Conformal immersions of warped products

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Abstract: We prove a decomposition theorem for conformal immersions $f\colon M^n \to \mathbb{R}^N$ into Euclidean space of a warped product of Riemannian manifolds $M^n := M_0 \times_\rho \Pi_{i=1}^k M_i$ of dimension $n \geq 3$ under the assumption that the second fundamental form $\alpha\colon TM \times TM \to T^\perp M$ of f satisfies $\alpha|_{TM_i \times TM_j} = 0$ for $i \neq j$. It generalizes the corresponding theorem of Nölker for isometric immersions as well as our previous result on conformal immersions of Riemannian products. In particular, we determine all conformal representations of Euclidean space of dimension $n \geq 3$ as a warped product of Riemannian manifolds. As a consequence, we classify the conformally flat warped products.

1 Introduction

It is well-known that the early developments of differential geometry are closely related to mathematical cartography. After the impossibility of ideal map projections of the sphere was settled by Euler, *infinitesimal similarity* was realized to be the primary feature of a cartographic map. This explains the interest that the study of conformal mappings of surfaces into the plane has attracted on many of the great differential geometers since the second half of the eighteenth century. For instance, as early as 1779 Lagrange solved the problem of determining all *tracés géographiques* (conformal mappings into the plane) of a rotation surface in $\mathbb{R}^3$ such that meridians and parallels are mapped into circles or straight lines (cf. [**Da**], vol I, p. 236).

A higher dimensional version of the problem of studying conformal mappings of rotation surfaces into the plane is that of investigating the conformal local diffeomorphisms of warped products of Riemannian manifolds onto open subsets of Euclidean space. In the isometric case, this problem was completely solved by Nölker [**No**], who was able to give an explicit description of all the so-called *warped product representations* of (open subsets of) Euclidean space.

The rigidity of conformal mappings in dimension $n \geq 3$ allows us to obtain a full solution of the problem also in the conformal setting. In contrast to the case $n = 2$, it turns out that the analogous to the additional condition in Lagrange's result is automatically satisfied when $n \geq 3$, namely, any conformal local diffeomorphism of a (locally conformally flat) warped product of dimension $n \geq 3$ into Euclidean space maps the leaves of the product foliation induced by each factor onto open subsets of spheres or affine subspaces.

As a consequence of our result, we are able to determine all locally conformally flat warped products of dimension $n \geq 3$ (with arbitrarily many factors), extending the known classification of locally conformally flat Riemannian products (cf. [**La**]).

Nölker's classification of warped product representations of Euclidean space is, in fact, a rather special case of a general decomposition theorem for isometric immersions of warped products into space forms whose second fundamental forms are adapted to their product nets, in the sense that each subbundle correspondent to a factor is invariant by all shape operators. The latter generalizes a well-known result for isometric immersions of Riemannian products into Euclidean space due to Moore [**Mo**], and has proven to be a very useful tool in the study of submanifolds of space forms.

Accordingly, our main result is a complete description of all conformal immersions of a warped product of dimension $n \geq 3$ into Euclidean space under the same assumption on the second fundamental form as in Nölker's result. This also generalizes our previous result in [$\mathbf{To}_1$] for conformal immersions of Riemannian products of dimension $n \geq 3$, which can be regarded as a local classification of higher dimensional analogues of isothermic surfaces. Our strategy here, however, is quite different from that in [$\mathbf{To}_1$], and can be roughly summarized as follows. We first prove a lemma of independent interest on mutually orthogonal families of spheres on Euclidean space which allows us, by means of an additional result on extrinsic spheres of Euclidean submanifolds, to reduce the problem to the isometric case and then apply Nölker's theorem.

Acknowledgment. After a preliminary version of this paper was completed, we learned that M. Brozos-Vázquez, E. García-Río and R. Vázquez-Lorenzo had independently obtained in [**BGV**] a local classification of locally conformally flat warped products similar to ours (Corollary 11) by a different method based on direct computation of the Weyl curvature tensor.

2 Preliminaries

2.1 Orthogonal nets and product manifolds

A *net* on a connected C^∞-manifold M is a splitting $TM = \oplus_{i=0}^k E_i$ by a family of integrable subbundles. The canonical net on a product manifold $M = \Pi_{i=0}^k M_i$ is called the *product net*. If M is a Riemannian manifold and the subbundles are mutually orthogonal then the net is said to be an *orthogonal net*. An orthogonal net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is called a *TP-net* if E_i is umbilical and $E_i^\perp$ is integrable for every $i = 0, \ldots, k$. Recall that a subbundle E of TM is *umbilical* if there exists a vector field η in $E^\perp$ such that

$$\langle \nabla_X Y, Z\rangle = \langle X, Y\rangle\langle \eta, Z\rangle \text{ for all } X, Y \in \Gamma(E),\ Z \in \Gamma(E^\perp).$$

Here and in the sequel, the space of smooth local sections of a vector bundle E over M is denoted by $\Gamma(E)$. The vector field η is called the *mean curvature normal* of E. If, in addition,

$$\langle \nabla_X \eta, Z \rangle = 0 \quad \text{for all } X \in \Gamma(E),\ Z \in \Gamma(E^\perp),$$

then E is said to be *spherical*. If E is umbilical and its mean curvature normal vanishes identically, then it is called *totally geodesic* (or *auto-parallel*). An umbilical distribution is automatically integrable, and the leaves are umbilical submanifolds of M. When E is totally geodesic or spherical, its leaves are totally geodesic or spherical submanifolds, respectively. By a *spherical submanifold*, or an *extrinsic sphere*, we mean an umbilical submanifold whose mean curvature vector is parallel with respect to the normal connection. If an orthogonal net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is such that E_a is spherical and $E_a^\perp$ is totally geodesic for every $a = 1, \ldots, k$, then $\mathcal{E}$ is called a *WP-net*.

The terminologies TP-net and WP-net are explained by their connections with twisted and warped product metrics, respectively. Recall that a metric $\langle\ ,\ \rangle$ on a product manifold $M = \Pi_{i=0}^k M_i$ is called a *twisted product metric* if there exist Riemannian metrics $\langle\ ,\ \rangle_i$ on M_i, $i = 0, \ldots, k$, and a C^∞ *twist-function* $\rho = (\rho_0, \ldots, \rho_k)\colon M \to \mathbb{R}_+^{k+1}$ such that $\langle\ ,\ \rangle = \sum_{i=0}^k \rho_i^2 \pi_i^* \langle\ ,\ \rangle_i$, where $\pi_i\colon M \to M_i$ is the projection of M onto M_i. Then $(M, \langle\ ,\ \rangle)$ is said to be a *twisted product* and is denoted by ${}^\rho\Pi_{i=0}^k (M_i, \langle\ ,\ \rangle_i)$. When $\rho_1, \ldots, \rho_k$ are independent of $M_1, \ldots, M_k$, that is, there exist $\tilde{\rho}_i \in C^\infty(M_0)$ such that $\rho_a = \tilde{\rho}_a \circ \pi_0$ for $a = 1, \ldots, k$, and, in addition, ρ_0 is identically 1, then $\langle\ ,\ \rangle$ is called a *warped product metric* and $(M, \langle\ ,\ \rangle) := (M_0, \langle\ ,\ \rangle_0) \times_{\tilde{\rho}} \Pi_{a=1}^k (M_a, \langle\ ,\ \rangle_a)$ a *warped product* with *warping function* $\tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_k)$. If there exists $\bar{p} \in M$ such that $\tilde{\rho}_a(\pi_0(\bar{p})) = 1$ for $a = 1, \ldots, k$, then the representation of M as a warped product is said to be *normalized* with respect to $\bar{p}$.

Proposition 1 **[MRS]** *On a connected product manifold $M =: \Pi_{i=0}^k M_i$ the product net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is a TP-net (resp., WP-net) with respect to a Riemannian metric $\langle\ ,\ \rangle$ on M if and only if $\langle\ ,\ \rangle$ is a twisted product (resp., warped product) metric on M.*

An orthogonal net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ on a Riemannian manifold is a *conformal warped product net*, or a CWP-net for short, if for $a = 1, \ldots, k$ it holds that

$$E_a \text{ and } E_a^\perp \text{ are umbilical and } \langle \nabla_{X_{\perp_a}} \eta_a, X_a \rangle = \langle \nabla_{X_a} H_a, X_{\perp_a} \rangle$$

for all $X_a \in \Gamma(E_a)$ and $X_{\perp_a} \in \Gamma(E_a^\perp)$, where H_a and η_a are the mean curvature normals of E_a and $E_a^\perp$, respectively. We have the following immediate consequence of the definition.

Proposition 2 *If $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is a CWP-net then for each $a = 1, \ldots, k$ one of E_a or $E_a^\perp$ being spherical implies the same for the other.*

The terminology is justified by the next result.

Proposition 3 [To$_2$] *On a connected simply connected product manifold $M = \Pi_{i=0}^k M_i$ the product net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is a CWP-net with respect to a Riemannian metric $\langle\ ,\ \rangle$ on M if and only if $\langle\ ,\ \rangle$ is conformal to a warped product metric.*

We conclude this subsection with the following observation for later use.

