Conformal de Rham decomposition of Riemannian manifolds.

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Abstract: We prove conformal versions of the local decomposition theorems of de Rham and Hiepko of a Riemannian manifold as a Riemannian or a warped product of Riemannian manifolds. Namely, we give necessary and sufficient conditions for a Riemannian manifold to be locally conformal to either a Riemannian or a warped product. We also obtain other related de Rham-type decomposition theorems. As an application, we study Riemannian manifolds that admit a Codazzi tensor with two distinct eigenvalues everywhere.

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§1. Introduction.

An important initial step in understanding a complicated mathematical object is to decompose it into simpler "irreducible" components. In differential geometry, a fundamental result in this direction is the decomposition theorem of de Rham, which gives necessary and sufficient conditions for a Riemannian manifold to split, both locally and globally, into a Riemannian product of Riemannian manifolds [4]. Several significant generalizations of de Rham's theorem have subsequently been obtained. For instance, a similar characterization was given by Hiepko [8] of Riemannian manifolds that split as a warped product of Riemannian manifolds. More recently, a suitable setting for treating such decomposition results was introduced in [11] by defining the notions of *netted manifolds* and of *net morphisms* between them.

A *net* on a connected C^∞-manifold M is a splitting $TM = \oplus_{i=1}^k E_i$ of the tangent bundle of M by a family of integrable subbundles. If M is a Riemannian manifold and the subbundles are mutually orthogonal then the net is said to be an *orthogonal net*.

The canonical net on a product manifold $M = \Pi_{i=1}^k M_i$ is called the *product net*. A C^∞-map ψ: $M \to N$ between two *netted manifolds* $(M, \mathcal{E})$, $(N, \mathcal{F})$, that is, C^∞-manifolds M, N equipped with nets $\mathcal{E} = (E_i)_{i=1,\dots,k}$ and $\mathcal{F} = (F_i)_{i=1,\dots,k}$, respectively, is called a *net morphism* if $\psi_* E_i(p) \subset F_i(\psi(p))$ for all $p \in M$, $1 \le i \le k$, or equivalently, if for any $p \in M$ the restriction $\psi|_{L_i^{\mathcal{E}}(p)}$ to the leaf of E_i through p is a C^∞-map into the leaf $L_i^{\mathcal{F}}(\psi(p))$ of F_i through $\psi(p)$. The net morphism ψ is said to be a *net isomorphism* if, in addition, it is a diffeomorphism and ψ^{-1} is also a net morphism. A net $\mathcal{E}$ on M is said to be *locally decomposable* if for every point $p \in M$ there exist a neighborhood U of p in M and a net isomorphism ψ from $(U, \mathcal{E}|_U)$ onto a product manifold $\Pi_{i=1}^k M_i$. The map ψ^{-1}: $\Pi_{i=1}^k M_i \to U$ is called a *product representation* of $(U, \mathcal{E}|_U)$.

A general problem in this context is to determine necessary and sufficient conditions for an orthogonal net to admit locally a product representation whose induced metric is of some nice particular type, for instance a Riemannian product or a warped product of Riemannian metrics on the factors as in de Rham and Hiepko theorems, respectively.

In this article we consider this problem from a conformal point of view. Our main results are conformal versions of the local decomposition theorems of de Rham and Hiepko. Namely, we find necessary and sufficient conditions for an orthogonal net to admit locally a product representation whose induced metric is *conformal* to either a Riemannian product or a warped product of Riemannian metrics on the factors. We also solve the above problem for other natural types of metrics and orthogonal nets.

As an application, we consider the problem of determining the restrictions that are imposed on a Riemannian manifold by the existence of a Codazzi tensor with exactly two distinct eigenvalues (cf. [5] and [6]). We start from the observation that the orthogonal net determined by the eigenbundles of such a Codazzi tensor admits locally a product representation whose induced metric is a twisted product of Riemannian metrics on the factors, and study further properties of the twisted product metric under additional assumptions on the Codazzi tensor. In particular, we determine all Riemannian manifolds that carry Codazzi tensors with exactly two distinct eigenvalues, both of which are constant along the corresponding eigenbundles, as well as the tensors themselves.

In a forthcoming paper [13] we study the problem posed by Burstall [2] of developing a theory of isothermic Euclidean submanifolds of higher dimensions and codimensions. Recall that a surface in Euclidean three-space is called *isothermic* if, away from umbilic points, its curvature lines form an isothermic net, that is, there exist locally conformal coordinates that diagonalize the second fundamental form. Studying conformal decomposition theorems of Riemannian manifolds was in part motivated by the problem of looking for a suitable extension of the intrinsic notion of an isothermic net. In fact, our conformal version of the local de Rham theorem can be seen as a generalization of a classical characterization of isothermic nets of curves in terms of their geodesic curvatures [3].

To conclude this introduction, we point out that the results of this paper, as well as their proofs, remain valid for pseudo–Riemannian manifolds.

§2. Twisted and warped products.

In this section we recall from [9] some basic definitions and results on warped and twisted products. If $M = \Pi_{i=0}^{k} M_i$ is the product of C^∞-manifolds $M_0, \ldots, M_k$, then $\langle\ ,\ \rangle$ is called a *twisted product metric* on M if there exist Riemannian metrics $\langle\ ,\ \rangle_i$ on M_i, $0 \le i \le k$, and a C^∞ *twist-function* $\rho = (\rho_0, \ldots, \rho_k)$: $M \to \mathbb{R}_+^{k+1}$ such that

$$\langle\ ,\ \rangle = \sum_{i=0}^{k} \rho_i^2 \pi_i^* \langle\ ,\ \rangle_i,$$

where π_i: $M \to M_i$ denotes the canonical projection. Then $(M, \langle\ ,\ \rangle)$ is said to be a *twisted product* and is denoted by ${}^\rho\Pi_{i=0}^{k}(M_i, \langle\ ,\ \rangle_i)$. When $\rho_1, \ldots, \rho_k$ are independent of $M_1, \ldots, M_k$, that is, there exist $\tilde{\rho}_i \in C^\infty(M_0)$ such that $\rho_i = \tilde{\rho}_i \circ \pi_0$ for $i = 1, \ldots, k$ and, in addition, ρ_0 is identically one, then $\langle\ ,\ \rangle$ is called a *warped product metric* and $(M, \langle\ ,\ \rangle) := (M_0, \langle\ ,\ \rangle_0) \times_\rho \Pi_{i=1}^{k}(M_i, \langle\ ,\ \rangle_i)$ a *warped product* with *warping function* $\rho = (\rho_1, \ldots, \rho_k)$. If ρ_i is identically 1 for $i = 0, \ldots, k$, the metric $\langle\ ,\ \rangle$ is the usual *Riemannian product metric*, in which case $(M, \langle\ ,\ \rangle)$ is called a *Riemannian product*.

The next result from [9] relates the Levi-Civita connections of a twisted product metric and the corresponding Riemannian product metric on a product manifold.

Proposition 1 [9] *Let* $(M, \langle\ ,\ \rangle) =^\rho \Pi_{i=0}^{k}(M_i, \langle\ ,\ \rangle_i)$ *be a twisted product with twist function* $\rho = (\rho_0, \ldots, \rho_k)$ *and product net* $\mathcal{E} = (E_i)_{i=0,\ldots,k}$, *let* ∇ *and* $\tilde{\nabla}$ *be the Levi-Civita connections of* $\langle\ ,\ \rangle$ *and of the product metric* $\langle\ ,\ \rangle^\sim$, *respectively, and let* $U_i = -\nabla(\log \circ \rho_i)$, $0 \le i \le k$, *where the gradient is calculated with respect to* $\langle\ ,\ \rangle$. *Then*

$$\nabla_X Y = \tilde{\nabla}_X Y + \sum_{i=0}^{k}(\langle X^i, Y^i\rangle U_i - \langle X, U_i\rangle Y^i - \langle Y, U_i\rangle X^i). \tag{1}$$

Here and throughout the paper, if $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is an orthogonal net on a Riemannian manifold then writing a vector field with a superscript i (resp., $\perp_i$) indicates taking the E_i-component (resp., $E_i^\perp$-component) of that vector field. Moreover, we always denote sections of E_i (resp., $E_i^\perp$) by X_i and Y_i (resp., $X_{\perp_i}$ and $Y_{\perp_i}$).

