ h. If $d=1$, it follows from Corollary 2.8 that $m_1(X,0)=\mu(p|_{X,0})$. Hence, it makes sense the following definition.

Definition 3.1. We define the *first polar multiplicity* of a reduced curve $(X,0)\subset(\mathbb{C}^n,0)$ as

$$m_1(X,0)=\mu(p|_{X,0}),$$

where $p:\mathbb{C}^n\to\mathbb{C}$ is a generic linear projection.

We recall that the multiplicity of a d-dimensional complex analytic set germ $(X,0)$ is defined as

$$m_0(X,0)=e(\mathfrak{m}_{X,0};\mathcal{O}_{X,0}),$$

where $\mathfrak{m}_{X,0}$ is the maximal ideal of $\mathcal{O}_{X,0}$. If we assume $(X,0)\subset(\mathbb{C}^n,0)$, the multiplicity is computed by taking a reduction of $\mathfrak{m}_{X,0}$ which is generated by d generic linear forms $p_1,\dots,p_d:\mathbb{C}^n\to\mathbb{C}$ (see [12]). Hence,

$$m_0(X,0)=e(\mathfrak{m}_{X,0};\mathcal{O}_{X,0})=e(\langle p_1,\dots,p_d\rangle;\mathcal{O}_{X,0})=\deg(p|_{X,0}),$$

where $p=(p_1,\dots,p_d):\mathbb{C}^n\to\mathbb{C}^d$ and the last equality follows from Samuel formula. In the case of a curve ($d=1$), we obtain the following immediate consequence of Theorem 2.6.

Corollary 3.2. *Given a reduced curve $(X,0)$, we have*

$$m_1(X,0)=\mu(X,0)+m_0(X,0)-1.$$

Moreover, if $(X,0)$ is smoothable and X_t is a smooth deformation,

$$\chi(X_t)=m_0(X,0)-m_1(X,0).$$

Remark 3.3. If $(X,0)$ is a d-dimensional ICIS, a recursive application of the Lê-Greuel formula gives that

$$m_0(X,0)-m_1(X,0)+\cdots+(-1)^d m_d(X,0)=1+(-1)^d\mu(X,0)=\chi(X_t),$$

where $m_i(X,0)$ is the i^{th} polar multiplicity and X_t is a smooth Milnor fibre (see [6] and [9]). Note that these formulas coincide with the formulas of Corollary 3.2 if $d=1$.

By a (flat) family of space curves we mean a reduced 2-dimensional complex analytic set germ $(\mathcal{X},0)\subset(\mathbb{C}\times\mathbb{C}^n,0)$ such that the projection $\pi:(\mathcal{X},0)\to(\mathbb{C},0)$, given by $\pi(t,x)=t$ is flat. We denote by $X_t\subset\mathbb{C}^n$ the space curve defined by $X_t=p_2(\pi^{-1}(t))$, where $p_2:\mathbb{C}\times\mathbb{C}^n\to\mathbb{C}^n$ is the projection $p_2(t,x)=x$. We assume that $(X_0,0)$ is also reduced. We say that $(\mathcal{X},0)$ is a (flat) deformation of $(X_0,0)$. We also assume that the family is localized at the origin, so that for each t we can consider the singularity $(X_t,0)$.

Definition 3.4. Let $(\mathcal{X},0)$ be a deformation of a reduced curve $(X,0)$.

(1) We say that $(\mathcal{X},0)$ is *good* if there is a representative $\mathcal{X}$ defined in $D\times U$, where D,U are open neighbourhoods of the origin in $\mathbb{C},\mathbb{C}^n$ respectively, such that $X_t\setminus\{0\}$ is smooth, for any $t\in D$.
(2) We say that $(\mathcal{X},0)$ is *μ-constant* if $\mu(X_t,0)$ is constant.
(3) We say that $(\mathcal{X},0)$ is *topologically trivial* if there is a homeomorphism germ $\Phi:(\mathcal{X},0)\to(\mathbb{C}\times X_0,0)$ such that $\pi\circ\Phi=\pi$.
(4) We say that $(\mathcal{X},0)$ is *Whitney equisingular* if there is a representative which admits a regular stratification so that the parameter axis $S=D\times\{0\}\subset\mathbb{C}\times\mathbb{C}^n$ is a stratum.

