

# RIEMANNIAN $G$ -MANIFOLDS AS EUCLIDEAN SUBMANIFOLDS

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ABSTRACT. We survey on some recent developments on the study of Riemannian  $G$ -manifolds as Euclidean submanifolds.

## 1. INTRODUCTION

Let  $M^n$  be a complete Riemannian manifold of dimension  $n$  acted on by a connected closed subgroup  $G$  of its isometry group  $Iso(M^n)$ . We refer to  $(M^n, G)$ , or simply to  $M^n$  for short, as a *Riemannian  $G$ -manifold*. The codimension of a maximal dimensional  $G$ -orbit is called the  *$G$ -cohomogeneity* of  $M^n$ . In particular,  $M^n$  is  *$G$ -homogeneous* if it has  $G$ -cohomogeneity zero, that is, if  $G$  acts transitively on  $M^n$ .

Riemannian  $G$ -manifolds have been extensively studied from various points of view. In particular, those of low cohomogeneity have been used as a rich source of examples of Riemannian manifolds whose metrics have several important geometric properties, including metrics of positive sectional curvatures, Einstein and Ricci-flat metrics, and metrics with exceptional holonomy groups.

An interesting and natural problem is to study Riemannian  $G$ -manifolds as submanifolds of Euclidean space  $\mathbb{R}^N$ . These include submanifolds that are invariant by the action of a closed subgroup of the isometry group of  $\mathbb{R}^N$ . By a result of Moore [Mo], every compact homogeneous Riemannian manifold  $M^n$  can be realized in this way, namely,  $M^n$  admits an isometric embedding into some Euclidean space that is equivariant with respect to its full isometry group. On the other hand, a theorem of Kobayashi [Ko] states that an isometric immersion  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 2$ , of a compact homogeneous Riemannian manifold must be equivariant with respect to the full isometry group of  $M^n$ , thus implying  $f$  to be an embedding of  $M^n$  as a round hypersphere of  $\mathbb{R}^{n+1}$ .

The aim of this paper is to survey on some recent developments on this problem, mostly for the case of Riemannian  $G$ -manifolds as Euclidean hypersurfaces.

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## 2. HOMOGENEOUS HYPERSURFACES

The study of Riemannian  $G$ -manifolds as Euclidean submanifolds was initiated by the following result of Kobayashi [Ko] for compact homogeneous hypersurfaces.

**Theorem 1.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$  be an isometric immersion of a compact homogeneous Riemannian manifold. Then  $f$  embeds  $M^n$  as a round sphere.*

The main step in the proof of Kobayashi's theorem is to show that  $Iso(M^n)$  can be realized as a group of extrinsic isometries of Euclidean space, i.e., that there exists a Lie-group homomorphism  $\Phi: Iso(M^n) \rightarrow Iso(\mathbb{R}^{n+1})$  with respect to which  $f$  is *equivariant*, in the sense that  $f \circ g = \Phi(g) \circ f$  for all  $g \in Iso(M^n)$ . As in several subsequent results, this is accomplished by establishing the rigidity of  $f$ . Recall that an isometric immersion  $f: M^n \rightarrow \mathbb{R}^{n+1}$  is *rigid* if any other isometric immersion  $\tilde{f}: M^n \rightarrow \mathbb{R}^{n+1}$  differs from  $f$  by a rigid motion of  $\mathbb{R}^{n+1}$ . Under the assumptions of Kobayashi's theorem, one can start by using the well-known fact that any compact Euclidean hypersurface has a point with strictly positive sectional curvatures (and hence with nonvanishing principal curvatures) and then use the homogeneity of  $M^n$  to conclude that this must hold everywhere on  $M^n$ . For  $n \geq 3$ , the local rigidity theorem of Cartan-Beez-Killing can thus be applied: a Euclidean hypersurface of dimension  $n \geq 3$  is rigid whenever it has at least three nonzero principal curvatures everywhere. Applying this result to the pair of isometric immersions  $f$  and  $f \circ g$ , for each fixed  $g \in Iso(M^n)$ , one gets a rigid motion  $\tilde{g}$  such that  $f \circ g = \tilde{g} \circ f$ . Once we have this, it is not difficult to prove that the correspondence  $g \mapsto \tilde{g}$  defines a Lie-group homomorphism  $\Phi: Iso(M^n) \rightarrow Iso(\mathbb{R}^{n+1})$  such that  $f \circ g = \Phi(g) \circ f$  for all  $g \in Iso(M^n)$ . Moreover, the subgroup  $\Phi(Iso(M^n))$  of  $Iso(\mathbb{R}^{n+1})$  being compact, it has a fixed point by a well-known result of Cartan. This implies that  $\Phi(Iso(M^n))$  is contained in (a conjugacy class of) the orthogonal group  $O(n+1)$ . Since the image  $f(M^n)$  lies in a  $\Phi(Iso(M^n))$ -orbit, it must be a round hypersphere of  $\mathbb{R}^{n+1}$ . Finally, that  $f$  is an embedding follows by a standard covering map argument.

