PSEUDO-PARALLEL LAGRANGIAN SUBMANIFOLDS IN COMPLEX SPACE FORMS

PABLO M. CHACÓN AND GUILLERMO A. LOBOS

Abstract. In this work we study pseudo-parallel Lagrangian submanifolds in a complex space form. We give several general properties pseudo-parallel submanifolds. For the 2-dimensional case, we show that any minimal Lagrangian surface is pseudo-parallel. We also give examples of non minimal pseudo-parallel Lagrangian surfaces. Here we prove a local classification of the pseudo-parallel Lagrangian surfaces. In particular, semi-parallel Lagrangian surfaces are totally geodesic or flat. Finally, we give examples of pseudo-parallel Lagrangian surfaces which are not semi-parallel.

1. Introduction

An isometric immersion $f : M^n \to \tilde{M}^N$ of an n-dimensional Riemannian manifold into a N-dimensional Riemannian manifold with metric $\langle , \rangle$ is called *pseudo-parallel* if its second fundamental form α satisfies:

$$\overline{R}(X, Y) \cdot \alpha = \varphi \, X \wedge Y \cdot \alpha, \tag{1.1}$$

for some real valued smooth function φ on M and for any X and Y vectors tangent to M, where $\overline{R}$ is the curvature operator of the Van der Waerden-Bortolotti connection $\overline{\nabla}$ of f and $X \wedge Y$ is the operator given by $(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$. So, M will be referred as a φ-pseudo-parallel submanifold of $\tilde{M}$.

Pseudo-parallel immersions have been studied in [1], [2], [21] and in [23] for the case of $\tilde{M}$ be of constant sectional curvature. In real space forms, the pseudo-parallelism condition is the extrinsic analogue of *pseudo-symmetry* in the sense of Deszcz [18]. Recall that a manifold M is pseudo-symmetric if the curvature tensor R of M satisfies:

$$R(X, Y) \cdot R = \psi \, X \wedge Y \cdot R, \tag{1.2}$$

2000 *Mathematics Subject Classification.* 53C40.

The first author is partially supported by MEC/FEDER Grant No. MTM2004-04934-C04-02, Spain. The second author is partially supported by CAPES/MEDC Grand 052-03, Brazil.

for any X and Y vectors tangent to M, being ψ some real valued smooth function on M. So, in ambients with constant curvature, any φ-pseudo-parallel submanifold is intrinsicly a φ-pseudo-symmetric manifold, see [1]. However, in general, pseudo-symmetry does not imply pseudo-parallelism, see [1].

Pseudo-symmetric manifolds have been studied by many authors from several viewpoints but there is not available a complete classification yet [3]. When $\psi = 0$ in (1.2), M is called a *semi-symmetric* manifold and they were classified in [28].

A *semi-parallel* immersion [17] is determined by $\varphi = 0$ in equation (1.1). Recall that a *parallel* immersion is defined by $\overline{\nabla}\alpha = 0$. Thus, pseudo-parallel immersions are a generalization of semi-parallel immersions and consequently a generalization of parallel immersions.

It is well-known the classification of parallel immersions in a real space form. However, the classification of semi-parallel immersions in real space forms is still an open problem although several authors have obtained important advances. We can refer to [24] for a survey. Examples of pseudo-parallel immersions in real space forms which are not semi-parallel have been shown in [1], [2] and [23].

Pseudo-parallel hypersurfaces of a real space form were classified in [2]. They are either quasi-umbilical hypersurfaces or cyclids of Dupin. Recently a generalization of this result has been obtained in [23], where a local classification of pseudo-parallel submanifolds with flat normal bundle of a real space form has been proved. For $n = 2$, in [21] it was given a local classification of pseudo-parallel surfaces. In particular, a constant pseudo-parallel surface with non-flat normal bundle in a 4-dimensional real space form is a piece of a Veronese surface.

First results on pseudo-parallel immersions in almost complex manifolds with constant holomorphic sectional curvature $4c$, appear in [22] where a local classification for pseudo-parallel real hypersurfaces has been given. Essentially, pseudo-parallel real hypersurfaces M^{2n-1}, $n \geq 2$, of a complex space form $\tilde{M}^n(4c)$ are either tubes over a totally geodesic $\mathbb{C}P^{n-1}$ or horospheres in $\mathbb{C}H^{n-1}$ or tubes over a totally geodesic $\mathbb{C}H^{n-1}$.

On the other hand, Lagrangian submanifolds of complex space form has been deeply studied from the decade of the 70's. A survey of the principal results in theory of Lagrangian submanifolds can be found in [9]. Since there is no a complete classification of Lagrangian submanifolds, it is natural to study the Lagrangian submanifolds with some additional properties. Therefore, the aim of this paper is to study

Lagrangian submanifolds M^n of a complex n-dimensional space form $\tilde{M}^n(4c)$ which are pseudo-parallel.

First we prove several properties of pseudo-parallel Lagrangian submanifolds of a complex space form. In the specific case of H-umbilical submanifolds, we are able to prove that the notion of pseudo-parallelism agrees with that of semi-parallelism. In the 2-dimensional case, we show here that a minimal Lagrangian surface M^2 of $\tilde{M}^2(4c)$ is φ-pseudo parallel with $\varphi = \frac{3}{2}K$, where K is the Gauss curvature of M^2. Moreover, we give examples of pseudo-parallel Lagrangian surfaces which are not minimal.

We also give a local classification of pseudo-parallel Lagrangian surfaces as: either totally geodesic or flat non totally geodesic, or minimal with non constant Gauss curvature which is not semi-parallel. In particular, from the classification follows that semi-parallel Lagrangian surfaces of a complex space form are (locally) totally geodesic or flat. Recently, it have been given the local classification of Lagrangian surfaces of constant curvature in a complex space form, see [10], [11], [12] and [13]. Combining these results, we can give a full local classification of the semi-parallel Lagrangian surfaces into a complex space form. Finally, here we give some examples of pseudo-parallel surfaces which are not semi-parallel.

Acknowledgments. The first author would like to thank the second author and their family for the hospitality during his stay in Department of Mathematics of the University of Salamanca.

