A NOTE ON THE THEOREMS OF LUSTERNIK-SCHNIRELMANN AND BORSUK-ULAM

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Abstract

Let p be a prime number and X a simply-connected CW-complex equipped with a free $\mathbb{Z}_p$-action generated by $f_p : X \to X$ and let $\alpha : S^{2n-1} \to S^{2n-1}$ be a homeomorphism generating a free $\mathbb{Z}_p$-action on the $(2n-1)$-sphere, whose orbit space is some Lens space. Then we prove that, under some homotopic conditions on X, there exists an equivariant map $F : (S^{2n-1}, \alpha) \to (X, f_p)$. As applications, we derive new versions of generalized Lusternik-Schnirelmann and Borsuk-Ulam theorems.

Keywords: Borsuk-Ulam Theorem, Lusternik-Schnirelmann Theorem, free actions, equivariant maps.

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1 Introduction

Let X be a simply connected CW-complex equipped with a free $\mathbb{Z}_p$ action (p a prime number) generated by $f_p : X \to X$. Given $l = (l_1, l_2, \ldots, l_n) \in \mathbb{Z}^n$ such that for each $j = 1, 2, \ldots n$, p does not divide l_j, consider the free $\mathbb{Z}_p$ action on S^{2n-1} generated by $\alpha_{p,l} : S^{2n-1} \to S^{2n-1}$, $\alpha_p(z_1, z_2, \ldots, z_n) = (\mathrm{e}^{\frac{2\pi i l_1}{p}} \cdot z_1, \mathrm{e}^{\frac{2\pi i l_2}{p}} \cdot z_2, \ldots, \mathrm{e}^{\frac{2\pi i l_n}{p}} \cdot z_n)$. Our main result is the following

Theorem 1 *Suppose that for each* $2 \leq q < m = 2n-1$

i) $\pi_q(X) = p \cdot \pi_q(X)$, *if q is odd, and*

ii) $\pi_q(X)$ *does not have elements of order p, if q is even.*

Then there exists an equivariant map $F : (S^m, \alpha_{p,l}) \to (X, f_p)$.

If $p = 2$, theorem 1 remains valid for any simply connected, paracompact, Hausdorff, locally path connected space X satisfying i) and ii), and for any m odd or even (α_2 is the antipodal map in any case).

This theorem provides the following versions of the Lusternik-Schnirelmann and Borsuk-Ulam theorems

Theorem 2 *Let X, $f_p : X \to X$ and m satisfying the same hypotheses of theorem 1. Then for each family $\mathcal{F} = \{C_0, \ldots C_k\}$ of $k+1$ sets covering X, each of which is either open or closed, and such that*

(1) $p = 2$ *and* $k \leq m$ *or*

(2) $p = 3$, *m is odd and* $k \leq m + 1$ *or*

(3) $p > 3$, *m is odd and* $\left(\frac{p-1}{2}\right)(k-2) + 2 \leq m$

there exists $C_{j_0} \in \mathcal{F}$ *such that* $f_p(C_{j_0}) \cap C_{j_0} \neq \emptyset$.

The case (1) remains true if X is a simply connected, paracompact, Hausdorff and locally path connected space, instead of a simply connected CW-complex.

Theorem 3 *Let X, $f_p : X \to X$ and m satisfying the same hypotheses of theorem 1.*

i) If $m \geq k(p-1)$, *then for each continuous map* $f : X \to \mathbb{R}^k$ *there exists* $x \in X$ *such that* $f(x) = f \circ (f_p)^j(x)$, $\forall\ 1 \leq j \leq p-1$.

ii) If $m \geq (k-1)(p-1)+1$, *then for each continuous map* $f : X \to \mathbb{R}^k$ *there exists* $x \in X$ *such that* $f(x) = f \circ f_p(x)$.