Lemma 4 *Let $\langle\ ,\ \rangle = \sum_{i=0}^k \rho_i^2 \pi_i^* \langle\ ,\ \rangle_i$ and $\langle\ ,\ \rangle^\sim = \sum_{i=0}^k \varphi_i^2 \pi_i^* \langle\ ,\ \rangle_i^\sim$ be twisted product metrics on a product manifold $M^n = \Pi_{i=0}^k M_i^{n_i}$. If $\langle\ ,\ \rangle^\sim = \psi^2 \langle\ ,\ \rangle$ for some $\psi \in C^\infty(M)$, then there exists $\psi_i \in C^\infty(M_i)$ such that $\psi\rho_i/\varphi_i = \psi_i \circ \pi_i$ and $\langle\ ,\ \rangle_i^\sim = \psi_i^2 \langle\ ,\ \rangle_i$ for $i = 0, \ldots, k$.*

Proof: Given $i \in \{0, \ldots, k\}$ and a local unit vector field $X_i \in \Gamma(TM_i)$ with respect to $\langle\ ,\ \rangle_i$, let $\tilde{X}_i$ be the lift of X_i to M. Then $\varphi_i^2(\langle X_i, X_i\rangle_i^\sim \circ \pi_i) = \langle \tilde{X}_i, \tilde{X}_i\rangle^\sim = \psi^2 \rho_i^2$. Therefore, there exists $\psi_i \in C^\infty(M_i)$ such that $\psi\rho_i/\varphi_i = \psi_i \circ \pi_i$ and $\langle X_i, X_i\rangle_i^\sim = \psi_i^2$ for any such X_i. It follows that $\langle\ ,\ \rangle_i^\sim = \psi_i^2 \langle\ ,\ \rangle_i$. ∎

2.2 The decomposition theorem of Nölker

Let $\mathbb{Q}_c^N$ denote a complete simply connected space form of dimension N and constant sectional curvature c. For a fixed point $\bar{p} \in \mathbb{Q}_c^N$, let $T_{\bar{p}}\mathbb{Q}_c^N = \oplus_{i=0}^k V_i$ be an orthogonal decomposition of $T_{\bar{p}}\mathbb{Q}_c^N$ into nontrivial subspaces, and let $z_1, \ldots, z_k \in V_0$ be such that $\langle z_a, z_b\rangle = -c$ for $a \neq b$. Let N_a, $a = 1, \ldots, k$, be the unique complete spherical submanifold of $\mathbb{Q}_c^N$ such that $V_a = T_{\bar{p}} N_a$ and whose mean curvature vector in $\mathbb{Q}_c^N$ at $\bar{p}$ is z_a. Regarding $\mathbb{Q}_c^N$ as an umbilical hypersurface of either Euclidean or Lorentzian $(N+1)$-space according as $c > 0$ or $c < 0$, respectively, in any case let $H_a = c\bar{p} - z_a$ be the opposite of the mean curvature vector of N_a in the flat ambient space and set $k_a = c + |z_a|^2 = |H_a|^2$. Then $H_1, \ldots, H_k \in \mathbb{R}\bar{p} \oplus V_0$ are pairwise orthogonal vectors, and k_a can be zero only if $c < 0$ or if $c = 0$ and $H_a = -z_a = 0$. In case $c = 0$, set

$$N_0 = \bar{p} - \sum_{k_a > 0} k_a^{-1} H_a + \{p \in V_0 : \langle H_a, p\rangle > 0 \text{ for all } a \text{ with } k_a > 0\},$$

and define $\sigma_a \colon N_0 \to \mathbb{R}_+$ by $\sigma_a(p) = 1 + \langle H_a, p - \bar{p}\rangle$ for $a = 1, \ldots, k$. In case $c \neq 0$, set

$$N_0 = \mathbb{Q}_c^N \cap \{p \in \mathbb{R}\bar{p} \oplus V_0 : \langle H_a, p\rangle > 0 \text{ for all } a \text{ with } k_a > 0\}$$

and define $\sigma_a \colon N_0 \to \mathbb{R}_+$ by $\sigma_a(p) = \langle H_a, p\rangle$ for $a = 1, \ldots, k$. Notice that in both cases N_0 is an open subset of the totally geodesic submanifold of $\mathbb{Q}_c^N$ whose tangent space at $\bar{p}$ is V_0. Observe also that k can not exceed dim $N_0 + 1 =$ dim $V_0 + 1$ if either $c > 0$ or $c = 0$ and $H_a = 0$ for at most one $a \in \{1, \ldots, k\}$. Now define

$$G = \mathbb{R}^N \setminus (\bar{p} - \sum_{k_a > 0} k_a^{-1} H_a + \bigcup_{k_a > 0} (\mathbb{R}H_a \oplus V_a)^\perp)$$

if $c=0$, and otherwise set

$$G=\mathbb{Q}_c^N\setminus\bigcup_{k_a>0}(\mathbb{R}H_a\oplus V_a)^\perp.$$

Then the map $\Phi\colon N_0\times_\sigma \Pi_{a=1}^k N_a\to G$ given by

$$\Phi(p_0,\ldots,p_k)=p_0+\sum_{a=1}^k\sigma_a(p_0)(p_a-\bar p)$$

is an isometry, called the *warped product representation* of $\mathbb{Q}_c^N$ determined by the data $(\bar p, N_1,\ldots,N_k)$. Moreover, it was proved by Nölker [**No**] that any isometry of a warped product onto an open subset of $\mathbb{Q}_c^N$ is essentially given as the restriction of such a warped product representation. As a consequence, one easily deduces that a warped product $M^n=M_0^{n_0}\times_\rho\Pi_{a=1}^k M_a^{n_a}$ has constant sectional curvature c if and only if $M_0^{n_0}$ has constant sectional curvature c (if $n_0\geq 2$), the warping functions satisfy $\operatorname{Hess}\rho_a+c\rho_a\langle\ ,\ \rangle=0$ and $\langle\operatorname{grad}\rho_a,\operatorname{grad}\rho_b\rangle+c\rho_a\rho_b=0$, $a\neq b$, and each fiber $M_a^{n_a}$ has (constant) sectional curvature $|\operatorname{grad}\rho_a|^2+c\rho_a^2$ (if $n_a\geq 2$). In fact, for the domain of a warped product representation $\Phi\colon N_0\times_\sigma\Pi_{a=1}^k N_a\to G\subset\mathbb{Q}_c^N$, the preceding conditions on the warping functions just express the fact that they are given by the restrictions to $N_0\subset\mathbb{Q}_c^{n_0}$ of linear functions with pairwise orthogonal gradients in the underlying flat ambient space. In particular, we must have $k\leq n_0+1$ if either $c>0$ or if $c=0$ and at most one of the warping functions is constant.

Given immersions $f_i\colon M_i\to N_i$, $i=0,\ldots,k$, the map

$$f:=\Phi\circ(f_0\times\ldots\times f_k)\colon\ M:=\Pi_{i=0}^k M_i\to\mathbb{Q}_c^N$$

is also an immersion whose induced metric is the warped product of the metrics induced by $f_0,\ldots,f_k$, with warping function $\rho=(\rho_1,\ldots,\rho_k)$ given by $\rho_a=\sigma_a\circ f_0$, $a=1,\ldots,k$. It is called the *warped product of* $f_0,\ldots,f_k$.

We now state Nölker's decomposition theorem.

Theorem 5 [**No**] *Let $f\colon M^n:=M_0\times_\rho\Pi_{a=1}^k M_a\to\mathbb{Q}_c^N$ be an isometric immersion of a warped product whose second fundamental form is adapted to the product net $\mathcal{E}=(E_i)_{i=0,\ldots,k}$ of M^n, that is,*

$$\alpha(X_i,X_j)=0\quad\text{for all}\quad X_i\in\Gamma(E_i),\quad X_j\in\Gamma(E_j),\quad i\neq j.$$

Given $\bar p=(\bar p_0,\ldots,\bar p_k)\in M^n$, set $f_i=f\circ\tau_i^{\bar p}\colon M_i\to\mathbb{Q}_c^N$ for $i=0,\ldots,k$, where $\tau_i^{\bar p}(p_i)=(\bar p_0,\ldots,p_i,\ldots,\bar p_k)$, and let N_a be the spherical hull of f_a for $a=1,\ldots,k$. Then f_0 is an isometric immersion, f_a is a homothetical immersion for $a=1,\ldots,k$ with homothety factor $\rho_a(\bar p_0)$ and $(f(\bar p);N_1,\ldots,N_k)$ determines a warped product representation $\Phi\colon N_0\times_\sigma\Pi_{a=1}^k N_a\to\mathbb{Q}_c^N$ such that $f_0(M_0)\subset N_0$, $\rho_a=\rho_a(\bar p_0)(\sigma_a\circ f_0)$ and $f=\Phi\circ(f_0\times\cdots\times f_k)$, where f_i is regarded as a map into N_i for $i=0,\ldots,k$.

2.3 Moebius geometry in the light cone

We now briefly describe the model of Euclidean space as an umbilical hypersurface of the light cone of Minkowski space. We refer to [**H-J**] for further details.

Let $\mathbb{L}^{N+2}$ be the $(N+2)$–dimensional Minkowski space endowed with a Lorentz scalar product of signature $(+,\ldots,+,-)$, and let

$$\mathbb{V}^{N+1} = \{p \in \mathbb{L}^{N+2} \colon \langle p,p\rangle = 0\}$$

denote the light cone in $\mathbb{L}^{N+2}$. For any $w \in \mathbb{V}^{N+1}$, the intersection

$$\mathbb{E}^N = \{p \in \mathbb{V}^{N+1} \colon \langle p,w\rangle = 1\}$$

of the light cone with the affine hyperplane $\langle p,w\rangle = 1$ is a model of the N–dimensional Euclidean space. Namely, fix $p_0 \in \mathbb{E}^N$ and a linear isometry $A\colon \mathbb{R}^N \to \mathrm{span}\{p_0,w\}^\perp$. Then we have an isometry $\Psi\colon \mathbb{R}^N \to \mathbb{E}^N$ given by $x \in \mathbb{R}^N \ \to\ A(x) + p_0 - (1/2)\|x\|^2 w$.