2. Notations and Preliminaries

Let $\big(\tilde{M}^N, \langle , \rangle\big)$ be an N-dimensional Riemannian manifold and M^n be an n-dimensional submanifold of $\tilde{M}^N$. On M^n, we have a metric induced by $\tilde{M}$ that becomes $f : M^n \to \tilde{M}^N$ into an isometric immersion.

In this work we denote by $X, Y, Z, W \ldots$ vectors tangent to M, by $\xi, \eta \ldots$ vectors normal to M and by $U, V \ldots$ generic vectors tangent to $\tilde{M}$, unless be mentioned otherwise.

As usual, ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of M and $\tilde{M}$, respectively. Both connections are related by the Gauss formula:

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y),$$

where α is the second fundamental form. Similarly, the Weingarten formula is:

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where A_ξ is the shape operator (which is auto-adjoint) in the direction ξ and D is the normal connection on M. The shape operator and the second fundamental form are related by:

$$\big\langle \alpha(X,Y),\xi\big\rangle = \big\langle A_\xi X, Y\big\rangle.$$

Following the convection of [15], the curvature $\tilde{R}$ of $\tilde{\nabla}$ is defined by:

$$\tilde{R}(U_1,U_2)V = [\nabla_{U_1},\nabla_{U_2}]V - \nabla_{[U_1,U_2]}V,$$

and the sectional curvature of a non-degenerate plane spanned by $\{U,V\}$ is given by $\langle \tilde{R}(U,V)V,U\rangle/(\|U\|^2\|V\|^2 - \langle U,V\rangle^2)$.

If R and R^D denote, respectively, the Riemannian curvature tensors of ∇ and D, then the Gauss equation and the Ricci equation are:

$$\begin{aligned}\big\langle \tilde{R}(X,Y)Z,W\big\rangle =&\big\langle R(X,Y)Z,W\big\rangle + \big\langle \alpha(X,Z),\alpha(Y,W)\big\rangle\\ &- \big\langle \alpha(X,W),\alpha(Y,Z)\big\rangle,\\ \big\langle \tilde{R}(X,Y)\xi,\eta\big\rangle =&\big\langle R^D(X,Y)\xi,\eta\big\rangle + \big\langle\, [A_\xi, A_\eta]\, X, Y\big\rangle.\end{aligned}$$

The Codazzi equation of the immersion is:

$$(\tilde{R}(X,Y)Z)^\perp = (\overline{\nabla}_X\alpha)(Y,Z) - (\overline{\nabla}_Y\alpha)(X,Z),$$

where $\overline{\nabla} = \nabla \bigoplus D$ is the Van der Waerden-Bortolotti connection. The covariant derivative $\overline{\nabla}\alpha$ of the second fundamental form α is given by:

$$(\overline{\nabla}_X\alpha)(Y,Z) = D_X\alpha(Y,Z) - \alpha(\nabla_X Y,Z) - \alpha(Z,\nabla_X Z).$$

The operators $\overline{R}(X,Y)$, from the curvature of $\overline{\nabla}$, and $X\wedge Y$ can be extended as derivations of tensor fields in the usual way. So, it following easily from the fundamental equations that $\overline{R}(X,Y)\cdot\alpha$ and $X\wedge Y\cdot\alpha$ of expression (1.1) are given by:

(2.1)

$$\begin{aligned}\big(\overline{R}(X,Y)\cdot\alpha\big)\,(Z,W) =&R^D(X,Y)\big(\alpha(Z,W)\big) - \alpha\big(R(X,Y)Z,W\big)\\ &- \alpha\big(Z,R(X,Y)W\big),\\ (X\wedge Y\cdot\alpha)(Z,W) =& - \alpha\big((X\wedge Y)Z,W\big) - \alpha\big(Z,(X\wedge Y)W\big)\\ =& - \langle Y,Z\rangle\alpha(X,W) + \langle X,Z\rangle\alpha(Y,W)\\ &- \langle Y,W\rangle\alpha(Z,X) + \langle X,W\rangle\alpha(Z,Y).\end{aligned}$$

In (1.2), $R(X,Y)$ acts as a derivation similarly.

An almost complex structure on $\tilde{M}$ is a tensor field J of type (1,1) such that $J^2 = -Id$. If J is an isometry, i.e. $\langle U,V\rangle = \langle JU,JV\rangle$, we say that $\tilde{M}^m$ is an almost Hermitian manifold, where m is the complex dimension of $\tilde{M}$. The most interesting Hermitian manifolds are, maybe, the Kaehlerian manifolds defined by $\tilde{\nabla}J = 0$, (i.e. the almost complex structure is parallel).

Two important properties of the curvature tensor of a Kaehlerian manifold are:

$$\tilde{R}(JU,JV)=\tilde{R}(U,V) \quad \text{and} \quad \tilde{R}(U_1,U_2)JV=J\tilde{R}(U_1,U_2)V.$$

The holomorphic sectional curvature of an almost Hermitian manifold is the restriction of the sectional curvature to holomorphic planes (that spanned by U and JU) of the tangent space. A manifold $\tilde{M}$ has constant holomorphic sectional curvature $4c$ if c is a constant such that $\langle\tilde{R}(U,JU)JU,U\rangle=4c\|U\|^4$ for any tangent vector U of $\tilde{M}$. The curvature tensor of a space of constant holomorphic sectional curvature $4c$, $\tilde{M}(4c)$, is given by:

$$\tilde{R}(U_1,U_2)V=c\big((U_1\wedge U_2)V+(JU_1\wedge JU_2)V+2\langle JU_1,U_2\rangle JV\big).$$

Finally, for a complex space form we mean a Kaehlerian manifold, complete, simply connected, with constant holomorphic sectional curvature. So, a complex space form is isometric to either the complex projective space $\mathbb{C}P^m(4c)$ if $c>0$, or the complex Euclidean space $\mathbb{C}^m$ if $c=0$, or the complex projective hyperbolic space $\mathbb{H}P^m(4c)$ if $c<0$.