Theorem 4 *Let X be a paracompact, Hausdorff, locally path connected and simply connected space and let $f_2 : X \to X$ be an involution without fixed points, both satisfying the hypotheses of theorem 1. If Y is a separable metric space with topological dimension* $\dim(Y) \leq \frac{m-1}{2}$, *then for any map* $f : X \to Y$, *there exists* $x \in X$ *such that* $f(x) = f \circ f_2(x)$.

In [7] M. Izydorek and J. Jaworowski constructed for each k and $n \leq 2k-1$ a map f from the n-sphere S^n into a specific contractible k-dimensional complex Y such that $f(x) \neq f(-x)$, for all $x \in S^n$. Thus the upper bound for dimension in theorem 4 is sharp for general cases.

Theorem 3 generalizes the result of Cohen and Connet [5].

Generalizations of the same nature of the Borsuk-Ulam Theorem with "nice" topological spaces X and Y satisfying some homological conditions can be found in [9] and [4].

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2 Proof of Theorem 1

Let p be a prime number and $l = (l_1, l_2, \ldots, l_n) \in \mathbb{Z}_n$ such that $p \nmid l_j$ for all $j = 1, 2, \ldots n$. The map $\alpha_{p,l} : S^{2n-1} \to S^{2n-1}$, given by $\alpha_{p,l}(z_1, z_2, \ldots, z_n) = (\mathrm{e}^{\frac{2\pi i l_1}{p}} \cdot z_1, \mathrm{e}^{\frac{2\pi i l_2}{p}} \cdot z_2, \ldots, \mathrm{e}^{\frac{2\pi i l_n}{p}} \cdot z_n)$ generates a free $\mathbb{Z}_p$-action on S^{2n-1}, whose orbit space is the generalized Lens space $L_p(l)$ of type $(p; l_1, l_2, \ldots l_n)$. Let us denote by $\xi_{p,l}$ the principal $\mathbb{Z}_p$-bundle $\mathbb{Z}_p \cdots S^{2n-1} \to L_p(l)$.

We have the following inclusions

$$\begin{array}{ccccccccc}
\mathbb{Z}_p & & \mathbb{Z}_p & & \mathbb{Z}_p & & \cdots & & \mathbb{Z}_p \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
S^1 & \to & S^3 & \to & S^5 & \to & \cdots & \to & S^\infty \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow{}^{g} \\
L_p(l_1) & \to & L_p(l_1, l_2) & \to & L_p(l_1, l_2, l_3) & \to & \cdots & \to & L_p(\infty)
\end{array}$$

diagram 1

where the bundle $\gamma_{\mathbb{Z}_p} : \mathbb{Z}_p \cdots S^\infty \xrightarrow{g} L_p(\infty)$ is the universal principal $\mathbb{Z}_p$-bundle and $L_p(\infty)$ is the infinity Lens space, which is an Eilenberg-Mac Lane space $K(\mathbb{Z}_p, 1)$. So there exists a characteristic class $c_1(\gamma_{\mathbb{Z}_p}) = c_1 \in H^1(L_p(\infty); \mathbb{Z}_p)$ corresponding to the inverse of the Hurewicz isomorphism $\mathcal{H}_p : \pi_1(L_p(\infty)) \to H_1(L_p(\infty))$. Thus for every principal $\mathbb{Z}_p$-bundle over a paracompact Hausdorff space $B(\xi)$, $\xi = f_\xi^!(\gamma_{\mathbb{Z}_p}) : \mathbb{Z}_p \cdots E(\xi) \to B(\xi)$, where $f_\xi^!(\gamma_{\mathbb{Z}_p})$ is the pull-back of the bundle $\gamma_{\mathbb{Z}_p}$ by the classifying map $f_\xi : B(\xi) \to B\mathbb{Z}_p$, we have the characteristic class $c_1(\xi) = f_\xi^*(c_1) \in H^1(B(\xi); \mathbb{Z}_p)$. It is well known that this characteristic class satisfies

Lemma 1 *If ξ and η are principal $\mathbb{Z}_p$-bundles over the same CW-complex base space B, then*

$$\xi \equiv \eta \iff c_1(\xi) = c_1(\eta)$$

In particular $c_1(\xi) = 0$ if and only if ξ is trivial.

