# $\mathbf{Z_2^k}$-actions fixing $\mathbf{RP^2 \cup RP^{even}}$

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**Abstract** This paper determines, up to equivariant cobordism, all manifolds with $Z_2^k$-action whose fixed point set is $RP^2 \cup RP^n$, where $n > 2$ is even.

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## 1 Introduction

Suppose $M$ is a smooth and closed manifold and $T : M \to M$ is a smooth involution defined on $M$. It is well known that the fixed point set of $T$, $F$, is a finite and disjoint union of closed submanifolds of $M$. For a given $F$, a basic problem in this context is the classification, up to equivariant cobordism, of the pairs $(M,T)$ for which the fixed point set is $F$. For related results, see for example [2], [3], [4], [10], [15], [6, Theorem 27.6], [1, Page 309], [16], [17] and [18]. Specifically, in [2], D. C. Royster studied this problem with $F$ being the disjoint union of two real projective spaces, $F = RP^m \cup RP^n$ (for $F = RP^n$, the classification was established in [6] and [16]), establishing the results via a case-by-case method depending on the parity of $m$ and $n$, with special arguments when one of the components is $RP^0 = \{point\}$, and leaving open the case in which $m$ and $n$ are even, with the exception of $(m,n) = (0, even)$ (Royster remarked that the methods applied in [2] are not sufficient to handle the case $(m,n) = (even, even)$ with $m, n > 0$). If $m = n =$ even, one knows from [1] that $(M,T)$ is an equivariant boundary when $dim(M) \geq 2n$; this case was completed in [3], where it was shown that $(M,T)$ also is a boundary when $n \leq dim(M) < 2n$. To understand the case $(m,n) = (0, even)$ and also to explain the goal of this paper, for any $m$ and $n$ consider the involution $(RP^{m+n+1}, T_{m,n})$ defined in homogeneous coordinates by

$$T_{m,n}[x_0, x_1, ..., x_{m+n+1}] = [-x_0, -x_1, ..., -x_m, x_{m+1}, ..., x_{m+n+1}] .$$

The fixed set of $T_{m,n}$ is $RP^m \cup RP^n$. From $T_{m,n}$, it may be possible to obtain other involutions fixing $RP^m \cup RP^n$: in general, for a given involution $(W,T)$

with fixed set $F$ and with $W$ being a boundary, the involution $\Gamma(W,T) = (\frac{S^1 \times W}{-Id \times T}, \tau)$ is equivariantly cobordant to an involution fixing $F$; here, $S^1$ is the 1-sphere, $Id$ is the identity map and $\tau$ is the involution induced by $c \times Id$, where $c$ is the complex conjugation (see [7]). If $\frac{S^1 \times W}{-Id \times T}$ is a boundary, we can repeat the process taking $\Gamma^2(W,T)$, and so on. If $F$ is nonbounding, this process finishes, that is, there exists the first natural number $r \geq 1$ for which the underlying manifold of $\Gamma^r(W,T)$ is nonbounding; this follows from the 5/2-theorem of J. Boardman of [5] and its strengthened version of [1]. In particular, if $m$ and $n$ are even and $m < n$, $RP^m \cup RP^n$ does not bound and $RP^{m+n+1}$ bounds, so this number makes sense for $(RP^{m+n+1}, T_{m,n})$ and we call it $h_{m,n}$. In [2], Royster proved the following

**Theorem.** *Let $(M,T)$ be an involution fixing $\{point\} \cup RP^n$, where $n$ is even. Then $(M,T)$ is equivariantly cobordant to $\Gamma^j(RP^{n+1}, T_{0,n})$ for some $0 \leq j \leq h_{0,n}$.*

Later, in [14], R. E. Stong and P. Pergher determined the value of $h_{0,n}$, thus answering the question posed by Royster in [2; page 271]: writing $n = 2^p q$ with $p \geq 1$ and $q \geq 1$ odd, they showed that $h_{0,n} = 2$ if $p = 1$ and $h_{0,n} = 2^p - 1$ if $p > 1$.

In this paper, we contribute to this problem by solving the case $(m,n) = (2, even)$. Specifically, we will prove the following

**Theorem 1.** *Let $(M,T)$ be an involution fixing $RP^2 \cup RP^n$, where $M$ is connected and $n \geq 4$ is even. Then, if $n > 4$, $(M,T)$ is equivariantly cobordant to $\Gamma^j(RP^{n+3}, T_{2,n})$ for some $0 \leq j \leq h_{2,n}$; if $n = 4$, $(M,T)$ is either equivariantly cobordant to $\Gamma^j(RP^7, T_{2,4})$ for some $0 \leq j \leq h_{2,4}$, or equivariantly cobordant to $\Gamma^2(RP^3, T_{0,2}) \cup (RP^5, T_{0,4})$.*

In addition, we generalize the result of Stong and Pergher of [14], calculating the general value of $h_{m,n}$ (which, in particular, makes numerically precise the statement of Theorem 1).

**Theorem 2.** *For $m,n$ even, $0 \leq m < n$, write $n - m = 2^p q$ with $p \geq 1$ and $q \geq 1$ odd. Then $h_{m,n} = 2$ if $p = 1$ and $h_{m,n} = 2^p - 1$ if $p > 1$.*

Finally, we also extend the results for $Z_2^k$-actions. This extension is automatic from the combination of the above results and the case $F = RP^{even}$ with a recent paper [12]. The details concerning this extension will be given in Section 4. Sections 2 and 3 will be devoted, respectively, to the proofs of Theorems 1 and 2.

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## 2 Involutions fixing $RP^2 \cup RP^{even}$

We start with an involution $(M,T)$ fixing $RP^2 \cup RP^n$, where $M$ is connected and $n \geq 4$ is even, and first establish some notations. We will always use $\lambda_r \to RP^r$ to denote the canonical line bundle over $RP^r$. Denote by $\alpha \in H^1(RP^2, Z_2)$ and $\beta \in H^1(RP^n, Z_2)$ the generators of the 1-dimensional $Z_2$-cohomology. The model involution $(RP^{n+3}, T_{2,n})$ fixes $RP^2 \cup RP^n$ with normal bundles $(n+1)\lambda_2 \to RP^2$ and $3\lambda_n \to RP^n$. The total Stiefel-Whitney classes are $W((n+1)\lambda_2) = (1+\alpha)^{n+1}$, $W(3\lambda_n) = (1+\beta)^3$. Denote by $\eta \to RP^2$ and $\xi \to RP^n$ the normal bundles of $RP^2$ and $RP^n$ in $M$. To prove Theorem 1, it suffices to prove the following