The extension of Kobayashi's theorem to the noncompact case came soon afterwards with the contribution by Nagano and Takahashi [NT], who proved that if  $f: M^n \rightarrow \mathbb{R}^{n+1}$  is an isometric immersion of a homogeneous Riemannian manifold then  $M^n$  is isometric to  $\mathbb{S}^m \times \mathbb{R}^{n-m}$ ,  $0 \leq m \leq n$ , provided that the rank  $\tau(f)$  of the second fundamental form of  $f$  is different from 2 at some point. Moreover,  $f$  splits as  $f = i \times id$  if  $\tau(f) \geq 3$  at some (and hence at any) point of  $M^n$ , where  $id: \mathbb{R}^{n-m} \rightarrow \mathbb{R}^{n-m}$  is the identity map and  $i: \mathbb{S}^m \rightarrow \mathbb{R}^{m+1}$ ,  $m \geq 3$ , is an umbilical inclusion, and otherwise it is a cylinder over a plane curve. The restriction on  $\tau(f)$  was later removed by Harle [Ha], as a consequence of a more general result on complete Euclidean hypersurfaces with constant scalar curvature. A simpler proof of Harle's theorem making use of the Gauss parameterization of hypersurfaces whose second fundamental forms have constant rank was subsequently given by Dajczer-Gromoll [DG<sub>2</sub>].

## 3. HOMOGENEOUS SUBMANIFOLDS OF CODIMENSION TWO

Non-equivariant isometric immersions  $f: M^n \rightarrow \mathbb{R}^{n+\ell}$ ,  $\ell \geq 2$ , of homogeneous Riemannian manifolds can be easily constructed by taking, for instance, a composition  $f = h \circ i$  of an umbilical inclusion  $i: \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$  with an isometric immersion  $h: V \rightarrow \mathbb{R}^{n+\ell}$  of an open subset  $V \subset \mathbb{R}^{n+1}$  containing  $i(\mathbb{S}^n)$ . This indicates that classifying homogeneous Euclidean submanifolds of higher codimension should be much harder than in the hypersurface case. Nevertheless, for compact submanifolds of dimension  $n \geq 5$  and codimension two this was accomplished by Castro-Noronha [CN]:

**Theorem 2.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+2}$ ,  $n \geq 5$ , be an isometric immersion of a compact homogeneous Riemannian manifold. Then one of the following possibilities holds:*

- (i)  *$f$  is equivariant with respect to an orthogonal representation*

$$\Phi: Iso(M^n) \rightarrow O(n+2),$$

*and hence  $f$  is a homogeneous isoparametric hypersurface of  $\mathbb{S}^{n+1}$ ;*

- (ii)  *$M^n$  is isometric to  $\mathbb{S}^n$ ;*

- (iii)  *$M^n$  is isometrically covered by  $\mathbb{R} \times \mathbb{S}^{n-1}$ .*

The authors have actually been able to obtain the same conclusion by assuming, instead of compactness of  $M^n$ , that  $M^n$  has at least one point where the index of relative nullity of  $f$ , that is, the dimension of the kernel of its second fundamental form, is not greater than  $n - 5$ .