If $p = 2$ the base space B in the above lemma can be a paracompact, Hausdorff, path connected and locally path connected, instead of a CW-complex.

The bundle $\xi_{p,l}$ is classified by the inclusion $J_{p,l} : L_p(l) \to L_p(\infty)$.

Let $\xi = \xi(X, f_p)$ be the principal $\mathbb{Z}_p$-bundle $\mathbb{Z}_p \cdots X \to X/f_p$ and let $f_\xi : X/f_p \to L_p(\infty)$ be its classifying map. To prove theorem 1 we construct a map $f_m : L_p(l) \to X/f_p$ which lifts to a bundle map from $\xi_{p,l}$ to $\xi(X, f_p)$.

$$\begin{array}{ccc} \mathbb{Z}_p & & \mathbb{Z}_p \\ \vdots & & \vdots \\ S^m & \xrightarrow{F_m} & X \\ \downarrow & & \downarrow \\ L_p(l) & \xrightarrow{f_m} & X/f_p \end{array}$$

diagram 2

To do this, we need the following lemma due to P. Olum [11].

Lemma 2 *: Let Y be a CW-complex, $A \subseteq Y$ be a path connected subcomplex, $Y_2 = A \cup Y^{(2)}$ (where $Y^{(2)}$ is the 2-skeleton of Y) and Z any topological space. If $f_0 : A \to Z$ is a continuous map and $\psi : \pi_1(Y, y_0) \to \pi_1(Z, z_0)$ is a homomorphism then, there exists a continuous map $f_2 : Y_2 \to Z$ such that $f_2|_A = f_0$ and $(f_2)_\# = \psi \iff (f_0)_\# = \psi \circ (J_A)_\#$, where $J_A : A \to Y$ is the inclusion.*

If the isomorphisms $h_W : H^1(W; \mathbb{Z}_p) \to \mathrm{Hom}_{\mathbb{Z}}(H_1(W; \mathbb{Z}); \mathbb{Z}_p)$, given by the universal-coefficient theorem are denoted by $h_W(z)(\omega) = \langle z, \omega \rangle$, then for $c_1 \in H^1(L_p(\infty); \mathbb{Z}_p)$ and for each $\omega \in H_1(X/f_p; \mathbb{Z})$ we have

$$\langle c_1, (f_\xi)_*(\omega) \rangle = \langle (f_\xi)^*(c_1), \omega \rangle = \langle c_1(\xi(X, f_p)), \omega \rangle$$

The total space X of the $\mathbb{Z}_p$-bundle $\xi(X, f_p)$ is path connected, hence $\xi(X, f_p)$ is not trivial, so by lemma 1 $c_1(\xi(X, f_p)) \neq 0$, which implies that $(f_\xi)_* \neq 0$.

Since X is simply-connected we conclude that $\pi_1(X/f_p) \cong \mathbb{Z}_p$, so by the Hurewicz isomorphism $\mathcal{H}_{f_p} : \pi_1(X/f_p) \to H_1(X/f_p;\mathbb{Z})$ we can find an element $e \in \pi_1(X/f_p)$ such that $(f_\xi)_* \circ \mathcal{H}_{f_p}(e) \neq 0$.