An n-dimensional submanifold M^n of an almost Hermitian complex manifold $\tilde{M}^m$ is said to be *totally real* if $J(T_pM)\subset(T_pM)^\perp$ for all $p\in M^n$. A totally real submanifold M^n of $\tilde{M}^m$ is said to be *Lagrangian* when $n=m$. The following relations are well-known for Lagrangian submanifolds of a Kaehlerian manifold (see for example [5]):

$$D_XJY=J\nabla_XY, \tag{2.2}$$

$$JA_{JX}Y=\alpha(X,Y)=JA_{JY}X, \tag{2.3}$$

$$\langle\alpha(X,Y),JZ\rangle=\langle\alpha(Y,Z),JX\rangle=\langle\alpha(Z,X),JY\rangle. \tag{2.4}$$

Note that from (2.2) follows $R^D(X,Y)JZ=JR(X,Y)Z$. Moreover, from the Gauss equation we have:

$$R(X,Y)=\tilde{R}(X,Y)+[A_{JX},A_{JY}]. \tag{2.5}$$

3. Pseudo-parallel Lagrangian Submanifolds

From the pseudo-parallelism condition (1.1), see also equations (2.1), and the basic concepts introduced in last section, a Lagrangian submanifold M^n of $\tilde{M}^n$ will be a φ-pseudo-parallel if and only if the following equation holds:

$$\begin{aligned} R(X,Y)A_{JW}Z=&A_{JW}R(X,Y)Z+A_{JZ}R(X,Y)W\\ &-\varphi\langle Y,W\rangle A_{JZ}X+\varphi\langle X,W\rangle A_{JZ}Y\\ &-\varphi\langle Y,Z\rangle A_{JW}X+\varphi\langle X,Z\rangle A_{JW}Y.\end{aligned} \tag{3.1}$$

If $\varphi = 0$ in (3.1), then the submanifold M^n of $\tilde{M}^n$ is called *semi-parallel*. As a *parallel* submanifold, $\overline{\nabla}\alpha = 0$ (in particular, *totally geodesic* submanifold, $\alpha = 0$), is semi-parallel, it is obvious that also is a pseudo-parallel submanifold.

Proposition 3.1. *Let M^n be a φ-pseudo-parallel Lagrangian submanifold of $\tilde{M}^n$. If there is another smooth function ψ that also satisfies (3.1), then $\varphi = \psi$ at least on $M - V$, where $V = \{p \in M \mid \alpha_p \equiv 0\}$.*

Proof. If φ and ψ are two smooth functions that satisfy (3.1), then we have:

$$\begin{aligned}(\varphi - \psi)\big(&\langle Y, W\rangle A_{JZ}X - \langle X, W\rangle A_{JZ}Y \\ &+ \langle Y, Z\rangle A_{JW}X - \langle X, Z\rangle A_{JW}Y\big) = 0,\end{aligned} \tag{3.2}$$

for all X, Y, Z and W vectors tangent to M. For $X = Z = W$ in such a way X and Y are orthonormal, from (3.2) we get:

$$(\varphi - \psi)A_{JX}Y = 0. \tag{3.3}$$

Now, when $X = Z$, $Y = W$ and X and Y are orthonormal, we obtain from (3.2):

$$(\varphi - \psi)(A_{JX}X - A_{JY}Y) = 0, \tag{3.4}$$

Let $p \in M$ such that $\varphi(p) \neq \psi(p)$. By (3.3) and (3.4) we know that:

$$A_{JX}Y = 0 \quad \text{and} \quad A_{JX}X = A_{JY}Y$$

for any $X \perp Y$. From here and (2.3), we get that

$$A_{JX}X = \langle A_{JX}X, X\rangle X = \langle A_{JY}Y, X\rangle X = 0.$$

So, $A_{JX} \equiv A_{JY} \equiv 0$ and consequently:

$$\{p \in M | \varphi(p) \neq \psi(p)\} \subseteq V.$$

This proves the Proposition. □

Accordingly, the function of pseudo-parallelism is unique where the second fundamental form does not vanish. In particular, if $V \neq \emptyset$ the function φ of (3.1) is completely determined.

From now on, we assume that $\tilde{M}^n(4c)$ is a complex n-dimensional space form of constant holomorphic curvature $4c$ and that M^n is a Lagrangian submanifold of $\tilde{M}^n(4c)$.

Proposition 3.2. *Let M^n be a pseudo-parallel Lagrangian submanifold of $\tilde{M}^n(4c)$ with mean curvature vector $H = \frac{1}{n}\mathrm{trace}(\alpha)$. Then:*

$$R(X, Y)JH = 0,$$

for all X, Y vectors tangent to M.

Proof. Let $\{e_1, ..., e_n\}$ be an orthonormal frame of M and Z a unit tangent vector of T_pM. By (2.4) we get:

$$\langle J\alpha(e_i, e_i), Z\rangle = -\langle \alpha(e_i, Z), Je_i\rangle = -\langle A_{Je_i}e_i, Z\rangle.$$

So,

$$JH = \frac{1}{n}\text{trace}(\alpha) = -\frac{1}{n}\sum_{i=1}^{n} A_{Je_i}e_i. \tag{3.5}$$

Consider now that the frame diagonalizes A_{JZ}, being $\{\lambda_i\}_{i=0}^{n}$ the eigenvalues. So, by (3.1) and (2.3):

$$\begin{aligned}
\langle R(X,Y)JH, Z\rangle &= -\frac{1}{n}\sum_{i=1}^{n}\langle R(X,Y)A_{Je_i}e_i, Z\rangle \\
&= \frac{2}{n}\sum_{i=1}^{n}\big(\varphi\langle A_{Je_i}(X\wedge Y)e_i, Z\rangle - \langle A_{Je_i}R(X,Y)e_i, Z\rangle\big) \\
&= \frac{2}{n}\sum_{i=1}^{n}\big(\varphi\langle (X\wedge Y)e_i, A_{JZ}e_i\rangle - \langle R(X,Y)e_i, A_{JZ}e_i\rangle\big) \\
&= \frac{2}{n}\sum_{i=1}^{n}\lambda_i\big(\varphi\langle (X\wedge Y)e_i, e_i\rangle - \langle R(X,Y)e_i, e_i\rangle\big) = 0.
\end{aligned}$$

□

Recall that a Lagrangian submanifold M^n of $\tilde{M}^n(4c)$ is said to be λ-*isotropic* if there exists a function $\lambda : M^n \to \mathbb{R}$ such that:

$$\|\alpha(X,X)\|^2 = \lambda^2(p),$$

for all unit vector $X \in T_pM$ and for all $p \in M^n$. In particular, if λ is constant then M^n is called λ-*isotropic constant* [27].

