# DIMENSIONS OF FIXED POINT SETS OF INVOLUTIONS

**PERGHER, PEDRO L. Q. ; FIGUEIRA, FÁBIO G.**

ABSTRACT. Suppose the fixed point set $F$ of a smooth involution $T : M \to M$ on a smooth, closed and connected manifold $M$ decomposes into two components $F^n$ and $F^2$ of dimensions $n$ and 2, respectively, with $n > 2$ odd. We show that the codimension $k$ of $F^n$ is small if the normal bundle of $F^2$ does not bound; specifically, we show that $k \leq 3$ in this case. In the more general situation where $F$ is not a boundary, $n$ (not necessarily odd) is the dimension of a component of $F$ of maximal dimension and $k$ is the codimension of this component, and fixed components of all dimensions $j$, $0 \leq j \leq n$, may occur, a theorem of Boardman gives that $k \leq \frac{3}{2}n$.

In addition, we show that this bound can be improved to $k \leq 1$ (hence $k = 1$) for some specific values of $n$ and some fixed stable cobordism classes of the normal bundle of $F^2$ in $M$; further, we determine in these cases the equivariant cobordism class of $(M, T)$.

## 1. **Introduction**

Let $F$ be a disjoint (finite) union of smooth and closed manifolds and $M$ be a smooth and closed manifold equipped with a smooth involution $T : M \to M$ whose fixed point set is $F$. Suppose that $F$ is not a boundary. If $n$ is the dimension of a component of $F$ of maximal dimension and $k$ is the codimension of this component, then $k \leq \frac{3}{2}n$; this follows from the famous $\frac{5}{2}$-theorem of J. Boardman of [4] and its strengthened version of [2] (in fact, more generally [2] says that this is valid when the normal bundle of $F$ in $M$ is not a boundary).

The general bound $k \leq \frac{3}{2}n$ can be improved in certain particular situations. For example, if $F$ has constant dimension $n$, then $k \leq n$ (see [2]). Recently, R. E. Stong and P. Pergher considered the situation in which $F$ has the form $F = F^n \cup \{point\}$, where $F^n$ is a manifold of dimension $n > 0$ [7]. Writing

1991 *Mathematics Subject Classification.* (2.000 Revision) Primary 57R85; Secondary 57R75.

*Key words and phrases.* involution, fixed data, Stiefel-Whitney class, characteristic number, Wu class, stable cobordism class, low codimension, projective space bundle, equivariant cobordism class.

The first author was partially supported by CNPq and FAPESP..

$n = 2^p q$, where $p \geq 0$ and $q$ is odd, they showed that in this case the bound in question can be improved to $k \leq n + p - q + 1$ if $p \leq q$, and $k \leq n + 2^{p-q}$ if $p > q$. Further, they showed that these bounds cannot be improved, in the sense that for each $n \geq 1$ there exists a special $F^n$ and an involution $(M, T)$ fixing $F^n \cup \{point\}$ so that the codimension of $F^n$ is maximal. The bounds above have some special features: for $n$ even, the corresponding bounds increase with $n$, in the sense that, for each fixed $p \geq 1$, the bound corresponding to $n = 2^p q$ is strictly increasing as a function of $q$ (for $n = 2^p$, it is precisely the Boardman's bound, but in all other cases, it is a smaller bound). However, an intriguing fact occurs when $n$ is odd: in this case, the bound is constant and quite low: $n + p - q + 1 = q + 0 - q + 1 = 1$. A consequence of this fact is that, if $(M, T)$ is an involution having fixed point set of the form $F = F^n \cup \{point\}$ with $n$ odd, then $(M, T)$ is equivariantly cobordant to the involution $(\mathbb{R}\mathrm{P}^{n+1}, T)$, where $\mathbb{R}\mathrm{P}^{n+1}$ is the $(n+1)$-dimensional real projective space and $T[x_0, x_1, \ldots, x_{n+1}] = [-x_0, x_1, \ldots, x_{n+1}]$ (these consequences for $n$ odd were obtained by D. Royster in [3]).