Hyperspheres in $\mathbb{R}^N$ can be nicely described in $\mathbb{E}^N$ as follows. Given a hypersphere S in $\mathbb{E}^N$ with (constant) mean curvature h with respect to a unit normal vector field n, then $v = n_p + hp \in \mathbb{L}^{N+2}$ is a constant space-like unit vector such that $\langle v,p\rangle = 0$ for all $p \in S$. Thus, the (nonoriented) sphere S is given in the model by $S = \mathbb{E}^N \cap \{v\}^\perp$. Since $h = \langle v,w\rangle$, then v represents a hyperplane in $\mathbb{E}^N$ if and only if $\langle v,w\rangle = 0$.

The intersection angle of two (oriented) hyperspheres has also a simple description in this model: given hyperspheres $S_i = \mathbb{E}^N \cap \{v_i\}^\perp$ with unit normal vectors n_p^i, $1 \le i \le 2$, at a common point p, their intersection angle at p is given by $\langle n_p^1, n_p^2\rangle = \langle v_1,v_2\rangle$. Thus S_1 and S_2 intersect orthogonally if and only if $\langle v_1,v_2\rangle = 0$.

A hypersphere $S = \mathbb{E}^N \cap \{v\}^\perp$ has center at $q_0 \in \mathbb{E}^N$ and mean curvature $h \neq 0$ if and only if $v = hq_0 + (2h)^{-1}w$. This follows from $\langle v,w\rangle = h$, $\langle v,v\rangle = 1$ and the fact that $\mathrm{span}\{q_0,w\}^\perp \subset \{v\}^\perp$, the latter being due to the fact that any hyperplane through q_0 is orthogonal to S.

Since any m-dimensional sphere S in $\mathbb{E}^N$ is the intersection of $(N-m)$ mutually orthogonal hyperspheres $S_i = \mathbb{E}^N \cap \{v_i\}^\perp$, $\langle v_i,v_j\rangle = \delta_{ij}$ for $1 \le i,j \le N-m$, it follows that $S = \mathbb{E}^N \cap V^\perp$, where V is the space–like $(N-m)$-dimensional subspace of $\mathbb{L}^{N+2}$ spanned by $v_1,\ldots,v_{N-m}$. Therefore, the space of (non-oriented) m-dimensional spheres in $\mathbb{E}^N$ is naturally identified with the Grassmann manifold $G_+(N{-}m, m{+}2)$ of space–like $(N-m)$-dimensional subspaces of $\mathbb{L}^{N+2}$. The m–dimensional affine subspaces in $\mathbb{E}^N$ are given by those space-like $(N-m)$–dimensional subspaces of $\mathbb{L}^{N+2}$ orthogonal to w.

We say that two m_i–dimensional spheres or affine subspaces S_i in $\mathbb{R}^N$, $1 \le i \le 2$, with $m_1 + m_2 \ge N$, intersect orthogonally if their normal spaces H_1 and H_2 in $\mathbb{R}^N$, respectively, at one (and hence at any) point in $S_1 \cap S_2$ satisfy $H_1 \subset H_2^\perp$. If S_i is given in the model $\mathbb{E}^N$ by $S_i = \mathbb{E}^N \cap V_i^\perp$, $1 \le i \le 2$, then this is equivalent to $V_1 \subset V_2^\perp$.

The relative position of spheres and affine subspaces can be easily recovered from the corresponding space like subspaces. For instance, let H_i, $1 \le i \le 2$, be m_i–dimensional affine subspaces in $\mathbb{R}^N$ with, say, $m_1 \ge m_2$. Let H_i be given in the model $\mathbb{E}^N$ by

$H_i = \mathbb{E}^N \cap V_i^\perp$, where V_i is a $(N - m_i)$-dimensional space-like subspace with $w \in V_i^\perp$, $1 \le i \le 2$. Then H_1 is parallel to H_2, meaning that H_1 contains a parallel translate of H_2, if and only if $V_1 \subset V_2 \oplus \mathrm{span}\{w\}$. In fact, H_1 is parallel to H_2 if and only if every affine subspace orthogonal to H_2 is also orthogonal to H_1, which is equivalent to $(V_2 \oplus \mathrm{span}\{w\})^\perp \subset V_1^\perp$. Arguing in a similar way, it follows that an m_1–dimensional sphere $S_1 = \mathbb{E}^N \cap V_1^\perp$ is tangent at $p \in \mathbb{E}^N$ to an m_2–dimensional sphere or affine subspace $S_2 = \mathbb{E}^N \cap V_2^\perp$, $m_1 \ge m_2$, if and only if $V_1 \subset V_2 \oplus \mathrm{span}\{p\}$.

Given a conformal immersion $G\colon M^n \to \mathbb{V}^{N+1}$ with conformal factor $\varphi \in C^\infty(M^n)$, for any $\mu \in C^\infty(M^n)$ the map $G_\mu\colon M^n \to \mathbb{V}^{N+1}$, $p \mapsto \mu(p)G(p)$, is also conformal with conformal factor $\mu\varphi$. Therefore, any conformal immersion $f\colon M^n \to \mathbb{R}^N$ with conformal factor $\varphi \in C^\infty(M^n)$ gives rise to an isometric immersion $\mathcal{I}(f) = (\Psi \circ f)_{\varphi^{-1}}\colon M^n \to \mathbb{V}^{N+1}$. Conversely, if $F\colon M^n \to \mathbb{V}^{N+1}$ is an isometric immersion whose image does not intersect $\mathbb{R}_w = \{tw : t > 0\}$, define $\mathcal{C}(F) =: M^n \to \mathbb{R}^N$ by $\Psi \circ \mathcal{C}(F) = \Pi \circ F$, where $\Pi = \Pi_w\colon \mathbb{V}^{N+1} \setminus \mathbb{R}_w \to \mathbb{E}^N$ is the projection onto $\mathbb{E}^N$ given by $\Pi(x) = x/\langle x, w\rangle$. Since Π is conformal with conformal factor $\varphi_\Pi(x) = \langle x, w\rangle^{-1}$, then $\mathcal{C}(F)$ is also conformal with conformal factor $\varphi_\Pi \circ F = \langle F, w\rangle^{-1}$.

In particular, conformal transformations of $\mathbb{R}^N$ are linearized in this model: any $T \in \mathbb{O}_1(N+2)$ gives rise to a conformal (Moebius) transformation $\mathcal{T} = \mathcal{C}(T \circ \Psi)$ of $\mathbb{R}^N$ and, conversely, any Moebius transformation of $\mathbb{R}^N$ is given in this way by means of some $T \in \mathbb{O}_1(N+2)$. For instance, if T is the reflection

$$T(p) = p - 2\langle p, v\rangle v \tag{1}$$

with respect to the hyperplane in $\mathbb{L}^{N+2}$ orthogonal to the unit space-like vector v, $\langle v, w\rangle \neq 0$, then $I = \mathcal{C}(T \circ \Psi)$ is the inversion with respect to the hypersphere $S = \mathbb{E}^N \cap \{v\}^\perp$. On the other hand, if $T \in \mathbb{O}_1(N+2)$ satisfies $T(w) = \lambda w$ for some $\lambda \in \mathbb{R}$ then there exists a similarity $\mathcal{H}$ of $\mathbb{R}^N$ of ratio λ such that $\mathcal{C}(T \circ \Psi) = \mathcal{H}$, i.e., $\Psi \circ \mathcal{H} = \lambda T \circ \Psi$. In particular, the isometries of $\mathbb{R}^N$ are given by those T that fix w.

3 The main lemma

The geometric nature of a family $\mathcal{F} = (H^\lambda)_{\lambda \in \Lambda}$ of at least two affine subspaces in $\mathbb{R}^N$ of a fixed dimension ℓ can be captured from the type of a subspace $V_{\mathcal{F}}$ of $\mathbb{L}^{N+2}$ naturally associated to it. Namely, for each $\lambda \in \Lambda$ let V^λ be the space-like subspace of $\mathbb{L}^{N+2}$ such that $H^\lambda = \mathbb{E}^N \cap V^{\lambda\perp}$, and let $V_{\mathcal{F}}$ be the subspace spanned by all V^λ, $\lambda \in \Lambda$. As $\mathcal{F}$ has at least two elements and each V^λ has dimension $N - \ell$, the dimension $m+1$ of $V_{\mathcal{F}}$ is at least $N - \ell + 1$. Since $w \in V_{\mathcal{F}}^\perp$, the subspace $V_{\mathcal{F}}$ is either degenerate or space-like. We say, accordingly, that $\mathcal{F}$ is a *light-like* or a *space-like* family.

If $\mathcal{F}$ is space-like with $m+1 \le N$, then $\bigcap_{\lambda\in\Lambda} H^\lambda = \bigcap_{\lambda\in\Lambda}(\mathbb{E}^N \cap V^{\lambda\perp}) = \mathbb{E}^N \cap V_{\mathcal{F}}^\perp := \mathcal{H}_{\mathcal{F}}$ is an affine subspace of dimension $N - m - 1$ (a point if $m+1 = N$). Conversely, if $\bigcap_{\lambda\in\Lambda} H^\lambda = \tilde{\mathcal{H}}$ is non-empty, then $\mathcal{F}$ is space-like and $\tilde{\mathcal{H}} = \mathcal{H}_{\mathcal{F}}$.

If $\mathcal{F}$ is light-like with $m+1 \leq N$, write $V_{\mathcal{F}} = U \oplus \mathrm{span}\{w\}$, where U is a (non-unique) space-like subspace of dimension m, and let $\mathbb{R}^{N-m}_{\mathcal{F}}$ be the $(N-m)$–dimensional subspace of $\mathbb{R}^N$ that is parallel to $\mathcal{H} = \mathbb{E}^N \cap U^{\perp}$. Since $V^{\lambda} \subset V_{\mathcal{F}} = U \oplus \mathrm{span}\{w\}$ for every $\lambda \in \Lambda$, it follows that H^{λ} is parallel to $\mathbb{E}^N \cap U^{\perp} = \mathcal{H}$, and hence to $\mathbb{R}^{N-m}_{\mathcal{F}}$. Moreover, $\mathbb{R}^{N-m}_{\mathcal{F}}$ is the maximal subspace with this property, for if $\mathbb{R}^s = \mathbb{E}^N \cap \tilde{U}^{\perp}$ is a subspace of $\mathbb{R}^N$ such that every $H^{\lambda} \in \mathcal{F}$ is parallel to $\mathbb{R}^s$, then $V^{\lambda} \subset \tilde{U} \oplus \mathrm{span}\{w\}$ for every $\lambda \in \Lambda$. Therefore $V_{\mathcal{F}} \subset \tilde{U} \oplus \mathrm{span}\{w\}$, which implies that $\mathbb{R}^s \subset \mathbb{R}^{N-m}_{\mathcal{F}}$.