The condition for the family to be good means that the curves X_t have isolated singularity "uniformly", that is, in a neighbourhood U which does not depend on t. It follows that any topologically trivial family is good. Moreover, according to the first isotopy lemma of Thom, any Whitney equisingular family is topologically trivial. The converse is not true, see [3].

The main problem in equisingularity theory is to find numerical invariants whose constancy in the family implies topological triviality or Whitney equisingularity. In the case of families of space curves $(\mathcal{X},0)$ the answer to this problem is given by the Milnor number and the multiplicity (see [1] and [3]):

(1) If $(\mathcal{X},0)$ is μ-constant, then it is good.

(2) $(\mathcal{X},0)$ is topologically trivial if and only if it is μ-constant.
(3) $(\mathcal{X},0)$ is Whitney equisingular if and only if it is μ-constant and $m_0(X_t,0)$ is also constant.

Since $\mu(X_t,0)$ and $m_0(X_t,0)$ are both upper semi-continuous, we have that their constancy is equivalent to the constancy of $m_1(X_t,0)$ by Corollary 3.2. Thus, we can control the Whitney equisingularity with just one invariant.

Corollary 3.5. *A family of space curves $(\mathcal{X},0)$ is Whitney equisingular if and only if $m_1(X_t,0)$ is constant.*

Remark 3.6. Again this result can be extended to higher dimensions if we restrict ourselves to ICIS. Let X_t be a family of d-dimensional ICIS. According to [6], it is Whitney equisingular if and only if all the polar multiplicities are constant. By using the formula of Remark 3.3, we can reduce the number of invariants. If d is odd, it is Whitney equisingular if and only if the odd polar multiplicities $m_1(X_t,0), m_3(X_t,0),\dots$ are constant. If d is even, then it is Whitney equisingular if and only if $\mu(X_t,0)$ and $m_1(X_t,0), m_3(X_t,0),\dots$ are constant (see [9]).

4. Equisingularity of families of functions on space curves

In the last section, we consider families of functions on space curve germs. Given $f:(X,0)\to(\mathbb{C},0)$ we will take deformations of both $(X,0)$ and f. We will use the Milnor number $\mu(f)$ to characterize topological triviality and Whitney equisingularity in a similar way to the case of families of curve germs. We also note that when working with deformations of map germs f_t, it is more helpful to consider the associated unfolding $F(t,x)=(t,f_t(x))$.

Definition 4.1. Let $f:(X,0)\to(\mathbb{C},0)$ be a finite function germ on a reduced curve $(X,0)$. An *unfolding* of f is a map germ $F:(\mathcal{X},0)\to(\mathbb{C}\times\mathbb{C},0)$ such that $(\mathcal{X},0)$ is a deformation of $(X,0)$, F is given by $F(t,x)=(t,f_t(x))$ and $f_0=f$. We will say that F is *origin preserving* if $f_t(0)=0$ for any t, so that we have an induced family of function germs $f_t:(X_t,0)\to(\mathbb{C},0)$. We will assume that all the unfoldings are origin preserving, unless otherwise specified.

Let $f:\mathbb{C}^n\to\mathbb{C}^p$ be a complex analytic map and let $A\subset\mathbb{C}^n$, $A'\subset\mathbb{C}^p$ be subsets such that $f(A)\subseteq A'$. A *stratification* of $f:A\to A'$ is a pair $(\mathcal{A},\mathcal{A}')$ of stratifications of A,A' respectively such that f maps strata submersively to strata. The stratification $(\mathcal{A},\mathcal{A}')$ is said to be *regular* if $\mathcal{A},\mathcal{A}'$ satisfy the Whitney regularity conditions and if any stratum $Y\in\mathcal{A}$ satisfies Thom's condition A_f over any other stratum $X\in\mathcal{A}$.