The objective of this paper is to study this phenomenon when we replace $\{point\}$ by any 2-dimensional and closed manifold $F^2$. Suppose $(M, T)$ an involution pair with fixed point set of the form $F = F^n \cup F^2$, and denote by $\eta$ and $\mu$ the normal bundles of $F^n$ and $F^2$, respectively, in $M$. If $\mu$ over $F^2$ and $\mu'$ over $F'^2$ are cobordant as bundles over 2-dimensional and closed manifolds, then there exists an involution $(N, T')$ cobordant to $(M, T)$ and with fixed data $\eta \cup \mu'$ (see [6]). Thus $\mu$ must be considered up to cobordism. If $\mu$ is a boundary, then it can be equivariantly removed to give a new involution, equivariantly cobordant to $(M, T)$ and with fixed point set $F^n$ (see again [6]). Thus, it is reasonable to suppose that $\mu$ does not bound (this is the case when $F = F^n \cup \{point\}$). If $k$ is the codimension of $F^n$, $\mu$ then represents a nonzero element of the bordism group of $(n+k-2)$-dimensional real vector bundles over 2-dimensional and closed manifolds, $\mathscr{N}_2(BO(n+k-2))$. Considering $n \geq 3$ and $k \geq 1$, one has $n + k - 2 \geq 2$, and since any bundle with dimension $\geq 2$ over a 2-dimensional manifold is stably cobordant (that is, cobordant modulo Whitney sums of trivial vector bundles) to a bundle with dimension $\leq 2$, this bordism group can be considered up to stability. We will see in Section 2, Lemma 2.1, that $\mathcal{N}_2(BO(n+k-2))$ has dimension 3 as a $Z_2$-vector space and thus contains seven nonzero (stable) cobordism classes. Each one of these nonzero classes will be identified by a nonzero list of three mod 2 characteristic numbers, $(a_1, a_2, a_3)$, coming from the list of characteristic classes $(w_1^2, v_2, v_1^2)$. The stable cobordism classes corresponding to the lists $(1,0,0)$, $(1,0,1)$, $(1,1,0)$, $(1,1,1)$, $(0,0,1)$, $(0,1,0)$ and $(0,1,1)$ will be denoted by $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$, $\beta_5$, $\beta_6$ and $\beta_7$, respectively (see Lemma 2.1 for an explicit description of these classes).

Now, for a given vector bundle $\eta$ over a closed manifold $F$, consider $\mathbb{R}\mathrm{P}(\eta)$ the real projective space bundle associated to $\eta$, and denote by $\lambda$ the line bundle over the total space of $\mathbb{R}\mathrm{P}(\eta)$ associated to the double covering $S(\eta) \to \mathbb{R}\mathrm{P}(\eta)$, $S(\eta)$ the sphere bundle of $\eta$. If $\mathbb{R}$ is the one-dimensional trivial vector bundle over $F$, one has the special involution $T_\eta$ defined on the total space of $\mathbb{R}\mathrm{P}(\eta \oplus \mathbb{R})$, given by $t_\eta[v, r] = [v, -r]$. The fixed data of $T_\eta$ consists of $F$ with normal bundle $\eta$, and of the total space of $\mathbb{R}\mathrm{P}(\eta)$ with normal bundle $\lambda$. We denote this involution pair by $(\mathbb{R}\mathrm{P}(\eta \oplus \mathbb{R}), T_\eta)$. If $\eta$ over $F$ and $\eta'$ over $F'$ are cobordant, then $\lambda$ (over the total space of $\mathbb{R}\mathrm{P}(\eta)$) and $\lambda'$ (over the total space of $\mathbb{R}\mathrm{P}(\eta')$) are cobordant. From [6], it follows that $(\mathbb{R}\mathrm{P}(\eta \oplus \mathbb{R}), T_\eta)$ and $(\mathbb{R}\mathrm{P}(\eta' \oplus \mathbb{R}), T_{\eta'})$ are equivariantly cobordant.

Returning to our stable cobordism classes $\beta_i$, for each $n \geq 3$ and $1 \leq i \leq 7$ choose any $(n-1)$-dimensional vector bundle $\mu^{n-1}$ over $F^2$ which is a representative for $\beta_i$. Then $(\mathbb{R}\mathrm{P}(\mu^{n-1} \oplus \mathbb{R}), T_{\mu^{n-1}})$ is an involution defined on a $(n+1)$-dimensional manifold whose fixed point set has the form $F = F^n \cup F^2$, and with the normal bundle over $F^2$ being $\mu^{n-1}$; further, the equivariant cobordism class of $(\mathbb{R}\mathrm{P}(\mu^{n-1} \oplus \mathbb{R}), T_{\mu^{n-1}})$ is determined by $\beta_i$. We will prove the following

**Theorem.** *Let $(M, T)$ be an involution having fixed point set of the form $F = F^n \cup F^2$, where $n \geq 3$ is odd and the normal bundle $\mu$ over $F^2$ does not bound.*