Proposition 3.3. [26] *Let $n \geq 3$ and M^n be a λ-isotropic Lagrangian submanifold of $\tilde{M}^n(4c)$. Then λ is constant and M^n is parallel.*

In [20], semi-parallel isotropic Lagrangian submanifolds were studied. From Proposition 3.3 the hypothesis of semi-parallelism in [20] can be dropped for $n \geq 3$, but not for $n = 2$.

Now, recall that a non totally geodesic Lagrangian submanifold M^n of $\tilde{M}^n(4c)$ is H-*umbilical* if its Weingarten operators takes the following form:

$$\begin{aligned}
&A_{Je_1}e_1 = \lambda e_1, \quad A_{Je_2}e_2 = ... = A_{Je_n}e_n = \mu e_1, \\
&A_{Je_1}e_j = \mu e_j, \quad A_{Je_j}e_k = 0, \quad 2 \leq j \neq k \leq n,
\end{aligned} \tag{3.6}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field [7].

Proposition 3.4. *If $n \geq 3$ and M^n is a Lagrangian H-umbilical submanifold of $\tilde{M}^n(4c)$, then M^n is pseudo-parallel if and only if it is semi-parallel and $c \geq 0$.*

Proof. Let M^n be a pseudo-parallel Lagrangian submanifold of $\tilde{M}^n(4c)$ with associated function φ. From (2.5) and Proposition 3.2 we have:

$$[A_{JX}, A_{JY}]JH = c\big(\langle X, JH\rangle Y - \langle Y, JH\rangle X\big). \tag{3.7}$$

If we consider $\{e_1, ..., e_n\}$ a local orthonormal frame as in (3.6) and we use $X = e_1$ and $Y = e_2$ in (3.7), we get:

$$(\lambda + (n-1)\mu)(c + (\mu - \lambda)\mu) = 0. \tag{3.8}$$

Now, if we set $Y = W = e_i$ in (3.1), and sum over $i = 1, ..., n$, we obtain:

$$\begin{aligned} &c\big(n\langle Z, JH\rangle X + A_{JZ}X\big) + n(c-\varphi)\big(A_{JZ}X + \langle X, Z\rangle JH\big) = \\ &\sum_{i=1}^{n}([A_{JX}, A_{Je_i}]A_{JZ}e_i - A_{Je_i}[A_{JX}, A_{Je_i}]Z - A_{JZ}[A_{JX}, A_{Je_i}]e_i), \end{aligned} \tag{3.9}$$

for any X and Z. If we put $X = Z = e_1$ in (3.9):

$$\varphi(\lambda - \mu) = (\lambda - 2\mu)(c + (\lambda - \mu)\mu). \tag{3.10}$$

If we substitute $X = e_1$ and $Z = e_2$ in (3.9), thus we have:

$$\mu(n\varphi - (n+1)c + (n+1)\mu(\mu - \lambda)) = 0. \tag{3.11}$$

Finally, if we put $X = e_1$ and $Y = Z = W = e_2$ in (3.1), we obtain

$$\mu(2\varphi - 3(c + (\lambda - \mu)\mu) = 0. \tag{3.12}$$

So, using that when $n \geq 3$ there are not totally umbilical Lagrangian submanifolds with $\alpha \neq 0$ in $\tilde{M}^n(4c)$ (see [6]), we obtain from (3.8), (3.10), (3.11), (3.12), after algebraic manipulations, that $\varphi = 0$ and $c = n\mu^2 \geq 0$. □

When $n \geq 3$, from the proof above it follows that pseudo-parallel Lagrangian H-umbilical submanifolds only can happen in $\mathbb{C}^n$ or $\mathbb{C}P^n(4c)$ and its Weingarten operators takes the form (3.6) for $\mu \equiv 0$ or $\lambda = (1-n)\mu \not\equiv 0$, respectively.

Recall that a Lagrangian submanifold is *minimal* if its mean curvature vector $H \equiv 0$. Hence and from the classification of Lagrangian H-umbilical submanifolds given in [7] and [8], we obtain the following corollary:

Corollary 3.5. *Let $f : M^n \to \tilde{M}^n(4c)$ be a Lagrangian H-umbilical isometric immersions. If f is pseudo-parallel then:*

i) *$c > 0$ and there exist a unit speed Legendre curve $z(x) = (z_1(x), z_2(x)) : I \longrightarrow S^3(c) \subset \mathbb{C}^2$ such that up to rigid motions of $\mathbb{C}P^n(4c)$, $f = \pi \circ g$ is a minimal immersion defined by $g(x, y_1, ..., y_n) = (z_1(x), z_2(x)y_1, ..., z_2(x)y_n)$ where $y_1^2 + ... + y_n^2 = 1$ and π is the projection of Hopf's fibration; or*

ii) *$c = 0$ and up to rigid motions of $\mathbb{C}^n$, f is a Lagrangian H-umbilical isometric immersion of a open portion of a flat twisted product manifold ${}_\sigma\mathbb{R} \times \mathbb{R}^{n-1}$ with twisted product metric given by $\langle , \rangle = \sigma^2 dx_1^2 + \sum_{j=2}^n dx_j^2$, where the twisted function is the form $\sigma = a(x_1) + \sum_{j=2}^n b_j(x_1)x_j$ for some functions $a, b_1, ..., b_n$ of x_1.*

Proposition 3.6. *A Lagrangian submanifold M^n of constant sectional curvature c_1 in $\tilde{M}^n(4c)$ is pseudo-parallel if and only if it is flat or totally geodesic.*

Proof. Let M^n be a pseudo-parallel Lagrangian submanifold of $\tilde{M}^n(4c)$ with associated function φ. We have $R(X,Y) = c_1 X \wedge Y$, and from Proposition 3.2, we get that $c_1 = 0$ or $H = 0$.