**Lemma.** *If $n > 4$, then $W(\eta) = (1+\alpha)^{n+1}$ and $W(\xi) = (1+\beta)^3$; if $n = 4$, then either $W(\eta) = (1+\alpha)^5$ and $W(\xi) = (1+\beta)^3$, or $W(\eta) = 1+\alpha$ and $W(\xi) = 1+\beta$.*

In fact, suppose the lemma is true, and denote by $R$ the trivial one-dimensional vector bundle over any base space. Set $k = dim(\eta)$, $l = dim(\xi)$, that is, $k = dim(M) - 2$, $l = dim(M) - n \geq 1$. First consider $n > 4$. For $0 \leq j \leq h_{2,n}$, the involution $\Gamma^j(RP^{n+3}, T_{2,n})$ is equivariantly cobordant to an involution with fixed data $((n+1)\lambda_2 \oplus jR \to RP^2) \cup (3\lambda_n \oplus jR \to RP^n)$ (see [7]). Using the notations $W = 1 + w_1 + w_2 + ...$ for Stiefel-Whitney classes and $\binom{a}{b}$ for binomial coefficients mod 2, note that $w_3(\xi) = \binom{3}{3}\beta^3 = \beta^3 \neq 0$ and thus $l \geq 3$. Then $\eta \cup \xi$ and $((n+1)\lambda_2 \oplus (l-3)R) \cup (3\lambda_n \oplus (l-3)R)$ are cobordant because they have the same characteristic numbers. If $l \leq 3 + h_{2,n}$, one then has from [6] that $(M,T)$ and $\Gamma^{l-3}(RP^{n+3}, T_{2,n})$ are equivariantly cobordant, proving the result. By contradiction, suppose then $l > 3 + h_{2,n}$. Again from [6], $((n+1)\lambda_2 \oplus (l-3)R) \cup (3\lambda_n \oplus (l-3)R)$ is the fixed data of an involution $(W,S)$, and by removing sections if necessary we can suppose, with no loss, that $dim(W) = n + h_{2,n} + 4$ (see [6, Theorem 26.4]). Let $(N, T\prime)$ be an involution cobordant to $\Gamma^{h_{2,n}}(RP^{n+3}, T_{2,n})$ and with fixed data $((n+1)\lambda_2 \oplus h_{2,n}R) \cup (3\lambda_n \oplus h_{2,n}R)$; one knows that $N$ is not a boundary. Then $\Gamma(N, T\prime) \cup (W,S)$ is cobordant to an involution with fixed data $R \to N$, and from [6] $R \to N$ then is a boundary, which is impossible. Now suppose $n = 4$. The case $W(\eta) = (1+\alpha)^5$ and $W(\xi) = (1+\beta)^3$ is included in the above approach, hence suppose $W(\eta) = 1+\alpha$

and $W(\xi) = 1+\beta$. Since $h_{0,2} = 2$, the involution $\Gamma^2(RP^3, T_{0,2})$ is cobordant to an involution with fixed data $(5R \to \{point\}) \cup (\lambda_2 \oplus 2R \to RP^2)$. Then the involution $\Gamma^2(RP^3, T_{0,2}) \cup (RP^5, T_{0,4})$ is cobordant to an involution $(W^5, T)$ with fixed data $(\lambda_2 \oplus 2R \to RP^2) \cup (\lambda_4 \to RP^4)$, and the total Stiefel-Whitney classes are $W(\lambda_2 \oplus 2R) = 1+\alpha$, $W(\lambda_4) = 1+\beta$. Because $h_{0,2} = 2$, the underlying manifold of $\Gamma^2(RP^3, T_{0,2})$ does not bound; since $RP^5$ bounds, $W^5$ does not bound. By contradiction, suppose $l \geq 2$. Using the hypothesis, [6] and removing sections if necessary, we can suppose with no loss that $(M, T)$ has fixed data $(\lambda_2 \oplus 3R \to RP^2) \cup (\lambda_4 \oplus R \to RP^4)$. Using the same above argument for $\Gamma(W^5, T) \cup (M, T)$, we conclude that $R \to W$ is a boundary, which is false. Then $l = 1$ and $(M, T)$ and $(W^5, T)$ (hence $\Gamma^2(RP^3, T_{0,2}) \cup (RP^5, T_{0,4})$) have fixed data with same characteristic numbers.

In order to prove the lemma, we will intensively use the following basic fact from [6]: the projective space bundles $RP(\eta)$ and $RP(\xi)$, with the standard line bundles $\lambda \to RP(\eta)$ and $\nu \to RP(\xi)$, are cobordant as elements of the bordism group $\mathcal{N}_{k+1}(BO(1))$. Then any class of dimension $k+1$, given by a product of the classes $w_i(RP(\eta))$ and $w_1(\lambda)$, evaluated on the fundamental homology class $[RP(\eta)]$, gives the same characteristic number as the one obtained by the corresponding product of the classes $w_i(RP(\xi))$ and $w_1(\nu)$, evaluated on $[RP(\xi)]$. To evaluate characteristic numbers, the following formula of Conner will be useful (see [9; Lemma 3.1]): if $\pi : \mu \to N$ is any $r$-dimensional vector bundle, $c$ is the first Stiefel-Whitney class of the standard line bundle over $RP(\mu)$, $\overline{W}(\mu) = 1+\overline{w}_1(\mu)+\overline{w}_2(\mu)+...$ is the dual Stiefel-Whitney class defined by $W(\mu)\overline{W}(\mu) = 1$ and $\alpha \in H^*(N, Z_2)$, then $c^j\pi^*(\alpha)[RP(\mu)] = \overline{w}_{j-r+1}(\mu)\alpha[N]$ when $j \geq r-1$. In this context, our numerical arguments will always be considered modulo 2. Write $W(\lambda) = 1+c$ and $W(\nu) = 1+d$ for the Stiefel-Whitney classes of $\lambda$ and $\nu$. The structure of the Grothendieck ring of orthogonal bundles over real projective spaces says that $W(\eta) = (1+\alpha)^p$ and $W(\xi) = (1+\beta)^q$ for some $p, q \geq 0$. From [6, 23.3], one then has

$$W(RP(\eta)) = (1+\alpha)^3 (\sum_{i=0}^{2} (1+c)^{k-i} \binom{p}{i} \alpha^i)$$

and

$$W(RP(\xi)) = (1+\beta)^{n+1} (\sum_{i=0}^{l} (1+d)^{l-i} \binom{q}{i} \beta^i) \ ,$$

where here we are suppressing bundle maps.