*1) If $k$ is the codimension of $F^n$, then $k \leq 3$.*

*2) If either $n \equiv 3$ mod 4 and $\mu$ represents $\beta_2, \beta_3, \beta_5$ or $\beta_6$, or $n \equiv 1$ mod 4 and $\mu$ represents $\beta_1, \beta_2, \beta_6$ or $\beta_7$, then $k = 1$ and in each case $(M, T)$ is equivariantly cobordant to $(\mathbb{R}\mathrm{P}(\mu \oplus \mathbb{R}), T_\mu)$.*

*3) For the pairs $(n, \beta_i)$ not included in 2), the bound $k \leq 3$ cannot be improved.*

A remark in the end of the paper will clarify the cases $F = F^n \cup F^2$ with $n$ even and $F = F^n \cup F^1$. We will see that small codimensions occur when $F = F^n \cup F^1$ and $n$ is even, but do not occur when $F = F^n \cup F^2$ and $n$ is even, and when $F = F^n \cup F^1$ and $n$ is odd.

## 2. Proof of the main result.

This section will be devoted to the proof of the theorem stated in Section 1.

**Lemma 2.1.** *For $p \geq 2$ the dimension of $\mathcal{N}_2(BO(p))$ as a $Z_2$-vector space is* 3.

**Proof.** As before mentioned, besides the proof we present an explicit description of each nonzero element of $\mathcal{N}_2(BO(p))$. For any $p$-dimensional vector bundle $\mu$ over a closed 2-dimensional manifold $F^2$, one lets $\mathbb{W}(F^2) = 1 + w_1 + w_2$ and $\mathbb{W}(\mu) = 1 + v_1 + v_2$ be the Stiefel-Whitney classes of $F^2$ and $\mu$, respectively. The element $[\mu] \in \mathcal{N}_2(BO(p))$ is determined by its characteristic numbers, which are obtained by evaluating the cohomology classes $w_1^2$, $w_2$, $v_1^2$, $v_2$ and $w_1 v_1$ on the fundamental homology class $[F^2]$. Now $F^2$ is either a boundary or cobordant to $\mathbb{R}\mathrm{P}^2$. Since $\mathbb{R}\mathrm{P}^2$ and any manifold which bounds satisfy $w_1^2 = w_2$, this also is true for $F^2$. Let $U = 1 + u$ be the Wu class of $F^2$; one knows that $u = w_1$. Then $Sq^1(v_1) = uv_1 = w_1 v_1$, where $Sq$ is the Steenrod operation; but also $Sq^1(v_1) = v_1^2$, which gives $v_1^2 = w_1 v_1$. In this way, the above characteristic numbers are reduced to the ones obtained from $w_1^2$ $(= w_2)$, $v_2$ and $v_1^2$ $(= w_1 v_1)$. This gives a dimension of at most 3 for $\mathcal{N}_2(BO(p))$. To end the proof, it suffices to describe examples realizing each possible nonzero list of mod 2 numbers $(a_1, a_2, a_3)$, corresponding to $(w_2, v_1^2, v_2)$. Consider $\xi$ the canonical line bundle over $\mathbb{R}\mathrm{P}^2$ . Then one has the bundles:

**1)** the 0-dimensional bundle over $\mathbb{R}\mathrm{P}^2$, with $w_1^2 \neq 0,\ v_2 = 0$ and $v_1^2 = 0$, and thus representing $\beta_1$;

**2)** $\xi$ over $\mathbb{R}\mathrm{P}^2$, with $w_1^2 \neq 0,\ v_2 = 0$ and $v_1^2 \neq 0$, representing $\beta_2$;

**3)** $\xi \oplus \xi$ over $\mathbb{R}\mathrm{P}^2$, with $w_1^2 \neq 0,\ v_2 \neq 0$ and $v_1^2 = 0$, representing $\beta_3$;

**4)** $\xi \oplus \xi \oplus \xi$ over $\mathbb{R}\mathrm{P}^2$, with $w_1^2 \neq 0,\ v_2 \neq 0$ and $v_1^2 \neq 0$, representing $\beta_4$.