The main tool in the proof is a rigidity result for Euclidean submanifolds due to do Carmo-Dajczer [dCD], which implies that a Euclidean  $n$ -dimensional submanifold of dimension  $n \geq 5$  and codimension two is rigid whenever its index of relative nullity is everywhere less than or equal to  $n - 5$  and there are at least three nonzero principal curvatures in every normal direction. Using this result, the authors prove that a compact homogeneous Euclidean submanifold of codimension two is rigid if it has at least one point where the index of relative nullity is not greater than  $n - 5$ , unless at every point one can find an orthonormal basis  $\xi, \eta$  of the normal space at that point such that the shape operators  $A_\eta$  and  $A_\xi$  satisfy  $\text{rank } A_\eta \leq 2$  and  $g_* \circ A_\xi = A_\xi \circ g_*$  for every intrinsic isometry  $g$  of the submanifold. Rigidity forces the submanifold to be extrinsically homogeneous, as previously discussed, giving the first possibility in the statement. The proof proceeds by considering separately the cases in which the rank of  $A_\eta$  is either 1 or 2. The former leads to the two remaining possibilities in the statement, whereas in the most delicate case where  $\text{rank } A_\eta = 2$  it is shown that the submanifold is an extrinsic product  $\mathbb{S}^2 \times \mathbb{S}^{n-2}$ .

As far as we know, it remains an open problem to determine the cases in which a homogeneous Euclidean submanifold of codimension greater than two can fail to be extrinsically homogeneous. In the next sections we go back to the case of  $G$ -hypersurfaces, but allow the intrinsic  $G$ -action to be no longer transitive.

4. COMPLETE  $G$ -HYPERSURFACES OF COHOMOGENEITY ONE

In this section we discuss Riemannian  $G$ -manifolds of cohomogeneity one isometrically immersed in Euclidean space as hypersurfaces. We start with the simplest examples:

**4.1. Hypersurfaces of revolution.** Consider a curve  $\gamma$  that is either contained in the interior of a half-space  $\mathbb{R}_+^2 \subset \mathbb{R}^{n+1}$  or meets its boundary  $\mathbb{R}$  orthogonally. Now let the subgroup  $G$  of  $SO(n+1)$  that fixes  $\mathbb{R}$  act on  $\gamma$ . This gives rise to a  $G$ -invariant hypersurface  $M^n$  of  $\mathbb{R}^{n+1}$ , called a *hypersurface of revolution* with  $\gamma$  as profile. The group  $G$  acts on  $M^n$  with one-codimensional round spheres as principal orbits. In particular, they are *umbilical* submanifolds of  $M^n$ , i.e., at any point their principal curvatures coincide. It was shown by Podestá-Spiro [PS] that this property characterizes hypersurfaces of revolution among compact hypersurfaces of cohomogeneity one with dimension  $n \geq 4$ :

**Theorem 3.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 4$ , be an isometric immersion of a compact Riemannian manifold. Assume that a closed connected subgroup  $G$  of  $Iso(M^n)$  acts on  $M^n$  with cohomogeneity one. If the  $G$ -principal orbits are umbilical in  $M^n$  then  $f$  is a hypersurface of revolution.*

For any manifold  $M^n$  of  $G$ -cohomogeneity one, and any open interval  $I \subset \Omega \setminus \partial\Omega$  in the interior of the orbit space  $\Omega$  (which is diffeomorphic to either  $S^1$ ,  $\mathbb{R}$ ,  $[0, 1]$  or  $(0, 1)$ ), the inverse image  $\pi^{-1}(I)$  by the quotient map is  $G$ -equivariantly diffeomorphic to  $I \times G/H$ , where  $H$  denotes an isotropy subgroup of principal type (see, e.g., [AA]). One of the main ingredients in Podestá-Spiro's proof of Theorem 3 is the observation that umbilicity of the principal orbits implies the metric induced on  $I \times G/H$  by this diffeomorphism to be a *warped product* metric  $\langle \cdot, \cdot \rangle = dt^2 + \phi(t)^2 \langle \cdot, \cdot \rangle_0$ .