An orthogonal net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ on a Riemannian manifold M is called a *TP-net* if E_i is umbilical and $E_i^\perp$ is integrable for $i = 0, \ldots, k$. Recall that a subbundle E of TM is *umbilical* if there exists a vector field η in $E^\perp$ such that

$$\langle \nabla_X Y, Z\rangle = \langle X, Y\rangle\langle \eta, Z\rangle \ \text{ for all } \ X, Y \in E,\ Z \in E^\perp.$$

The vector field η is called the *mean curvature normal* of E. If, in addition,

$$\langle \nabla_X \eta, Z\rangle = 0 \ \text{ for all } \ X \in E,\ Z \in E^\perp,$$

then E is said to be *spherical*. If E is umbilical and its mean curvature normal vanishes identically, then it is called *totally geodesic* (or *auto-parallel*). An umbilical distribution is automatically integrable, and the leaves are umbilical submanifolds of M. When E is totally geodesic or spherical, then its leaves are totally geodesic or spherical submanifolds, respectively. By a *spherical submanifold* we mean an umbilical submanifold whose mean curvature vector is parallel with respect to the normal connection.

An orthogonal net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is called a *WP-net* if E_i is spherical and $E_i^\perp$ is totally geodesic for $i = 1, \ldots, k$. In a WP-net the subbundle E_0 is automatically totally geodesic and $E_0^\perp$ is integrable (see Proposition 3 in [9] or Lemma 5 below); in particular, every WP-net is also a TP-net.

It follows from (1) that

$$(\nabla_{X_i} Y_i)^{\perp_i} = \langle X_i, Y_i \rangle U_i^{\perp_i}, \tag{2}$$

and

$$(\nabla_{X_{\perp_i}} Y_{\perp_i})^i = \sum_{j \neq i} \langle X_{\perp_i}^j, Y_{\perp_i}^j \rangle U_j^i. \tag{3}$$

Equation (2) implies that E_i is umbilical with mean curvature normal $H_i = U_i^{\perp_i}$. Thus the product net of a twisted product is a TP-net. For a warped product, we have that $H_i = U_i$ for $i = 1, \ldots, k$, because ρ_i depends only on M_0, and that $E_i^\perp$ is totally geodesic, as follows from (3). Then $\langle \nabla_{X_i} H_i, X_{\perp_i} \rangle = \langle \nabla_{X_{\perp_i}} H_i, X_i \rangle = 0$, where the first equality holds because H_i is a gradient vector field. Thus E_i is spherical, and hence $\mathcal{E}$ is a WP-net. The converses also hold (see Proposition 4 of [9]):

Proposition 2 [9] *On a connected product manifold $M =: \Pi_{i=0}^k M_i$ the product net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is a TP-net (resp., WP-net) with respect to a Riemannian metric $\langle\ ,\ \rangle$ on M if and only if $\langle\ ,\ \rangle$ is a twisted (resp., warped) product metric on M.*

The following result from [9] (see Corollary 1 of [9]) contains the local version of Hiepko's decomposition theorem [8].

Theorem 3 [9] *Let $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ be a TP-net (resp., WP-net) on a Riemannian manifold M. Then, for every point $p \in M$ there exists a local product representation ψ: $\Pi_{i=0}^k M_i \to U$ of $\mathcal{E}$ with $p \in U \subset M$, which is an isometry with respect to a twisted product (resp., warped product) metric on $\Pi_{i=0}^k M_i$.*

Theorem 3 is a consequence of Proposition 2 and the following basic criterion for local decomposability of a net on an arbitrary C^∞-manifold (cf. Theorem 1 of [11]).

Proposition 4 [11] *A net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ on a C^∞-manifold is locally decomposable if and only if $E_i^\perp := \oplus_{j \neq i} E_i$ is integrable for $i = 0, \ldots, k$.*

§3. Quasi-warped products.

We say that a Riemannian metric on a product manifold $M = \Pi_{i=0}^k M_i$ is a *quasi-warped product metric* if it is a twisted product metric with twist function $\rho = (\rho_0, \ldots, \rho_k)$ and, in addition, ρ_0 is identically 1 and ρ_i depends only on M_0 and M_i for $i = 1, \ldots, k$. In this section we characterize the orthogonal nets that admit locally a product representation whose induced metric is either isometric or conformal to a quasi-warped product metric. We start with a few preliminary facts.

Lemma 5 *Let $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ be an orthogonal net on a Riemannian manifold such that E_i and $E_i^\perp$ are umbilical for $i = 1, \ldots, k$. Then $E_0^\perp$ is integrable and E_0 is umbilical with mean curvature normal $H_0 = \sum_{i=1}^k \eta_i$, where η_i is the mean curvature normal of $E_i^\perp$. Therefore $\mathcal{E}$ is a TP-net. Moreover, if $E_i^\perp$ is spherical for $i = 1, \ldots, k$ then the same holds for E_0.*

Proof: First notice that $[X_i, Y_i] \in E_i \subset E_0^\perp$ for $i = 1, \ldots, k$, because E_i is umbilical, and hence integrable. Now, using that $E_i^\perp$ is umbilical with mean curvature normal η_i for $i = 1, \ldots, k$ we have for all $i, j = 1, \ldots, k$ with $i \neq j$ that

$$\langle \nabla_{X_j} X_i, X_0 \rangle = -\langle X_i, \nabla_{X_j} X_0 \rangle = -\langle X_j, X_0 \rangle \langle \eta_i, X_i \rangle = 0, \tag{4}$$

thus $\nabla_{X_j} X_i \in E_0^\perp$. Therefore $E_0^\perp$ is integrable. That E_0 is umbilical with mean curvature normal $H_0 = \sum_{i=1}^k \eta_i$ follows from

$$(\nabla_{X_0} Y_0)^{\perp_0} = \sum_{i=1}^k (\nabla_{X_0} Y_0)^i = \sum_{i=1}^k \langle X_0, Y_0 \rangle \eta_i = \langle X_0, Y_0 \rangle \sum_{i=1}^k \eta_i.$$

Finally, since $E_j^\perp$ is umbilical we have that $(\nabla_{X_0} \eta_i)^j = 0$ for $i, j = 1, \ldots, k$ with $i \neq j$, and hence

$$(\nabla_{X_0} H_0)^{\perp_0} = \sum_{i=1}^k (\nabla_{X_0} \eta_i)^{\perp_0} = \sum_{i=1}^k (\nabla_{X_0} \eta_i)^i.$$

We conclude that E_0 is spherical whenever $E_i^\perp$ is spherical for $i = 1, \ldots, k$. ∎

Lemma 6 *Let $(M, \langle\, , \,\rangle) = {}^\rho\Pi_{i=0}^k (M_i, \langle\, , \,\rangle_i)$ be a twisted product and let $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ be its product net. Then for every fixed $i \in \{0, \ldots, k\}$ the following are equivalent:*

(i) *ρ_j / ρ_ℓ does not depend on M_i for all $j, \ell \in \{0, \ldots, k\}$ with $j, \ell \neq i$;*

(ii) *$H_j^i = H_\ell^i$ for all $j, \ell \in \{0, \ldots, k\}$ with $j, \ell \neq i$, where $H_j = U_j^{\perp_j}$ ($U_j = -\nabla \log \circ \rho_j$), is the mean curvature normal of E_j;*

(iii) *$E_i^\perp$ is umbilical.*

If these assertions are true, then the mean curvature normal η_i of $E_i^\perp$ coincides with the vectors H_j^i, $j \neq i$.

Proof: We have that $H_j^i - H_\ell^i = -(\nabla(\log \circ(\rho_j/\rho_\ell)))^i$, which gives the equivalence between (i) and (ii). The equivalence between (ii) and (iii) follows from (3). ∎

Proposition 7 *Let $M = \Pi_{i=0}^k M_i$ be a connected product manifold and $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ its product net. Then E_i is umbilical and $E_i^\perp$ is totally geodesic (resp., E_i and $E_i^\perp$ are umbilical) for $i = 1, \ldots, k$ with respect to a Riemannian metric $\langle\ ,\ \rangle$ on M if and only if $\langle\ ,\ \rangle$ is (resp., $\langle\ ,\ \rangle$ is conformal to) a quasi-warped product metric.*

Proof: Assume that $\langle\ ,\ \rangle = \varphi^2 \langle\ ,\ \rangle^\sim$ is conformal to a quasi-warped product metric $\langle\ ,\ \rangle^\sim = \pi_0^* \langle\ ,\ \rangle_0 + \sum_{i=1}^k \tilde{\rho}_i^2 \pi_i^* \langle\ ,\ \rangle_i$. Then it is a twisted product metric with twist function $\rho = (\varphi, \varphi\tilde{\rho}_1, \ldots, \varphi\tilde{\rho}_k)$. Therefore E_i is umbilical for $i = 1, \ldots, k$ (in fact for $i = 0, \ldots, k$) and, by Lemma 6, the same holds for $E_i^\perp$. On the other hand, if $\langle\ ,\ \rangle$ is a quasi-warped product metric, then (3) implies that $E_i^\perp$ is totally geodesic for $i = 1, \ldots, k$.