Now consider $\xi \oplus \mathbb{R}$ over $\mathbb{R}\mathrm{P}^1$, where again $\xi$ denotes the canonical line bundle and $\mathbb{R}$ the trivial one-dimensional bundle; let $\lambda$ be the usual line bundle over the total space $K^2$ of $\mathbb{R}\mathrm{P}(\xi \oplus \mathbb{R})$. Note that $K^2$ is a closed 2-dimensional manifold, and one has the following bundles with its respective characteristic numbers, obtained from standard computations in the cohomology of $K^2$:

**5)** $\lambda$ over $K^2$, with $w_1^2 = 0,\ v_2 = 0$ and $v_1^2 \neq 0$, representing $\beta_5$;

**6)** $\lambda \oplus \lambda$ over $K^2$, with $w_1^2 = 0,\ v_2 \neq 0$ and $v_1^2 = 0$, representing $\beta_6$;

**7)** $\lambda \oplus \lambda \oplus \lambda$ over $K^2$, with $w_1^2 = 0,\ v_2 \neq 0$ and $v_1^2 \neq 0$, representing $\beta_7$.

If $p \geq 2$, each one of the above examples is stably cobordant to a $p$-dimensional bundle over a 2-dimensional manifold, which gives the result. $\square$

**Lemma 2.2.** *Let $\eta^k$ be any $k$-dimensional vector bundle over a closed $n$-dimensional manifold $F^n$. Let $(M, T)$ be an involution where $M$ is $(n+k)$-dimensional, and suppose that the fixed data of $T$ consists of $F^n$ with normal bundle $\eta^k$ and some $(n+k-1)$-dimensional manifold $P$ with (one-dimensional) normal bundle $\theta$ over $P$. Then $(M, T)$ is equivariantly cobordant to $(\mathbb{R}\mathrm{P}(\eta^k \oplus \mathbb{R}), T_{\eta^k})$.*

**Proof.** The fixed data of the involution $(M, T) \cup (\mathbb{R}\mathrm{P}(\eta^k \oplus \mathbb{R}), T_{\eta^k})$ is $\theta \cup \eta^k \cup \lambda \cup \eta^k$, where $\lambda$ is the usual line bundle over the total space of $\mathbb{R}\mathrm{P}(\eta^k)$. Then this involution is equivariantly cobordant to an involution $(N^{n+k}, S)$ with fixed data $\theta \cup \lambda$. But from [6] one knows that an involution with codimension one fixed point set bounds, which means that $\theta$ is cobordant to $\lambda$. Then $(M, T)$ and $(\mathbb{R}\mathrm{P}(\eta^k \oplus \mathbb{R}), T_{\eta^k})$ have cobordant fixed data, which ends the proof. $\square$

**Lemma 2.3.** *Suppose $(M, T)$ an involution having fixed point set of the form $F = F^n \cup F^2$, with $n$ odd. Let $\mu$ be the normal bundle of $F^2$ in $M$, and set*

$\mathbb{W}(F^2) = 1 + w_1 + w_2,\ \mathbb{W}(\mu) = 1 + v_1 + v_2$ *for the Stiefel-Whitney classes. If* $k$ *is the codimension of* $F^n$ *and* $k \geq 2$*, then*

$$\binom{n+1}{2}(w_1^2 + v_1^2) + v_2 + v_1^2 = 0,$$

*where* $\binom{}{}$ *means binomial coefficient modulo 2.*

**Proof.** We need the following basic fact from [6]: if $\eta$ denotes the normal bundle over $F^n$ and $E_\eta$ and $E_\mu$ denote the total spaces of the projective space bundles $\mathbb{R}\mathrm{P}(\eta)$ and $\mathbb{R}\mathrm{P}(\mu)$, respectively, then the usual line bundles $\lambda$ over $E_\eta$ and $\nu$ over $E_\mu$ are cobordant as elements of $\mathscr{N}_{n+k-1}(BO(1))$. Then any class of dimension $n+k-1$, given by a product of the classes $w_i(E_\eta)$ and $w_1(\lambda)$, evaluated on the fundamental homology class $[E_\eta]$, gives the same characteristic number as the one obtained by the corresponding product of the classes $w_i(E_\mu)$ and $w_1(\nu)$, evaluated on $[E_\mu]$. Write $\mathbb{W}(F^n) = 1 + \theta_1 + \cdots + \theta_n$, $\mathbb{W}(\eta) = 1 + u_1 + \cdots + u_k$ and $\mathbb{W}(\lambda) = 1 + c$ for the Stiefel-Whitney classes of $F^n$, $\eta$ and $\lambda$, respectively. From [1] one knows that

$$\mathbb{W}(E_\eta) = (1 + \theta_1 + \cdots + \theta_n)\{(1+c)^k + (1+c)^{k-1}u_1 + \cdots + (1+c)u_{k-1} + u_k\},$$

where here we are suppressing bundle maps. One has

$$w_1(E_\eta) + \binom{k}{1}c = \theta_1 + u_1 + \binom{k}{1}c + \binom{k}{1}c = \theta_1 + u_1.$$