The assumption of umbilicity of the principal orbits in Podestá-Spiro's theorem has been weakened in subsequent developments. Namely, still in the compact case the same conclusion was obtained by assuming, instead, that  $n \geq 4$  and all  $G$ -principal orbits have constant sectional curvature [AMN], and that  $n \geq 5$  and all  $G$ -principal orbits have positive sectional curvatures [CN].

The noncompact case of Podestá-Spiro's result was studied by Mercuri-Seixas in [MS]. Here, even for cohomogeneity one isometric actions of compact Lie groups it is easy to construct non-rotational examples. For instance, consider the standard action of  $O(n)$  on  $\mathbb{R}^n$  and isometrically immerse  $\mathbb{R}^n$  into  $\mathbb{R}^{n+1}$  as a cylinder over a plane curve. Moreover, asking the hypersurface not to be everywhere flat is not enough, as pointed out in [MS]. For actions of compact groups, the right assumption turns out to be that no connected component of the flat part of the hypersurface be unbounded. The case of complete noncompact cohomogeneity one hypersurfaces under the action of a closed and noncompact Lie group (with umbilical principal orbits) was also considered in [MS].

**4.2. Standard examples.** More general examples of Euclidean  $G$ -hypersurfaces of cohomogeneity one, with  $G$  compact and connected, are produced as follows. Start with a cohomogeneity two closed connected subgroup  $G \subset SO(n+1)$ , so that the orbit space  $\mathbb{R}^{n+1}/G$  is a two dimensional manifold, possibly with boundary. Now consider the hypersurface  $M^n$  of  $\mathbb{R}^{n+1}$  given by the inverse image under the canonical projection onto  $\mathbb{R}^{n+1}/G$  of a curve that is either contained in the interior of  $\mathbb{R}^{n+1}/G$  or meets its boundary orthogonally. We call  $M^n$  a *standard example*.

The natural question that emerges is whether, under reasonable global assumptions, the standard examples comprise all Euclidean hypersurfaces of cohomogeneity one. The answer is given by the following result of [MPST]:

**Theorem 4.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$  be an isometric immersion of a complete Riemannian manifold acted on with cohomogeneity one by a compact connected subgroup of  $Iso(M^n)$ . If either  $n \geq 3$  and  $M^n$  is compact, or if  $n \geq 5$  and the connected components of the flat part of  $M^n$  are bounded, then  $f$  is either rigid or a hypersurface of revolution. In particular, it is a standard example.*

The proof of Theorem 4 relies on a global rigidity result due to Sacksteder [Sa], which states that a compact Euclidean hypersurface  $f: M^n \rightarrow \mathbb{R}^{n+1}$  of dimension  $n \geq 3$  is rigid whenever the subset of totally geodesic points of  $f$  does not disconnect  $M^n$ . In fact, the arguments in an unpublished proof of Sacksteder's theorem due to Ferus actually show more than the preceding statement. Namely, if  $f, \tilde{f}: M^n \rightarrow \mathbb{R}^{n+1}$  are isometric immersions of a complete Riemannian manifold of dimension  $n \geq 3$  such that there exists no complete leaf of dimension  $n-1$  or  $n-2$  of the relative nullity distribution of  $f$ , then it is shown that their second fundamental forms are related by  $\alpha_{\tilde{f}}(x) = \pm\phi(\alpha_f(x))$  at every  $x \in M^n$ , where  $\phi$  is a vector bundle isometry between the normal bundles of  $f$  and  $\tilde{f}$ . This allows one to prove the following general result [MPST], [MT]:

**Theorem 5.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 3$ , be a complete hypersurface. Assume that there exists no complete leaf of relative nullity of  $f$  of dimension  $n-1$  or  $n-2$ . Then there exists a Lie-group homomorphism  $\Phi: Iso^0(M^n) \rightarrow Iso(\mathbb{R}^{n+1})$  of the identity component of  $Iso(M^n)$  into  $Iso(\mathbb{R}^{n+1})$  such that  $f \circ g = \Phi(g) \circ f$  for all  $g \in Iso^0(M^n)$ .*