Conversely, assume that E_i and $E_i^\perp$ are umbilical for $i = 1, \ldots, k$ with respect to a Riemannian metric $\langle\ ,\ \rangle$ on M. Then $\mathcal{E}$ is a TP-net by Lemma 5, and hence $\langle\ ,\ \rangle$ is a twisted product metric $\langle\ ,\ \rangle = \sum_{i=0}^k \rho_i^2 \pi_i^* \langle\ ,\ \rangle_i$ by Proposition 2. Moreover, by Lemma 6 we have that $\tilde{\rho}_i = \rho_i/\rho_0$ only depends on M_i and M_0 for $i = 1, \ldots, k$. Thus $\langle\ ,\ \rangle^\sim = \pi_0^* \langle\ ,\ \rangle_0 + \sum_{i=1}^k \tilde{\rho}_i^2 \pi_i^* \langle\ ,\ \rangle_i$ is a quasi-warped product metric and $\langle\ ,\ \rangle = \rho_0^2 \langle\ ,\ \rangle^\sim$. If, in addition, $E_i^\perp$ is totally geodesic for $i = 1, \ldots, k$, then (3) implies that ρ_0 only depends on M_0, and hence $\langle\ ,\ \rangle = \pi_0^* \langle\ ,\ \rangle_0^* + \sum_{i=1}^k \rho_i^2 \pi_i^* \langle\ ,\ \rangle_i$ is a quasi-warped product metric, where $\langle\ ,\ \rangle_0^* = \tilde{\rho}_0^2 \langle\ ,\ \rangle_0$ for $\tilde{\rho}_0 \in C^\infty(M_0)$ such that $\tilde{\rho}_0 \circ \pi_0 = \rho_0$. ∎

Remark 8 In Proposition 7 we have that E_i is umbilical for $i = 1, \ldots, k$ and that $E_i^\perp$ is umbilical for $i = 0, \ldots, k$ (and hence also E_0 is umbilical by Lemma 5) with respect to a Riemannian metric $\langle\ ,\ \rangle$ on M if and only if $\langle\ ,\ \rangle$ is conformal to a quasi-warped product metric $\langle\ ,\ \rangle^\sim = \pi_0^* \langle\ ,\ \rangle_0 + \sum_{i=1}^k \tilde{\rho}_i^2 \pi_i^* \langle\ ,\ \rangle_i$ and, in addition, $\tilde{\rho}_i/\tilde{\rho}_j$ does not depend on M_0 for $i, j = 1, \ldots, k$, or equivalently, there exists $Y_0 \in E_0$ (not necessarily a gradient vector field) such that $(\nabla \log \circ \tilde{\rho}_i)^0 = Y_0$ for $i = 1, \ldots, k$. This follows by applying Lemma 6 once more for $i = 0$.

Theorem 9. *Let $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ be an orthogonal net on a Riemannian manifold M. Assume that E_i is umbilical and $E_i^\perp$ is totally geodesic (resp., E_i and $E_i^\perp$ are umbilical) for $i = 1, \ldots, k$. Then for every point $p \in M$ there exists a local product representation ψ: $\Pi_{i=0}^k M_i \to U$ of $\mathcal{E}$ with $p \in U \subset M$, which is an isometry (resp., a conformal diffeomorphism) with respect to a quasi-warped product metric on $\Pi_{i=0}^k M_i$.*

Proof: By Lemma 5, the net $\mathcal{E}$ is a TP-net, and thus it is locally decomposable by Proposition 4. Let ψ: $\Pi_{i=0}^k M_i \to U$ be a local product representation with $p \in U \subset M$ and let $\Pi_{i=0}^k M_i$ be endowed with the metric induced by ψ. Then the product net of $\Pi_{i=0}^k M_i$ satisfies the same properties as $\mathcal{E}$ and Proposition 7 concludes the proof. ∎

§4. Local conformal versions of de Rham and Hiepko theorems.

In this section we prove our main results. Namely, we derive conformal versions of the local decomposition theorems of de Rham and Hiepko and prove some related decomposition results.

We say that an orthogonal net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ on a Riemannian manifold is a *conformal warped product net*, or a CWP-net for short, if for $i = 1, \ldots, k$ it holds that

$$E_i \text{ and } E_i^\perp \text{ are umbilical and } \langle \nabla_{X_{\perp_i}} \eta_i, X_i \rangle = \langle \nabla_{X_i} H_i, X_{\perp_i} \rangle, \tag{5}$$

where H_i and η_i are the mean curvature normals of E_i and $E_i^\perp$, respectively. If, in addition, also $E_0^\perp$ is umbilical, then we say that $\mathcal{E}$ is a *conformal product net*, or a CP-net for short. We first observe a few elementary facts on CP-nets and CWP-nets.

Proposition 10 *(i) For a CP-net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ condition (5) holds also for $i = 0$;*

(ii) If $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is an orthogonal net such that E_i and $E_i^\perp$ are spherical for $i = 1, \ldots, k$ then it is a CWP-net. If, in addition, also $E_0^\perp$ is spherical then it is a CP-net.

(iii) If $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is a CP-net (resp., a CWP-net), then for each $i = 0, \ldots, k$ (resp., $i = 1, \ldots, k$) one of E_i and $E_i^\perp$ being spherical implies the same for the other.

Proof: If $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is a CP-net, then we have from Lemma 5 that E_0 is umbilical with mean curvature normal $H_0 = \sum_{i=1}^k \eta_i$, where η_i is the mean curvature normal of $E_i^\perp$. Thus, in order to prove (i) it remains to verify that $\langle \nabla_{X_{\perp_0}} \eta_0, X_0 \rangle = \langle \nabla_{X_0} H_0, X_{\perp_0} \rangle$, where η_0 is the mean curvature normal of $E_0^\perp$. First recall that $H_i^0 = \eta_0$ for $i = 1, \ldots, k$ by Lemma 6, where H_i is the mean curvature normal of E_i. Then,

$$\begin{aligned} \langle \nabla_{X_0} H_0, X_{\perp_0} \rangle &= \textstyle\sum_{i=1}^k \langle \nabla_{X_0} \eta_i, X_{\perp_0} \rangle = \sum_{i=1}^k \langle \nabla_{X_0} \eta_i, X_{\perp_0}^i \rangle = \sum_{i=1}^k \langle \nabla_{X_{\perp_0}^i} H_i, X_0 \rangle \\ &= \textstyle\sum_{i=1}^k \langle \nabla_{X_{\perp_0}^i} H_i^0, X_0 \rangle = \sum_{i=1}^k \langle \nabla_{X_{\perp_0}^i} \eta_0, X_0 \rangle = \langle \nabla_{X_{\perp_0}} \eta_0, X_0 \rangle, \end{aligned}$$

where in the second equality we have used that $(\nabla_{X_0} \eta_i)^j = 0$ for $i, j = 1, \ldots, k$ with $i \neq j$ because $E_j^\perp$ is umbilical and in the fourth one that $\nabla_{X_{\perp_0}^i} H_i^j = 0$ for $i, j = 1, \ldots, k$ with $i \neq j$ because $E_0^\perp$ is umbilical. Clearly, condition (5) is satisfied if both E_i and $E_i^\perp$ are spherical. Moreover, if (5) holds then one of E_i or $E_i^\perp$ being spherical implies the same for the other. Taking into account (i) and the last statement in Lemma 5, all the assertions in (ii) and (iii) follow. ∎

Proposition 11 *On a connected and simply connected product manifold* $M = \Pi_{i=0}^k M_i$ *the product net* $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ *is a* CWP*-net (resp.,* CP*-net) with respect to a Riemannian metric* $\langle\ ,\ \rangle$ *on* M *if and only if* $\langle\ ,\ \rangle$ *is conformal to a warped product metric (resp., to a Riemannian product metric) on* M*.*