Then $(w_1(E_\eta) + \binom{k}{1}c)^{n+1} = 0$, since it has dimension $n+1$ and comes from the cohomology of $F^n$. Since $k \geq 2,\ n+k-1 \geq n+1$ and we can form the class

$$\left(w_1(E_\eta) + \binom{k}{1}c\right)^{n+1} \cdot c^{k-2},$$

which gives a zero characteristic number for $\lambda$. Therefore the corresponding characteristic number for $\nu$ is also zero. Setting $\mathbb{W}(\nu) = 1 + d$, this number is

$$\left(w_1(E_\mu) + \binom{k}{1}d\right)^{n+1} \cdot d^{k-2}\ [E_\mu].$$

One has

$$\mathbb{W}(E_\mu) = (1 + w_1 + w_2)\{(1+d)^{n+k-2} + (1+d)^{n+k-3}v_1 + (1+d)^{n+k-4}v_2\}$$

and

$$w_1(E_\mu) + \binom{k}{1}d = w_1 + v_1 + \binom{n+k-2}{1}d + \binom{k}{1}d.$$

Since $n$ is odd, $n+k-2 \neq k \mod 2$, which gives $w_1(E_\mu) + \binom{k}{1}d = w_1 + v_1 + d$. Because $n+1$ is even and any class with dimension greater than 2 from the cohomology of $F^2$ is zero, one has

$$\left(w_1(E_\mu) + \binom{k}{1}d\right)^{n+1} d^{k-2} = (w_1 + v_1 + d)^{n+1} d^{k-2}$$

$$= d^{n+k-1} + \binom{n+1}{1} d^{n+k-2}(w_1 + v_1) + \binom{n+1}{2} d^{n+k-3}(w_1^2 + v_1^2)$$

$$= d^{n+k-1} + \binom{n+1}{2} d^{n+k-3}(w_1^2 + v_1^2).$$

Denoting by $\overline{W}(\mu) = \dfrac{1}{W(\mu)} = 1+\overline{v}_1+\overline{v}_2$ the dual Stiefel-Whitney class of $\mu$, one has from [5] that $d^{n+k-1}[E_\mu] = \overline{v}_2[F^2]$ and $d^{n+k-3}(w_1^2+v_1^2)[E_\mu] = (w_1^2+v_1^2)[F^2]$. Since $\overline{v}_2 = v_1^2 + v_2$, the result follows. □

Now we have in hand the necessary tools to prove our theorem. One has an involution fixing $F^n \cup F^2$, with $n \geq 3$ odd and the normal bundle $\mu$ over $F^2$ being nonbounding, and wants first to show that $k \leq 3$, where $k$ is the codimension of $F^n$. Our strategy will consist in showing, with the technique used in Lemma 2.3, that if $k > 3$ then all characteristic numbers of $\mu$ are zero, giving the contradiction.

We formally introduce the class

$$\widetilde{W} = \frac{\mathbb{W}}{(1+c)^k} = \frac{1 + W_1 + W_2 + \cdots}{(1+c)^k} = 1 + \widetilde{W}_1 + \widetilde{W}_2 + \cdots$$

Each $\widetilde{W}_i$ is a polynomial in the classes $W_j$ and $c$, which means that these classes can be used to yield characteristic numbers. In particular, if $k > 3$, $n+k-1 \geq n+3$ and we can form the classes $\widetilde{W}_1^{n-1}\widetilde{W}_2^2 c^{k-4}$ and $\widetilde{W}_1^{n-1}(\widetilde{W}_2^2 + \widetilde{W}_1^4)c^{k-4}$, which give characteristic numbers. The next step is to compute these numbers on $F^n$ and $F^2$. We repeat the notation used in the proof of Lemma

2.3 for the suitable characteristic classes. On $F^n$, $\widetilde{W}$ is

$$\widetilde{W} = (1+\theta_1+\cdots+\theta_n)\left(1+\frac{u_1}{1+c}+\frac{u_2}{(1+c)^2}+\cdots+\frac{u_k}{(1+c)^k}\right).$$

One has $\widetilde{W}_1 = \theta_1 + u_1$ and $\widetilde{W}_2 = u_1c + u_2 + \theta_1 u_1 + \theta_2$.

