CANTOR SINGULAR CONTINUOUS SPECTRUM FOR OPERATORS ALONG INTERVAL EXCHANGE TRANSFORMATIONS

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Abstract. It is shown that Schrödinger operators, with potentials along the shift embedding of Lebesgue almost every interval exchange transformations, have Cantor spectrum of measure zero and pure singular continuous for Lebesgue almost all points of the interval.

1. Introduction and Main Results

In [7] the spectrum $\sigma(H_\omega)$ of the discrete Schrödinger operators $H_\omega : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$,

$$(H_\omega\psi)_j = \psi_{j+1} + \psi_{j-1} + \omega_j\psi_j, \tag{1}$$

with $\omega = (\omega_j)_{j\in\mathbb{Z}}$ a sequence of real numbers (the so-called potential) modulated along the shift embedding of interval exchange transformations (iets) [10, 11] was investigated. There it was proved the presence of pure singular continuous spectrum for H_ω, for the shift associated with a dense set of iets and for a.e. points of the interval (a.e. with no specification means *almost everywhere* with respect to Lebesgue measure). In this work we give a step further by showing that the above mentioned results hold for Lebesgue almost every iets. The proof of absence of eigenvalues involves a rather different argument, mainly the Rauzy induction map. As a by-product of our a.e. results, which are summarized in Theorem 1 ahead, there is a set of interesting results in the literature ready to be applied, which will lead to Cantor spectrum of zero Lebesgue measure. We next recall some notations and a description of the iets necessary to state and prove our results. Let π be a permutation of the symbols $\{1, 2, \ldots, n\}$ and

$$\mathbf{a} = \{a = a_0 < a_1 < a_2 < \cdots < a_n = b\}$$

be a partition of the interval $[a, b)$. An iet $E\colon [a, b) \rightarrow [a, b)$ is associated to the pair $(\pi, \mathbf{a})$ by cutting the interval $[a, b)$ in n semi-open intervals

$$I_1 := [a_0, a_1), I_2 := [a_1, a_2), \ldots, I_n := [a_{n-1}, a_n)$$

(which are naturally ordered from left to right) and then exchanging their positions according to the permutation π in such a way that the interval in the i^{th} position, I_i, is translated to the $\pi(i)^{th}$ position (from left to right). In this way the transformation obtained is of the form

$$E(x) = x + d_i, \quad x \in I_i, \quad i = 1, 2, \ldots, n,$$

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for some numbers $d_1, \ldots, d_n$. We will consider the symbolic coding, that is, the map $\mathcal{A}_E : [a, b) \to \{1, 2, \ldots, n\}$ given by

$$\mathcal{A}_E(x) = i \quad \text{if and only if} \quad x \in I_i,$$

and write simply $\mathcal{A}$ in situations where E is clear from the context. In this work we consider mostly interval exchanges of the form $E : [0, b) \to [0, b), b > 0$. E is said to be normalized if $b = 1$. Denote by P_n the set of all permutations of the symbols $\{1, 2, \ldots, n\}$ and by G_n the set of *irreducible* permutations of P_n, i.e, those permutations π for which $\pi(\{1, 2, \ldots, k\}) \neq \{1, 2, \ldots, k\}$ unless $k = n$. Δ^{n-1} will denote the standard simplex of $\mathbb{R}^n$, i.e.,

$$\Delta^{n-1} := \left\{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n_+ : \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\right\}.$$

Λ_n will denote the set of all partitions of $[0, 1)$. In the standard simplex Δ^{n-1} we will consider the $(n-1)$-dimensional Lebesgue measure. There relations $\lambda_i = a_i - a_{i-1}$, $1 \leq i \leq n$ determine a one-to-one correspondence between partitions $\mathbf{a} \in \Lambda_n$ and vectors $\lambda \in \Delta^{n-1}$.