If $\mathcal{F}$ is a space-like (resp., light-like) family of affine subspaces of dimension ℓ in $\mathbb{R}^N$, then we call the affine subspace $\mathcal{H}_{\mathcal{F}}$ (resp., the subspace $\mathbb{R}^{N-m}_{\mathcal{F}}$) the *axis* of $\mathcal{F}$.

Now let $\mathcal{F} = (S^{\lambda})_{\lambda \in \Lambda}$ be a family of at least two spheres (which may include affine subspaces) of dimension ℓ in $\mathbb{R}^N$. As before, for each $\lambda \in \Lambda$ let V^{λ} be the space-like subspace of $\mathbb{L}^{N+2}$ such that $S^{\lambda} = \mathbb{E}^N \cap V^{\lambda \perp}$ and let $V_{\mathcal{F}}$ be the subspace spanned by all V^{λ}, $\lambda \in \Lambda$. Then $V_{\mathcal{F}}$ can now be either a space-like, degenerate or time-like subspace of dimension $m+1 \geq N-\ell+1$. We say accordingly that $\mathcal{F}$ is a *space-like*, *light-like* or *time-like* family. If $V_{\mathcal{F}}$ is time-like and $w \in V_{\mathcal{F}}$, then $\mathcal{H} = \mathbb{E}^N \cap V_{\mathcal{F}}$ is an affine subspace in $\mathbb{R}^N$ (a point $\bar{q}$ if $\ell = N-1$ and $V_{\mathcal{F}}$ assumes its minimum dimension $m+1=2$) and every $S^{\lambda} \in \mathcal{F}$ is orthogonal to $\mathcal{H}$, for $V^{\lambda} \subset V_{\mathcal{F}}$ (when $m+1=2$ then $\mathcal{F}$ is a family of hyperspheres centered at $\bar{q}$). Moreover, there exists no proper subspace $\tilde{\mathcal{H}}$ of $\mathcal{H}$ such that every $S^{\lambda} \in \mathcal{F}$ is orthogonal to $\tilde{\mathcal{H}}$. Otherwise, if $\tilde{\mathcal{H}} = \mathbb{E}^N \cap \tilde{V}$ with $\tilde{V} \subset V_{\mathcal{F}}$ time-like, we would have $V^{\lambda} \subset \tilde{V}$ for every $\lambda \in \Lambda$, contradicting the fact that $V_{\mathcal{F}}$ is the subspace spanned by all V^{λ}, $\lambda \in \Lambda$. We say that $\mathcal{F}$ is a *standard time-like family* of spheres in $\mathbb{R}^N$ and call $\mathcal{H} = \mathcal{H}_{\mathcal{F}}$ the *axis* of $\mathcal{F}$.

We are now able to prove the following characterization of mutually orthogonal families of spheres in Euclidean space, which is the main tool in the proof of Theorem 7.

Lemma 6 *Let $\mathcal{F}_a = (S^{\lambda}_a)_{\lambda \in \Lambda}$, $a = 1, \ldots, k$, be families of spheres or affine subspaces in $\mathbb{R}^N$, each of which with at least two elements, such that every $S^{\lambda}_a \in \mathcal{F}_a$ has codimension $n_a \geq 1$, $\sum_{a=1}^k n_a = m \leq N$, and S^{λ}_a is orthogonal to S^{β}_b for all $\lambda, \beta \in \Lambda$ and for all $a, b = 1, \ldots, k$ with $a \neq b$. Then one of the following holds:*

(a) *After reordering the families, there exist $s \in \{0, \ldots, k\}$ with $k - s \leq N - m$, an orthogonal decomposition $\mathbb{R}^N = \oplus_{a=1}^{k+1} \mathbb{R}^{m_a}$ with $m_a \geq n_a$ for $a = 1, \ldots, s$, $m_a \geq n_a + 1$ for $a = s+1, \ldots, k$ and $m_{k+1} \geq 0$, and $(N - m_a)$–dimensional affine subspaces $\mathcal{H}_a$ parallel to $\mathbb{R}^{m_a \perp}$, $s+1 \leq a \leq k$, such that, up to an inversion in $\mathbb{R}^N$ with respect to a sphere of unit radius, $\mathcal{F}_a$ is a light-like family of affine subspaces with axis $\mathbb{R}^{m_a \perp}$ for $a = 1, \ldots, s$ and a space-like family of affine subspaces with axis $\mathcal{H}_a$ for $a = s+1, \ldots, k$.*

(b) *$k \leq N - m + 1$ and there exist an orthogonal decomposition $\mathbb{R}^{N+1} = \oplus_{a=1}^k \mathbb{R}^{m_a}$ with $m_a \geq n_a + 1$, space-like families of affine subspaces $\hat{\mathcal{F}}_a$ with axis $\mathcal{H}_{\hat{\mathcal{F}}_a} = \mathbb{R}^{m_a \perp}$, $1 \leq a \leq k$, and a stereographic projection $\mathcal{P}$: $\mathbb{S}^N(c) \setminus \{z\} \to \mathbb{R}^N$, where $\mathbb{R}^N$ is identified with the tangent space of $\mathbb{S}^N(c)$ at $-z$, such that each $S^{\lambda}_a \in \mathcal{F}_a$ is given by $S^{\lambda}_a = \mathcal{P}(\hat{S}^{\lambda}_a \cap \mathbb{S}^N(c))$ for some $\hat{S}^{\lambda}_a \in \hat{\mathcal{F}}_a$.*

(c) $k \leq N - m + 2$ *and for exactly one* $b \in \{1, \dots, k\}$ *there exist an orthogonal decomposition* $\mathbb{R}^N = \oplus_{a=1}^{k+1} \mathbb{R}^{m_a}$ *with* $m_b \geq n_b - 1$, $m_a \geq n_a + 1$ *for* $a = 1, \dots, k$ *with* $a \neq b$ *and* $m_{k+1} \geq 0$, $(N - m_a)$*–dimensional affine subspaces* $\mathcal{H}_a$ *parallel to* $\mathbb{R}^{m_a \perp}$, $1 \leq a \neq b \leq k$, *and an* m_b*–dimensional affine subspace* $\mathcal{H}_b \subset \bigcap_{a \neq b} \mathcal{H}_a$ *parallel to* $\mathbb{R}^{m_b}$ *(which reduces to a point* $\bar{q}$ *if* $m_b = 0$*), such that, up to an inversion in* $\mathbb{R}^N$ *with respect to a sphere of unit radius,* $\mathcal{F}_b$ *is a standard time-like family of spheres or affine subspaces with axis* $\mathcal{H}_{\mathcal{F}_b} = \mathcal{H}_b$ *(which is a family of hyperspheres centered at* $\bar{q}$ *if* $m_b = 0$*) and* $\mathcal{F}_a$ *is a space-like family of affine subspaces with axis* $\mathcal{H}_{\mathcal{F}_a} = \mathcal{H}_a$ *for* $a = 1, \dots, k$ *with* $a \neq b$.

Proof: Write $S_a^\lambda = \mathbb{E}^N \cap V_a^{\lambda \perp}$, where V_a^λ is a space-like subspace of $\mathbb{L}^{N+2}$ of dimension n_a and let $V_a = V_{\mathcal{F}_a}$ be the subspace of $\mathbb{L}^{N+2}$ spanned by all V_a^λ, $\lambda \in \Lambda$. Since each $\mathcal{F}_a$ has at least two elements, the dimension of V_a is at least $n_a + 1$. On the other hand, since $S_a^\lambda \in \mathcal{F}_a$ is orthogonal to every $S_b^\beta \in \mathcal{F}_b$ for all $a, b = 1, \dots, k$ with $a \neq b$, we have that $V_a^\lambda \subset V_b^{\beta \perp}$ for every $\lambda, \beta \in \Lambda$. Therefore $V_a \subset V_b^\perp$ for all $a, b = 1, \dots, k$ with $a \neq b$. Then one of the following holds:

(i) V_a is either degenerate or space-like for $a = 1, \dots, k$ and if V_a is space-like for every $a = 1, \dots, k$ then $V = \oplus_{a=1}^k V_a$ has dimension at most N.

(ii) V is a space-like subspace of dimension $N + 1$.

(iii) There exists (exactly one) $b \in \{1, \dots, k\}$ such that V_b is time-like.