If $c_1 = 0$, we have from (3.1) that $\varphi = 0$ or:

$$A_{JZ}(X \wedge Y)W + A_{JW}(X \wedge Y)Z = 0 \tag{3.13}$$

for any X, Y, Z and W. If $\varphi \neq 0$, then from (3.13), after some algebraic manipulation, we obtain $A_{JX} \equiv 0$, for any X. So M will be totally geodesic.

Now, if $c_1 \neq 0$ equation (3.1) yields:

$$c_1(X \wedge Y)A_{JW}Z = (c_1 - \varphi)(A_{JW}(X \wedge Y)Z + A_{JZ}(X \wedge Y)W), \tag{3.14}$$

for any X, Y, Z and W. If we set $Y = W = e_i$ in (3.14), and sum over $i = 1, ..., n$, using that $H = 0$, we obtain $\varphi = \frac{c_1(n+1)}{n}$ or $A_{JZ}X = 0$, for any X and Z. The second case means that M is totally geodesic. Assume in the following $\varphi = \frac{c_1(n+1)}{n}$. Use (3.14) with $Y = Z = W$ and $X \perp Y$ to obtain:

$$X \wedge Y(A_{JY}Y) = -\frac{2}{n}A_{JY}X. \tag{3.15}$$

Hence, $\langle A_{JY}X, Z\rangle = 0$ for any $Z \perp X, Y$. From the scalar product of (3.15) with Y we get $\langle A_{JY}X, Y\rangle = 0$ and by symmetry $\langle A_{JX}Y, X\rangle = 0$. Consequently $A_{JY}X = 0$. With that and the scalar product of (3.15) with X we obtain $\langle A_{JY}Y, Y\rangle = 0$. In order to conclude $A_{JY}Y = 0$, recall that $\langle A_{JY}Y, X\rangle = 0$. Therefore, in this case, M^n is totally geodesic.

Finally, if M is flat or totally geodesic, trivially it will be pseudo-parallel. □

Corollary 3.7. *A Lagrangian submanifold M^n of constant sectional curvature $c_1 \neq 0$ in $\tilde{M}^n(4c)$ is pseudo-parallel if and only if it is totally geodesic.*

Corollary 3.8. *A flat Lagrangian submanifold M^n in $\tilde{M}^n(4c)$ is semi-parallel.*

Conjecture: If $n \geq 3$ there are not pseudo-parallel Lagrangian submanifolds except the semi-parallel ones.

4. Pseudo-parallel Lagrangian Surfaces

Let M^2 be a Lagrangian submanifold in $\tilde{M}^2(4c)$ and consider $\{X, Y\}$ an orthonormal frame of M^2. Note that $R(X,Y)X = -KY$ and $R(X,Y)Y = KX$ where K is the Gaussian curvature of M^2. So, the condition of pseudo-parallelism (3.1) is equivalent to:

$$\begin{aligned} &R(X,Y)A_{JX}X = -R(X,Y)A_{JY}Y = 2(\varphi - K)A_{JX}Y, \\ &R(X,Y)A_{JX}Y = (\varphi - K)(A_{JY}Y - A_{JX}X). \end{aligned} \tag{4.1}$$

Remark 4.1. We know that there is no totally umbilical Lagrangian submanifolds except the totally geodesic ones [6]. Then, from (4.1) we obtain that a Lagrangian surface M^2 of $\tilde{M}^2(4c)$ with Gaussian curvature $K = 0$ is semi-parallel. Therefore, any flat Lagrangian surface is an example of pseudo-parallel Lagrangian surface.

Example 4.2. If $c > 0$ it was shown in [21] that the minimal flat torus $T : \mathbb{R}^2 \to S^5(c)$ given by:

$$\begin{aligned} T(x,y) \;=\; & \frac{2}{\sqrt{6c}}(\cos u \cos v, \cos u \sin v, \frac{\sqrt{2}}{2}\cos 2u, \\ & \sin u \cos v, \sin u \sin v, \frac{\sqrt{2}}{2}\sin 2u), \end{aligned}$$

where $u = \sqrt{\frac{c}{2}}x$, $v = \frac{\sqrt{6c}}{2}y$, is (up to isometry of $S^5(c)$) the only pseudo-parallel immersion of a 5-dimensional real space form with φ constant which is not semi-parallel. We will see later that there is no pseudo-parallel Lagrangian surfaces of $\tilde{M}^n(4c)$, non semi-parallel, with φ constant.

On the other hand, if we compose $T : \mathbb{R}^2 \to S^5(c)$ with the Hopf fibration $\pi : S^5(c) \to \mathbb{C}P^2(4c)$, we obtain a minimal flat Lagrangian surface which is parallel [19]. Hence $\pi \circ T$ is pseudo-parallel and also is semi-parallel. Therefore, the Hopf fibration does not preserve, in general, the strictly pseudo-parallel condition in a real space form.