Proof: If $\langle\ ,\ \rangle^{\sim} = \pi_0^*\langle\ ,\ \rangle_0^{\sim} + \sum_{i=1}^k \tilde{\rho}_i^2 \pi_i^*\langle\ ,\ \rangle_i^{\sim}$ is a warped product metric on M and $\langle\ ,\ \rangle = \varphi^2\langle\ ,\ \rangle^{\sim}$ for some $\varphi \in C^\infty(M)$ then $\langle\ ,\ \rangle$ is a twisted product metric with twist function $\rho = (\rho_0, \ldots, \rho_k) = (\varphi, \varphi\tilde{\rho}_1, \ldots, \varphi\tilde{\rho}_k)$. Set $W = -\nabla \log \circ \varphi$, $\tilde{U}_i = -\nabla \log \circ \tilde{\rho}_i \in E_0$ and $U_i = -\nabla \log \circ \rho_i = \tilde{U}_i + W$, $1 \le i \le k$. Then, for $i = 1, \ldots, k$ we have that E_i is umbilical with respect to $\langle\ ,\ \rangle$ with mean curvature normal $H_i = U_i^{\perp_i} = \tilde{U}_i + W^{\perp_i}$. Moreover, it follows from Lemma 6 that $E_i^\perp$ is also umbilical with mean curvature normal $\eta_i = W^i$. Therefore, for any $i \in \{1, \ldots, k\}$ we have

$$\begin{aligned} \langle \nabla_{X_{\perp_i}} \eta_i, X_i \rangle &= \langle \nabla_{X_{\perp_i}} W, X_i \rangle - \langle \nabla_{X_{\perp_i}} W^{\perp_i}, X_i \rangle \\ &= \langle \nabla_{X_{\perp_i}} W, X_i \rangle - \langle X_{\perp_i}, W^{\perp_i} \rangle \langle X_i, W^i \rangle. \end{aligned} \tag{6}$$

On the other hand, using that $\tilde{U}_i \in E_0$ is a gradient vector field we have

$$\langle \nabla_{X_i} \tilde{U}_i, X_{\perp_i} \rangle = \langle \nabla_{X_{\perp_i}} \tilde{U}_i, X_i \rangle = \langle X_{\perp_i}, \tilde{U}_i \rangle \langle W^i, X_i \rangle,$$

and hence

$$\begin{aligned} \langle \nabla_{X_i} H_i, X_{\perp_i} \rangle &= \langle \nabla_{X_i} W, X_{\perp_i} \rangle - \langle \nabla_{X_i} W^i, X_{\perp_i} \rangle + \langle \nabla_{X_i} \tilde{U}_i, X_{\perp_i} \rangle \\ &= \langle \nabla_{X_i} W, X_{\perp_i} \rangle - \langle X_i, W^i \rangle \langle H_i, X_{\perp_i} \rangle + \langle X_{\perp_i}, \tilde{U}_i \rangle \langle W^i, X_i \rangle \\ &= \langle \nabla_{X_i} W, X_{\perp_i} \rangle - \langle X_i, W^i \rangle \langle W^{\perp_i}, X_{\perp_i} \rangle. \end{aligned} \tag{7}$$

Since W is a gradient vector field, we conclude from (6) and (7) that (5) holds, and thus $\mathcal{E}$ is a CWP-net with respect to $\langle\ ,\ \rangle$. Moreover, if $\langle\ ,\ \rangle^{\sim}$ is a Riemannian product metric on M, then also $E_0^\perp$ is umbilical with respect to $\langle\ ,\ \rangle$ with mean curvature normal $\eta_0 = W^0$. Therefore $\mathcal{E}$ is a CP-net with respect to $\langle\ ,\ \rangle$.

We now prove the converse. If $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is a CWP-net with respect to $\langle\ ,\ \rangle$ then it is also a TP-net by Lemma 5, thus $\langle\ ,\ \rangle$ is a twisted product metric $\langle\ ,\ \rangle = \sum_{i=0}^k \rho_i^2 \pi_i^*\langle\ ,\ \rangle_i$ by Proposition 2. Moreover, by Lemma 6 we have that ρ_i/ρ_0 only depends on M_0 and M_i for $i = 1, \ldots, k$.

CLAIM: There exist $\tilde{\varphi}_i \in C^\infty(M_i)$ and $\tilde{\psi}_i \in C^\infty(M_0)$ such that $\rho_i/\rho_0 = (\tilde{\varphi}_i \circ \pi_i)(\tilde{\psi}_i \circ \pi_0)$ for $i = 1, \ldots, k$.

Assuming the claim, we conclude that $\langle\ ,\ \rangle$ is conformal to the warped product metric $\langle\ ,\ \rangle^{\sim} = \pi_0^*\langle\ ,\ \rangle_0 + \sum_{i=1}^k \psi_i^2 \pi_i^*\langle\ ,\ \rangle_i^{\sim}$ with conformal factor ρ_0, where $\psi_i = \tilde{\psi}_i \circ \pi_0$ and $\langle\ ,\ \rangle_i^{\sim} = \tilde{\varphi}_i^2\langle\ ,\ \rangle_i$ for $i = 1, \ldots, k$.

Proof of the claim: Let $U_i = -\nabla \log \circ \rho_i$. Then $(U_i - U_0) = (U_i - U_0)^i + (U_i - U_0)^0$, and the claim is equivalent to $(U_i - U_0)^i$ (and hence also $(U_i - U_0)^0$) being a gradient vector field. Since M is simply connected, this is in turn equivalent to $\langle\nabla_X(U_i-U_0)^i, Y\rangle$ being symmetric in X and Y. We now verify that this is indeed the case. First, using that $E_i^\perp$ is umbilical with mean curvature normal η_i we have

$$\langle\nabla_{X_{\perp_i}}(U_i-U_0)^i, Y_{\perp_i}\rangle = -\langle X_{\perp_i}, Y_{\perp_i}\rangle\langle\eta_i, (U_i-U_0)^i\rangle = \langle\nabla_{Y_{\perp_i}}(U_i-U_0)^i, X_{\perp_i}\rangle.$$

For $X = X_i$ and $Y = Y_i$, using that E_i is umbilical with mean curvature normal H_i, the symmetry follows from

$$\begin{aligned}\langle\nabla_{X_i}(U_i-U_0)^i, Y_i\rangle &= \langle\nabla_{X_i}(U_i-U_0), Y_i\rangle - \langle\nabla_{X_i}(U_i-U_0)^0, Y_i\rangle \\ &= \langle\nabla_{X_i}(U_i-U_0), Y_i\rangle + \langle X_i, Y_i\rangle\langle H_i, (U_i-U_0)^0\rangle\end{aligned}$$

and the fact that U_i-U_0 is a gradient vector field. Finally, we consider the case $X = X_{\perp_i}$ and $Y = X_i$. On one hand, using that the mean curvature normal of $E_i^\perp$ is given by $\eta_i = H_0^i = U_0^i$ by Lemma 6, and that $U_i^{\perp_i} = H_i$, we have

$$\begin{aligned}\langle\nabla_{X_{\perp_i}}(U_i-U_0)^i, X_i\rangle &= \langle\nabla_{X_{\perp_i}}U_i, X_i\rangle - \langle\nabla_{X_{\perp_i}}H_i, X_i\rangle - \langle\nabla_{X_{\perp_i}}\eta_i, X_i\rangle \\ &= \langle\nabla_{X_{\perp_i}}U_i, X_i\rangle - \langle X_{\perp_i}, H_i\rangle\langle\eta_i, X_i\rangle - \langle\nabla_{X_{\perp_i}}\eta_i, X_i\rangle. \qquad (8)\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle\nabla_{X_i}(U_i-U_0)^i, X_{\perp_i}\rangle &= \langle\nabla_{X_i}U_i, X_{\perp_i}\rangle - \langle\nabla_{X_i}H_i, X_{\perp_i}\rangle - \langle\nabla_{X_i}\eta_i, X_{\perp_i}\rangle \\ &= \langle\nabla_{X_i}U_i, X_{\perp_i}\rangle - \langle\nabla_{X_i}H_i, X_{\perp_i}\rangle - \langle X_i, \eta_i\rangle\langle X_{\perp_i}, H_i\rangle. \qquad (9)\end{aligned}$$

It follows from (5) and the fact that U_i is a gradient vector field that the right-hand-sides of (8) and (9) coincide, and the proof of the claim is completed.