Identify the product $P_n \times \Delta^{n-1}$ with the set of all interval exchanges of n intervals. For a (fixed) irreducible permutation $\pi \in G_n$, denote by $E(\pi)$ the set of all iets $E\colon [0, 1) \to [0, 1)$ with permutation π. We identify the metric spaces Δ^{n-1} and $E(\pi)$ by the homeomorphism $\Delta^{n-1} \ni \lambda \mapsto E_\lambda := (\pi, \lambda)$. Denote the E_λ-orbit of $x \in [0, 1)$ by

$$\mathcal{O}_\lambda(x) = \left\{E^k_\lambda(x) : k \in \mathbb{Z}\right\}.$$

Let $\Sigma_n := \{1, 2, \ldots, n\}^{\mathbb{Z}}$. Associated to each orbit is $\phi_\lambda(x) \in \Sigma_n$ given by $\phi_\lambda(x) := \mathcal{A}(\mathcal{O}_\lambda(x))$; i.e., $\phi_\lambda(x)$ is a natural coding of the E_λ-orbit of x by assigning to each entry of this orbit the number of the interval which contains it. The same concept is defined for intervals: take numbers $j, k \in \mathbb{Z}$ with $j \leq k$, suppose that $I \subset [a, b)$ is a nonempty interval (which may be reduced to a point) such that, for all integer $i \in [j, k]$, $E^i|_I$ is continuous; then the sequence

$$\mathcal{A}(E^j(I))\, \mathcal{A}(E^{j+1}(I)) \cdots \mathcal{A}(E^k(I))$$

will be said to be the *E-itinerary* of I associated to $[j, k]$. Set

$$\Omega_\lambda = \text{closure } \{\phi_\lambda([0, 1))\}$$

in Σ_n, so that Ω_λ with the left shift dynamics is a subshift over the alphabet $\{1, 2, \ldots, n\}$. Given $\omega \in \Sigma_n$ and an injective map $V : \{1, 2, \cdots, n\} \to \mathbb{R}$, consider the potential $V(\omega) := (V(\omega_j))_{j \in \mathbb{Z}}$ and the operator $H_{V(\omega)}$ as in (1). E_λ is aperiodic if no sequence in Ω_λ is periodic. A nonempty set in a metric space is a *Cantor set* if it is closed with empty interior and no isolated points.

Theorem 1. *Fix $\pi \in G_n$. Given V as above, there is a subset $\mathcal{F} \subset E(\pi)$ of full Lebesgue measure so that:*

(i) *each $E_\lambda \in \mathcal{F}$ is minimal, aperiodic and uniquely ergodic;*
(ii) *for each $E_\lambda \in \mathcal{F}$, the spectrum of $H_{V(\omega)}$ in (1) is the same for all $\omega \in \Omega_\lambda$ and it is a Cantor set of zero Lebesgue measure.*
(iii) *for each $E_\lambda \in \mathcal{F}$ the corresponding Schrödinger operators (1) with potentials $V(\phi_\lambda(x))$ have pure singular continuous spectrum for a.e. $x \in [0, 1)$.*

We recall that for $n = 2$ and $\pi(1, 2) = (2, 1)$ there is only one discontinuity point $a_1 \in [0, 1)$ and the system is reduced to rotations of the circle by the angle $(1 - a_1)$. In this case the potentials Ω_E are the Sturmian sequences [1, 5], which include the well-known Fibonacci substitution sequence [13, 16]. Hence, the potentials generated by iets are natural generalizations of Sturmian potentials, one of the standard models of one-dimensional quasicrystals. We refer the reader to [7] for additional comments and to [6], as well as references therein, for related examples of Cantor zero measure spectrum. We close this section with an open question we have found interesting. Although almost every iets are minimal and uniquely ergodic [12, 17], there are cases of minimal iets with more than one ergodic component ($n/2$ is an upper bound for the number of ergodic probability measures [11, 12]), so it is natural to ask if the characterization of the spectrum in Theorem 1 holds in such cases.