Assume first that (i) holds. After reordering the families, there exist $s \in \{0, \dots, k\}$ and $\zeta \in \mathbb{V}^{N+1}$ such that V_a orthogonally decomposes as $V_a = U_a \oplus \operatorname{span}\{\zeta\}$ for $a = 1, \dots, s$ and V_a is space-like for $a = s + 1, \dots, k$. In the latter case set $U_a = V_a$. Then U_a is a space-like subspace for every $a = 1, \dots, k$, whose dimension m_a is at least n_a for $a = 1, \dots, s$ and at least $n_a + 1$ for $a = s + 1, \dots, k$. By taking a multiple of ζ we may assume either that $\zeta = w$ or that $\zeta \in \mathbb{E}^N$, that is, $\langle \zeta, w \rangle = 1$. In the first case, define the affine subspaces $\mathcal{H}_a$ by $\mathcal{H}_a = \mathbb{E}^N \cap U_a^\perp$, $1 \leq a \leq k$, let $\mathbb{R}^{m_a}$ be the subspace of $\mathbb{R}^N$ such that $\mathbb{R}^{m_a \perp}$ is parallel to $\mathcal{H}_a$, and let $\mathbb{R}^{m_{k+1}} = \bigcap_{a=1}^k \mathbb{R}^{m_a \perp}$. Then $\mathbb{R}^N = \oplus_{a=1}^{k+1} \mathbb{R}^{m_a}$ is an orthogonal decomposition as in the statement and $\mathcal{F}_a$, $1 \leq a \leq s$ (respectively, $s + 1 \leq a \leq k$), is a light-like (respectively, space-like) family of affine subspaces with axis $\mathbb{R}^{m_a \perp}$ (respectively, $\mathcal{H}_a$).

Now suppose $\zeta \in \mathbb{E}^N$. Let I be the inversion in $\mathbb{R}^N$ determined by $T \in \mathbb{O}_1(N+2)$ as in (1) with $v = \zeta + (1/2)w$, so that $w = -2T(\zeta)$. Arguing as before with U_a replaced by $T(U_a)$, $1 \leq a \leq k$, we obtain an orthogonal decomposition $\mathbb{R}^N = \oplus_{a=1}^{k+1} \mathbb{R}^{m_a}$ as in the statement and affine $(N - m_a)$–dimensional subspaces $\mathcal{H}_a$ parallel to $\mathbb{R}^{m_a \perp}$, $1 \leq a \leq k$, such that I takes $\mathcal{F}_a$, $1 \leq a \leq s$ (respectively, $s+1 \leq a \leq k$) into a light-like (respectively, space-like) family of affine subspaces with axis $\mathbb{R}^{m_a \perp}$ (respectively, $\mathcal{H}_a$).

In case (ii), we must have $n + k + 1 = \sum_{a=1}^k (n_a + 1) \leq \dim V = N + 1$, and hence $k \leq N - n$. Consider the model $\mathbb{E}^{N+1} \subset \mathbb{L}^{N+3}$ of $\mathbb{R}^{N+1}$, identify $\mathbb{L}^{N+2}$ with $e_0^\perp$, where e_0

is a unit space-like vector orthogonal to w, and let $\mathbb{R}^N$ be represented by $\mathbb{E}^N = \mathbb{E}^{N+1} \cap e_0^\perp$. For each $S_a^\lambda = \mathbb{E}^N \cap V_a^{\lambda\perp} \in \mathcal{F}_a$, let $\hat{S}_a^\lambda$ be the unique sphere in $\mathbb{R}^{N+1}$ that intersects $\mathbb{R}^N$ orthogonally along S_a^λ, that is, $\hat{S}_a^\lambda = \mathbb{E}^{N+1} \cap V_a^{\lambda\perp}$, where V_a^λ is now regarded as a subspace of $\mathbb{L}^{N+3}$. Then we have an orthogonal decomposition $\mathbb{L}^{N+3} = \oplus_{a=1}^k V_a \oplus \mathbb{L}^2$, where $\mathbb{L}^2$ is the time-like plane spanned by e_0 and the one-dimensional orthogonal complement $V^\perp$ of V in $\mathbb{L}^{N+2}$. Let $\zeta \in \mathbb{E}^{N+1} \cap \mathbb{L}^2$ and extend ζ to a pseudo-orthonormal basis $\{\zeta, \eta\}$ of $\mathbb{L}^2$. Define

$$v = \frac{1}{\langle \zeta, e_0 \rangle}\zeta + \frac{1}{2}\langle \zeta, e_0 \rangle w. \tag{2}$$

Then $\langle v, e_0 \rangle = 1 = \langle v, v \rangle$, and hence $q_0 = v - e_0$ belongs to $\mathbb{E}^{N+1}$ and satisfies $q_0 \in \mathbb{R}^N \cap S$, where $S = \mathbb{E}^{N+1} \cap \{v\}^\perp$. Therefore S and $\mathbb{R}^N$ are tangent at q_0. Consider the inversion I in $\mathbb{R}^{N+1}$ determined by $T \in \mathbb{O}_1(N+3)$ as in (1), with v given by (2). Then $w = -2\langle \zeta, e_0 \rangle^{-2} T(\zeta) \in T(V)^\perp$ and $\bar{q} = -(\langle \zeta, e_0 \rangle^2/2) T(\eta) \in \mathbb{E}^{N+1} \cap T(V)^\perp$, and hence I takes $\hat{\mathcal{F}}_i$ into a space-like family of affine subspaces with axis $\mathcal{H}_{\hat{\mathcal{F}}_a} = \bar{q} + \mathbb{R}^{m_a \perp} : \mathbb{E}^{N+1} \cap T(V_a)^\perp$ for $a = 1, \ldots, k$. Define $\mathbb{S}^N(c) = \mathbb{E}^{N+1} \cap \{T(e_0)\}^\perp$. Since $e_0 = \alpha\eta + (2\alpha)^{-1}\zeta$, with $\alpha = \langle e_0, \zeta \rangle$, it follows that $T(e_0) = \beta\bar{q} + (2\beta)^{-1} w$, with $\beta = -2\alpha\langle \zeta, e_0 \rangle^{-2}$, and thus $\mathbb{S}^N(c)$ has center at $\bar{q}$. Moreover, we have that $I(\mathbb{R}^N) = \mathbb{S}^N(c) \setminus \{z\}$, where z is the center of S, and the restriction of I itself to $\mathbb{S}^N(c) \setminus \{z\}$ can be regarded as a stereographic projection $\mathcal{P}$ of $\mathbb{S}^N(c) \setminus \{z\}$ onto $\mathbb{R}^N$. Finally, we have that $\mathcal{P}(I(\hat{S}_a^\lambda) \cap \mathbb{S}^N(c)) = \hat{S}_a^\lambda \cap \mathbb{R}^N = S_a^\lambda$ for every $\hat{S}_a^\lambda \in \hat{\mathcal{F}}_a$, and we may assume that $\bar{q}$ is the origin in $\mathbb{R}^{N+1}$.

Finally, if *(iii)* holds, we have that $n + k + 1 = \sum_{a=1}^k (n_a + 1) \leq \dim V \leq N + 2$, thus $k \leq N - n + 1$. Assume first that $w \in V_b$ and set $m_b = \dim V_b - 2 \geq n_b - 1$. Define the affine subspaces $\mathcal{H}_a$ by $\mathcal{H}_a = \mathbb{E}^N \cap V_a^\perp$ for $a = 1, \ldots, k$ with $a \neq b$, and set $\mathcal{H}_b = \mathbb{E}^N \cap V_b$. Then $\mathcal{H}_b$ is either an affine subspace of dimension m_b if $m_b > 0$ or a single point $\bar{q} \in \mathbb{R}^N$ otherwise, in which case n_b must be equal to one, and the affine subspace $\mathcal{H}_a$ has codimension $m_a \geq n_a + 1$ and contains $\mathcal{H}_b$ for every $a = 1, \ldots, k$ with $a \neq b$. Let $\mathbb{R}^{m_a}$ be the subspace of $\mathbb{R}^N$ such that $\mathbb{R}^{m_a \perp}$ is parallel to $\mathcal{H}_a$ and define $\mathbb{R}^{m_b}$ as the subspace parallel to the affine subspace $\mathcal{H}_b$ if $m_b > 0$, or the trivial subspace otherwise. Then we obtain an orthogonal decomposition $\mathbb{R}^N = \oplus_{a=1}^{k+1} \mathbb{R}^{m_a}$ as in the statement such that $\mathcal{F}_b$ is a standard time-like family of spheres or affine subspaces with axis $\mathcal{H}_{\mathcal{F}_b} = \mathcal{H}_b$ (which is a family of hyperspheres centered at $\bar{q}$ if $m_b = 0$) and $\mathcal{F}_a$ is a space-like family of affine subspaces with axis $\mathcal{H}_{\mathcal{F}_a} = \mathcal{H}_a$ for $a = 1, \ldots, k$ with $a \neq b$.

If $w \notin V_b$, choose $\zeta \in \mathbb{E}^N \cap V_b$ and let I be the inversion in $\mathbb{R}^N$ determined by $T \in \mathbb{O}_1(N+2)$ as in (1) with $v = \zeta + (1/2)w$. Then $w = -2T(\zeta) \in T(V_b)$ and we can argue as before with V_b, V and V_a replaced by $T(V_b)$, $T(V)$ and $T(V_a)$, respectively. We obtain affine subspaces $\mathcal{H}_a$, $1 \leq a \neq b \leq k$, of codimension $m_a \geq n_a + 1$ and either an affine subspace $\mathcal{H}_b$ of dimension $m_b = \dim V_b - 2 \geq n_b - 1$ if $m_b > 0$ or a point $\bar{q}$ if $m_b = 0$, and an orthogonal decomposition $\mathbb{R}^N = \oplus_{a=1}^{k+1} \mathbb{R}^{m_a}$ such that I takes $\mathcal{F}_b$ into a standard time-like family of spheres or affine subspaces with axis $\mathcal{H}_{\mathcal{F}_b} = \mathcal{H}_b$ and $\mathcal{F}_a$ into a space-like family of affine subspaces with axis $\mathcal{H}_{\mathcal{F}_a} = \mathcal{H}_a$ for $a = 1, \ldots, k$ with $a \neq b$. ∎

4 The main result

We are now in a position to state and prove our main result. For $M = \Pi_{i=0}^k M_i$, we denote by $T_b\colon M \to M_b \times \Pi_{i\neq b} M_i$ the map $T_b(p_0, \ldots, p_k) = (p_b, (p_0, \ldots, \hat{p}_b, \ldots, p_k))$, where the hat indicates a missing element.