Now assume that $\mathcal{E}$ is a CP-net with respect to $\langle\ ,\ \rangle$, that is, also $E_0^\perp$ is umbilical. We may assume that $k \geq 2$. It follows from Lemma 6 that ψ_i/ψ_1 does not depend on M_0 for $i = 2, \ldots, k$, thus there exist $a_i \neq 0$ such that $\psi_i = a_i\psi_1$ for $i = 2, \ldots, k$. Therefore $\langle\ ,\ \rangle^\sim$ is conformal to the Riemannian product metric $\langle\ ,\ \rangle^* = \sum_{i=0}^k \pi_i^*\langle\ ,\ \rangle_i^*$ with conformal factor ψ_1, where $\langle\ ,\ \rangle_1^* = \langle\ ,\ \rangle_1^\sim$, $\langle\ ,\ \rangle_i^* = a_i^2\langle\ ,\ \rangle_i^\sim$ for $i = 2, \ldots, k$ and $\langle\ ,\ \rangle_0^* = \tilde{\psi}_1^{-2}\langle\ ,\ \rangle_0$. ∎

Arguing as in the proof of Theorem 9, we obtain from Proposition 11 the following conformal versions of the local de Rham and Hiepko theorems.

Theorem 12 *Let $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ be a CWP-net (resp., CP-net) on a Riemannian manifold M. Then for every point $p \in M$ there exists a local product representation $\psi\colon \Pi_{i=0}^k M_i \to U$ of $\mathcal{E}$ with $p \in U \subset M$, which is a conformal diffeomorphism with respect to a warped (resp., Riemannian) product metric on $\Pi_{i=0}^k M_i$.*

Given a connected product manifold $M = \Pi_{i=0}^k M_i$ endowed with a metric $\langle\ ,\ \rangle$ that is conformal to a warped product metric on M, we now investigate under what conditions the subbundles E_i and $E_i^\perp$ of the product net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ of M are spherical with respect to $\langle\ ,\ \rangle$ for a fixed $i \in \{1, \ldots, k\}$.

Lemma 13 *Let* $M = \Pi_{i=0}^k M_i$ *be a connected and simply connected product manifold and let* $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ *be its product net. Let* $\langle\ ,\ \rangle^\sim = \pi_0^*\langle\ ,\ \rangle_0^\sim + \sum_{i=1}^k \tilde{\rho}_i^2 \pi_i^* \langle\ ,\ \rangle_i^\sim$ *be a warped product metric on* M *and let* $\langle\ ,\ \rangle = \varphi^2 \langle\ ,\ \rangle^\sim$ *be conformal to* $\langle\ ,\ \rangle^\sim$ *with conformal factor* $\varphi \in C^\infty(M)$*. Then for each fixed* $i \in \{1, \ldots, k\}$ *the following assertions are equivalent:*

- (i) *Either* E_i *or* $E_i^\perp$ *is (and hence both* E_i *and* $E_i^\perp$ *are) spherical with respect to* $\langle\ ,\ \rangle$*;*
- (ii) *The vector field* $W = -\nabla \log \circ \varphi$ *satisfies*

$$\langle \nabla_{X_{\perp_i}} W, X_i \rangle = \langle X_{\perp_i}, W \rangle \langle X_i, W \rangle; \tag{10}$$

- (iii) *The conformal factor* $\varphi \in C^\infty(M)$ *satisfies Hess* $\varphi(X_i, X_{\perp_i}) = 0$*;*
- (iv) $\tilde{\rho}_i^{-1} \tilde{W}^i$ *is a gradient vector field, where* $\tilde{W} = \nabla \varphi^{-1}$*;*
- (v) *There exist* $\phi_i \in C^\infty(M_i)$ *and* $\phi_{\perp_i} \in C^\infty(M_{\perp_i} := M_0 \times \cdots \times \hat{M}_i \times \cdots \times M_k)$ *(the hat indicates that* M_i *is missing) such that* $\varphi^{-1} = \tilde{\rho}_i(\phi_i \circ \pi_i) + \phi_{\perp_i} \circ \pi_{\perp_i}$*, where* $\pi_{\perp_i} \colon M \to M_{\perp_i}$ *denotes the canonical projection.*

Proof: We have from the beginning of the proof of Proposition 11 that E_i and $E_i^\perp$ are umbilical with respect to $\langle\ ,\ \rangle$ with mean curvature normals $H_i = \tilde{U}_i + W^{\perp_i}$ and $\eta_i = W^i$, respectively, where $\tilde{U}_i = -\nabla \log \circ \tilde{\rho}_i \in E_0$. The equivalence between (i) and (ii) then follows from (6) and (7). Now, we have

$$\begin{aligned} \nabla_X W &= -\nabla_X(\varphi^{-1}\nabla\varphi) = -X(\varphi^{-1})\nabla\varphi - \varphi^{-1}\nabla_X\nabla\varphi \\ &= \varphi^{-2}\langle \nabla\varphi, X\rangle \nabla\varphi - \varphi^{-1}\nabla_X\nabla\varphi \\ &= \langle W, X\rangle W - \varphi^{-1}\nabla_X\nabla\varphi. \end{aligned}$$

Thus

$$\langle \nabla_{X_{\perp_i}} W, X_i \rangle - \langle X_{\perp_i}, W \rangle \langle X_i, W \rangle = -\varphi^{-1} \text{Hess } \varphi(X_{\perp_i}, X_i),$$

and the equivalence between (ii) and (iii) follows. We now prove that (ii) is equivalent to the symmetry of $\langle \nabla_X \tilde{\rho}_i^{-1} \tilde{W}^i, Y \rangle$ with respect to X and Y. Since M is simply connected, this implies the equivalence between (ii) and (iv). From $\tilde{W} = \varphi^{-1} W$ we obtain

$$\nabla_X \tilde{W} = \varphi^{-1}(\langle W, X\rangle W + \nabla_X W),$$

and hence

$$\begin{aligned} \langle \nabla_{X_{\perp_i}} \tilde{W}^i, X_i \rangle &= \langle \nabla_{X_{\perp_i}} \tilde{W}, X_i \rangle - \varphi^{-1} \langle W^{\perp_i}, X_{\perp_i} \rangle \langle W^i, X_i \rangle \\ &= \varphi^{-1} \langle \nabla_{X_{\perp_i}} W, X_i \rangle, \end{aligned} \tag{11}$$

where in the first equality we have used that $E_i^{\perp}$ is umbilical with mean curvature normal W^i. Since

$$X_{\perp_i}(\tilde{\rho}_i^{-1}) = \tilde{\rho}_i^{-1} X_{\perp_i}(-\log \circ \tilde{\rho}_i) = \tilde{\rho}_i^{-1}\langle \tilde{U}_i, X_{\perp_i}\rangle,$$

we obtain from (11) that

$$\begin{aligned}\langle \nabla_{X_{\perp_i}} \tilde{\rho}_i^{-1} \tilde{W}^i, X_i\rangle &= \tilde{\rho}_i^{-1}\langle \tilde{U}_i, X_{\perp_i}\rangle\langle \tilde{W}^i, X_i\rangle + \tilde{\rho}_i^{-1}\varphi^{-1}\langle \nabla_{X_{\perp_i}} W, X_i\rangle \\ &= \tilde{\rho}_i^{-1}\varphi^{-1}(\langle \tilde{U}_i, X_{\perp_i}\rangle\langle W^i, X_i\rangle + \langle \nabla_{X_{\perp_i}} W, X_i\rangle). \qquad (12)\end{aligned}$$

On the other hand, since E_i is umbilical with mean curvature normal $H_i = \tilde{U}_i + W^{\perp_i}$ we have

$$\begin{aligned}\langle \nabla_{X_i} \tilde{W}^i, X_{\perp_i}\rangle &= \langle X_i, \tilde{W}^i\rangle\langle X_{\perp_i}, H_i\rangle \\ &= \varphi^{-1}(\langle X_i, W^i\rangle\langle X_{\perp_i}, W^{\perp_i}\rangle + \langle X_i, W^i\rangle\langle X_{\perp_i}, \tilde{U}_i\rangle).\end{aligned}$$

Using that $\tilde{\rho}_i$ only depends on M_0, we obtain

$$\langle \nabla_{X_i} \tilde{\rho}_i^{-1} \tilde{W}^i, X_{\perp_i}\rangle = \tilde{\rho}_i^{-1}\varphi^{-1}(\langle X_i, W^i\rangle\langle X_{\perp_i}, W^{\perp_i}\rangle + \langle X_i, W^i\rangle\langle X_{\perp_i}, \tilde{U}_i\rangle). \qquad (13)$$