**Fact 1.** *$p$ and $q$ are odd; in particular, $w_1(\eta) = \alpha$ and $w_1(\xi) = \beta$.*

**Proof.** One has $w_1(RP(\eta)) = \binom{k}{1}c + \alpha + \binom{p}{1}\alpha$ and $w_1(RP(\xi)) = \binom{l}{1}d + \beta + \binom{q}{1}\beta$. Since $k+2 = l+n$ and $n$ is even, $\binom{k}{1} = \binom{l}{1}$, and thus $w_1(RP(\eta)) + \binom{k}{1}c = (\binom{p}{1}+1)\alpha$ and $w_1(RP(\xi)) + \binom{l}{1}d = (\binom{q}{1}+1)\beta$ are corresponding characteristic classes. Because $n > 2$, it follows that

$$0 = (\binom{p}{1}+1)\alpha^n c^{l-1}[RP(\eta)] = (\binom{q}{1}+1)\beta^n d^{l-1}[RP(\xi)] = (\binom{q}{1}+1)\beta^n[RP^n] = \binom{q}{1}+1 ,$$

which gives that $q$ is odd. Also $\binom{p}{1}+1 = (\binom{p}{1}+1)\alpha^2 c^{k-1}[RP(\eta)] = (\binom{q}{1}+1)\beta^2 d^{k-1}[RP(\xi)] = 0$, and $p$ is odd. □

**Fact 2.** *If $l = 1$, then $n = 4$, $W(\eta) = 1+\alpha$ and $W(\xi) = 1+\beta$.*

**Proof.** Since $l = 1$ and $w_1(\xi) = \beta$, $W(\xi) = 1+\beta$. Then the involution $(M,T) \cup (RP^{n+1}, T_{0,n})$ is cobordant to an involution with fixed data $(\eta \to RP^2) \cup ((n+1)R \to \{point\})$. From [2] and the fact that $h_{0,2} = 2$, $W(\eta) = 1+\alpha$ and $n = 4$. □

Fact 2 reduces our lemma to the following assertion: if $l > 1$, then $W(\eta) = (1+\alpha)^{n+1}$ and $W(\xi) = (1+\beta)^3$; so, we assume throughout the remainder of this section that $l > 1$. Note that $(1+\alpha)^{n+1} = (1+\alpha)^3$ if $\binom{n}{2} = 1$ and $(1+\alpha)^{n+1} = 1+\alpha$ if $\binom{n}{2} = 0$. Denote by $r$ the greatest power of 2 that appears in the 2-adic expansion of $n$; that is, $4 \le 2^r \le n < 2^{r+1}$. We can assume $q < 2^{r+1}$ and $p < 4$. Then Facts 3 and 4 below show that $W(\eta) = (1+\alpha)^{n+1}$:

**Fact 3.** *If $\binom{n}{2} = 1$, then $p = 3$.*

**Fact 4.** *If $\binom{n}{2} = 0$, then $p = 1$.*

Set $p' = 4 - p$, $q' = 2^{r+1} - q$. Then the dual Stiefel-Whitney classes of $\eta$ and $\xi$ are given by $\overline{W}(\eta) = (1+\alpha)^{p'}$, $\overline{W}(\xi) = (1+\beta)^{q'}$. Since $p$ and $q$ are odd, $p'$ and $q'$ are odd; further, $\binom{p}{2} + \binom{p'}{2} = 1$ and $\binom{q}{2^u} + \binom{q'}{2^u} = 1$ for each $1 \le u \le r$. Now we prove Fact 3. We will use several times the fact that a binomial coefficient $\binom{a}{b}$ is nonzero modulo 2 if and only if the 2-adic expansion of $b$ is a subset of the 2-adic expansion of $a$. We have $n = 4j+2$, with $j \ge 1$, and want to show that $p = 3$; since $p < 4$ is odd, it suffices to show that $\binom{p}{2} = 1$, or equivalently that $\binom{p'}{2} = 0$. Suppose by contradiction that $\binom{p'}{2} = 1$. By Conner's formula, $c^{k+1}[RP(\eta)] = \binom{p'}{2}\alpha^2[RP^2] = \binom{p'}{2} = d^{k+1}[RP(\xi)] = \binom{q'}{4j+2}$. Then $\binom{q'}{4j+2} = 1$ and consequently $\binom{q'}{2} = 1$. We formally introduce the class (with $l-1 \ge 1$)

$$\widetilde{W}(RP(\ )) = \frac{W(RP(\ ))}{(1+c)^{l-1}}.$$

Since $k = l + 4j$ and $p$ and $q$ are odd, on $RP^2$ this class is

$$\widetilde{W}(RP(\eta)) = (1+\alpha)^3(1+c^4)^j(1+c+\alpha+(1+c)^{-1}\binom{p}{2}\alpha^2),$$

and on $RP^n$ it is

$$\widetilde{W}(RP(\xi)) = (1+\beta)^{4j+3}(1+d+\beta+(1+d)^{-1}\binom{q}{2}\beta^2+(1+d)^{-2}\binom{q}{3}\beta^3+...).$$

Then $\widetilde{w}_3(RP(\eta)) = \alpha^2 c + \binom{p}{2}\alpha^2 c = \binom{p'}{2}\alpha^2 c = \alpha^2 c$, and since $\binom{q}{2}+\binom{q}{3} = 0$ because $q$ is odd, $\widetilde{w}_3(RP(\xi)) = \binom{q'}{2}\beta^2 d = \beta^2 d$. Now we observe that, if $a$ and $b$ are one-dimensional cohomology classes, then by the Cartan formula one has $Sq^{2^u}(a^{2^u}b) = a^{2^{u+1}}b$, where $Sq$ is the Steenrod operation and $u \geq 1$. Also one has, by the Wu and Cartan formulae, that $Sq^i$ evaluated on a product of characteristic classes gives a polynomial in the characteristic classes. Then

$$Sq^{2^{r-1}}(...(Sq^4(Sq^2(\alpha^2 c)))...) = \alpha^{2^r}c \quad \text{and} \quad Sq^{2^{r-1}}(...(Sq^4(Sq^2(\beta^2 d)))...) = \beta^{2^r}d$$

are corresponding classes on $RP^2$ and $RP^n$. Using the Conner formula and the fact that $2^r \geq 4$, one then has

$$0 = (\alpha^{2^r}c)c^{4j+1-2^r+l-1}[RP(\eta)] = (\beta^{2^r}d)d^{4j+1-2^r+l-1}[RP(\xi)] = \binom{q'}{4j+2-2^r}.$$