2. Proof of Theorem 1

The proof of this theorem will be reduced to the proof that, given an irreducible permutation $\pi \in G_n$, for almost every iet the corresponding Schrödinger operator $H_{V(\phi(x))}$ has no eigenvalues for Lebesgue almost every $x \in [0, 1)$. In this section we clarify such statement and next sections are devoted to the related proof of absence of eigenvalues. Given a subshift Ω over the finite alphabet $\mathcal{B}$ (the dynamics is always given by the left shift), Boshernitzan [2, 3] introduced a condition which later was called condition (B) in [6]. The set of finite words in Ω is

$$W = W(\Omega) = \{\omega(j)\cdots\omega(j+n-1) : j \in \mathbb{Z}, n \in \mathbb{N}, \omega \in \Omega\},$$

and for $w \in W$ denote by $|w|$ its length and by

$$V_w = V_w(\Omega) = \{\omega \in \Omega : \omega(1)\cdots\omega(|w|) = w\}$$

the cylinders. For each invariant probability measure μ on Ω set

$$\eta_\mu(n) = \min\{\mu(V_w) : w \in W, |w| = n\}.$$

The subshift Ω satisfies the *condition* (B) if there exists an ergodic probability measure μ on Ω with

$$\limsup_{n\to\infty} n\,\eta_\mu(n) > 0.$$

This condition was shown by Boshernitzan [4] to imply unique ergodicity for minimal subshifts, and in the particular case of iets also by Veech [18]. Recall also the following basic result:

Lemma 1. [10] *If E_λ is minimal, then Ω_λ is a minimal subshift, i.e., every $\omega \in \Omega_\lambda$ has dense orbit in Ω_λ.*

With respect to the spectrum of discrete Schrödinger operators, the following important result was proved in [6]:

Theorem 2. *Let Ω be a minimal subshift which satisfies condition (B). If Ω is aperiodic, then there exists a Cantor set $\Sigma \subset \mathbb{R}$ of Lebesgue measure zero so that the spectrum $\sigma(H_\omega) = \Sigma$ for every $\omega \in \Omega$.*

For iets Boshernitzan [4] has proved

Theorem 3. *Let $\pi \in G_n$. Then for Lebesgue almost every $\lambda \in \Delta^{n-1}$ the subshift Ω_λ satisfies condition (B).*

Hence, by merging the above theorems one concludes part of Theorem 1. In fact, we have:

Theorem 4. *Let $\pi \in G_n$ and V as in Theorem 1. Then there is a subset $\mathcal{L} \subset E(\pi)$ of full Lebesgue measure so that:*

(i) *each $E_\lambda \in \mathcal{L}$ is minimal, aperiodic and uniquely ergodic;*
(ii) *for each $E_\lambda \in \mathcal{L}$ the spectrum of $H_{V(\omega)}$ in (1) is the same for all $\omega \in \Omega_\lambda$, and it is a Cantor set of zero Lebesgue measure.*
(iii) *for each $E_\lambda \in \mathcal{L}$ the corresponding Schrödinger operators (1) with potentials $V(\phi_\lambda(x))$ have no absolutely continuous spectrum for all $x \in [0,1)$.*

Proof. (i) It is well known that the set of minimal iets has full Lebesgue measure, and each of them is necessarily aperiodic. By Theorem 3 the set of iets that satisfy condition (B) also has full measure. Define $\mathcal{L}$ as the intersection of such sets. Then $\mathcal{L}$ has full Lebesgue measure and each of them is uniquely ergodic [4]. (ii) By minimality of $E_\lambda \in \mathcal{L}$ the spectrum is the same set for all $\omega \in \Omega_\lambda$ and by Theorem 2 it is a Cantor set of zero Lebesgue measure. (iii) By (ii) the spectrum has zero Lebesgue measure and so no absolutely continuous component. □

Now, in order to complete the proof of Theorem 1 it is enough to show that, given $\pi \in G_n$, there is a set $\mathcal{P} \subset E(\pi)$ of full Lebesgue measure so that for each $E_\lambda \in \mathcal{P}$ the corresponding Schrödinger operators $H_{V(\phi_\lambda(x))}$ have no eigenvalues for Lebesgue almost every $x \in [0,1)$; thus only singular continuous spectrum remains. Then the set $\mathcal{F}$ in Theorem 1 can be defined by the intersection of this set $\mathcal{P}$ with $\mathcal{L}$. Our final task is to exclude eigenvalues. An important tool to exclude eigenvalues for a given operator H_ω, $\omega \in \Sigma_n$, is the Delyon-Petritis [8] version of an argument of Gordon [9], by means of suitable local word repetitions.