Notice that Theorem 5 easily implies the results of Kobayashi and, more generally, of Nagano-Takahashi mentioned in Section 1. Namely, if  $f: M^n \rightarrow \mathbb{R}^{n+1}$  is an isometric immersion of a homogeneous Riemannian manifold such that  $\tau(f) \neq 2$  at some point, then either  $M^n$  is flat, and hence  $f$  is a cylinder over a plane curve by a well-known theorem of Hartman-Nirenberg, or  $\tau(f) \geq 3$  and Theorem 5 applies: it follows that  $f(M^n)$  is an orbit of a cohomogeneity one subgroup of  $Iso^0(\mathbb{R}^{n+1})$ , in which case  $f$  is an isoparametric hypersurface of  $\mathbb{R}^{n+1}$ , whence an extrinsic product  $\mathbb{S}^m \times \mathbb{R}^{n-m}$ ,  $m \geq 3$ .

The proof of Theorem 5 is simple enough to be included here in almost full detail: given  $g \in Iso^0(M^n)$ , let  $\alpha_{f \circ g}$  denote the second fundamental form of  $f \circ g$ . On one hand, using that  $g$  is an isometry we have

$$\alpha_{f \circ g}(x)(X, Y) = \alpha_f(gx)(g_*X, g_*Y)$$

for every  $g \in Iso^0(M^n)$ ,  $x \in M^n$  and  $X, Y \in T_x M^n$ . In particular, this implies that the map  $\Theta_x: Iso^0(M^n) \rightarrow \text{Sym}(T_x M^n \times T_x M^n \rightarrow T_x^\perp M_f^n)$  into the vector space of symmetric bilinear maps of  $T_x M^n \times T_x M^n$  into  $T_x^\perp M_f^n$ , given by

$$\Theta_x(g)(X, Y) = \phi_g(x)^{-1}(\alpha_{f \circ g}(x)(X, Y)) = \phi_g(x)^{-1}(\alpha_f(gx)(g_*X, g_*Y))$$

for all  $X, Y \in T_x M^n$ , is continuous. On the other hand, by the preceding remarks on Sacksteder's theorem, either  $\alpha_{f \circ g}(x) = \phi_g(x) \circ \alpha_f(x)$  or  $\alpha_{f \circ g}(x) = -\phi_g(x) \circ \alpha_f(x)$ , where  $\phi_g$  is a vector bundle isometry between  $T^\perp M_f^n$  and  $T^\perp M_{f \circ g}^n$ . Thus  $\Theta_x$  is a continuous map taking values in  $\{\alpha_f(x), -\alpha_f(x)\}$ , whence must be constant by the connectedness of  $Iso^0(M^n)$ . Since  $\Theta_x(\text{id}) = \alpha_f(x)$ , it follows that  $\alpha_{f \circ g}(x) = \phi_g(x) \circ \alpha_f(x)$  for every  $g \in Iso^0(M^n)$  and  $x \in M^n$ . By the Fundamental Theorem of Hypersurfaces, for each  $g \in Iso^0(M^n)$  there exists  $\tilde{g} \in Iso(\mathbb{R}^{n+1})$  such that  $f \circ g = \tilde{g} \circ f$ . It now follows from standard arguments that  $g \mapsto \tilde{g}$  defines a Lie-group homomorphism  $\Phi: Iso^0(M^n) \rightarrow Iso(\mathbb{R}^{n+1})$ , whose image must lie in  $Iso^0(\mathbb{R}^{n+1})$  because it is connected.

As a consequence of Theorem 5, if  $B$  denotes the set of totally geodesic points of  $f: M^n \rightarrow \mathbb{R}^{n+1}$  as in the statement, then

$$(1) \quad \Phi(Iso^0(M^n)) \text{ has a fixed vector whenever } B \text{ is nonempty.}$$