Comparing (12) and (13) implies that $\langle \nabla_{X_{\perp_i}} \tilde{\rho}_i^{-1} \tilde{W}^i, X_i\rangle = \langle \nabla_{X_i} \tilde{\rho}_i^{-1} \tilde{W}^i, X_{\perp_i}\rangle$ is equivalent to (ii). Since

$$\langle \nabla_{X_{\perp_i}} \tilde{\rho}_i^{-1} \tilde{W}^i, Y_{\perp_i}\rangle = -\tilde{\rho}_i^{-1}\langle X_{\perp_i}, Y_{\perp_i}\rangle\langle W^i, \tilde{W}^i\rangle = \langle \nabla_{Y_{\perp_i}} \tilde{\rho}_i^{-1} \tilde{W}^i, X_{\perp_i}\rangle,$$

where we have used again that $E_i^{\perp}$ is umbilical with mean curvature normal W^i, and

$$\begin{aligned}\langle \nabla_{X_i} \tilde{\rho}_i^{-1} \tilde{W}^i, Y_i\rangle &= \tilde{\rho}_i^{-1}(\langle \nabla_{X_i} \tilde{W}, Y_i\rangle - \langle \nabla_{X_i} \tilde{W}^{\perp_i}, Y_i\rangle) \\ &= \tilde{\rho}_i^{-1}(\langle \nabla_{X_i} \tilde{W}, Y_i\rangle + \langle X_i, Y_i\rangle\langle H_i, \tilde{W}^{\perp_i}\rangle) = \langle \nabla_{Y_i} \tilde{\rho}_i^{-1} \tilde{W}^i, X_i\rangle,\end{aligned}$$

because $\tilde{W}$ is a gradient vector field and E_i is umbilical with mean curvature normal H_i, the proof of the equivalence between (ii) and (iv) is completed.

Finally, if (iv) holds let $\phi_i \in C^\infty(M_i)$ be such that $\tilde{\rho}_i^{-1}\tilde{W}^i = \nabla(\phi_i \circ \pi_i)$. Then $\nabla(\varphi^{-1} - \tilde{\rho}_i(\phi_i \circ \pi_i)) \in E_{\perp_i}$, hence there exists $\phi_{\perp_i} \in C^\infty(M_{\perp_i})$ such that $\varphi^{-1} - \tilde{\rho}_i(\phi_i \circ \pi_i) = \phi_{\perp_i} \circ \pi_{\perp_i}$. Conversely, assuming (v) we obtain $\tilde{W}^i = \tilde{\rho}_i \nabla(\phi_i \circ \pi_i)$, thus (iv) holds. ∎

Remark 14 The assumption that M is simply connected in Lemma 13 may be dropped if the assertions in (iv) and (v) are required to hold only locally.

Proposition 15 *Let $M = \Pi_{i=0}^k M_i$ be a connected and simply connected product manifold and let $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ be its product net. Then the following assertions on a Riemannian metric $\langle\ ,\ \rangle$ on M are equivalent:*

(i) E_i and $E_i^{\perp}$ are spherical for $i = 1, \ldots, k$ (resp., E_i and $E_i^{\perp}$ are spherical for $i = 1, \ldots, k$ and, in addition, $E_0^{\perp}$ is spherical) with respect to $\langle\ ,\ \rangle$;

(ii) $\langle\ ,\ \rangle = \varphi^2 \langle\ ,\ \rangle^{\sim}$ *is conformal to a warped product metric*

$$\langle\ ,\ \rangle^{\sim} = \pi_0^* \langle\ ,\ \rangle_0^{\sim} + \sum_{i=1}^{k} \tilde{\rho}_i^2 \pi_i^* \langle\ ,\ \rangle_i^{\sim}$$

(resp., a Riemannian product metric $\langle\ ,\ \rangle^{\sim}$*) on* M *with conformal factor* $\varphi \in C^\infty(M)$ *given by* $\varphi^{-1} = \phi_0 \circ \pi_0 + \sum_{i=1}^k \tilde{\rho}_i (\phi_i \circ \pi_i)$ *(resp.,* $\varphi^{-1} = \sum_{i=0}^k \phi_i \circ \pi_i$*), where* $\phi_i \in C^\infty(M_i)$ *for* $i = 0, \ldots, k$*.*

Proof: Assuming (i), we have from Proposition 11 that $\langle\ ,\ \rangle = \varphi^2 \langle\ ,\ \rangle^{\sim}$ is conformal to a warped product metric $\langle\ ,\ \rangle^{\sim} = \pi_0^* \langle\ ,\ \rangle_0^{\sim} + \sum_{i=1}^k \tilde{\rho}_i^2 \pi_i^* \langle\ ,\ \rangle_i^{\sim}$ (resp., a Riemannian product metric $\langle\ ,\ \rangle^{\sim}$) on M with conformal factor $\varphi \in C^\infty(M)$. Moreover, by Lemma 13 there exist $\phi_i \in C^\infty(M_i)$ such that $\tilde{\rho}_i^{-1} (\nabla \varphi^{-1})^i = \nabla (\phi_i \circ \pi_i)$ (resp., $(\nabla \varphi^{-1})^i = \nabla(\phi_i \circ \pi_i)$) for $i = 1, \ldots, k$. Therefore $\nabla(\varphi^{-1} - \sum_{i=1}^k \tilde{\rho}_i (\phi_i \circ \pi_i)) \in E_0$ (resp., $\nabla(\varphi^{-1} - \sum_{i=1}^k \phi_i \circ \pi_i) \in E_0$), which implies (ii). Conversely, if (ii) holds for a warped product metric $\langle\ ,\ \rangle^{\sim}$ on M, then (i) holds by the fact that (v) implies (i) in Lemma 13 for $i = 1, \ldots, k$. Moreover, if $\langle\ ,\ \rangle^{\sim}$ is a Riemannian product metric, then $\mathcal{E}$ is a CP-net by Proposition 11, thus also $E_0^\perp$ is umbilical. But $E_i^\perp$ being spherical for $i = 1, \ldots, k$ implies that E_0 is spherical by Lemma 5, and hence also $E_0^\perp$ is spherical by Proposition 10-(iii). ∎

Arguing again as in the proof of Theorem 9, we obtain from Proposition 15 the following local decomposition theorem of de Rham-type.

Theorem 16 *Let* $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ *be an orthogonal net on a Riemannian manifold* M *such that* E_i *and* $E_i^\perp$ *are spherical for* $i = 1, \ldots, k$ *(resp.,* E_i *is spherical for* $i = 1, \ldots, k$ *and* $E_i^\perp$ *is spherical for* $i = 0, \ldots, k$*). Then for every point* $p \in M$ *there exists a local product representation* ψ: $\Pi_{i=0}^k M_i \to U$ *of* $\mathcal{E}$ *with* $p \in U \subset M$*, which is a conformal diffeomorphism with respect to a warped (resp., Riemannian) product metric on* $\Pi_{i=0}^k M_i$ *whose conformal factor* $\varphi \in C^\infty(M)$ *is given by* $\varphi^{-1} = \phi_0 \circ \pi_0 + \sum_{i=0}^k \tilde{\rho}_i (\phi_i \circ \pi_i)$ *(resp.,* $\varphi^{-1} = \sum_{i=1}^k \phi_i \circ \pi_i$*), where* $\phi_i \in C^\infty(M_i)$ *for* $i = 0, \ldots, k$ *and* $\tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_k)$ *is the warping function.*

Remark 17 Classically, a one-parameter family of curves in a surface $S \subset \mathbb{R}^3$ is said to be an *isothermal family* if it is the family of u-coordinate curves of an isothermic coordinate system (u, v) on S. In this setting, Theorem 12 reduces to a well-known characterization of isothermal families of curves in terms of their geodesic curvatures and the geodesic curvatures of their orthogonal trajectories (cf. [3], vol. III, p. 154, eq.(36)). Theorem 12 and Proposition 10 also generalize the facts that orthogonal families of curves with constant geodesic curvature are isothermal, and that the curves of an isothermal family must have constant geodesic curvature if the same holds for their orthogonal trajectories (cf. [3], vol. III, p. 154). Finally, Theorem 16 extends the characterization of the first fundamental form of a surface that admits two orthogonal families of curves with constant geodesic curvature (cf. [1], p. 368, eq. (38)).

§5. Codazzi tensors.