Theorem 5. [8] *If for given $\omega \in \Sigma_n$ there exists a sequence $k_i \to \infty$ such that*

$$\omega_{j-k_i} = \omega_j = \omega_{j+k_i},$$

for all $1 \le j \le k_i$, then the Schrödinger operator H_ω in (1) has no eigenvalues.

Given an irreducible permutation π, the idea is to show that, for almost all $\lambda \in \Delta^{n-1}$ and E_λ, Theorem 5 applies to H_ω, $\omega = \phi_\lambda(x)$, with x in a set of total Lebesgue measure over $[0,1)$. In other words, we have to prove that for almost all $x \in [0,1)$ there is a sequence (r_k) of natural numbers, such that the itinerary of x associated to $[-r_k, 2r_k]$ if of the form

$$w_0w_1 \dots w_{r_k} \quad w_0w_1 \dots w_{r_k} \quad w_0w_1 \dots w_{r_k}, \quad w_i \in \{1,2,\dots,n\}.$$

Then we will prove

Propositon 1. *Fix an irreducible permutation $\pi \in G_n$. Then for almost all $\lambda \in \Delta^{n-1}$, the corresponding iet E_λ is minimal and, for a.e. $x \in [0,1)$, the coding $\phi_\lambda(x)$ satisfies the hypotheses of Theorem 5 and so, the operator $H_{\phi_\lambda(x)}$ has empty point spectrum.*

To prove this proposition we will use the well-known *Rauzy Renormalization Operator* in the space of interval exchange transformations. In the next section we will introduce the appropriate definitions and preliminary results; the end of the proof of this proposition is delayed until Section 5.

3. Rauzy's Renormalization

Let $E : [a,b) \to [a,b)$ be an iet and $J := [c,d)$ a proper subinterval of $[a,b)$. Let us denote by E_J the *Poincaré's first return map* of E to the interval J, that is, for

$x \in J$, $E_J(x)$ is given by the first point in the positive orbit of x (by E) that returns to the interval J. E_J is again an iet, that is, there is a partition $\mathbf{a}' = (a_0, a_1, \dots, a_p)$ of $[c, d)$ and a permutation $\pi' \in G_p$ such that $E_J = (\pi', \mathbf{a}')$. In general the number of intervals of continuity increases: if E exchanges n intervals then $p \geq n$. E_J will be called the *induced map of E on the interval J*. Let $I_1, I_2, \dots, I_p$ be the intervals of continuity of E_J. For each $1 \leq k \leq p$ there exists an integer $r_k > 0$ such that

$$E(I_k), E^2(I_k), \dots, E^{r_k}(I_k)$$

are all intervals disjoints of J whereas $E^{r_k+1}(I_k)$ is totally contained in J. By definition $E_J(I_k) = E^{r_k+1}(I_k)$. The number r_k is called the *return time of I_k to J*. We want to remark that, by the bijectivity of E, the whole interval $[a, b)$ is given by the union

$$[a, b) = \bigcup_{k=1}^{p} \bigcup_{j=1}^{r_k} E^j(I_k). \tag{2}$$

Rauzy's map. Take $(\pi, \lambda) \in G_n \times \Delta^{n-1}$ and let E be the (π, λ)-interval exchange.

Consider $\nu = \nu(\pi, \lambda)$ defined as the minimum between λ_n and $\lambda_{\pi^{-1}(n)}$ provided that these numbers are different. If E_J is the induced map of E on the interval $J := [0, 1 - \nu)$, $\nu > 0$, it is proved in [14, 15] that E_J is an iet of exactly n intervals associated to a new irreducible permutation $\pi' \in G_n$ and partition $\mathbf{b}' = (b_0, b_1, \dots, b_n)$ of J. Then, by normalizing E_J, we obtain a pair $(\pi', \lambda') \in G_n \times \Delta^{n-1}$. Indeed

$$\lambda' := \frac{1}{1 - \nu} (b_1, b_2 - b_1, \dots, b_n - b_{n-1}) \in \Delta^{n-1}.$$