The inclusion $J_2 : L_p(\infty)^{(2)} \to L_p(\infty)$ of the 2-skeleton of $L_p(\infty)$ induces an isomorphism $(J_2)_* : H_1(L_p(\infty)^{(2)};\mathbb{Z}) \to H_1(L_p(\infty);\mathbb{Z}) \cong \mathbb{Z}_p$. So there is some $u \in H_1(L_p(\infty)^{(2)};\mathbb{Z})$, $u \neq 0$ such that $(J_2)_*(u) = (f_\xi)_* \circ \mathcal{H}_{f_p}(e)$. Let $\mathcal{H}_{(2)} : \pi_1(L_p(\infty)^{(2)}) \to H_1(L_p(\infty)^{(2)};\mathbb{Z})$ be the Hurewicz homomorphism. There is $w \in \pi_1(L_p(\infty)^{(2)})$ such that $\mathcal{H}_{(2)}(w) = u$. Let us denote

$$\psi : \pi_1(L_p(\infty)) \to \pi_1(X/f_p)$$

the isomorphism such that $\psi(w) = \psi((J_2)_\#(w)) = e$, where $(J_2)_\#$ is the induced homomorphism on π_1.

Applying lemma 2 for $Z = X/f_p$, $Y = L_p(\infty)$ and $A = \{\text{point}\}$ the base point of $L_p(\infty)$, we find a continuous map $f_2 : L_p(\infty)^{(2)} \to X/f_p$ such that $\psi = (f_2)_\#$, the induced homomorphism on π_1 of f_2. Therefore

$$\begin{aligned}
\langle (f_2)^*(c_1(\xi(X,f_p))), u\rangle &= \langle c_1(\xi), (f_2)_*(u)\rangle = \left\langle c_1(\xi), (f_2)_*(\mathcal{H}_{(2)}(w))\right\rangle = \\
&= \left\langle c_1, \mathcal{H}_{f_p}((f_2)_\#(w))\right\rangle = \left\langle c_1(\xi), \mathcal{H}_{f_p}(\psi(w)\right\rangle = \\
&= \left\langle c_1(\xi), \mathcal{H}_{f_p}(e)\right\rangle = \left\langle (f_\xi)^*(c_1), \mathcal{H}_{f_p}(e)\right\rangle = \\
&= \left\langle c_1, (f_\xi)_* \circ \mathcal{H}_{f_p}(e)\right\rangle = \langle c_1, (J_2)_*(u)\rangle = \\
&= \langle (J_2)^*(c_1), u\rangle
\end{aligned}$$

and as $0 \neq u \in H_1(L_p(\infty);\mathbb{Z}) \cong \mathbb{Z}_p$ is a generator, it follows that

$$(f_\xi \circ f_2)^*(c_1) = (f_2)^* \circ (f_\xi)^*(c_1) = (f_2)^*(c_1(\xi(X,f_p))) = (J_2)^*(c_1)$$

Thus, by lemma 1, $(J_2)^!(\gamma_{\mathbb{Z}_p}) : \mathbb{Z}_p \cdots g^{-1}(L_p(\infty)^{(2)}) \xrightarrow{g} L_p(\infty)^{(2)}$ and $(f_\xi \circ f_2)^!(\gamma_{\mathbb{Z}_p})$ are isomorphic bundles, and we have

$$\begin{array}{ccccc}
\mathbb{Z}_p & & \mathbb{Z}_p & & \mathbb{Z}_p \\
\vdots & & \vdots & & \vdots \\
g^{-1}(L_p(\infty)^{(2)}) & \xrightarrow{F_2} & X & \xrightarrow{F_\xi} & S^\infty \\
\downarrow & & \downarrow & & \downarrow \\
L_p(\infty)^{(2)} & \xrightarrow{f_2} & X/f_p & \xrightarrow{f_\xi} & L_p(\infty)
\end{array}$$

diagram 3

To construct the maps F_m and f_m of diagram 2, we observe that if

$$(f_m)^*(c_1(\xi(X,\iota))) = J_{p,l}^*(c_1) = c_1(\xi_m) \quad (1)$$

where $J_{p,l} : L_p(l) \to L_p(\infty)$ is the inclusion, then by lemma 1 f_m is covered by a bundle map F_m. Since the characteristic classes $c_1(\xi)$ live in the first cohomology groups and the formula (1) is valid for f_2, we need only to extend the map f_2 to the m-skeleton ($m = 2n - 1$ if $p > 2$), $L_p(\infty)^{(m)} = L_p(l)$, of $L_p(\infty)$ to obtain such an f_m with the required property (1).