Theorem 7 *Let $f\colon M^n := M_0^{n_0} \times_\rho \Pi_{a=1}^k M_a^{n_a} \to \mathbb{R}^N$, $n \geq 3$, be a conformal immersion of a warped product whose second fundamental form is adapted to the product net of M. Then one of the following holds:*

(i) *There exist a warped product representation $\Phi\colon N_0 \times_\sigma \Pi_{a=1}^k N_a \to \mathbb{Q}_c^N$, $c \geq 0$, a conformal immersion $f_0\colon M_0 \to N_0$ with conformal factor φ_0 and homothetical immersions $f_a\colon M_a \to N_a$ with homothety factors μ_a, $1 \leq a \leq k$, such that $\varphi_0 \rho_a = \mu_a(\sigma_a \circ f_0)$ and $f = \mathcal{P} \circ \Phi \circ (f_0 \times \cdots \times f_k)$, where $\mathcal{P}$ is either a stereographic projection if $c > 0$ or an inversion in $\mathbb{R}^N$ otherwise.*

$$\begin{array}{ccc} M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k & \xrightarrow{\quad f \quad} & \mathbb{R}^N \\ \downarrow f_0 \qquad \downarrow f_1 \qquad \downarrow f_k & & \uparrow \mathcal{P} \\ N_0 \times_{\sigma_1} N_1 \times \cdots \times_{\sigma_k} N_k & \xrightarrow{\quad \Phi \quad} & \mathbb{Q}_c^N \end{array}$$

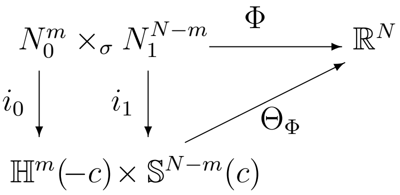
(ii) *$k \leq N - n + n_0 + 2$ and for exactly one $b \in \{1, \ldots, k\}$ there exist a warped product representation $\Phi\colon N_0 \times_\sigma \Pi_{a=1, a\neq b}^k N_a \to \mathbb{S}^{N-m}(c)$, an isometric immersion $f_b\colon M_b \to \mathbb{H}^m(-c)$, homothetical immersions $f_a\colon M_a \to N_a$ with homothety factors μ_a, $1 \leq a \neq b \leq k$, a conformal immersion $f_0\colon M_0 \to N_0$ with conformal factor ρ_b^{-1} and an inversion I in $\mathbb{R}^N$ such that $\rho_a = \mu_a \rho_b(\sigma_a \circ f_0)$ and*

$$f = I \circ \Theta \circ (f_b \times (\Phi \circ (f_0 \times \cdots \times \hat{f}_b \times \cdots \times f_k))) \circ T_b.$$

$$\begin{array}{ccc} M_b \times (M_0 \times_{\rho/\rho_b} \Pi_{a=1,a\neq b}^k M_a) & \xrightarrow{\quad f \quad} & \mathbb{R}^N \\ & \downarrow f_0 \times \cdots \times \hat{f}_b \times \cdots \times f_k & \\ \downarrow f_b & N_0 \times_\sigma \Pi_{a=1,a\neq b}^k N_a & \downarrow I \\ & \downarrow \Phi & \\ \mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) & \xrightarrow{\quad \Theta \quad} & \mathbb{R}^N \end{array}$$

In part *(ii)*, the map $\Theta\colon \mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) \to \mathbb{R}^N$ denotes the conformal diffeomorphism onto the complement of an $(m-1)$-dimensional affine subspace that arises by means of a warped product representation $\Phi\colon N_0^m \times_\sigma N_1^{N-m} \to \mathbb{R}^N$ as follows.

Lemma 8 *Let Φ: $N_0^m \times_\sigma N_1^{N-m} \to \mathbb{R}^N \setminus \mathcal{H}^{m-1}$ be a warped product representation. Then there exist conformal diffeomorphisms Θ_Φ: $\mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) \to \mathbb{R}^N \setminus \mathcal{H}^{m-1}$ and i_0: $N_0^m \to \mathbb{H}^m(-c)$ with conformal factors $\sigma \circ \pi_0$ and σ, respectively, and an isometry i_1: $N_1^{N-m} \to \mathbb{S}^{N-m}(c)$ such that $\Phi = \Theta_\Phi \circ (i_0 \times i_1)$.*



Proof: We assume for simplicity that Φ: $N_0^m \times_\sigma N_1^{N-m} \to \mathbb{R}^N$ is the warped product representation determined by $(\bar{p}, N_1)$, where $\bar{p}_i = c^{-1/2}\delta_{mi}$, $c > 0$, $N_1^{N-m} = \mathbb{S}^{N-m}(c)$ is the sphere through $\bar{p}$ whose tangent space at $\bar{p}$ is $\mathbb{R}^{m\perp}$ and whose mean curvature vector $-a$ at $\bar{p}$ is given by $a_i = c^{1/2}\delta_{mi}$, and $N_0^m = \mathbb{R}_+^m := \{X = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_m > 0\}$. Then σ: $\mathbb{R}_+^m \to \mathbb{R}$ is given by $\sigma(x) = c^{1/2}x_m$, and Φ: $\mathbb{R}_+^m \times_\sigma \mathbb{S}^{N-m}(c) \to \mathbb{R}^N \setminus \mathbb{R}^{m-1}$, where $\mathbb{R}^{m-1} = \{X = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_m = 0\}$, is the isometry

$$(X, Y) \mapsto (x_1, \ldots, x_{m-1}, c^{1/2}x_m Y), \quad \text{for } X \in \mathbb{R}_+^m \text{ and } Y \in \mathbb{S}^{N-m}(c). \tag{3}$$

Endowing $\mathbb{R}_+^m$ with the metric $\sigma^{-2}\langle\ ,\ \rangle$, where $\langle\ ,\ \rangle$ denotes the Euclidean metric, we obtain the half-space model of $\mathbb{H}^m(-c)$, and the metric induced by (3) is conformal to the product metric of $\mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c)$ with conformal factor $\sigma \circ \pi_0$. ∎

For the proof of Theorem 7 we also need the following fact on extrinsic spheres of Euclidean submanifolds.

Proposition 9 *Let f: $M^n \to \mathbb{R}^N$ and g: $L^k \to M^n$ be isometric immersions. Then the following assertions are equivalent:*

i) g is an extrinsic sphere whose mean curvature vector has length $1/r$, $r > 0$ (respectively, g is totally geodesic) and

$$\alpha_f(g_*X, Z) = 0 \ \textit{for all} \ X \in \Gamma(TL), \ \ Z \in \Gamma(T_g^\perp L). \tag{4}$$

ii) $f(g(L))$ is contained in a sphere of radius r (respectively, affine subspace) and dimension $(N - n + k)$ in $\mathbb{R}^N$ whose normal space along $f(g(L))$ is $f_*T_g^\perp L$.

Proof: Assume (i) and denote by H_g the mean curvature vector of g. Then

$$\alpha_g(X, Y) = \langle X, Y\rangle H_g \tag{5}$$

and

$$\langle \nabla_X H_g, Z\rangle = 0 \ \text{ for all } \ X \in \Gamma(TL), \ \ Z \in \Gamma(T_g^\perp L), \tag{6}$$

where ∇ denotes the Levi-Civita connection of M^n. In particular,

$$\nabla_X H_g = -r^{-2} g_* X \quad \text{for all} \quad X \in \Gamma(TL). \tag{7}$$

Set $h = f \circ g + r^2 f_* H_g$. Using (4) and (7), we obtain

$$h_* X = f_* g_* X + r^2(-r^{-2} f_* g_* X + \alpha_f(g_* X, H_g)) = 0$$

for all $X \in \Gamma(TL)$, whence $h = P_0 \in \mathbb{R}^N$ is a constant map. It follows that $f(g(L))$ is contained in a hypersphere of radius r centered at P_0. Moreover, let W be the subbundle of $T_g^\perp L$ orthogonal to H_g. By (5), for any $X, Y \in \Gamma(TL)$ and $Z \in \Gamma(W)$ we have

$$\langle \nabla_X Z, g_* Y \rangle = -\langle \nabla_X g_* Y, Z \rangle = \langle \alpha_g(X, Y), Z \rangle = 0,$$

whereas (6) implies that $\langle \nabla_X Z, H_g \rangle = 0$, and thus $\nabla_X Z \in \Gamma(W)$. Using (4) we obtain

$$\tilde{\nabla}_X f_* Z = f_* \nabla_X Z + \alpha_f(g_* X, Z) = f_* \nabla_X Z \in f_* W.$$

Hence $f_* W$ is a parallel subbundle in $\mathbb{R}^N$, and *ii*) follows. The proof for the case when g is totally geodesic is similar and easier.

Conversely, assume that $f(g(L))$ is contained in a sphere in $\mathbb{R}^N$ of dimension $(N-n+k)$ with center at P_0 and radius r whose normal space along $f(g(L))$ is $f_* T_g^\perp L$. Then, there exists a unit vector field $\eta \in T_g^\perp L$ such that $f \circ g + r\eta = P_0$. Moreover, if W is the subbundle of $T_g^\perp L$ orthogonal to η, then $f_* W$ is parallel in $\mathbb{R}^N$ along $f(g(L))$. Hence

$$f_* g_* X + r \tilde{\nabla}_X \eta = 0 \tag{8}$$

and

$$(\tilde{\nabla}_X f_* Z)_{(f_* W)^\perp} = 0 \quad \text{for all} \quad X \in \Gamma(TL), \quad Z \in \Gamma(W). \tag{9}$$

Taking the component in $T_f^\perp M$ of (8) and (9) yields (4). On the other hand, taking the inner product of (8) and (9) with $f_* g_* Y$, $Y \in TL$, implies that

$$\alpha_g(X, Y) = (1/r)\langle X, Y \rangle \eta \quad \text{for all} \quad X, Y \in \Gamma(TL),$$

and thus g is umbilical with mean curvature vector $H_g = (1/r)\eta$. Finally, parallelism of H_g in the normal connection of g follows by taking the η–component of (9). ∎

Proof of Theorem 7 : If $k = 1$, then the warped product metric

$$\langle\ ,\ \rangle = \pi_0^* \langle\ ,\ \rangle_0 + (\rho \circ \pi_0)^2 \pi_1^* \langle\ ,\ \rangle_1$$

of M^n is conformal to the Riemannian product metric $\langle\ ,\ \rangle^\sim = \pi_0^* \rho^{-2} \langle\ ,\ \rangle_0 + \pi_1^* \langle\ ,\ \rangle_1$ with conformal factor $\rho \circ \pi_0$, hence the conclusion follows by applying Theorem 5 in [**To**$_1$] to f: $(M^n, \langle\ ,\ \rangle^\sim) \to \mathbb{R}^N$.