Rauzy's Renormalization map is the association $(\pi, \lambda) \xrightarrow{R} (\pi', \lambda')$. We remark that such association is not defined when $\lambda_n = \lambda_{\pi^{-1}(n)}$. **The domain of R.** It is

proved in [14] that G_n is divided into several subsets called *Rauzy Classes* which are invariant by the process of induction just defined. Let us denote by $\mathcal{C}$ one of the Rauzy classes of G_n. Then if $\pi \in \mathcal{C}$ and $R(\pi, \lambda) = (\pi', \lambda')$ then $\pi' \in \mathcal{C}$. Set

$$\Delta_{\mathcal{C}}^{n-1} := \left\{ (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta^{n-1} : \lambda_n \neq \lambda_{\pi^{-1}(n)}, \, \pi \in \mathcal{C} \right\}.$$

Then the transformation $R(\pi, \lambda) = (\pi', \lambda')$ is well defined in $\mathcal{C} \times \Delta_{\mathcal{C}}^{n-1}$. In $\mathcal{C} \times \Delta^{n-1}$ there is a natural measure m which is the product of the counting measure in $\mathcal{C}$ and the $(n-1)$-dimensional Lebesgue measure in Δ^{n-1}. As the set $\Delta_{\mathcal{C}}^{n-1}$ have total measure in Δ^{n-1} (regarding the $(n-1)$-dimensional Lebesgue measure), the map R is defined m-almost everywhere in $\mathcal{C} \times \Delta^{n-1}$ and by abuse of language it is usually written

$$R : \mathcal{C} \times \Delta^{n-1} \to \mathcal{C} \times \Delta^{n-1}.$$

The Rauzy's transformation plays a central role in the ergodic theory of iets due to the following result [17, 12].

Theorem 6. *The Rauzy's operator is ergodic with respect to a measure absolute continuous with respect to m (the natural measure described above).*

Another related result proved in [17, 12] is

Theorem 7. *For a fixed permutation $\pi \in \mathcal{C}$ and for almost every vector $\lambda \in \Delta^{n-1}$, the R-orbit of E_λ is dense in $\mathcal{C} \times \Delta^{n-1}$.*

Let us show now a relation between the process of renormalization and the important property stated in Lemma 1. Let $E\colon [0,1) \to [0,1)$ be an interval exchange and let E_J be the induced map of E on $J = [a,b) \subset [0,1)$. Suppose that $I \subset J$ is an interval of continuity of E_J and that, for some $x \in I$, it happens that $E_J(x) \in I$ and $E_J^{-1}(x) \in I$. If r is the return time of I to J, the itinerary of every point in I associated to $[0,r]$ is given by the word

$$w_0 w_1 \dots w_r := \mathcal{A}(I)\mathcal{A}(E(I))\mathcal{A}(E^2(I))\dots\mathcal{A}(E^r(I)).$$

As $E_J(x) \in I$ and $E_J^{-1}(x) \in I$, the itinerary of x associated to $[-r, 2r]$ will be given by the word

$$w_0 w_1 \dots w_r \quad w_0 w_1 \dots w_r \quad w_0 w_1 \dots w_r. \tag{3}$$

A point $x \in [a,b)$ with this property will be called *a candidate point in the interval* $[a,b)$ *for the length* r. $C_N([a,b))$ will be the set of candidate points in $[a,b)$ for the lengths $r \leq N$. Note that if there is a nested sequence of intervals shrinking to a point x,

$$[a_1, b_1) \supset [a_2, b_2) \supset \dots [a_k, b_k) \supset \dots$$

and such that x is a candidate point for each $[a_k, b_k)$ for the length r_k, then necessarily $r_k \to \infty$ as $k \to \infty$, and the itinerary of x associated to its whole orbit, $\phi_E(x)$, is given by a sequence that satisfies the hypotheses of Lemma 1. From now on, π_n we will denote the permutation in G_n given by

$$\pi_n(j) = n - j + 1, \quad j = 1, 2, \dots, n.$$

Let us denote by $|C|$ the Lebesgue measure of a measurable set $C \subset \mathbb{R}$.