Since $\binom{q'}{4j+2} = 1$ and $2^r$ belongs to the 2-adic expansion of $4j+2$, also $\binom{q'}{4j+2-2^r} = 1$, which is impossible. Hence Fact 3 is proved. To prove Fact 4, we can consider $n = 4j$ with $j \geq 1$; in this case, to show that $p = 1$, it suffices to show that $\binom{p'}{2} = 1$, and again by contradiction we suppose $\binom{p'}{2} = 0$. Then $\binom{p}{2} = 1$ and $k = l + 4j - 2$ gives

$$\widetilde{W}(RP(\eta)) = (1+\alpha)^3((1+c)^{4j-1}+(1+c)^{4j-2}\alpha+(1+c)^{4j-3}\alpha^2)$$

and $\widetilde{w}_2(RP(\eta)) = c^2 + \alpha^2 + c\alpha$. Also

$$\widetilde{W}(RP(\xi)) = (1+\beta)^{4j+1}(1+d+\beta+(1+d)^{-1}\binom{q}{2}\beta^2+(1+d)^{-2}\binom{q}{3}\beta^3+...)$$

and $\widetilde{w}_2(RP(\xi)) = \binom{q}{2}\beta^2 + \beta d + \beta^2$. Let $2^t$ be the lesser power of 2 of the 2-adic expansion of $n = 4j$ ($2^t \geq 4$). For $t \leq x \leq r$ and with the same preceding tools, we then get

$$Sq^{2^{x-1}}(...(Sq^4(Sq^2(\widetilde{w}_2(RP(\eta))c)))...)c^{4j+l-2^x-2}[RP(\eta)] =$$
$$(c^{2^x}c+\alpha^{2^x}c+c^{2^x}\alpha)c^{4j+l-2^x-2}[RP(\eta)] = \binom{p'}{2}+0+\binom{p'}{1} = 1 =$$
$$Sq^{2^{x-1}}(...(Sq^4(Sq^2(\widetilde{w}_2(RP(\xi))d)))...)d^{4j+l-2^x-2}[RP(\xi)] =$$
$$(\binom{q}{2}\beta^{2^x}d+\beta d^{2^x}+\beta^{2^x}d)d^{4j+l-2^x-2}[RP(\xi)] =$$
$$\binom{q}{2}\binom{q'}{4j-2^x}+\binom{q'}{4j-1}+\binom{q'}{4j-2^x} = \binom{q'}{2}\binom{q'}{4j-2^x}+\binom{q'}{4j-1},$$

$$0 = \binom{p'}{2} = c^{k+1}[RP(\eta)] = d^{k+1}[RP(\xi)] = \binom{q'}{4j} \quad \text{and}$$

$$\widetilde{w}_2(RP(\eta))c^{4j+l-3}[RP(\eta)] = \binom{p'}{2}+1+\binom{p'}{1} = 0 = \widetilde{w}_2(RP(\xi)d^{4j+l-3}[RP(\xi)] =$$
$$\binom{q}{2}\binom{q'}{4j-2}+\binom{q'}{4j-1}+\binom{q'}{4j-2} = \binom{q'}{2}\binom{q'}{4j-2}+\binom{q'}{4j-1}.$$

That is, we get the equations: i) $0=\binom{q'}{4j}$, ii) $0=\binom{q'}{2}\binom{q'}{4j-2}+\binom{q'}{4j-1}$ and iii) $1=\binom{q'}{2}\binom{q'}{4j-2^x}+\binom{q'}{4j-1}$. By using equations ii) and iii), we conclude that $\binom{q'}{2}=1$ and $\binom{q'}{4j-2^x}\neq\binom{q'}{4j-2}$. Suppose $t<r$. If $\binom{q'}{4j-2^r}=1$, equation i) and the fact that $2^r$ belongs to the 2-adic expansion of $4j$ imply that $2^r$ is the only power of 2 of the 2-adic expansion of $4j$ that does not belong to the 2-adic expansion of $q'$. Hence $\binom{q'}{4j-2^t}=0$, which is a contradiction. Then $\binom{q'}{4j-2^r}=\binom{q'}{4j-2^t}=0$. In this case, equation i) and $\binom{q'}{4j-2}=1$ give that $2^t$ is the only power of 2 of the 2-adic expansion of $4j$ that does not belong to the 2-adic expansion of $q'$, which gives the contradiction $\binom{q'}{4j-2^t}=1$. Now suppose $t=r$, that is, $n=4j=2^r$. One has

$$(\widetilde{w}_2(RP(\eta))^2c^{2^r+l-5}[RP(\eta)] = \binom{p'}{2}+0+1 = 1 = (\widetilde{w}_2(RP(\xi))^2d^{2^r+l-5}[RP(\xi)] =$$
$$\binom{q}{2}\binom{q'}{2^r-4}+\binom{q'}{2^r-2}+\binom{q'}{2^r-4} = \binom{q'}{2}\binom{q'}{2^r-4}+\binom{q'}{2^r-2} = \binom{q'}{2^r-4}+\binom{q'}{2^r-2}.$$

Since $\binom{q'}{2}=1$, $\binom{q'}{2^r-4}=\binom{q'}{2^r-2}$, which gives a contradiction. Thus Fact 4 is proved.

Now we prove that $q=3$. To do this, first we prove

**Fact 5.** $\binom{q}{2}=1$*; in particular,* $q\geq 3$.

**Proof.** As before, first consider $n=4j+2$, with $j\geq 1$. In this case, we know that $0=\binom{p'}{2}=\binom{q'}{4j+2}$, $\widetilde{w}_2(RP(\eta))=\binom{p}{2}\alpha^2+\alpha c=\alpha^2+\alpha c$ and $\widetilde{w}_2(RP(\xi))=\binom{q}{2}\beta^2+\beta d$. Then

$$(\widetilde{w}_2(RP(\eta)))^2c^{4j+l-3}[RP(\eta)] = 1 = (\widetilde{w}_2(RP(\xi)))^2d^{4j+l-3}[RP(\xi)] =$$
$$\binom{q}{2}\binom{q'}{4j-2}+\binom{q'}{4j}.$$

Since $\binom{q}{2}+\binom{q'}{2}=1$ and 2 belongs to the 2-adic expansion of $4j-2$, one has that $\binom{q}{2}\binom{q'}{4j-2}=0$, and thus $\binom{q'}{4j}=1$. Now $\binom{q'}{4j+2}=0$ and $\binom{q'}{4j}=1$ imply that $\binom{q'}{2}=0$, and thus $\binom{q}{2}=1$. Since $q$ is odd, this means that $q\geq 3$.