To do this, let (Y, A) be a CW pair and Z an n-simple space, $n \geq 1$. For an abelian group G, let $K = K(G, n)$ be an Eilenberg-MacLane space and $i_n \in H^n(K; G)$ an n-characteristic element.

Given $u_n \in H^n(Z; G)$ there exists a map $\varphi : Z \to K$ such that $\varphi^*(i_n) = u_n$. The induced homomorphism $\varphi_* : \pi_n(Z) \to \pi_n(K)$ provides the homomorphisms

$$\varphi^q_{u_n} : H^q(Y, A; \pi_n(Z)) \to H^q(Y, A; \pi_n(K)), \quad \forall\ q \in \mathbb{N}$$

Let $f_A : A \to Z$ be a map, $Y^{(q)}$ the q-skeleton of Y and $Y_q = Y^{(q)} \cup A$.

With these notations we have the following (see [3])

Proposition 1 *If $\varphi^{n+1}_{u_n}$ is a monomorphism and $g_n : Y_n \to Z$ is an extension of f_A, then there exists $g_{n+1} : Y_{n+1} \to Z$ such that $g_{n+1}|_{Y_{n-1}} = g_n|_{Y_{n-1}}$ if and only if $\delta f_A^*(u) = 0$, where δ is the coboundary operator.*

Using proposition 1 with $Z = X/f_p$, $(Y, A) = (L_p(\infty), \emptyset)$ and starting with $g_2 = f_2 : L_p(\infty)^{(2)} \to X/f_p$, since $\pi_q(X/f_p) \cong \pi_q(X)$ for all $q \geq 2$ we have that if

$$\varphi^{q+1}_{u_q} : H^{q+1}(L_p(\infty); \pi_q(X)) \to H^{q+1}(L_p(\infty); \pi_q(K(\mathbb{Z}_p, q))$$

is injective for all $2 \leq q < m$ then we can conclude that there exists a map $f_m : L_p(l) \to X/f_p$ such that $f_m|_{L_p(\infty)^{(2)}} = f_2$ and therefore satisfying the formula (1). By the universal coefficient theorem for cohomology we have

$$H^{q+1}(L_p(\infty); \pi_q(X)) \cong \begin{cases} \mathrm{H}om_{\mathbb{Z}}(\mathbb{Z}_p, \pi_q(X)) & \text{if } q \text{ is even} \\ \mathrm{E}xt(\mathbb{Z}_p, \pi_q(X)) & \text{if } q \text{ is odd} \end{cases}$$

but from the definitions of Hom and Ext we have

$$\mathrm{H}om_{\mathbb{Z}}(\mathbb{Z}_p, \pi_q(X)) \cong \{\alpha \in \pi_q(X) : p\alpha = 0\}$$

and

$$\mathrm{E}xt(\mathbb{Z}_p, \pi_q(X)) \cong \pi_q(X)/p\pi_q(X)$$

Therefore if X satisfies the hypotheses i) and ii) then $H^{q+1}(L_p(\infty); \pi_q(X)) = 0$ thus $\varphi_{u_q}^{q+1}$ is injective for all $2 \leq q < m$ and the theorem 1 follows.

3 Proof of Theorem 2

The classical Lusternik-Schnirelmann Theorem says the following

Theorem LS: *Let $m \geq k$ and let $H_0, H_1, \ldots, H_k$ be closed sets such that $S^m = \cup_{j=0}^k H_j$, then there exists $j_0 \in \{0, 1, \ldots, k\}$ with $H_{j_0} \cap -H_{j_0} \neq \emptyset$*

In 1979, Steinlein in [12] proved the following

Theorem S: *Let p be a prime number and $\alpha_p : S^m \to S^m$ a continuous map generating a free $\mathbb{Z}_p$-action on S^m. Let $m, k \in \mathbb{N}$ be such that m is odd and*

$$m \geq \begin{cases} k-1 & , \text{ if } p = 3 \\ \left(\frac{p-1}{2}\right)(k-2) + 2 & , \text{ if } p > 3 \end{cases}$$