Proposition 4.3. *A minimal Lagrangian surface M of $\tilde{M}^2(4c)$ is φ-pseudo-parallel where φ is given by:*

$$\varphi(p) = \frac{3}{2}K(p), \quad for\ all\ p \in M.$$

Proof. From [26], we know that the minimality of M implies that M is λ-isotropic. Take an orthonormal local frame $\{e_1, e_2\}$ of M and denote $\alpha_{ij} = \alpha(e_i, e_j)$. Minimality provides $\alpha_{11} = -\alpha_{22}$. If $X = \cos\theta e_1 + \sin\theta e_2$, we have:

$$\begin{aligned} \lambda^2 = \|\alpha(X,X)\|^2 &= \lambda^2\cos^4\theta + 2\langle\alpha_{11},\alpha_{12}\rangle\cos(2\theta)\sin(2\theta) \\ &\quad + (\|\alpha_{12}\|^2 - \frac{1}{2}\lambda^2)\sin^2(2\theta) + \lambda^2\sin^4\theta. \end{aligned} \tag{4.2}$$

Since λ is independent of θ, we obtain from (4.2):

$$0 = \frac{d}{d\theta}\|\alpha(X,X)\|^2|_{\theta=0} = 4\langle\alpha_{11},\alpha_{12}\rangle. \tag{4.3}$$

Choose $\theta = \frac{\pi}{4}$ in (4.2) and use (4.3) to obtain $\lambda^2 = \|\alpha_{12}\|^2$. From here and with the Gauss equation we also obtain:

$$K = c - 2\lambda^2. \tag{4.4}$$

On the other hand, $H = 0$, so:

$$\begin{aligned} R(e_1,e_2)A_{Je_1}e_1 &= \sum_{j=1}^{2}\langle A_{Je_1}e_1, e_j\rangle R(e_1,e_2)e_j \\ &= -\langle\alpha_{11}, Je_1\rangle Ke_2 + \langle A_{Je_1}e_2, e_1\rangle Ke_1 \\ &= \langle\alpha_{22}, Je_1\rangle Ke_2 + \langle A_{Je_1}e_2, e_1\rangle Ke_1 = KA_{Je_1}e_2. \end{aligned} \tag{4.5}$$

Recall (3.5) to get easily $R(e_1,e_2)A_{Je_2}e_2 = -KA_{Je_1}e_2$ too. Analogously, we obtain:

$$\begin{aligned} R(e_1,e_2)A_{Je_1}e_2 &= -\langle A_{Je_1}e_2, e_1\rangle Ke_2 + \langle A_{Je_1}e_2, e_2\rangle Ke_1 \\ &= K\big(\langle A_{Je_2}e_2, e_1\rangle e_1 - \langle A_{Je_1}e_1, e_2\rangle e_2\big) \\ &= \frac{1}{2}K(A_{Je_2}e_2 - A_{Je_1}e_1). \end{aligned} \tag{4.6}$$

Therefore, from (4.1), (4.4), (4.5) and (4.6), we concluded that M is φ-pseudo-parallel with $\varphi = \frac{3}{2}K = \frac{3}{2}(c - 2\lambda^2)$. □

The converse from the Proposition 4.3 is false in any ambient space $\tilde{M}^2(4c)$. That is, there are pseudo-parallel Lagrangian surfaces which are not minimal, as we show in the following examples.

Example 4.4. Let $M^2 = {}_\sigma I_1 \times_\sigma I_2$ the flat *twisted product* of two open intervals I_1 and I_2, where σ is a positive function on differential manifold $I_1 \times I_2$ endowed with the twisted product metric $\langle , \rangle = \sigma(dx_1^2 + dx_2^2)$. The Lagrangian immersion $f : M^2 \to \mathbb{C}^2$ given by:

$$f(x_1, x_2) = (ae^{ix_1}, ae^{ix_2}),$$

where $a > 0$ and $\sigma(x_1, x_2) = a^2$; and $g : M^2 \to \mathbb{C}^2$ is given by:

$$g(x_1, x_2) = \frac{1}{\sqrt{2}\,|\zeta|} e^{\zeta u}\big(\cos(|\zeta|\, v), \sin(|\zeta|\, v)\big),$$

where $u = x_1 + x_2$, $v = x_1 - x_2$, $\zeta = b + \frac{1}{2}i$, $b \neq 0$ and $\sigma(x_1, x_2) = e^{2b(x_1+x_2)}$, is semi-parallel with $K = 0$ and $\|H\|^2 = \frac{\sigma}{4} \neq 0$. Thus, by Remark 4.1, f is pseudo-parallel but not minimal. This Lagrangian immersion appears in [8], [14] and [25].

Example 4.5. For each $b > 0$ the immersion $f_b : \mathbb{R} \times \mathbb{R} \to \mathbb{C}P^2(4c)$ defined by:

$$\begin{aligned} f_b(s,t) &= \Bigg[\frac{e^{ibs}}{2a}\bigg(\frac{e^{ias}}{2\sqrt{c}}((a-b)(e^{\gamma it} + e^{-\gamma it}) + \frac{e^{-ais}}{\sqrt{c}}(b+a), \\ &\qquad \frac{e^{ias}}{2}(e^{\gamma it} + e^{-\gamma it}) - \frac{e^{-ias}}{\sqrt{c}}, \frac{\sqrt{a}e^{ias}}{\sqrt{2(a+b)}}(e^{\gamma it} - e^{-\gamma it})\bigg)\Bigg], \end{aligned}$$

where $a = \sqrt{b^2 + c}$ and $\gamma = \sqrt{2a(a+b)}$, is semi-parallel with $K = 0$ and, moreover, f_b is not parallel. So, H is non-constant. This example is also a Lagrangian H-umbilical immersion and appears in [7].

Example 4.6. Let $b \in \mathbb{R}$ such that $c = -b^2$, the immersion $l_b : \mathbb{R} \times \mathbb{R} \to \mathbb{C}H^2(4c)$ given by:

$$l_b(x,y) = \left[\frac{e^{i\sqrt{-c}x}}{2}\left(\frac{1}{\sqrt{-c}} - ix + \frac{\sqrt{-c}}{2}y^2, x + \frac{i}{2}y^2, y\right)\right],$$

is semi-parallel, with $K = 0$ and $H \neq 0$. So, l_b is pseudo-parallel but not minimal. This example is also a Lagrangian H-umbilical immersion and appears in [7].

Now we give our main result: the classification of pseudo-parallel Lagrangian surfaces.

Theorem 4.7. *Let M be a φ-pseudo-parallel Lagrangian surface in $\tilde{M}^2(4c)$. Then, there exist an open dense subset of M on which M is locally one of the following possibilities:*

1. *Totally geodesic,*
2. *Flat with $\varphi = 0$ (semi-parallel) and non totally geodesic,*

(3) *Minimal with* $\varphi = \frac{3}{2}K \neq 0$.