Now suppose $n = 4j$, with $j \geq 1$. In this case, one has $\binom{p'}{2} = 1$, $\widetilde{w}_3(RP(\eta)) = c^3 + \binom{p'}{2}\alpha^2 c = c^3 + \alpha^2 c$ and $\widetilde{w}_3(RP(\xi)) = \binom{q}{2}\beta^2 d$. Then

$$\begin{aligned}
&Sq^{2^{r-1}}(...(Sq^4(Sq^2(\widetilde{w}_3(RP(\eta))))...)c^{4j+l-2^r-2}[RP(\eta)] = \\
&(c^{2^r}c + \alpha^{2^r}c)c^{4j+l-2^r-2}[RP(\eta)] = \binom{p'}{2} = 1 = \\
&Sq^{2^{r-1}}(...(Sq^4(Sq^2(\binom{q}{2}\beta^2 d)))...)d^{4j+l-2^r-2}[RP(\xi)] = \\
&(\binom{q}{2}\beta^{2^r}d)d^{4j+l-2^r-2}[RP(\xi)] = \binom{q}{2}\binom{q'}{4j-2^r}.
\end{aligned}$$

Thus $\binom{q}{2} = 1$, and Fact 5 is proved. □

To end our task, we will show that $q \leq 3$. The strategy will consist in finding nonzero characteristic numbers coming from characteristic classes involving $\alpha^{q-1}$. To do this, we need the following

**Fact 6.** $n + l - 1 > 2(q-1)$.

**Proof.** First suppose $n = 4j+2$, $j \geq 1$. From the proof of Fact 5, $\binom{q'}{4j} = 1$, and thus $\binom{q'}{2^r} = 1$ and $\binom{q}{2^r} = 0$. Since $q < 2^{r+1}$, $q < 2^r < 4j+2$. In particular, $w_q(\xi) = \alpha^q \neq 0$ and $q \leq l$. Then $n+l-1 = 4j+2+l-1 > 2q-1 > 2(q-1)$. Now suppose $n = 4j$, $j \geq 1$. In this case, $\binom{p'}{2} = 1 = \binom{q'}{4j}$, so the argument is the same. □

Fact 6 says that we can consider characteristic numbers coming from classes involving $\widetilde{w}_2^{q-1}$; in this direction, first consider $n = 4j+2$, $j \geq 1$. In this case, $\widetilde{w}_2(RP(\eta)) = \binom{p}{2}\alpha^2 + \alpha c = \alpha(\alpha + c)$ and $\widetilde{w}_2(RP(\xi)) = \binom{q}{2}\beta^2 + \beta d = \beta(\beta + d)$. Thus

$$(\alpha^{q-1}(\alpha+c)^{q-1}c^{4j+l-2q+3})[RP(\eta)] = (\beta^{q-1}(\beta+d)^{q-1}d^{4j+l-2q+3})[RP(\xi)].$$

By Conner's formula, the last term is the coefficient of $\beta^{4j+2}$ in $\beta^{q-1}(1+\beta)^{q-1}(1+\beta)^{q'}$. If $n = 4j$, $j \geq 1$, similarly one has $\widetilde{w}_2(RP(\eta)) + c^2 = (c^2 + \binom{p}{2}\alpha^2 + \alpha c) + c^2 = \alpha c$, $\widetilde{w}_2(RP(\xi)) + d^2 = \binom{q'}{2}\beta^2 + \beta d + d^2 = (\beta + d)d$,

$$((\alpha^{q-1}c^{q-1})c^{4j+l-2q+1})[RP(\eta)] = ((\beta+d)d)^{q-1}d^{4j+l-2q+1})[RP(\xi)],$$

and the last term is the coefficient of $\beta^{4j}$ in $(1+\beta)^{q-1}(1+\beta)^{q'}$. Since $(1+\beta)^{q-1}(1+\beta)^{q'} = (1+\beta)^{-1}$, these numbers have value 1, which means that $\alpha^{q-1} \neq 0$ and $q - 1 \leq 2$, thus ending the proof.

# 3 Calculation of $h_{m,n}$

Denote by $\mathcal{W}_r$ the underlying manifold of $\Gamma^r(RP^{m+n+1}, T_{m,n})$, and by $\mathcal{P}_r$ the total space of the iterated fibration

$$RP((m+1)\mu_r \oplus (n+1)R) \to RP(\lambda_1 \oplus (r-1)R) \to RP^1,$$

where $\mu_r$ is the standard line bundle over $RP(\lambda_1 \oplus (r-1)R)$.

**Lemma** *$\mathcal{W}_r$ is cobordant to $\mathcal{P}_r$.*

**Proof.** If $(W, T)$ is a free involution and $\lambda \to W/T$ is the usual line bundle, the sphere bundle $S(\lambda \oplus R)$ with the antipodal involution in the fibers can be identified to the free involution $(\frac{W \times S^1}{T \times c}, \tau)$, where $c$ is the complex conjugation and $\tau$ is induced by $Id \times -Id$. Starting with $(S^1, -Id)$ and by iteratively applying this fact, we can see that $\mathcal{W}_r$ is diffeomorphic to the total space of the iterated fibration $RP((m+1)\xi_r \oplus (n+1)R) \to RP(\xi_{r-1} \oplus R) \to \ldots \to RP(\xi_2 \oplus R) \to RP(\xi_1 \oplus R) \to RP^1$; here, $\xi_1 = \lambda_1$ and $\xi_i$ is the standard line bundle over $RP(\xi_{i-1} \oplus R)$, for each $i > 1$. From [6], one knows that $\mathcal{N}_*(BO(1))$ is a free $\mathcal{N}_*$-module, where $\mathcal{N}_*$ is the unoriented cobordism ring, with one generator $X_j$ in each dimension $j \geq 0$; these generators are characterized by the fact that $c^j[V^j] = 1$, where $\lambda \to V^j$ is a representative of $X_j$ and $c$ is the first Whitney class of $\lambda$. Further, it was shown in [8, Theorem 24.5] that there is a unique basis $\{X_j\}_{j=0}^{\infty}$ for $\mathcal{N}_*(BO(1))$ which satisfies: i) $\triangle(X_j) = X_{j-1}$, $j \geq 1$, where $\triangle : \mathcal{N}_j(BO(1)) \to \mathcal{N}_{j-1}(BO(1))$ is the Smith homomorphism; ii) if $\lambda \to V^j$ is a representative of $X_j$ for $j \geq 1$, then $V^j$ bounds. Also it was shown in [8, Theorem 24.5] that $X_1 = [\xi_1 \to RP^1]$ and $X_j = [\xi_j \to RP(\xi_{j-1} \oplus R)]$ for $j \geq 2$. For $j \geq 1$, set $Y_j = [\mu_j \to RP(\lambda_1 \oplus (j-1)R)]$. One has $c^j[RP(\lambda_1 \oplus (j-1)R)] = \overline{w_1}(\lambda_1)[S^1] = 1$, $Y_1 = X_1$ and $\triangle([\mu_j \to RP(\lambda_1 \oplus (j-1)R)]) = [\mu_{j-1} \to RP(\lambda_1 \oplus (j-2)R)]$ for $j \geq 2$; further, every projective space bundle over $S^1$ bounds (see [7, Lemma 2.2]). By the uniqueness, $Y_j = X_j$ for $j \geq 1$, and the result follows. □