In this section we apply our results to study Riemannian manifolds that carry a Codazzi tensor with exactly two distinct eigenvalues everywhere. Recall that a symmetric tensor Φ is said to be a *Codazzi tensor* if $(\nabla_X\Phi)Y = (\nabla_Y\Phi)X$ for all $X, Y \in TM$, where $(\nabla_X\Phi)Y = \nabla_X\Phi Y - \Phi\nabla_X Y$. We start with the following basic result due to Reckziegel [12], a short proof of which is included for the sake of completeness.

Proposition 18 *Let Φ be a Codazzi tensor on a Riemmanian manifold M, and let $\lambda \in C^\infty(M)$ be an eigenvalue of Φ such that $E_\lambda = \ker(\lambda I - \Phi)$ has constant rank r. Then*

(i) *E_λ is an umbilical distribution with mean curvature normal η given by*

$$(\lambda I - \Phi)\,\eta = (\nabla\,\lambda)_{E_\lambda^\perp}. \tag{14}$$

(ii) *If $r \geq 2$ then λ is constant along E_λ.*

(iii) *If λ is constant along E_λ then E_λ is spherical.*

Proof: Let $T \in E_\lambda$ and $X \in TM$. Taking the inner product of both sides of $(\nabla_X\Phi)T = (\nabla_T\Phi)X$ with $S \in E_\lambda$ yields

$$(\lambda I - \Phi)\nabla_T S = \langle T, S\rangle\nabla\,\lambda - T(\lambda)S. \tag{15}$$

Since $\lambda I - \Phi$ vanishes on E_λ, we have that $(\lambda I - \Phi)\nabla_T S = (\lambda I - \Phi)(\nabla_T S)_{E_\lambda^\perp} \in E_\lambda^\perp$. Therefore, comparing the components in $E_\lambda^\perp$ of both sides of (15) yields

$$(\lambda I - \Phi)(\nabla_T S)_{E_\lambda^\perp} = \langle T, S\rangle(\nabla\,\lambda)_{E_\lambda^\perp}, \tag{16}$$

and (i) follows. Now assume that $r \geq 2$. Since the left-hand-side of (15) is in $E_\lambda^\perp$, it follows that $\langle T, S\rangle(\nabla\,\lambda)_{E_\lambda} = T(\lambda)S$. Then, for any $T \in E_\lambda$, choosing $0 \neq S \in E_\lambda$ orthogonal to T yields $T(\lambda) = 0$. To prove (iii), from $T(\lambda) = 0$ for all $T \in E_\lambda$ we have

$$\begin{aligned}\langle\nabla_T\eta, (\lambda I - \Phi)X\rangle &= T\langle(\lambda I - \Phi)\eta, X\rangle - \langle\eta, \nabla_T(\lambda I - \Phi)X\rangle\\ &= TX(\lambda) - \lambda\langle\nabla_T X, \eta\rangle + \langle\nabla_T\Phi X, \eta\rangle.\end{aligned}$$

Using that $\nabla_T\Phi X = \nabla_X\Phi T - \Phi\nabla_X T + \Phi\nabla_T X$ we obtain

$$\langle\nabla_T\Phi X, \eta\rangle = \langle(\lambda I - \Phi)\eta, \nabla_X T\rangle + \langle\Phi\eta, \nabla_T X\rangle.$$

Therefore

$$\langle\nabla_T\eta, (\lambda I - \Phi)X\rangle = TX(\lambda) - \langle(\lambda I - \Phi)\eta, [T, X]\rangle = TX(\lambda) - [T, X](\lambda) = 0. \quad \blacksquare$$

Here and in the sequel, writing a vector subbundle as a subscript of a vector field indicates taking its component in that subbundle. From now on we consider Codazzi tensors with exactly two distinct eigenvalues λ and μ everywhere, and always denote sections of the corresponding eigenbundles E_λ and E_μ by X and Y, respectively.

Theorem 19 *Let M be a connected Riemannian manifold and let Φ be a Codazzi tensor on M with exactly two distinct eigenvalues λ and μ everywhere. Let E_λ and E_μ be the corresponding eigenbundles. Then, for every point $p \in M$ there exists a local product representation ψ: $M_0 \times M_1 \to U$ of (E_λ, E_μ) with $p \in U \subset M$, which is an isometry with respect to a twisted product metric $\langle\ ,\ \rangle$ on $M_0 \times M_1$. Moreover,*

(i) *$\langle\ ,\ \rangle$ is conformal to a Riemannian product metric if and only if*

$$2\beta X(\alpha)Y(\alpha) + \alpha X(\alpha)Y(\beta) + \alpha Y(\alpha)X(\beta) - \alpha\beta \mathit{Hess}\, \alpha(X, Y) = 0, \tag{17}$$

where $\alpha = \dfrac{1}{2}(\mu + \lambda)$ *and* $\beta = \dfrac{\mu - \lambda}{\mu + \lambda}$.

(ii) *If either*

$$2X(\mu)Y(\mu) - X(\mu)Y(\lambda) + (\lambda - \mu)\mathit{Hess}\, \mu(X, Y) = 0 \tag{18}$$

or

$$2X(\lambda)Y(\lambda) - X(\mu)Y(\lambda) - (\lambda - \mu)\mathit{Hess}\, \lambda(X, Y) = 0, \tag{19}$$

then (shrinking U if necessary) the assertions below are equivalent:

(a) *$\langle\ ,\ \rangle$ is conformal to a Riemannian product metric on $M_0 \times M_1$;*

(b) *$\langle\ ,\ \rangle = \varphi^2\langle\ ,\ \rangle^\sim$, where $\langle\ ,\ \rangle^\sim$ is a Riemannian product metric on $M_0 \times M_1$ and $\varphi^{-1} = \phi_0 \circ \pi_0 + \phi_1 \circ \pi_1$ for $\phi_i \in C^\infty(M_i)$, $0 \le i \le 1$.*

(c) *Both (18) and (19) hold.*

(iii) *Equation (18) (resp., (19)) is satisfied if λ (resp., μ) is constant along E_λ (resp., E_μ), in particular if E_λ (resp., E_μ) has rank at least two. Therefore, the assumption in (ii) is always satisfied if $n \ge 3$. Moreover, if both λ and μ are constant along the corresponding eigenbundles, then the functions $\phi_i \in C^\infty(M_i)$, $0 \le i \le 1$, in (ii)–(b) can be chosen so that $\psi_*^{-1}\Phi\psi_* = (\phi_1 \circ \pi_1)\Pi_0 - (\phi_0 \circ \pi_0)\Pi_1$, where Π_0 and Π_1 denote the orthogonal projections onto the subbundles E_0 and E_1, respectively, of the product net (E_0, E_1) of $M_0 \times M_1$.*

Proof: By Proposition 18, E_λ and E_μ are umbilical with mean curvature normals

$$\eta = (\lambda - \mu)^{-1}(\nabla\, \lambda)_{E_\mu} \quad \text{and} \quad \zeta = (\mu - \lambda)^{-1}(\nabla\, \mu)_{E_\lambda}. \tag{20}$$

Thus (E_λ, E_μ) is a TP-net and the first assertion follows from Theorem 3. Now we have

$$\langle \nabla_X \eta, Y\rangle = X(\lambda - \mu)^{-1}Y(\lambda) + (\lambda - \mu)^{-1}\langle \nabla_X(\nabla\lambda)_{E_\mu}, Y\rangle. \tag{21}$$

Using that E_λ is umbilical with mean curvature normal η we obtain

$$\begin{aligned} \langle \nabla_X(\nabla\lambda)_{E_\mu}, Y\rangle &= \langle \nabla_X \nabla\lambda, Y\rangle - \langle \nabla_X(\nabla\lambda)_{E_\lambda}, Y\rangle \\ &= \mathrm{Hess}\, \lambda(X, Y) - \langle X, \nabla\lambda\rangle\langle \eta, Y\rangle \\ &= \mathrm{Hess}\, \lambda(X, Y) - (\lambda - \mu)^{-1}X(\lambda)Y(\lambda). \end{aligned} \tag{22}$$

Substituting (22) into (21) yields

$$\langle \nabla_X \eta, Y\rangle = -(\lambda-\mu)^{-2}(2X(\lambda)Y(\lambda) - X(\mu)Y(\lambda) - (\lambda-\mu)\mathrm{Hess}\,\lambda(X,Y)). \tag{23}$$