Proof. If X is orthogonal to Y, from (4.1) we deduce:

$$\begin{aligned} KJH = 0; \qquad \varphi JH = 0; \\ (2\varphi - 3K)\langle A_{JX}X, Y\rangle = 0. \end{aligned} \tag{4.7}$$

Consider the open subsets $U_1 = \{p \in M \mid \alpha_p \neq 0\}$ and $U_2 = \{p \in M \mid K(p) \neq 0\}$ and denote $U_0 = M \setminus \big(\mathrm{fr}(U_1) \cup \mathrm{fr}(U_1 \cap U_2)\big)$ where fr indicates the border of the subset. Observe that U_0 is an open and dense (possibly non connected) subset of M. So, $V_1 = \{p \in U_0 \,|\, \alpha_p = 0\}$ and $V_2 = \{p \in U_0 \,|\, K(p) = 0\}$ are open (and close) subsets of U_0. Thus, if $V_1 \neq \emptyset$, M will be totally geodesic in a open neighborhood of U_0. Notice that $V_2 \cap (U_0 \setminus V_1)$ is an open subset of U_0. So, if in a point $p \in U_0$ we have $\alpha_p \neq 0$ and $K(p) = 0$ we will can find an open neighborhood with the same conditions. From (4.7), $\alpha_p \neq 0$ and $K(p) = 0$ implies $\varphi(p) = 0$. In this case M is locally a flat semi-parallel immersion. Finally, if $p \in V_1 \cap V_2$, that is $\alpha_p \neq 0$ and $K(p) \neq 0$, equations (4.7) provides $H(p) = 0$ and from Proposition 4.3 we have $\varphi|_{V_1 \cap V_2} = \frac{3}{2}K \neq 0$. □

Notice that equations (4.7) also gives that there is no an open subset where H and K are both different from zero.

From Theorem 4.7, we know that semi-parallel Lagrangian surfaces in $M^2(4c)$ are totally geodesic or flat. Recently, a local classification of Lagrangian surfaces of constant curvature in complex space form has been given in [10], [11] and [13]. With the help of Proposition 3.6, we know that there are the following number of families of semi-parallel immersions: 9 families for $c = 0$ [11], 6 families [10] for $c = 1$ and 17 families [13] for $c = -1$.

Now, we give explicit examples of pseudo-parallel Lagrangian surface of $\tilde{M}^2(4c)$ with $c \neq 0$ all due by [4].

Example 4.8. For each $\rho \in (0, \frac{\pi}{2})$, the map $g_\rho : \mathbb{R} \times S^1(1) \to \mathbb{C}P^2(4)$ defined by:

$$g_\rho(s, z) = \left[\left(\sin\theta(s) e^{-ia\int_0^s \frac{dt}{\sin^3\theta(t)}} z, \cos\theta(s) e^{ia\int_0^s \frac{\tan^2\theta(t)dt}{\sin^3\theta(t)}}\right)\right],$$

where θ is the only solution of the O.D.E.

$$\theta'' \sin\theta\cos\theta = (1 - (\theta')^2)(2\cos^2\theta - \sin^2\theta), \;\; \theta(0) = \rho, \;\; \theta'(0) = 0,$$

and $a = \cos\rho \sin^2\rho$, provides a minimal Lagrangian surface which is not totally geodesic [4]. By Proposition 4.3, g_ρ is a pseudo-parallel

Lagrangian surface with $\varphi = \frac{3}{2}(4 - 2\sin^2\theta)$. If $\theta(s) = \arctan\sqrt{2}$ (constant), then the Lagrangian surface is parallel. Therefore, if $\theta(s)$ is not constant, g_ρ is not semi-parallel by Theorem 4.7.

Example 4.9. For each $\rho > 0$, the map $f_\rho : \mathbb{R} \times_\tau S^1(1) \to \mathbb{C}H^2(-4)$ defined by:

$$f_\rho(s,z) = \left[\left(\sinh\theta(s)e^{ia\int_0^s \frac{dt}{\sinh^3\theta(t)}} z, \cosh\theta(s)e^{ia\int_0^s \frac{\tanh^2\theta(t)dt}{\sinh^3\theta(t)}}\right)\right],$$

where θ is the only solution of the O.D.E.

$$\theta''\sinh\theta\cosh\theta = (1-(\theta')^2)(\sinh^2\theta + 2\cosh^2\theta), \quad \theta(0) = \rho, \quad \theta'(0) = 0,$$

and $a = \cosh\rho\sinh^2\rho$, provides a minimal Lagrangian surface which is not totally geodesic [4]. By Proposition 4.3, f_ρ is a pseudo-parallel Lagrangian immersion with $\varphi = -\frac{3}{2}(4 + \frac{2}{\sinh^6\theta})$, and by Theorem 4.7 f_ρ is not semi-parallel.

Example 4.10. Also in [4], for each $\rho > 0$, the map $h_\rho : \mathbb{R} \times_\tau \mathbb{R}H^1(-1) \to \mathbb{C}H^2(-4)$ given by:

$$h_\rho(s,z) = \left[\left(\sinh\theta(s)e^{ia\int_0^s \frac{\coth^2\theta(t)dt}{\cosh^3\theta(t)}}, \cosh\theta(s)e^{ia\int_0^s \frac{dt}{\cosh^3\theta(t)}} z\right)\right],$$

where θ is the only solution of the O.D.E.

$$\theta''\sinh\theta\cosh\theta = (1-(\theta')^2)(\cosh^2\theta + 2\sinh^2\theta), \quad \theta(0) = \rho, \quad \theta'(0) = 0,$$

and $a = \sinh\rho\cosh^2\rho$, provides a minimal Lagrangian surface which is not totally geodesic. By Proposition 4.3, h_ρ is a pseudo-parallel with $\varphi = \frac{3}{2}(-4 + \frac{2}{\cosh^6\theta})$, and by Theorem 4.7 h_ρ is not semi-parallel.