Thus $\widetilde{W}_1^{n-1}\widetilde{W}_2^2 = (\theta_1+u_1)^{n-1}(u_1^2c^2 + u_2^2 + \theta_1^2u_1^2 + \theta_2^2)$ and

$$\widetilde{W}_1^{n-1}(\widetilde{W}_2^2 + \widetilde{W}_1^4) = (\theta_1+u_1)^{n-1}(u_1^2c^2 + u_2^2 + \theta_1^2u_1^2 + \theta_2^2 + \theta_1^4 + u_1^4).$$

Each term of these classes has a factor of dimension at least $n+1$ from the cohomology of $F^n$, and thus the characteristic numbers $\widetilde{W}_1^{n-1}\widetilde{W}_2^2c^{k-4}[E_\eta]$ and $\widetilde{W}_1^{n-1}(\widetilde{W}_2^2+\widetilde{W}_1^4)c^{k-4}[E_\eta]$ are zero.

On $F^2$, $\widetilde{W}$ is

$$\widetilde{W} = (1+w_1+w_2)\{(1+d)^{n-2} + (1+d)^{n-3}v_1 + (1+d)^{n-4}v_2\}.$$

Because $n$ is odd, one has $\widetilde{W}_1 = w_1 + v_1 + \binom{n-2}{1}d = w_1+v_1+d$ and

$$\begin{aligned}\widetilde{W}_2 &= w_2 + v_2 + w_1v_1 + \binom{n-2}{1}dw_1 + \binom{n-3}{1}dv_1 + \binom{n-2}{2}d^2 \\ &= w_2+v_2+w_1v_1+dw_1+\binom{n-2}{2}d^2.\end{aligned}$$

Thus, by dimensional reasons,
$\widetilde{W}_1^4 = d^4$, $\widetilde{W}_2^2 = d^2w_1^2 + \binom{n-2}{2}d^4$ and $\widetilde{W}_1^{n-1} = d^{n-1} + \binom{n-1}{2}d^{n-3}(w_1^2+v_1^2)$.

It follows that

$$\widetilde{W}_1^{n-1}\widetilde{W}_2^2d^{k-4} = d^{n+k-3}w_1^2 + \binom{n-2}{2}d^{n+k-1} + \binom{n-1}{2}\binom{n-2}{2}d^{n+k-3}(w_1^2+v_1^2).$$

Since $n$ is odd, $\binom{n-1}{2}\binom{n-2}{2} \equiv 0$ and thus

$$\widetilde{W}_1^{n-1}\ \widetilde{W}_2^2\ d^{k-4} = d^{n+k-3}\ w_1^2\ +\ \binom{n-2}{2}d^{n+k-1}.$$

Also

$$\begin{aligned}\widetilde{W}_1^{n-1}(\widetilde{W}_2^2+\widetilde{W}_1^4)d^{k-4} &= d^{n+k-1} + d^{n+k-3}\ w_1^2 + \\ &+\binom{n-2}{2}d^{n+k-1} + \binom{n-1}{2}d^{n+k-3}(w_1^2+v_1^2).\end{aligned}$$

Putting together the computations on $F^n$ and $F^2$ and using again [5], we obtain the following system of equations:

$$\begin{cases} w_1^2 + \binom{n-2}{2}\overline{v}_2 = 0 \\ \\ w_1^2 + \binom{n-2}{2}\overline{v}_2 + \overline{v}_2 + \binom{n-1}{2}(w_1^2 + v_1^2) = 0 \end{cases}$$

Since $k > 3$, $k \geq 2$ and thus Lemma 2.3 applies to give $\binom{n+1}{2}(w_1^2 + v_1^2) + v_2 + v_1^2 = 0$. First, suppose $\binom{n+1}{2} \equiv 0$. Then $v_1^2 = v_2$ and, since $n$ is odd, $\binom{n-2}{2} \equiv 0$ and $\binom{n-1}{2} \equiv 1$. On the other hand, if $\binom{n+1}{2} \equiv 1$, then $w_1^2 = v_2$, $\binom{n-2}{2} \equiv 1$ and $\binom{n-1}{2} \equiv 0$. With these data, it is easy to check that, in both cases, $w_1^2 = v_1^2 = v_2 = 0$ is the unique solution for the above system, and part 1) of the theorem is ended.