Definition 4.2. Let $F:(\mathcal{X},0)\to(\mathbb{C}\times\mathbb{C},0)$ be an unfolding of a finite function germ $f:(X,0)\to(\mathbb{C},0)$ on a reduced curve $(X,0)$.

(1) We say that F is *good* if there is a representative defined in $D\times U$, where D,U are open neighbourhoods of the origin in $\mathbb{C},\mathbb{C}^n$ respectively, such that $X_t\setminus\{0\}$ is smooth and f_t is regular on $X_t\setminus\{0\}$, for any $t\in D$.
(2) We say that F is *μ-constant* if $\mu(f_t,0)$ is constant.
(3) We say that F is *topologically trivial* if there are homeomorphism map germs:

$$\Phi:(\mathcal{X},0)\to(\mathbb{C}\times X,0),\qquad \Phi(t,x)=(t,\phi_t(x)),$$
$$\Psi:(\mathbb{C}\times\mathbb{C},0)\to(\mathbb{C}\times\mathbb{C},0),\qquad \Psi(t,y)=(t,\psi_t(y)),$$

such that $F=\Psi\circ G\circ\Phi$, where $G(t,x)=(t,f(x))$ is the trivial unfolding of f.

(4) We say that F is *Whitney equisingular* if there is a representative which admits a regular stratification so that the parameter axes $S = D \times \{0\} \subset \mathbb{C} \times \mathbb{C}^n$ and $T = D \times \{0\} \subset \mathbb{C} \times \mathbb{C}$ are strata.

Again, by using an appropriate version of Thom's second isotopy lemma for complex analytic maps, it follows that any Whitney equisingular unfolding is topologically trivial (see [5, theorem 6.1]). In the case of functions on space curve germs the relations between these concepts are summarized in the following theorem.

Theorem 4.3. *Let $F : (\mathcal{X}, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ be an unfolding of a finite function germ $f : (X, 0) \to (\mathbb{C}, 0)$ on a reduced curve $(X, 0)$.*

(1) *If F is μ-constant, then it is good.*
(2) *F is topologically trivial if and only if it is μ-constant.*
(3) *F is Whitney equisingular if and only if it is μ-constant and $m_0(X_t, 0)$ is also constant.*

Proof. The first part follows from the fact that the Milnor number $\mu(f)$ is preserved under simultaneous deformations of both f and $(X, 0)$ (see [13]). Let $F : \mathcal{X} \to \mathbb{C} \times \mathbb{C}$ be a representative defined in $D \times U$, where D, U are open neighbourhoods of the origin in $\mathbb{C}, \mathbb{C}^n$ respectively. By shrinking D and U if necessary, we can assume that for any $t \in D$,

$$\mu(f) = \sum_{x \in X_t} \mu(f_t, x).$$

If F is μ-constant, then $\mu(f) = \mu(f_t, 0)$ and hence, $\mu(f_t, x) = 0$ for any $x \in X_t \setminus \{0\}$ and any $t \in D$. But this implies by Corollary 2.7 that $X_t \setminus \{0\}$ is smooth and f_t is regular on $X_t \setminus \{0\}$.

Let us see part (2). If F is topologically trivial, then $(\mathcal{X}, 0)$ is a topologically trivial family of space curves and hence, $\mu(X_t, 0)$ is constant. Moreover, there are homeomorphisms Φ, Ψ such that $\Phi(t, x) = (t, \phi_t(x))$, $\Psi(t, y) = (t, \psi_t(y))$ and $F = \Psi \circ (\mathrm{id} \times f) \circ \Phi$. For any t, this means that $f_t = \psi_t \circ f \circ \phi_t$, which implies $\deg(f_t, 0) = \deg(f, 0)$ and hence, $\mu(f_t, 0) = \mu(f, 0)$ by Theorem 2.6.