With the above lemma in hand, Theorem 2 can now be paraphrased as

**Theorem 2.** *For $m, n$ even, $0 \leq m < n$, write $n - m = 2^p q$ with $p \geq 1$ and $q \geq 1$ odd. Then,*

*a) if $p = 1$, $\mathcal{P}_1$ bounds and $\mathcal{P}_2$ does not bound;*

*b) if $p > 1$, $\mathcal{P}_r$ bounds for each $1 \leq r \leq 2^p - 2$ and $\mathcal{P}_{2^p-1}$ does not bound.*

Denote by $\alpha \in H^1(RP^1, Z_2)$ the generator and by $\theta_r \to \mathcal{P}_r$ the standard line bundle; set $W(\mu_r) = 1 + c$ and $W(\theta_r) = 1 + d$. The following lemma, which follows from Conner's formula, will be useful in our computations:

**Lemma.** *i) For $f + g + h = m + n + 1 + r$, $c^f(c+d)^g d^h[\mathcal{P}_r]$ is the coefficient of $c^r$ in $\frac{c^f(1+c)^g}{(1+c)^{m+1}}$.*

*ii) For $f+g+h=m+n+r$, $\alpha c^f(c+d)^g d^h[\mathcal{P}_r]$ is the coefficient of $c^r$ in $\dfrac{c^{f+1}(1+c)^g}{(1+c)^{m+1}}$.*

If $M$ is a closed manifold and $(1+t_1)(1+t_2)...(1+t_l)$ is the factored form of $W(M)$, one has the $s$-class $s_j$ given by the polynomial in the classes of $M$ corresponding to the symmetric function $t_1^j+t_2^j+...+t_l^j$. Since

$$W(\mathcal{P}_r)=(1+c+\alpha)(1+c)^{r-1}(1+c+d)^{m+1}(1+d)^{n+1},$$

$c^i=0$ if $i>r$ and $\alpha^i=0$ if $i>1$, the $s$-class $s_{m+n+1+r}$ of $\mathcal{P}_r$ then is

$$s_{m+n+1+r}=(c+\alpha)^{m+n+1+r}+(r-1)c^{m+n+1+r}+(m+1)(c+d)^{m+n+1+r}+ (n+1)d^{m+n+1+r}=(c+d)^{m+n+1+r}+d^{m+n+1+r}.$$

Using part i) of the above lemma and the fact that

$$\frac{1}{(1+c)^{m+1}}=1+\sum_{i=1}^{r}\binom{m+i}{i}c^i$$

in $H^*(\mathcal{P}_r,Z_2)$, one then has

$$s_{m+n+1+r}[\mathcal{P}_r]=\text{coefficient of } c^r \text{ in } (1+c)^{n+r}\ +\ \text{coefficient of } c^r \text{ in } \frac{1}{(1+c)^{m+1}}=\binom{n+r}{r}+\binom{m+r}{r}.$$

Because $n=2^pq+m$ and $q$ is odd, one then gets

$$s_{m+n+1+2^p}[\mathcal{P}_{2^p}]=\binom{n+2^p}{2^p}+\binom{m+2^p}{2^p}=1.$$

It follows that $\mathcal{P}_{2^p}$ does not bound; because $\mathcal{P}_1$ is a projective space bundle over $S^1$ and hence a boundary, this in particular proves part a) of Theorem 2. So we can assume from now that $p>1$ and $r<2^p$. Using again $n=2^pq+m$, we rewrite $W(\mathcal{P}_r)$ as

$$W(\mathcal{P}_r)=(1+c+\alpha)(1+c)^{r-1}(1+c+d(c+d))^{m+1}(1+d^{2^p})^q.$$

Then a general characteristic number of $\mathcal{P}_r$ is a sum of terms of the form $\alpha^e c^f(d(c+d))^g d^{2^p h}[\mathcal{P}_r]$, where $e+f+2g+2^p h=m+n+1+r$ and either $e=0$ or $e=1$. Since by the above lemma $\alpha^e c^f(d(c+d))^g d^{2^p h}[\mathcal{P}_r]=c^{f+1}(d(c+d))^g d^{2^p h}[\mathcal{P}_r]$, we can assume $e=0$. Thus, to prove the first statement of part b) of Theorem 2, it suffices to show that $c^f(d(c+d))^g d^{2^p h}[\mathcal{P}_r]=0$ when $f+2g+2^p h=m+n+1+r$ and $r<2^p-1$. Since $c^f=0$ if $f>r$, we assume

$f \leq r$ and thus $0 \leq r - f < 2^p - 1$. Take $s > p$ with $2^s > m+1$; in particular, $2^s > 2^p > r$ and $\frac{1}{(1+c)^{m+1}} = (1+c)^{2^s-m-1}$. Then

$$c^f(d(c+d))^g d^{2^p h}[\mathcal{P}_r] = \text{coefficient of } c^r \text{ in } \frac{c^f(1+c)^g}{(1+c)^{m+1}} =$$
$$\text{coefficient of } c^r \text{ in } c^f(1+c)^g(1+c)^{2^s-m-1} =$$
$$\binom{2^s+g-m-1}{r-f} = \binom{2^{p-1}(2^{s-p+1}+q-h)+\frac{r-f+1}{2}-1}{r-f}.$$

Write $r - f + 1 = 2^t a$, where $a$ is odd. Since $r - f + 1 = 2g + 2^p h - m - n$ is even and $r - f + 1 < 2^p$, one has $1 \leq t \leq p-1$. Then $2^{t-1}$ belongs to the 2-adic expansion of $r - f$ and does not belong to the 2-adic expansion of $2^{p-1}(2^{s-p+1}+q-h)+\frac{r-f+1}{2}-1$, which means, as required, that the above number is zero.