To prove part 2), it suffices to note that $n \equiv 3 \bmod 4$ implies $\binom{n+1}{2} \equiv 0 \bmod 2$, thus by Lemma 2.3 $v_2 = v_1^2$ when $k \geq 2$; also $n \equiv 1 \bmod 4$ implies $\binom{n+1}{2} \equiv 1 \bmod 2$, thus Lemma 2.3 says that $w_1^2 = v_2$ when $k \geq 2$. Now for $\beta_2, \beta_3, \beta_5$ and $\beta_6$ one has $v_2 \neq v_1^2$, and for $\beta_1, \beta_2, \beta_6$ and $\beta_7$ one has $w_1^2 \neq v_2$. This means that for all pairs $(n, \beta_i)$ listed in the theorem one has $k = 1$. The fact that, in each case, $(M, T)$ is equivariantly cobordant to $(\mathbb{R}\mathrm{P}(\mu \oplus \mathbb{R}), T_\mu)$, is a direct application of Lemma 2.2.

Finally, we prove part 3). We need to exhibit maximal examples for the pairs $(n, \beta_1)$, $(n, \beta_4)$, $(n, \beta_7)$ with $n \equiv 3 \bmod 4$, and $(n, \beta_3)$, $(n, \beta_4)$, $(n, \beta_5)$ with $n \equiv 1 \bmod 4$. First note that if $(M^m, T)$ and $(N^m, S)$ are involutions having fixed point set of the form $F^n \cup F^2$, and in such a way that $\mu$ over $F^2$ represents $\beta_i$ for $(M^m, T)$ and $\beta_j$ for $(N^m, S)$, then $(M^m, T) \cup (N^m, S)$ has still fixed point set of the form $F^n \cup F^2$ with $\mu$ over $F^2$ representing $\beta_i + \beta_j$. Taking into account that the sum of cobordism classes of bundles is compatible with the sum mod 2 of the corresponding characteristic numbers, it is easy to check that $\beta_1 + \beta_4 = \beta_7$ and $\beta_3 + \beta_4 = \beta_5$. Thus, it suffices to exhibit maximal examples for the pairs $(n, \beta_1)$ with $n \equiv 3 \bmod 4$, $(n, \beta_3)$ with $n \equiv 1 \bmod 4$ and $(n, \beta_4)$ for any $n$ odd. Consider the involution $(\mathbb{R}\mathrm{P}^{n-1}, T)$ given by $T[x_0, x_1, \ldots, x_{n-1}] = [-x_0, x_1, \ldots, x_{n-1}]$. Then $T$ fixes $\mathbb{R}\mathrm{P}^{n-2} \cup \{point\}$. Next, consider $(\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2, S)$, where $S(x, y, z) = (T(x), z, y)$. Then

$S$ fixes $(\mathbb{R}\mathrm{P}^{n-2} \times \mathbb{R}\mathrm{P}^2) \cup \mathbb{R}\mathrm{P}^2$, and the normal bundle over $\mathbb{R}\mathrm{P}^2$, which is stably cobordant to the tangent bundle over $\mathbb{R}\mathrm{P}^2$, represents $\beta_4$. This gives examples $(M^{n+3}, T)$ fixing $F^n \cup F^2$ with the normal bundles over $F^2$ representing $\beta_4$ for every $n \geq 2$, in particular for $n \geq 3$ odd.

Now consider $(\mathbb{R}\mathrm{P}^{n+3}, T)$, where

$$T[x_0, x_1, x_2, x_3, \ldots, x_{n+3}] = [-x_0, -x_1, -x_2, x_3, \ldots, x_{n+3}].$$

The fixed data of $T$ consists of $\mathbb{R}\mathrm{P}^n$ with normal bundle $3\xi$ and $\mathbb{R}\mathrm{P}^2$ with normal bundle $(n+1)\xi$, where $l\xi$ means the Whitney sum of $l$ copies of the corresponding canonical line bundles. If $n$ is odd, one can write $n+1 = 4x + r$, where $r = 0$ or 2. Denoting by $\alpha \in H^1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}_2)$ the generator, one has $\mathbb{W}((n+1)\xi) = (1+\alpha)^{n+1} = (1+\alpha)^r$. If $n \equiv 3 \mod 4$, then $n+1 \equiv 0 \mod 4$ and $r = 0$, which means that $(n+1)\xi$ over $\mathbb{R}\mathrm{P}^2$ represents $\beta_1$. If $n \equiv 1 \mod 4$, then $n+1 \equiv 2 \mod 4$ and $r = 2$, which means that $(n+1)\xi$ over $\mathbb{R}\mathrm{P}^2$ represents $\beta_3$. Thus our task is completed. $\square$