Lemma 2. *Let* $E : [0,1) \to [0,1)$ *be an iet and let* E_J *be the induced map of* E *on the interval* $J = [a,b) \subset [0,1)$. *Suppose that* E_J *is associated with the permutation* π_n *and that the first interval of continuity of* E_J *is of the form* $I_1 = [a, b-\delta)$, *where* $0 < \delta < (b-a)/4$. *Let* r_1 *be the return time of* I_1 *to* J. *Then the Lebesgue measure of* $C_{r_1}([a,b))$ *satisfies*

$$\frac{|C_{r_1}([a,b))|}{b-a} \geq 1 - \frac{3\delta}{b-a}. \tag{4}$$

Proof. E_J restricted to I_1 should be of the form $E_J(x) = x + \delta$ (because $\pi_n(1) = n$). Observe that the interval $L := [a+\delta, b-2\delta)$ is made of candidate points of length r_1. Indeed $K := [a+\delta, b)$ is the last interval of continuity of E_J^{-1} and $E_J^{-1}(x) = x - \delta$, $x \in K$. Then

$$E_J(L) = [a+2\delta, b-\delta) \quad \text{and} \quad E_J^{-1}(L) = [a, b-3\delta)$$

are both intervals contained in I_1. In this way, the Lebesgue measure of $C_{r_1}([a,b))$ is greater than $|L| \geq (b-a) - 3\delta$. □

4. A Key Example

In this section we present an example that will play a key role in the proof of Proposition 1. Let $q > 1$ be a natural number and consider the partitions

$\mathbf{a} = (a_0, a_1, \cdots, a_n)$ and $\mathbf{b} = (b_0, b_1, \cdots, b_n)$ of $[0, 1)$:

$$a_k = \begin{cases} 1 - q^{-k}, & 0 \le k < n, \\ 1, & k = n, \end{cases}$$
$$b_k = \begin{cases} 0, & k = 0, \\ q^{-(n-k)}, & 0 < k < n, \\ 1, & k = n. \end{cases} \tag{5}$$

$$\mathbf{a} := \left(0, 1 - q^{-1}, 1 - q^{-2}, \ldots, 1 - q^{-(n-1)}, 1\right),$$

$$\mathbf{b} = \left(0, q^{-(n-1)}, q^{-(n-2)}, \ldots, q^{-1}, 1\right).$$

F_q will be the interval exchange associated to the pair $(\pi_n, \mathbf{a})$. Then F_q^{-1} is associated to $(\pi_n, \mathbf{b})$. Let $I_k := [a_{k-1}, a_k)$ and $J_k := [b_{k-1}, b_k)$, $k = 1, \ldots, n$, be the intervals of continuity of F_q and F_q^{-1} respectively. We will see that F_q has the property that each of the induced maps on I_k satisfies the conditions in Lemma 2. Note that F_q and F_q^{-1} are explicitly given by the formulas

$$F_q(x) = \begin{cases} x - a_{k-1} + b_{n-k}, & x \in I_k,\ 1 < k < n, \\ x - a_{n-1}, & x \in I_n, \end{cases}$$
$$F_q^{-1}(x) = \begin{cases} x + a_{n-k} - b_{k-1}, & x \in J_k,\ 1 < k < n, \\ x - a_1, & x \in J_n. \end{cases} \tag{6}$$

Lemma 3. *Let $F_{q,k}$ be the induced map of F_q on I_k, $k = 1, \ldots, n$. Then if C_k is the set of candidate points in I_k,*

$$\frac{|C_k|}{|I_k|} \ge 1 - \frac{3}{q-1}.$$

Proof. It is not hard to check the following facts: **1.** The induced map of F_q on the interval $[a_k, 1)$, $k = 1, 2, \ldots, n-1$ is given by the permutation π_{n-k+1} and the partition of $[a_k, 1)$,

$$(a_k, a_{k+1}, a_{k+2}, \ldots, a_{n-1}, a_n).$$

2. The induced map of E on $I_k = [a_{k-1}, a_k)$, $k = 1, 3, \ldots, n-1$, is associated to the pair $(\pi_{n-k+1}, \mathbf{c}_k)$ where $\mathbf{c}_k$ is the partition of I_k :

$$\mathbf{c}_k = \left(a_{k-1}, a_k - q^{-k}, a_k - q^{-(k+1)}, \ldots, a_k - q^{-(n-1)}, a_k\right).$$