Example 4.11. Finally, for each $\rho > 0$, the map $l_\rho : \mathbb{R}\times\mathbb{R} \to \mathbb{C}H^2(-4)$ given by:

$$\begin{aligned} l_\rho(s,t) &= \Bigg[e^{iB_3(s)}\Bigg(r(s)t, \frac{1 + r(s)^2(|x|^2 - 1 - 2iB_5(s))}{2r(s)}, \\ &\qquad \frac{1 + r(s)^2(|x|^2 + 1 - 2iB_5(s))}{2r(s)}\Bigg)\Bigg], \end{aligned}$$

where $B_n(s) = \rho^n \int_0^s \frac{du}{r(u)^n}$ and $r(s) = \rho\cosh^{\frac{1}{3}}(3s)$, provides a minimal Lagrangian surface [4]. Hence, by the Proposition 4.3 and Theorem 4.7, we conclude that h_ρ is a pseudo-parallel Lagrangian surface which is not semi-parallel, because the Gaussian curvature is non-constant.

From [16], [19], and Theorem 4.7, we obtain the following corollary.

Corollary 4.12. *Let $f : M^2 \to \tilde{M}^2(4c)$ be a completed pseudo-parallel Lagrangian immersion. If φ and H are constants, then f is totally geodesic or $c > 0$ and, up to holomorphic isometry, f is the immersion given in the Example 4.2.*

References

[1] A.C. Asperti, G. A. Lobos and F. Mercuri, *Pseudo-parallel immersions in space forms*, Mat. Contemp., **17** (1999), 59–70.
[2] A. C. Asperti, G. A. Lobos and F. Mercuri, *Pseudo-parallel submanifolds of a space form*, Adv. Geo., **2** (2002) 57–71.
[3] M. Belkhelfa, R. Deszcz, M. Glogowska, M. Hotlós, D. Kowalczyk and L. Verstraelen, *On some type of curvature conditions*, Banach Center Publ., **57** (2002), 179–194.
[4] I. Castro, C. Montealegre and F. Urbano, *Minimal Lagrangian submanifolds in the complex hyperbolic space*, Illinois J. Math., **46** (2002), 695–721.
[5] B. Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc., **193** (1974), 257–266.
[6] B.Y. Chen and K. Ogiue, *Two theorems on Kaehler manifolds*, Michigan Math. J., **21** (1974), 225–229.
[7] B. Y. Chen, *Interaction of Legendre curves and Lagrangian submanifolds*, Israel J. Math., **99** (1997), 69–108.
[8] B. Y. Chen, *Complex extensors and Lagrangian submanifolds in complex Euclidean spaces*, Tôhoku Math. J., **49** (1997), 277–297.
[9] B. Y. Chen, *Riemannian geometry of Lagrangian submanifolds*, Taiwanese J. Math., **5** (2001), 681–723.
[10] B. Y. Chen, *Classification of Lagrangian surfaces of constant curvature in complex projective plane*, J. Geom. Phys., **53** (2005), 428–460.
[11] B. Y. Chen, *Classification of Lagrangian surfaces of constant curvature in the complex Euclidean plane*, Proc. Edinb. Math. Soc., **48** (2005), 337–364.
[12] B. Y. Chen, *Maslovian Lagrangian surfaces of constant curvature in complex projective or complex hyperbolic planes*, Math. Nachr., **278** (2005), 1242–1281.
[13] B. Y. Chen, *Classification of Lagrangian surfaces of constant curvature in complex hyperbolic plane*, J. Geom. Phys., **55** (2005), 399–439.
[14] B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Lagrangian isometric immersions of a real-space-form $M^n(c)$ into a complex-space-form $\tilde{M}^n(4c)$*, Math. Proc. Camb. Phil. Soc., **124** (1998), 107–125.
[15] M. Dajczer, Submanifolds and isometric immersions, Publish or Perish Inc., Houston TX, 1990.
[16] M. Dajczer and R. Tojeiro, *Flat totally real submanifolds of $\mathbb{C}P^n$ and the symmetric generalized wave equation*, Tôhoku Math. J., **47** (1995), 117–123.
[17] J. Deprez, *Semi-parallel surfaces in euclidean space*, J. Geom., **25** (1985), 192–200.
[18] R. Deszcz, *On pseudosymmetric space*, Bull. Belg. Math. Soc., **44** (1992), 1–34.
[19] O. Kasabov, *Constant isotropic totally real surfaces in 4-dimensional complex space forms*, Tensor (N. S.) **51** (1992), 22–25.
[20] O. Kasabov, *Totally real semiparallel isotropic submanifolds of complex space forms*, Tensor (N. S.) **56** (1995), 12–18.

[21] G. A. Lobos, *Pseudo-parallel surfaces in space forms*, Differential Geometry. Valencia, 2001. World Sci. Publishing, River Edger, NJ (2002),pp. 197–204.

[22] G. A. Lobos and M. Ortega, *Pseudo-parallel real hypersurfaces in complex space forms*, Bull. Korean Math. Soc., **41** (2004), 609–618.

[23] G. A. Lobos and R. Tojeiro, *Pseudo-parallel submanifolds with flat normal bundle of space forms*, Glasg. Math. J., **48** (2006), 171–177.

[24] Ü. Lumiste, *Submanifold with parallel fundamental form*, Handbook of differential geometry, Vol. I, Amsterdam: North-Holland, (2000),pp. 777–864.

[25] I. Mihai and B. Rouxel, *Tensor product surfaces of Euclidean plane curves*, Results Math., **27** (1995), 308–123.

[26] S. Montiel and F. Urbano, *Isotropic Totally Real Submanifolds*, Math. Z., **199** (1988), 55–60.

[27] B. O'Neill, *Isotropic and Kaehler immersions*, Cand. J. Math., **17** (1965), 907–915.

[28] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R = 0$. I. The local version*, J. Diff. Geom., **17** (1982), 531–582.

Pablo M. Chacón
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Salamanca
Plaza de la Merced 1-4
37008, Salamanca
Spain
pmchacon@usal.es

Guillermo A. Lobos
Departamento de Matemática
Universidade Federal de São Carlos
13565-905, São Carlos, SP
Brazil
lobos@dm.ufscar.br