Assume now that F is μ-constant. Again by Theorem 2.6 this implies the constancy of $\mu(X_t, 0)$ and $\deg(f_t, 0)$. Since $(\mathcal{X}, 0)$ is a μ-constant family of space curves, it is also δ-constant and it admits a normalization in family (see [3] and [16]). This means that there is a map $N : (\mathbb{C} \times \bar{X}, \bar{0}) \to (\mathcal{X}, 0)$ given by $N(t, x) = (t, n_t(x))$ such that for any t, $n_t : (\bar{X}, \bar{0}) \to (X_t, 0)$ is the normalization of $(X_t, 0)$.

The composition $F \circ N : (\mathbb{C} \times \bar{X}, \bar{0}) \to (\mathbb{C} \times \mathbb{C}, 0)$ is an unfolding of a function multigerm $f \circ n_0 : (\bar{X}, \bar{0}) \to (\mathbb{C}, 0)$, whose domain $(\bar{X}, \bar{0})$ is smooth and with constant degree on each component. It is well known that such a family is analytically trivial, that is, there is a diffeomorphism multigerm $\bar{\Phi} : (\mathbb{C} \times \bar{X}, \bar{0}) \to (\mathbb{C} \times \bar{X}, \bar{0})$, defined by $\bar{\Phi}(t, x) = (t, \bar{\phi}_t(x))$ such that $f_t \circ n_t \circ \bar{\phi}_t = f \circ n_0$.

Now $\bar{\Phi}$ induces a unique map germ Φ such that the following diagram is commutative:

$$\begin{array}{ccccc}
(\mathbb{C} \times \bar{X}, \bar{0}) & \xrightarrow{N} & (\mathcal{X}, 0) & \xrightarrow{F} & (\mathbb{C} \times \mathbb{C}, 0) \\
{\scriptstyle \bar{\Phi}} \uparrow & & {\scriptstyle \Phi} \uparrow & & \\
(\mathbb{C} \times \bar{X}, \bar{0}) & \xrightarrow{\mathrm{id} \times n_0} & (\mathbb{C} \times X, 0) & &
\end{array}$$

In fact, Φ has the form $\Phi(t,x) = (t,\phi_t(x))$ and is defined outside the origin by $\phi_t = n_t \circ \bar{\phi}_t \circ n_0^{-1}$. It follows that Φ is a homeomorphism germ which trivializes F, that is, $f_t \circ \phi_t = f$, for any t.

We finish by showing part (3). If F is Whitney equisingular, then it is topologically trivial and hence μ-constant, by (2). Moreover, $(\mathcal{X},0)$ becomes a Whitney equisingular family of space curves which implies that $m_0(X_t,0)$ is also constant.

Conversely, assume that F is μ-constant and $m_0(X_t,0)$ is also constant. By (1), F is good and hence, there is a representative $F:\mathcal{X}\to\mathbb{C}\times\mathbb{C}$ defined in $D\times U$, where D,U are open neighbourhoods of the origin in $\mathbb{C},\mathbb{C}^n$ respectively, such that $X_t\setminus\{0\}$ is smooth and f_t is regular on $X_t\setminus\{0\}$.

We have now a stratification $(\mathcal{A},\mathcal{A}')$ of F given by

$$\mathcal{A}=\{\mathcal{X}\setminus S,S\},\quad \mathcal{A}'=\{F(\mathcal{X})\setminus T,T\}.$$

Since $F(\mathcal{X})$ is open, it is obvious that $\mathcal{A}'$ satisfies the Whitney regularity conditions. Moreover, F restricted to each stratum is a diffeomorphism, so that the Thom A_F condition is also satisfied trivially. Finally, $\mathcal{A}$ is also Whitney regular, since $(\mathcal{X},0)$ is a Whitney equisingular family of space curves. Just note that the constancy of $\mu(f_t,0)$ also implies the constancy of $\mu(X_t,0)$ by Theorem 2.6 and the upper semi-continuity of the invariants. □

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