Then for each covering $S^m = \cup_{j=0}^k H_j$ of the m-sphere S^m by $k+1$ closed sets, there exists some H_{j_0} such that $H_{j_0} \cap \alpha_p(H_{j_0}) \neq \emptyset$

In [6], J. E. Greene proved that in theorem LS each set H_j can be either open or closed. With a similar reasoning theorem S can be improved to

Theorem SG *Let p be a prime number and $\alpha_p : S^m \to S^m$ a continuous map generating a free $\mathbb{Z}_p$-action on S^m. Let $m, k \in \mathbb{N}$ be such that m is odd and*

$$m \geq \begin{cases} k-1 & , \text{ if } p = 3 \\ \left(\frac{p-1}{2}\right)(k-2) + 2 & , \text{ if } p > 3 \end{cases}$$

Then for each covering $S^m = \cup_{j=0}^k H_j$ of the m-sphere S^m by $k+1$ sets, each of which is either open or closed, there exists some H_{j_0} such that $H_{j_0} \cap \alpha_p(H_{j_0}) \neq \emptyset$

Proof: Following the reasoning of Greene we prove theorem SG by induction on the number t of closed sets in the cover of S^m. The case $t = 0$ correspond to a cover of S^m by open sets $H_0, H_1, \ldots, H_k$. Select a Lebesgue number for this cover, that is, a positive number λ such that for all $x \in S^m$, the closed ball $\bar{B}(x, \lambda)$ is contained in some H_j. By compacteness, there exists a finite collection of points $\{x_i\}$ such that the open balls $B(x_i, \lambda)$ cover S^m. For each j, let F_j denote the union of those $\bar{B}(x_i, \lambda)$ contained in H_j. Then F_j is closed, F_j is a subset of H_j for each j, and together the F_j cover S^m. Therefore, the theorem S implies that there exists some F_{j_0} such that $F_{j_0} \cap \alpha_p(F_{j_0}) \neq \emptyset$, and hence there exists some H_{j_0} such that $H_{j_0} \cap \alpha_p(H_{j_0}) \neq \emptyset$.

Thus we may assume that $0 < t < k+1$ and the theorem holds for fewer than t closed sets. We now show that it holds for t closed sets. Let $\mathcal{C}$ be a cover of S^m with $k+1$ sets, of which exactly t are closed and the remaining sets are open. Fix a closed set F in $\mathcal{C}$, and suppose that $F \cap \alpha_p(F) = \emptyset$. By normality, there exist open sets A and B such that $F \subset A$, $\alpha_p(F) \subset B$ and $A \cap B = \emptyset$. Let U denote the set $A \cap \alpha_p^{-1}(B)$. Then, U is open, $F \subset U$ and $U \cap \alpha_p(U) = \emptyset$. Therefore $\mathcal{C}' = (\mathcal{C} - \{F\}) \cup \{U\}$ is a cover of S^m with $k+1$ sets, of which exactly $t-1$ are closed and the remaining sets are open, so by the induction hypothesis some set H in the cover satisfies $H \cap \alpha_p(H) \neq \emptyset$ and by construction this H must be different from U, and hence some set H in the original cover must satisfy $H \cap \alpha_p(H) \neq \emptyset$. This complete the inductive step.

Now we are ready to prove theorem 2. Suppose that $X = \bigcup_{j=0}^{k} C_j$ is a covering of X by $k+1$ sets, each of which is either open or closed, and k satisfies the condition (1), (2) or (3) of theorem 2.