In particular, $F_{q,k}(x) = x + q^{-k}$ for each x in the first interval of continuity $I_{k,1}$; **3.** The induced map of F_q on the first interval $I_1 = [0, 1 - q^{-1}) = [a_0, a_1)$ is given by the pair $(\pi_n, \mathbf{c}_1)$ where $\pi_n = \pi$ and

$$\mathbf{c}_1 = \left(0, a_1 - q^{-1}, a_1 - q^{-2}, \ldots, a_1 - q^{-(n-1)}, a_1\right).$$

4. The induced map of F_q on the last interval $I_n = [a_{n-1}, a_n)$ is the identity map. In particular the last interval is all made of candidate points. **5.** $F_{q,k}$ restricted to the first interval of continuity

$$I_{k,1} := \left[a_{k-1}, a_k - q^{-k}\right)$$

is given by

$$F_{q,k}(x) = x + q^{-k}.$$

Then Lemma 2 applies to each map $F_{q,k}$, $1 \leq k < n$ (with $\delta = q^{-k}$) and therefore if C_k is the set of candidate points in I_k, then

$$\frac{|C_k|}{|I_k|} \geq 1 - \frac{3\delta}{|I_k|} \geq 1 - \frac{3q^{-k}}{(q-1)\, q^{-k}} = 1 - \frac{3}{q-1}.$$

Indeed, in this way the interval

$$L_k := \left[a_{k-1} + \frac{1}{q^k}, a_k - \frac{2}{q^k}\right)$$

$$= \left[1 - \frac{1}{q^{k-1}} + \frac{1}{q^k}, 1 - \frac{1}{q^k} - \frac{2}{q^k}\right)$$

which is contained in $I_{k,1}$, is made of candidate points and the length of L_k is $|I_k| - 3q^{-k}$. □

We say that two iets $E_1 : [a,b) \to [a,b)$ and $E_2 : [c,d) \to [c,d)$ are *equivalent* if they are conjugated by a translation of the real line, more precisely

$$E_2(x + c - a) = E_1(x) + c - a, \text{ for all } x \in [a,b).$$

In particular we should have $b - a = d - c$. In many situations we identify an interval exchange transformation with its equivalent model defined on $[0, b-a)$.

Lemma 4. *Let $E\colon [a,b) \to [a,b)$ be an iet and $J := [c,d) \subset [a,b)$. If $E(J)$ is also an interval, say K, then the induced maps of E on J and K are equivalent.*

Proof. As $E(J)$ is an interval, there is a number d such that, for all $x \in J$, $E(x) = x + d$. Put $K = [E(c), E(d))$ and l Let $J_k \subset J$ be an interval of continuity of E_J and let r_k be the return time of J_k to J, this means that

$$E(J_k), E^2(J_k), \ldots, E^{r_k}(J_k)$$

are all intervals disjoints of J and $E^{r_k+1}(J_k)$ is completely contained in J. By definition $E_J(J_k) = E^{r_k+1}(J_k)$. Observe that

$$E^{r_k+2}(J_k) = E(E^{r_k+1}(J_k)) = E_J(J_k) + d$$

is contained in K. Let $K_k := E(J_k) = J_k + d \subset K$. The successive iterates of K_k by E are (the same as)

$$E^2(J_k), E^3(J_k), \ldots, E^{r_k}(J_k), E^{r_k+1}(J_k)$$

and this intervals are disjoint of K whereas

$$E^{r_k+2}(J_k) = E_J(J_k) + d \subset K,$$

that is, K_k is an interval of continuity of E_K, with the same return time and more over,

$$E_K(K_k) = E^{r_k+2}(J_k) = E_J(K_k - d) + d.$$

As this happens for all the intervals of continuity of E_J and E_K, this maps are equivalent. □