Now assume that $k \geq 2$. Let $\mathcal{E} = (E_i)_{i=0,\dots,k}$ be the product net of M^n and let $\langle\ ,\ \rangle^*$ be the metric induced by f. Since $\langle\ ,\ \rangle^*$ is conformal to the warped product metric $\langle\ ,\ \rangle$ of M^n, the net $\mathcal{E}$ is a CWP-net with respect to $\langle\ ,\ \rangle^*$ by Proposition 3. Since $k \geq 2$, the rank of $E_i^\perp$ is at least 2 for $i = 0, \dots, k$. By Lemma 12 in [**To**$_1$], $E_a^\perp$ is spherical with respect to $\langle\ ,\ \rangle^*$ for $a = 1, \dots, k$, whence the same holds for E_a by Proposition 2. It follows from Proposition 9 that for each $a = 1, \dots, k$ and for each $p \in M^n$ there exists a sphere or affine subspace $S_a(p)$ (resp., $\tilde{S}_a(p)$) of dimension $(N-n_a)$ (resp., $(N-n+n_a)$) that contains the image $f(L_a^\perp(p))$ (resp., $f(L_a(p))$) of the leaf of $E_a^\perp$ (resp., E_a) through p, and whose normal space along $f(L_a^\perp(p))$ (resp., $f(L_a(p))$) is f_*E_a (resp., $f_*E_a^\perp$). In particular, the families $\mathcal{F}_a = (S_a(p))_{p\in M}$, $a = 1, \dots, k$, satisfy the assumptions of Lemma 6 with $m = n - n_0$.

Let us assume first that the statement in either part (a) or (b) of Lemma 6 holds. Let $\mathcal{P}$ denote either an inversion in $\mathbb{R}^N$ in case (a) or a stereographic projection of $\mathbb{S}^N(c) \setminus \{z\} \subset \mathbb{R}^{N+1}$ onto $\mathbb{R}^N$ in case (b). Set $g = \mathcal{P}^{-1} \circ f$. In case (a) (resp., case (b)), we have that $\mathcal{P}^{-1}(S_a(p))$ is an affine subspace in $\mathbb{R}^N$ (resp., a great sphere in $\mathbb{S}^N(c)$) that contains $g(L_a^\perp(p))$ and whose normal space along $g(L_a^\perp(p))$ is g_*E_a, $1 \leq a \leq k$. It follows from Proposition 9 that $L_a^\perp(p)$ is totally geodesic with respect to the metric $\langle\ ,\ \rangle^\sim$ induced by g, and hence $E_a^\perp$ is a totally geodesic subbundle with respect to $\langle\ ,\ \rangle^\sim$. On the other hand, since $\mathcal{P}^{-1}(\tilde{S}_a(p))$ is a sphere or an affine subspace (resp., a sphere) that contains $g(L_a(p))$ and whose normal space along $g(L_a(p))$ is $g_*E_a^\perp$, we also have from Proposition 9 that $L_a(p)$ is spherical with respect to $\langle\ ,\ \rangle^\sim$, and thus E_a is spherical with respect to $\langle\ ,\ \rangle^\sim$. Therefore $\langle\ ,\ \rangle^\sim$ is a warped product metric $\langle\ ,\ \rangle^\sim = \pi_0^*\langle\ ,\ \rangle_0^\sim + \sum_{a=1}^k (\tilde{\rho}_a \circ \pi_0)^2 \pi_a^*\langle\ ,\ \rangle_a^\sim$ by Proposition 1.

We now apply Nölker's theorem to $g\colon (M, \langle\ ,\ \rangle^\sim) \to \mathbb{Q}_c^N$, where $c = 0$ in case (a) and $c > 0$ in case (b). Fixed $\bar{p} \in M$, set $f_i = g \circ \tau_i^{\bar{p}}\colon M_i \to \mathbb{Q}_c^N$ for $i = 0, \dots, k$ and let N_a be the spherical hull of f_a for $a = 1, \dots, k$. Then $f_0\colon (M_0, \langle\ ,\ \rangle_0^\sim) \to \mathbb{Q}_c^N$ is an isometric immersion, $f_a\colon (M_a, \langle\ ,\ \rangle_a^\sim) \to \mathbb{Q}_c^N$ is a homothetical immersion with homothety factor $\theta_a = \tilde{\rho}_a(\bar{p}_0)$, $1 \leq a \leq k$, and there is a warped product representation $\Phi\colon N_0 \times_\sigma \Pi_{a=1}^k N_a \to \mathbb{Q}_c^N$ determined by $(f(\bar{p}); N_1, \dots, N_k)$ such that $f_0(M_0) \subset N_0$, $\tilde{\rho}_a = \theta_a(\sigma_a \circ f_0)$ and $g = \Phi \circ (f_0 \times \cdots \times f_k)$.

Finally, let $\varphi \in C^\infty(M)$ be such that $\langle\ ,\ \rangle^\sim = \varphi^2\langle\ ,\ \rangle$. It follows from Lemma 4 that there exist $\varphi_0 \in C^\infty(M_0)$ and $\lambda_a > 0$, $1 \leq a \leq k$, such that $\varphi = \varphi_0 \circ \pi_0$, $\lambda_a\tilde{\rho}_a = \varphi_0\rho_a$, $\langle\ ,\ \rangle_0^\sim = \varphi_0^2\langle\ ,\ \rangle_0$ and $\langle\ ,\ \rangle_a^\sim = \lambda_a^2\langle\ ,\ \rangle_a$. Hence $f_0\colon (M_0, \langle\ ,\ \rangle_0) \to N_0$ is conformal with conformal factor φ_0, $f_a\colon (M_a, \langle\ ,\ \rangle_a) \to N_a$ is homothetical with homothety factor $\mu_a = \theta_a\lambda_a$, $1 \leq a \leq k$, and $\varphi_0\rho_a = \mu_a(\sigma_a \circ f_0)$. This gives case (i) of the statement.

Now assume that the alternative (c) in Lemma 6 holds. Then $k \leq N - n + n_0 + 2$ and, arguing as in the preceding case, we obtain that $E_b^\perp$ is spherical and that $E_a^\perp$ is totally geodesic for $a, b = 1, \dots, k$ with $a \neq b$ with respect to the metric $\langle\ ,\ \rangle^\sim$ induced by $g = I \circ f$. Moreover, the image by I of each sphere or affine subspace $\tilde{S}_b(p)$ must be an affine subspace containing the axis $\mathcal{H}_b$ of $\mathcal{G}_b = (I(S_b(p)))_{p\in M}$. Since $I(\tilde{S}_b(p))$ contains $g(L_b(p))$ and has $g_*E_b^\perp$ as normal space along $g(L_b(p))$ for each $p \in M^n$, we also have from Proposition 9 that each leaf $L_b(p)$ is totally geodesic with respect to $\langle\ ,\ \rangle^\sim$. Thus, E_b is a totally geodesic subbundle with respect to $\langle\ ,\ \rangle^\sim$. By Proposition 1, $\langle\ ,\ \rangle^\sim$ is a warped–product metric $\langle\ ,\ \rangle^\sim = \pi_b^*\langle\ ,\ \rangle_b^\sim + (\rho\circ\pi_b)^2\pi_{\perp_b}^*\langle\ ,\ \rangle_{\perp_b}^\sim$ on M regarded as a product $M = M_b \times M_b^\perp$, where $M_b^\perp := M_0 \times \cdots \times \hat{M}_b \times \cdots \times M_k$ and $\pi_{\perp_b}\colon M \to M_b^\perp$ is the projection onto $M_b^\perp$. We can assume $\langle\ ,\ \rangle^\sim$ to be normalized with respect to some point $\bar{p} \in M$. Denote $\tilde{g}_b = g \circ \tau_b^{\bar{p}}\colon M_b \to \mathbb{R}^N$ and $\bar{g}_b = g \circ \tau_{\perp_b}^{\bar{p}}\colon M_b^\perp \to \mathbb{R}^N$, where $\tau_{\perp_b}^{\bar{p}}\colon M_b^\perp \to M$ is the canonical inclusion of $M_b^\perp$ into M as the leaf of $E_b^\perp$ through $\bar{p}$. Then $\tilde{g}_b\colon (M_b, \langle\ ,\ \rangle_b^\sim) \to \mathbb{R}^N$ and $\bar{g}_b\colon (M_b^\perp, \langle\ ,\ \rangle_{\perp_b}^\sim) \to \mathbb{R}^N$ are isometric immersions. Moreover, denoting by $\bar{N}_b$ the spherical hull of $\bar{g}_b$, it follows from Theorem 5 that $(g(\bar{p}), \bar{N}_b)$ determines a warped product representation $\Phi\colon \tilde{N}_b \times_\sigma \bar{N}_b \to \mathbb{R}^N$ such that $\tilde{g}_b(M_b) \subset \tilde{N}_b$, $\rho = \sigma \circ \tilde{g}_b$ and $g = \Phi \circ (\tilde{g}_b \times \bar{g}_b) \circ T_b$.