By theorem 1 for each $\alpha_{p,l} : S^m \to S^m$, generating a free $\mathbb{Z}_p$-action on S^m, and for each $f_p : X \to X$, generating a free $\mathbb{Z}_p$-action on X, there exists an equivariant continuous map from $(S^m, \alpha_{p,l})$ to (X, f_p). Note that, $(f_p)^{-1} = (f_p)^{p-1}$ also generates a free $\mathbb{Z}_p$-action on X, and analogously if $l = (l_1, \ldots, l_n)$ then $\alpha_{p,l}^{-1} = \alpha_{p,l'}$ $(l' = (p - l_1, p - l_2, \ldots, p - l_n))$ generates a free $\mathbb{Z}_p$-action on S^m. Then there exists an equivariant continuous map $F : (S^m, \alpha_{p,l}) \to (X, (f_p)^{-1})$. Thus we have that $S^m = \bigcup_{j=0}^{k} F^{-1}(C_j)$ is a covering of S^m by $k+1$ sets, each of which is either open or closed.

If $p \geq 3$ it follows from theorem SG that there exists some C_{j_0} such that $F^{-1}(C_{j_0}) \bigcap \alpha_{p,l'}(F^{-1}(C_{j_0})) \neq \emptyset$. This together the facts that $\alpha_{p,l'} = \alpha_{p,l}^{-1}$ and $F : (S^m, \alpha_{p,l}) \to (X, f_p)$ is equivariant implies that $C_{j_0} \bigcap f_p(C_{j_0}) \neq \emptyset$.

If $p = 2$, m can be even or odd, and in any case α_2 is the antipodal map. The same reasoning applies to the Greene's version of theorem LS.

4 Proof of Theorem 3

Here we need the following theorem, which follows from the works of H. J. Munkholm [10] and E. L. Lusk [8].

Theorem ML: *Let p be a prime number, $k, m \in \mathbb{N}$ and $\alpha : S^m \to S^m$ a continuous map generating a free $\mathbb{Z}_p$-action on S^m.*

a) If $m \geq k(p-1)$, then for each continuous map $h : S^m \to \mathbb{R}^k$, there exists an $x \in S^m$ with $h(x) = h(\alpha^j(x))$ for all j, $1 \leq j \leq p-1$.

b) If $m \geq (k-1)(p-1)+1$, then for each continuous map $h : S^m \to \mathbb{R}^k$, there exists an $x \in S^m$ with $h(x) = h(\alpha(x))$.

Now, to prove theorem 3 let (X, f_p) be a pair satisfying the hypotheses of theorem 3 and let $f : X \to \mathbb{R}^k$ be a continuous map, then by theorem 1 there exists a continuous equivariant map $F : (S^m, \alpha_{p,l}) \to (X, f_p)$. Thus $h = f \circ F : S^m \to \mathbb{R}^k$ is a continuous map.

If $m \geq (k-1)(p-1)+1$, it follows from item b) of theorem ML that there exists $y \in S^m$ such that $h(y) = h(\alpha_{p,l}(y))$. Then if $x = F(y) \in X$ we have

$$f(x) = h(y) = h(\alpha_{p,l}(y)) = f(F(\alpha_{p,l}(y))) = f(f_p(F(y))) = f(f_p(x))$$

If $m \geq k(p-1)$, it follows from item a) of theorem ML that there exists $y \in S^m$ such that $h(y) = h((\alpha_{p,l})^j(y))$, for all $j = 0, 1, \ldots p-1$. Then if $x = F(y) \in X$ we have in a simillar way as above that $f(x) = f((f_p)^j(x))$ for all $j = 1, \ldots p-1$.