5. Proof of Proposition 1

Let an irreducible permutation π and let $0 < \epsilon < 1$. Let $(q_m)_{m\geq 1}$ be an increasing sequence of natural numbers such that

$$\frac{3}{q_m - 1} < \frac{\epsilon}{2^m}, \quad m \geq 1.$$

Consider the iets F_{q_m} as defined in Section 4 and let

$$\mathbf{a}_m = (a_0^m, a_1^m, \dots, a_n^m)$$

be the correspondent partition of $[0, 1)$ associated to F_{q_m} as in (5). Observe that any iet which is sufficiently close to F_{q_m} enjoys the property given in Lemma 3, that is, there exist a number $\delta_m > 0$ such that if $E_{\mathbf{c}}$, the iet associated to $(\pi_n, \mathbf{c})$ and the partition $\mathbf{c} = (c_0, c_1, \dots, c_n)$ in Δ^{n-1} is such that

$$\max\{|c_i - a_i^m| : 1 \leq i \leq n\} < \delta_m, \tag{7}$$

then the set $C_{\mathbf{c},k}$ of candidate points in each $I_k := [c_{k-1}, c_k)$, $1 \leq k \leq n$ for $E_{\mathbf{c}}$ verify

$$\frac{|C_{\mathbf{c},k}|}{|I_k|} \geq 1 - \frac{3}{q_m - 1} \geq 1 - \frac{\epsilon}{2^m}. \tag{8}$$

Let $\pi \in G_n$ be an irreducible permutation. By Theorem 7, for almost all $\lambda \in \Delta^{n-1}$, the Rauzy orbit of $E_\lambda = (\pi, \lambda)$ is dense in $\mathcal{C} \times \Delta^{n-1}$. In this way there is a nested sequence of intervals

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m \dots$$

given by the Rauzy process of induction, and an increasing sequence $(n_m)_{m\geq 1}$ of natural numbers such that the induced map $E_m := R^{n_m}(E)$ of (π, λ) on J_m satisfies (7). So, if $I_1^m, I_2^m, \dots, I_n^m$ are the intervals of continuity of E_m in J_m and r_k^m is the return time of I_k^m to J_m, $1 \leq k \leq n$, then the set of candidate points in I_k^m for the length r_k^m has Lebesgue measure greater than $1 - \epsilon \cdot 2^{-m}$ times the measure of I_k^m. Let $E_{m,k}$ be the induced map of E_m on I_k^m, $1 \leq k \leq n$. Therefore the set of candidate points in I_k^m for the length N_m with

$$N_m = \max\{r_1^m, \dots, r_n^m\},$$

a set named here as $C_{N_m}(J_m)$, satisfies

$$|C_{N_m}(J_m)| \geq \left(1 - \frac{\epsilon}{2^m}\right) \cdot |J_m|. \tag{9}$$

By Lemma 4, the induce map of (π, λ) on each interval $E^j(I_k^m)$, $1 \leq j \leq r_k^m$ is equivalent to $E_{k,m}$ and this implies that the set of candidate point for the length r_k^m in the union

$$\bigcup_{j=1}^{r_k^m} E^j(I_k^m)$$

is greater than $1 - \epsilon \cdot 2^{-m}$ times the measure of this union. Using the fact that

$$[0, 1) = \bigcup_{k=1}^{m} \bigcup_{j=1}^{r_k^m} E^j(I_k^m),$$

(see relation (2)) we conclude that the set of candidate points in $[0,1)$ for the lengths $r \leq N_m$ with

$$N_m = \max\{r_1^m, \dots, r_n^m\}$$

(a set named here as $C_{N_m}([0,1))$), satisfies

$$|C_{N_m}([0,1))| \geq 1 - \frac{\epsilon}{2^m}. \tag{10}$$

Now it follows that the Lebesgue measure of the intersection

$$C_\epsilon := \bigcap_{m \geq 1} C_{N_m}([0,1))$$

is greater than $1 - \epsilon \sum_{m\geq 1} 2^{-m} = 1 - \epsilon$. Observe that the sequence of integers (n_m) can be chosen in such a way that (N_m) is a strictly increasing sequence and then all the points in C_ϵ satisfy the condition of Proposition 1. As the number $\epsilon > 0$ is arbitrary, Proposition 1 is proved.

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