Since $\langle\ ,\ \rangle^\sim$ is conformal to the product metric $\langle\ ,\ \rangle = \pi_b^*\langle\ ,\ \rangle_b + \pi_{\perp_b}^*\langle\ ,\ \rangle_{\perp_b}$, where $\langle\ ,\ \rangle_{\perp_b} = \pi_0^*(\rho_b^{-2}\langle\ ,\ \rangle_0) + \sum_{a=1,a\neq b}^k (\rho_a\circ\pi_0/\rho_b\circ\pi_0)^2\pi_a^*\langle\ ,\ \rangle_a$, it follows from Lemma 4 that there exists $\lambda > 0$ such that $\langle\ ,\ \rangle_b^\sim = \lambda^2(\sigma\circ\tilde{g}_b)^2\langle\ ,\ \rangle_b$ and $\langle\ ,\ \rangle_{\perp_b}^\sim = \lambda^2\langle\ ,\ \rangle_{\perp_b}$. By renormalizing the metric $\langle\ ,\ \rangle^\sim$ we may assume that $\lambda = 1$, and hence $\bar{g}_b$: $(M_b^\perp, \langle\ ,\ \rangle_{\perp_b}) \to \bar{N}_b$ is an isometric immersion and $\tilde{g}_b\colon (M_b, \langle\ ,\ \rangle_b) \to \tilde{N}_b$ is a conformal immersion with conformal factor $(\sigma\circ\tilde{g}_b)$. By Lemma 8, there exist conformal diffeomorphisms $\Theta\colon \mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) \to \mathbb{R}^N \setminus \mathcal{H}^{m-1}$ and $i_0\colon \tilde{N}_j \to \mathbb{H}^m(-c)$ with conformal factors $\sigma\circ\pi_0$ and σ, respectively, and an isometry $i_1\colon \bar{N}_b \to \mathbb{S}^{N-m}(c)$ such that $\Phi = \Theta \circ (i_0 \times i_1)$. It follows that $f_b := i_0 \circ \tilde{g}_b\colon (M_b, \langle\ ,\ \rangle_b) \to \mathbb{H}^m(-c)$ and $\hat{g}_b := i_1 \circ \bar{g}_b\colon (M_b^\perp, \langle\ ,\ \rangle_{\perp_b}) \to \mathbb{S}^{N-m}(c)$ are isometric immersions and

$$g = \Theta \circ (f_b \times \hat{g}_b) \circ T_b. \tag{10}$$

Finally, we apply Nölker's theorem to $\hat{g}_b$: $(M_b^\perp, \langle\ ,\ \rangle_{\perp_b}) \to \mathbb{S}^{N-m}(c)$. We obtain that $f_0 = \hat{g}_b \circ \tau_0^{\bar{p}}\colon (M_0, \rho_b^{-2}\langle\ ,\ \rangle_0) \to \mathbb{S}^{N-m}(c)$ is an isometric immersion and that $f_a = \hat{g}_b \circ \tau_a^{\bar{p}}\colon (M_a, \langle\ ,\ \rangle_a) \to \mathbb{S}^{N-m}(c)$ is a homothetical immersion with homothety factor $\mu_a = \rho_a(\bar{p}_0)\rho_b^{-1}(\bar{p}_0)$ for $a = 1, \ldots, k$ with $a \neq b$. Moreover, denoting by N_a the spherical hull of f_a, then $(\hat{g}_b(\bar{p}), N_1, \ldots, \hat{N}_b, \ldots, N_k)$ determines a warped product representation $\Phi\colon N_0 \times_\sigma \Pi_{a=1,a\neq b}^k N_a \to \mathbb{S}^{N-m}(c)$ such that $\rho_a\rho_b^{-1} = \mu_a(\sigma_a \circ f_0)$ and

$$\hat{g}_b = \Phi \circ (f_0 \times \cdots \times \hat{f}_b \times \cdots \times f_k). \tag{11}$$

We conclude from (10) and (11) that the statement in part (ii) holds. ∎

Remark 10 By the remarks on warped products of constant sectional curvature in Section 2.2, in part (i) of Theorem 7 we must have $k \leq \dim N_0 + 1 \leq N - n + n_0 + 1$ if either $c > 0$ or if $c = 0$ and at most one of the warping functions $\sigma_1, \ldots, \sigma_k$ is constant. Notice that this last condition must hold if one assumes that the warping functions $\rho_1, \ldots, \rho_k$ of M^n are such that ρ_a is not a constant multiple of ρ_b for $a \neq b$, which can always be accomplished by considering as a single factor the product of all factors correspondent to warping functions that are constant multiples of a fixed one (the metric of each factor being rescaled by the corresponding constant).

Setting $n = N$ in Theorem 7 yields the classification referred to in the introduction of conformal local diffeomorphisms of a warped product onto an open subset of $\mathbb{R}^n$, $n \geq 3$. It suffices to replace in the statement the expressions conformal, homothetic and isometric immersions by local conformal diffeomorphisms, local homotheties and local isometries, respectively. Clearly, no assumption besides $n \geq 3$ is needed in this case.

As a consequence, we obtain the following classification of locally conformally flat warped products.

Corollary 11 *Let* $(M^n, \langle\ ,\ \rangle) := (M_0^{n_0}, \langle\ ,\ \rangle_0) \times_\rho \Pi_{a=1}^k (M_a^{n_a}, \langle\ ,\ \rangle_a)$ *be a warped product constructed in one of the two following ways:*

(i) *Start with a warped product of constant sectional curvature* $c \geq 0$

$$(M^n, \langle\ ,\ \rangle^\sim) := (M_0^{n_0}, \langle\ ,\ \rangle_0^\sim) \times_{\tilde\rho} \Pi_{a=1}^k (M_a^{n_a}, \langle\ ,\ \rangle_a^\sim),$$

choose $\varphi_0 > 0 \in C^\infty(M_0)$ *and* $\mu_a > 0$ *for* $1 \leq a \leq k$, *and define* $\langle\ ,\ \rangle_0 = \varphi_0^{-2}\langle\ ,\ \rangle_0^\sim$, $\langle\ ,\ \rangle_a = \mu_a^{-2}\langle\ ,\ \rangle_a^\sim$, *and* $\rho_a = \mu_a \varphi_0^{-1}\tilde\rho_a$.

(ii) *Start with a warped product of constant sectional curvature* $c > 0$

$$(\tilde M^{n-n_b}, \langle\ ,\ \rangle^\sim) := (M_0^{n_0}, \langle\ ,\ \rangle_0^\sim) \times_{\tilde\rho} \Pi_{a=1, a\neq b}^k (M_a^{n_a}, \langle\ ,\ \rangle_a^\sim),$$

a Riemannian manifold $(M_b^{n_b}, \langle\ ,\ \rangle_b)$ *that has constant sectional curvature* $-c$ *if* $n_b \geq 2$, *choose* $\varphi_0 > 0 \in C^\infty(M_0)$ *and* $\mu_a > 0$ *for* $1 \leq a \neq b \leq k$, *and define* $\langle\ ,\ \rangle_0 = \varphi_0^{-2}\langle\ ,\ \rangle_0^\sim$, $\langle\ ,\ \rangle_a = \mu_a^{-2}\langle\ ,\ \rangle_a^\sim$, $\rho_b = \varphi_0^{-1}$ *and* $\rho_a = \mu_a\varphi_0^{-1}\tilde\rho_a$, $1 \leq a \neq b \leq k$.

Then M^n *is locally conformally flat. Conversely, if* M^n *is a locally conformally flat warped product of dimension* $n \geq 3$ *then its universal covering is given as in* (*i*) *or* (*ii*).

Proof: In (*i*), we have $\langle\ ,\ \rangle = \varphi_0^{-2}\langle\ ,\ \rangle^\sim$, hence $(M^n, \langle\ ,\ \rangle)$ is locally conformally flat because $\langle\ ,\ \rangle^\sim$ has constant sectional curvature. In (*ii*), we have $\langle\ ,\ \rangle = \varphi_0^{-2}\langle\ ,\ \rangle^*$, where $\langle\ ,\ \rangle^*$ is the product metric $\langle\ ,\ \rangle^* = \pi_b^*\langle\ ,\ \rangle_b + \pi_{\perp_b}^*\langle\ ,\ \rangle^\sim$ on M^n regarded as a product $M^n = M^{n_b} \times \tilde M^{n-n_b}$, which is well-known to be locally conformally flat (cf. [**La**]).

The converse follows from Theorem 7 (for $n = N$) and a well-known theorem due to Kuiper [**Ku**], according to which any simply-connected locally conformally flat n-dimensional manifold admits a global conformal immersion into $\mathbb{R}^n$. ∎

As a consequence of Corollary 11 and the remarks in Section 2.2, it follows that the base M^{n_0} of a locally conformally flat warped product $M_0^{n_0} \times_\rho \Pi_{a=1}^k M_a^{n_a}$ must also be locally conformally flat and that all fibers must have constant sectional curvature, at most one of which can be negative. Moreover, the number of factors can not exceed

$n_0 + 2$ if one assumes that ρ_a is a not a constant multiple of ρ_b for $a \neq b$ (cf. [**BGV**]). We also point out that the *complete* simply-connected locally conformally flat warped products are given by one of the constructions in Corollary 11 by choosing a positive smooth function φ_0 on an open simply-connected subset $(M_0^{n_0}, \langle\ ,\ \rangle_0^{\sim})$ of $\mathbb{Q}_c^m$, with $c \geq 0$ in case (i) and $c > 0$ in case (ii), such that $(M_0^{n_0}, \varphi_0^{-2}\langle\ ,\ \rangle_0^{\sim})$ is complete, and by taking all the remaining factors as complete simply connected space forms.

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