**Remark.** As mentioned in the introduction, in [7] Stong and Pergher constructed, for each $n \geq 1$, a specific involution $(M^{m(n)}, T_n)$ for which the fixed point set has the form $F^n \cup \{point\}$, and where, writing $n = 2^p q$ with $p \geq 0$ and $q$ odd,

$$m(n) = \begin{cases} 2n + p - q + 1 & \text{if } p \leq q, \text{ and} \\ 2n + 2^{p-q} & \text{if } p > q. \end{cases}$$

Given $n > 3$ even, consider the involution $(M^{m(n-2)} \times \mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2, T)$, where $T(x, y, z) = (T_{n-2}(x), z, y)$. The fixed point set of $T$ has the form $F^{n-2} \times \mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^2$ and evidently the normal bundle over $\mathbb{R}\mathrm{P}^2$ (which is stably cobordant to the tangent bundle over $\mathbb{R}\mathrm{P}^2$) is not a boundary. Since $n-2$ is even, $m(n-2)+4-n$ is not limited, and thus small codimensions do not occur for $F = F^n \cup F^2$ when $n$ is even.

In [8] (and in [9]), S. Kelton studied bounds for $m$ when $(M^m, T)$ is an involution having fixed point set of the form $F = F^n \cup \mathbb{R}\mathrm{P}^j$. Among the results one finds: suppose $(M^m, T)$ is an involution whose fixed point set has the form $F = F^n \cup \mathbb{R}\mathrm{P}^1$ and the normal bundle of $\mathbb{R}\mathrm{P}^1$ in $M^m$ is nonbounding. Then, if $n$ is odd, $m \leq m(n-1)+1$, and if $n$ is even, $m \leq m(n-1)+2$; further, these bounds are best possible. Since there exists an unique stable cobordism class of vector bundles over one-dimensional and closed manifolds, which is the class of the canonical line bundle over $\mathbb{R}P^1$, the above bounds apply to the case in which $(M^m, T)$ is an involution with fixed point set of the form $F^n \cup F^1$ and with the normal bundle over $F^1$ being nonbounding. Now, if $n$ is even, $m(n-1)+2-n=2$, and if $n$ is odd, $m(n-1)+1-n$ is not limited. Thus, for $F = F^n \cup F^1$, small codimensions occur when $n$ is even and do not occur when $n$ is odd.

**ACKNOWLEDGEMENTS.** We would like to express our sincere gratitude and indebtedness to Professor Robert E. Stong of the University of Virginia for the valuable help given by many crucial suggestions. We also thank the referee for valuable suggestions concerning the presentation of this paper.

## References

[1] A. Borel and F. Hirzebruch, *On characteristic classes and homogeneous spaces*, I, Amer. J. Math. 80, (1958), 458-538.

[2] C. Kosniowski and R. E. Stong, *Involutions and characteristic numbers*, Topology 17, (1978), 309-330.

[3] D. C. Royster, *Involutions fixing the disjoint union of two projective spaces*, Indiana Univ. Math. J. 29, (1980), 267-276.

[4] J. M. Boardman, *On manifolds with involution*, Bulletin Amer. Math. Soc. 73, (1967), 136-138.

[5] P. E. Conner, *The bordism class of a bundle space*, Michigan Math. J. 14, (1967), 289-303.

[6] P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Springer-Verlag, Berlin, (1964).

[7] P. L. Q. Pergher and R. E. Stong, *Involutions fixing* $\{point\} \cup F^n$, Transformation Groups 6, (2001), 78-85.

[8] Suzanne M. Kelton, *Involutions fixing* $\mathbb{R}P^j \cup F^n$, Doctoral Thesis, University of Virginia, (2001).

[9] Suzanne M. Kelton, *Involutions fixing* $\mathbb{R}P^j \cup F^n$, Topology Appl. 142, (2004), 197-203.

**Pedro L. Q. Pergher**
Departamento de Matemática
Universidade Federal de São Carlos
Caixa Postal 676
São Carlos, SP 13565-905, Brazil
E-mail: *pergher@dm.ufscar.br*

**Fábio G. Figueira**
Departamento de Matemática
Universidade Federal de São Carlos
Caixa Postal 676
São Carlos, SP 13565-905, Brazil
E-mail: *fabio@dm.ufscar.br*