Finally, we must to show that $\mathcal{P}_{2^p-1}$ does not bound. One has $w_2(\mathcal{P}_{2^p-1}) = \alpha c + \binom{m+1}{2}c^2 + d(c+d)$. We have seen above that $c^f(d(c+d))^g d^{2^p h}[\mathcal{P}_r] = 0$ for $f + 2g + 2^p h = m + n + 1 + r$ and $0 \leq r - f < 2^p - 1$; in particular, this is true for $r = 2^p - 1$ and $f > 0$. In this way,

$$w_2(\mathcal{P}_{2^p-1})^{\frac{m+n+2^p}{2}}[\mathcal{P}_{2^p-1}] = (d(c+d))^{\frac{m+n+2^p}{2}}[\mathcal{P}_{2^p-1}] =$$
$$\text{coefficient of } c^{2^p-1} \text{ in } \frac{(1+c)^{\frac{m+n+2^p}{2}}}{(1+c)^{m+1}} =$$
$$\text{coefficient of } c^{2^p-1} \text{ in } (1+c)^{\frac{n-m}{2}+2^{p-1}-1} = \binom{2^{p-1}q+2^{p-1}-1}{2^p-1} = 1,$$

and $\mathcal{P}_{2^p-1}$ does not bound.

# 4 $Z_2^k$-actions fixing $RP^2 \cup RP^{even}$

Let $F^n$ be a connected, smooth and closed $n$-dimensional manifold satisfying the following property, which we call *property* $\mathcal{H}$ : if $N^m$ is any smooth and closed $m$-dimensional manifold with $m > n$ and $T : N^m \to N^m$ is a smooth involution whose fixed point set is $F^n$, then $m = 2n$. From [1], this implies that $(N^m, T)$ is cobordant to the *twist involution* $(F^n \times F^n, t)$, given by $t(x, y) = (y, x)$. This concept was introduced and studied in [13], and it was inspired in [6; 27.6] (or [8; 29.2]), where it was shown that $RP^{even}$ has this property. In [12], we studied the equivariant cobordism classification of smooth actions $(M;\Phi)$ of the group $Z_2^k$ on closed and smooth manifolds $M$ for which the fixed point set $F$ of the action is the union $F = K \cup L$, where $K$ and $L$ are submanifolds of $M$ with property $\mathcal{H}$ and with $dim(K) < dim(L)$. We showed that, for this $F$, the $Z_2^k$-classification is completely determined by the corresponding $Z_2$-classification. Specifically, the equivariant cobordism classes of $Z_2^k$-actions fixing $K \cup L$ can be represented by a special set of $Z_2^k$-actions which are explicitly obtained from involutions fixing $K \cup L$, $K$ and $L$. Together with the results of Sections 2 and 3 and the case $F = RP^{even}$, this gives a precise cobordism description of the $Z_2^k$-actions fixing $RP^2 \cup RP^n$, where $n > 2$ is even; next we give this description. Here, $Z_2^k$ is considered as the group generated by $k$ commuting involutions $T_1, T_2, ..., T_k$. The *fixed data* of a $Z_2^k$-action $(M;\Phi)$, $\Phi = (T_1, T_2, ..., T_k)$, is $\eta = \bigoplus_\rho \varepsilon_\rho \to F$, where $F = \{x \in M \ / \ T_i(x) = x$ for all $1 \leq i \leq k\}$ is the fixed point set of $\Phi$ and $\eta = \bigoplus_\rho \varepsilon_\rho$ is the normal bundle of $F$ in $M$, decomposed into eigenbundles $\varepsilon_\rho$ with $\rho$ running through the $2^k - 1$ nontrivial irreducible representations of $Z_2^k$. A collection of $Z_2^k$-actions fixing $F$ can be obtained from an involution fixing $F$ through the following procedure: let $(W,T)$ be any involution. For each $r$ with $1 \leq r \leq k$, consider the $Z_2^k$-action $\Gamma_r^k(W,T)$, defined on the cartesian product $W^{2^{r-1}} = W \times ... \times W$ ($2^{r-1}$ factors), and described in the following inductive way: first set $\Gamma_1^1(W,T) = (W,T)$. Taking $k \geq 2$ and supposing by inductive hypothesis one has constructed $\Gamma_{k-1}^{k-1}(W,T)$, define $\Gamma_k^k(W,T) = (W^{2^{k-1}}; T_1, T_2, ..., T_k)$, where $(W^{2^{k-1}}; T_1, T_2, ..., T_{k-1}) = (W^{2^{k-2}} \times W^{2^{k-2}}; T_1, T_2, ..., T_{k-1}) = \Gamma_{k-1}^{k-1}(W,T) \times \Gamma_{k-1}^{k-1}(W,T)$ and $T_k$ acts switching $W^{2^{k-2}} \times W^{2^{k-2}}$. This defines $\Gamma_k^k(W,T)$ for any $k \geq 1$. Next, define $\Gamma_r^k(W,T) = (W^{2^{r-1}};\ T_1, T_2, ..., T_k)$ setting $(W^{2^{r-1}}; T_1, T_2, ..., T_r) = \Gamma_r^r(W,T)$ and letting $T_{r+1}, ..., T_k$ act trivially. If (W,T) fixes $F$ and if $\eta \to F$ is the normal bundle of $F$ in $W$, then $\Gamma_r^k(W,T)$ fixes $F$ and its fixed data consists of $2^{r-1}$ copies of $\eta$, $2^{r-1} - 1$ copies of the tangent bundle of $F$ and $2^k - 2^r$ copies of the zero-dimensional bundle over $F$. In particular, for the twist involution $(F \times F, t)$, $\Gamma_r^k(F \times F, t) = (F^{2^r}; T_1, T_2, ..., T_k)$, where $(T_1, T_2, ..., T_r)$ is the usual twist $Z_2^r$-action on $F^{2^r}$ which interchanges factors and $T_{r+1}, ..., T_k$ act trivially, with the fixed data having in this case $2^r - 1$ copies of the tangent bundle of $F$ and $2^k - 2^r$ zero bundles. In this special case, we allow $r$ to be zero, setting $\Gamma_0^k(F \times F, t) = (F; T_1, T_2, ..., T_k)$, where each $T_i$ is the identity involution.