A similar computation using that E_μ is umbilical with mean curvature normal ζ gives

$$\langle \nabla_Y \zeta, X\rangle = -(\lambda-\mu)^{-2}(2X(\mu)Y(\mu) - X(\mu)Y(\lambda) + (\lambda-\mu)\mathrm{Hess}\,\mu(X,Y)). \tag{24}$$

Hence

$$\begin{aligned}(\lambda-\mu)^2(\langle \nabla_X \eta, Y\rangle - \langle \nabla_Y \zeta, X\rangle) =\\ -2X(\lambda)Y(\lambda) + 2X(\mu)Y(\mu) + (\lambda-\mu)\mathrm{Hess}\,(\lambda+\mu)(X,Y).\end{aligned} \tag{25}$$

Now, a straightforward computation using that $\lambda = \alpha(1-\beta)$ and $\mu = \alpha(1+\beta)$ yields

$$X(\lambda)Y(\lambda) - X(\mu)Y(\mu) = -4\beta X(\alpha)Y(\alpha) - 2\alpha X(\alpha)Y(\beta) - 2\alpha Y(\alpha)X(\beta). \tag{26}$$

Substituting (26) into (25) and observing that $\lambda+\mu = 2\alpha$ and $\lambda-\mu = -2\alpha\beta$, we obtain that (E_λ, E_μ) is a CP-net if and only if (17) holds. The assertion in (i) now follows from Proposition 11. We now prove (ii). By (23) and (24), we have that (18) and (19) are equivalent to E_μ and E_λ being spherical, respectively. Therefore, if either (18) or (19) holds, then the following are equivalent by Proposition 10:

(a') (E_λ, E_μ) is a CP-net; (b') E_λ and E_μ are spherical; (c) Both (18) and (19) hold.

The proof of (ii) is completed by observing that (a') is equivalent to (a) by Proposition 11 and that (b') is equivalent to (b) by Proposition 15 (shrinking U if necessary).

If λ (resp., μ) is constant along E_λ (resp., E_μ), which is always the case by Proposition 18-(ii) if E_λ (resp., E_μ) has rank at least two, then E_λ (resp., E_μ) is spherical by Proposition 18-(iii). Therefore the first assertion in (iii) follows from (23) (resp., (24)). Now assume that both λ and μ are constant along the corresponding eigenbundles. Denote $\tilde{\phi}_1 = \lambda \circ \psi$ and $\tilde{\phi}_0 = \mu \circ \psi$ the eigenvalues of $\psi_*^{-1}\Phi\psi_*$. Then the eigenbundles of $\tilde{\phi}_0$ and $\tilde{\phi}_1$ are E_0 and E_1, respectively, thus we have from (20) that the mean curvature normals of E_0 and E_1 are given, respectively, by

$$\eta = (\tilde{\phi}_1 - \tilde{\phi}_0)^{-1}\nabla\,\tilde{\phi}_1 \quad \text{and} \quad \zeta = (\tilde{\phi}_0 - \tilde{\phi}_1)^{-1}\nabla\,\tilde{\phi}_0.$$

On the other hand, since (E_0, E_1) is the product net of $M_0 \times M_1$, we have

$$\eta = -(\nabla\,\log\circ\varphi)_{E_1} \quad \text{and} \quad \zeta = -(\nabla\,\log\circ\varphi)_{E_0}.$$

It follows that $\nabla\,\log\circ(\tilde{\phi}_1 - \tilde{\phi}_0) = -\nabla\,\log\circ\varphi$, thus $\varphi^{-1} = A(\tilde{\phi}_1 - \tilde{\phi}_0)$ for some $A \neq 0$. By rescaling the metric $\langle\ ,\ \rangle$ we may assume that $A = 1$. The proof is completed by defining $\phi_i \in C^\infty(M_i)$, $0 \le i \le 1$, such that $\tilde{\phi}_0 = -\phi_0 \circ \pi_0$ and $\tilde{\phi}_1 = \phi_1 \circ \pi_1$. ∎

Remark 20 In case Φ is the shape operator of a surface in $\mathbb{R}^3$ then (17) reduces to a criterion for the surface to be isothermic in terms of its principal curvatures. In particular, since it is clearly satisfied if $\lambda + \mu = 2\alpha$ is constant on M, it implies the well-known fact that surfaces with constant mean curvature are isothermic surfaces.

Corollary 21 *Let M be a connected Riemannian manifold and let Φ be a Codazzi tensor on M with exactly two distinct eigenvalues λ and μ everywhere. Let E_λ and E_μ be the corresponding eigenbundles. Assume that μ is constant along E_μ and that $\lambda = h(\mu)$ for some smooth real function h. Then one of the following possibilities holds:*

(i) *λ is constant along E_λ and for every point $p \in M$ there exists a local product representation ψ: $M_0 \times M_1 \to U$ of (E_λ, E_μ) with $p \in U \subset M$, which is an isometry with respect to a Riemannian product metric $\langle\ ,\ \rangle$ on $M_0 \times M_1$. Moreover, $\psi_*^{-1}\Phi\psi_* = A_0\Pi_0 + A_1\Pi_1$ for some $A_0, A_1 \in \mathbb{R}$.*

(ii) *for every point $p \in M$ there exists a local product representation ψ: $I \times M_1 \to U$ of (E_λ, E_μ) with $p \in U \subset M$, where $I \subset \mathbb{R}$ is an open interval, which is an isometry with respect to a warped product metric $\langle\ ,\ \rangle$ on $I \times M_1$ with warping function $\sigma = \tilde{\sigma} \circ \pi_0$, where $\tilde{\sigma} \in C^\infty(I)$. Moreover, $\psi_*^{-1}\Phi\psi_* = h(\tilde{\mu} \circ \pi_0)\Pi_0 + (\tilde{\mu} \circ \pi_0)\Pi_1$, where $\tilde{\mu} \in C^\infty(I)$ is determined (up to a constant) by*

$$\int \frac{d\tilde{\mu}}{h(\tilde{\mu}) - \tilde{\mu}} = \log \tilde{\sigma}. \tag{27}$$

Proof: Since μ is constant along E_μ then $(\nabla\,\mu)_{E_\mu} = 0$, and hence also $(\nabla\,\lambda)_{E_\mu} = 0$ by the assumption that $\lambda = h(\mu)$ for some smooth real function h. It follows from (20) that E_λ is totally geodesic. If also λ is constant along E_λ, we obtain in a similar way that E_μ is totally geodesic, thus (i) follows from the local de Rham theorem and the fact that both λ and μ are now constant on M. Otherwise, by Proposition 18-(ii) the distribution E_λ must have rank one. Since E_μ is spherical by Proposition 18-(iii), then (E_λ, E_μ) is a WP-net. Thus, the first assertion in (ii) holds by Theorem 3. Finally, let E_0 and E_1 be the eigenbundles of $\psi_*^{-1}\Phi\psi_*$ correspondent to its eigenvalues $\lambda \circ \psi$ and $\mu \circ \psi$, respectively. Since (E_0, E_1) is the product net of $I \times M_1$, the mean curvature normal of E_1 is given, on one hand, by $\zeta = -\nabla\,\log \circ \sigma$, and on the other hand by $\zeta = (\mu \circ \psi - \lambda \circ \psi)^{-1}(\nabla\,\mu \circ \psi)_{E_0} = (\mu \circ \psi - h(\mu \circ \psi))^{-1}\nabla\,\mu \circ \psi$, because $(\nabla\,\mu \circ \psi)_{E_1} = 0$. Writing $\mu \circ \psi = \tilde{\mu} \circ \pi_0$ for $\tilde{\mu} \in C^\infty(I)$, we conclude that $\tilde{\mu}$ and $\tilde{\sigma}$ are related by (27). ∎

Remarks 22 (i) Codazzi tensors with two eigenvalues have also been studied in [5] and [6]. In [6] it is shown that the existence of such a tensor on a Riemannian manifold imposes some restrictions on its curvature tensor. One can check that such restrictions follow from the form of the curvature tensor of a twisted product (cf. formula (3) in [9]). Corollary 21 is proved in [5] for Codazzi tensors with constant trace.

(ii) Part (ii) of Corollary 21 can be regarded as an intrinsic version of Theorem 4.2 in [7], which states that a hypersurface of dimension $n \geq 3$ of a space form with two distinct principal curvatures λ and μ, one of which has multiplicity one, must be a rotation hypersurface whenever $\lambda = h(\mu)$ for some smooth real function h. In fact, that result follows from Corollary 21 together with Nölker's decomposition theorem for isometric immersions of warped products into space forms [10].

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