5 Proof of Theorem 4

Here we use the following theorem due to Aarts, Fokkink and Vermeer [1]

Theorem AFV: *Let W be a paracompact, Hausdorff space such that* $\dim(W) \leq m$. *Suppose that α is a fixed point free involution of W. Then there exists a closed cover $\mathcal{C} = \{C_0, C_1, \ldots, C_k\}$ of W with $k \leq m+1$ sets such that $C_j \cap \alpha(C_j) = \emptyset$ for each $j = 0, 1, \ldots m+1$.*

To prove theorem 3, let us suppose by contradiction that $f(x) \neq f(f_2(x))$ for all $x \in X$. Let (W, τ) be a pair such that $W = Y \times Y - \triangle$ where $\triangle$ is the diagonal, and τ is the involution $\tau(x, y) = (y, x)$, then W is paracompact, Hausdorff, $\dim(W) \leq m-1$ and τ is a free continuous involution of W. By theorem AFV it follows that there exists a covering $W = \bigcup_{j=0}^{k} H_j$ of W by $k+1$ closed sets such that $H_j \cap \tau(H_j) = \emptyset$ for all $j = 0, 1, \dots k$ and $k \leq m$. By theorem 1 there exists an equivariant map $F : (S^m, \alpha_2) \to (X, f_2)$, and the map $g : (X, f_2) \to (W, \tau)$ given by $g(x) = (f(x), f(f_2(x)))$ is also an equivariant map, so $h = g \circ F : (S^m, \alpha_2) \to (W, \tau)$ is equivariant, therefore $S^m = \bigcup_{j=0}^{k} h^{-1}(H_j)$ is a covering of S^m by $k+1$ closed sets and since $H_j \cap \tau(H_j) = \emptyset$ for all $j = 0, 1, \dots k$, it follows that $h^{-1}(H_j) \cap \alpha_2(h^{-1}(H_j)) = \emptyset$ for all $j = 0, 1, \dots k$ and this contradicts the classical Lusternik-Schnirelmann theorem (theorem LS). Thus we conclude that there exists $x \in X$ such that $f(x) \neq f(f_2(x))$.

References

[1] Aarts, J. M., Fokkink, R. J., Vermeer, H. *Variations on a Theorem of Lusternik and Schnirelmann*, Topology, 35 (4) (1996), 1051-1056.

[2] Aarts, J. M. and Nishiura, T. *Dimension and Extension*, North Holland, Amsterdam, (1992).

[3] Biasi, C. and Libardi, A. K. M., *Some applications of obstruction theory*, Notas do Instituto de Ciências Exatas de São Carlos - USP (1988).

[4] Biasi, C. and Mattos, D., *A Borsuk-Ulam Theorem for compact Lie group actions*, Bull. Braz. Math. Soc., new series 37(1) (2006), 127-137.

[5] Cohen, F. and Connet, J. E. *A coincidence theorem related to the Borsuk-Ulam theorem*, Proc. Amer. math. Soc., vol. 44 (1), (1974), 218-220.

[6] Greene, J. E., *A new short proof of Kneser's conjecture*, Amer. Math. Monthly 109 (2002), no. 10, 918-920.

[7] Izydorek, M. and Jaworowski, J., *Antipodal coincidence for maps of spheres into complexes*, Proc. Amer. Math. Soc. 123 (1995), 1947-1950.

[8] Lusk, E. L. *The mod p Smith index and a generalized Borsuk-Ulam theorem*, Michigan Math. J. 22 (1975), 151-160.

[9] Mattos, D., Santos, E. L. and Pergher, P. L. Q., *A Borsuk-Ulam theorem for general spaces*, Arch. Math. (Basel) 81 (2003), 96-102.

[10] Munkholm, Hans J. *Borsuk-Ulam type theorems for proper $\mathbb{Z}_p$-actions on (mod p homology) n-spheres*, Math. Scand. 24, (1969), 167-185.

[11] Olum, P. *Obstructions to extensions and homotopies*, Ann. of Math. 52 (2), (1950), 1-50.

[12] Steinlein, H. *Some abstract generalizations of the Ljusternik-Schnirelmann-Borsuk covering theorem*, Pacific J. Math. 83 (1979), 285-296.

[13] Steinlein, H. *On the theorems of Borsuk-Ulam and Ljusternik-Schnirelmann-Borsuk*, Canad. Math. Bull. vol. 27 (2), (1984), 192-204.

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