Now, from a given $Z_2^k$-action $(M;\Phi)$, $\Phi = (T_1, ..., T_k)$, we can obtain a collection of new $Z_2^k$-actions, described as follows: first, each automorphism $\sigma : Z_2^k \to Z_2^k$ yields a new action given by $(M; \sigma(T_1), ..., \sigma(T_k))$; we denote this action by $\sigma(M;\Phi)$. The fixed data of $\sigma(M;\Phi)$ is obtained from the fixed data of $(M;\Phi)$ by a permutation of eigenbundles, obviously depending on $\sigma$. Next, it was shown in [11] that if $(M;\Phi)$ has fixed data $\bigoplus_\rho \varepsilon_\rho \to F$ and one of the eigenbundles $\varepsilon_\theta$ is isomorphic to $\varepsilon'_\theta \oplus R$, then there is an action $(N;\Psi)$ with fixed data $\bigoplus_\rho \mu_\rho \to F$, where $\mu_\rho = \varepsilon_\rho$ if $\rho \neq \theta$ and $\mu_\theta = \varepsilon'_\theta$. We say in this case that $(N;\Psi)$ is obtained from $(M;\Phi)$ *by removing one section* . Thus, the iterative process of removing sections may possibly enlarge the set $\{\sigma(M;\Phi),\, \sigma \in Aut(Z_2^k)\}$. Summarizing, from a given involution $(W,T)$ that fixes $F$, we obtain a collection of $Z_2^k$-actions fixing $F$ by applying the operations $\sigma\Gamma_r^k$ on $(W,T)$ and next by removing the (possible) sections from the resultant eigenbundles. The results of [12] say that when $F = K \cup L$, where $K$ and $L$ have property $\mathcal{H}$ and $dim(K) < dim(L)$, then, up to equivariant cobordism, all $Z_2^k$-actions fixing $F$ are obtained, with the above procedure, from involutions fixing $K \cup L$, $K$ and $L$. Together with the $Z_2$-classification obtained in Sections 2 and 3 and the case $F = RP^{even}$, this gives the following $Z_2^k$-classification for $F = RP^2 \cup RP^n$, where $n > 2$ is even (in our terminology, we agree that *the set obtained from* $(M;\Phi)$ *by removing sections* includes $(M;\Phi)$):

**Theorem.** *Let $(M;\Phi)$ be a $Z_2^k$-action fixing $RP^2 \cup RP^n$, where $n > 2$ is even. Then $(M;\Phi)$ is equivariantly cobordant to an action belonging to the set $A \cup B$, where the sets $A$ and $B$ are described below in terms of $n$:*

*i) $n-2 = 2^p q$, with $q$ odd and $p > 1$: $A = \emptyset$= the empty set; $B$= the set obtained from $\{\sigma\Gamma_r^k\Gamma^{2^p-1}(RP^{n+3}, T_{2,n}),\ \sigma \in Aut(Z_2^k),\ 1 \leq r \leq k\}$ by removing sections.*

*ii) $n-2 = 2q$, with $q$ odd, and $n$ is not a power of $2$: $A = \emptyset$; $B$= the set obtained from $\{\sigma\Gamma_r^k\Gamma^2(RP^{n+3}, T_{2,n}),\ \sigma \in Aut(Z_2^k),\ 1 \leq r \leq k\}$ by removing sections;*

*iii) $n = 2^t$ is a power of $2$ with $t \geq 3$: $A = \{\sigma\Gamma_r^k(RP^2 \times RP^2, twist) \cup \sigma'\Gamma_{r-t+1}^k(RP^{2^t} \times RP^{2^t}, twist),\ \sigma,\sigma' \in Aut(Z_2^k),\ t-1 \leq r \leq k\}$; $B$= the set obtained from $\{\sigma\Gamma_r^k\Gamma^2(RP^{2^t+3}, T_{2,2^t}),\ \sigma \in Aut(Z_2^k),\ 1 \leq r \leq k\}$ by removing sections (by dimensional reasons, in this case $A = \emptyset$ if $t-1 > k$);*

*iv) $n = 4$: $A = \{\sigma\Gamma_{r+1}^k(RP^2 \times RP^2, twist) \cup \sigma'\Gamma_r^k(RP^4 \times RP^4, twist),\ \sigma,\sigma' \in Aut(Z_2^k),\ 0 \leq r \leq k-1\} \cup \{\sigma\Gamma_r^k(W^5,T),\ \sigma \in Aut(Z_2^k),\ 1 \leq r \leq k\}$, where $(W^5,T) = \Gamma^2(RP^3, T_{0,2}) \cup (RP^5, T_{0,4})$; $B$= the set obtained from $\{\sigma\Gamma_r^k\Gamma^2(RP^7, T_{2,4}),\ \sigma \in Aut(Z_2^k),\ 1 \leq r \leq k\}$ by removing sections.*

## References


[1] C. Kosniowski and R. E. Stong, *Involutions and characteristic numbers*, Topology 17, (1978), 309-330.

[2] D. C. Royster, *Involutions fixing the disjoint union of two projective spaces*, Indiana Univ. Math. J. 29, (1980), 267-276.

[3] D. Hou and B. Torrence, *Involutions fixing the disjoint union of copies of even projective spaces*, Acta Math. Sinica (2) 12, (1996), 162-166.

[4] D. Hou and B. Torrence, *Involutions fixing the union of odd-dimensional projective spaces*, Canadian Math. Bull. 37, (1994), 66-74.

[5] J. M. Boardman, *On manifolds with involution*, Bull. Amer. Math. Soc. 73, (1968), 136-138.

[6] P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps* , Springer-Verlag, Berlin, (1964).

[7] P. E. Conner and E. E. Floyd, *Fibring within a cobordism class* , Michigan Math. J. 12, (1965), 33-47.

[8] P. E. Conner, *Differentiable Periodic Maps* , Lecture Notes in Mathematics 738, Springer-Verlag, Berlin, (1979).

[9] P. E. Conner, *The bordism class of a bundle space* , Michigan Math. J. 14, (1967), 289-303.

[10] P. L. Q. Pergher, *Involutions fixing an arbitrary product of spheres and a point*, Manuscripta Math. 89, (1996), 471-474.

[11] P. L. Q. Pergher, $(Z_2)^k$*-actions whose fixed data has a section* , Trans. Amer. Math. Soc. 353, (2001), 175-189.

[12] P. L. Q. Pergher and R. de Oliveira, *Commuting involutions whose fixed point set consists of two special components* , pre-print.

[13] P. L. Q. Pergher and R. de Oliveira, $(Z_2)^k$*-actions with a special fixed point set* , Fund. Mathem. 186, n. 2, (2005), 97-109.

[14] P. L. Q. Pergher and R. E. Stong, *Involutions fixing* $\{point\} \cup F^n$ , Transformation Groups 6, (2001), 78-85.

[15] R. E. Stong, *Involutions fixing products of circles*, Proc. Amer. Math Soc. (3) 119, (1993), 1005-1008.

[16] R. E. Stong, *Involutions fixing projective spaces*, Michigan Math. J. 13, (1966), 445-447.

[17] Zhi Lu, *Involutions fixing* $RP^{odd} \cup P(h,i)$*, I*, Transactions Amer. Math. Soc. 354, (2002), 4539-4570.

[18] Zhi Lu, *Involutions fixing* $RP^{odd} \cup P(h,i)$*, II*, Transactions Amer. Math. Soc. 356, (2004), 1291-1314.

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