Before we sketch a proof of this assertion, we first observe that it already implies the conclusion of Theorem 4 in the compact case. Namely, if  $G$  is a compact connected subgroup of  $Iso(M^n)$  acting with cohomogeneity one, then (1) and Sacksteder's theorem imply that  $\Phi(G)$  has a fixed vector  $v$ , hence every orbit of  $\Phi(G)$  through a point not in the line spanned by  $v$  is a hypersphere of an affine hyperplane orthogonal to  $v$ . Thus  $f$  is a hypersurface of revolution. If  $M^n$  is not compact, one still has to consider the case in which  $M^n$  carries a complete leaf of relative nullity of  $f$  of dimension  $n-2$ . Here a key rôle is played by the results of [CN], applied to the orbits of  $G$  as compact homogeneous submanifolds of  $\mathbb{R}^{n+1}$  with codimension two. In this case  $f$  turns out to be an extrinsic product  $\mathbb{S}^2 \times \mathbb{R}^{n-2}$ , hence rigid, with products  $\mathbb{S}^2 \times \mathbb{S}^{n-3}(r)$  as the  $G$ -principal orbits.

Now, to prove (1) one first uses the equivariance of  $f$  in order to show that  $B$  is invariant under  $Iso^0(M^n)$ . It then follows that the orbit of  $Iso^0(M^n)$  through a point  $p \in B$  is a connected subset of totally geodesic points, whence must be contained in a hyperplane  $\mathcal{H}$  that is tangent to  $f$  along it by a lemma of [DG<sub>1</sub>]. Therefore, a unit vector  $v$  orthogonal to  $\mathcal{H}$  spans  $T_{gp}^\perp M_f^n$  for every  $g \in Iso^0(M^n)$ . Since  $\tilde{g}T_p^\perp M_f^n = T_{gp}^\perp M_f^n$  for every  $\tilde{g} = \Phi(g) \in \Phi(Iso^0(M^n))$ , because  $f$  is equivariant with respect to  $\Phi$ , the connectedness of  $\Phi(Iso^0(M^n))$  implies that it must fix  $v$ .

5. COMPACT LOCALLY POLAR  $G$ -HYPERSURFACES

An isometric action of a compact Lie group  $G$  on a Riemannian manifold  $M^n$  is said to be *locally polar* if the distribution of normal spaces to principal orbits on the regular part of  $M^n$  is integrable. Then  $M^n$  is called a locally polar Riemannian  $G$ -manifold. For instance, any isometric action of cohomogeneity one is locally polar. It was shown by Heintze-Liu-Olmos [HLO], answering a question posed by Palais-Terng [PT], that for any complete locally polar Riemannian  $G$ -manifold there exists a connected complete immersed submanifold  $\Sigma$  of  $M^n$  that intersects orthogonally all  $G$ -orbits. Such a submanifold is called a *section*, and it is always a totally geodesic submanifold of  $M^n$ . The action is said to be *polar* if there exists a closed and embedded section. Clearly, for orthogonal representations there is no distinction between polar and locally polar actions, for in this case sections are just affine subspaces. Polar orthogonal representations have been classified by Dadok [D]. As a consequence of his classification, he obtained that every polar representation is orbit equivalent to (i.e., has the same orbits as) the isotropy representation of a semi-simple symmetric space.

It was shown in [BCO] (see Proposition 3.2.9) that if a closed subgroup of  $SO(N)$  acts polarly on  $\mathbb{R}^N$  and leaves invariant a submanifold  $f: M^n \rightarrow \mathbb{R}^N$ , then its restricted action on  $M^n$  is locally polar. It is not true, however, that every locally polar action on a Euclidean submanifold arises in this way, even assuming it to be compact. For instance, consider a compact submanifold  $f: M^n \rightarrow \mathbb{R}^{n+2}$  that is invariant under the action of one of the closed subgroups  $G \subset SO(n+2)$  that act non-polarly on  $\mathbb{R}^{n+2}$  with cohomogeneity three. Then the induced action of  $G$  on  $M^n$  has cohomogeneity one, whence is locally polar. Nevertheless, for compact hypersurfaces of dimension  $n \geq 3$  it was shown in [MT] that this is indeed the case.

**Theorem 6.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 3$ , be an isometric immersion of a compact Riemannian manifold. Assume that a closed connected subgroup  $G$  of  $Iso(M^n)$  acts locally polarly on  $M^n$  with cohomogeneity  $k$ . Then there exists an orthogonal representation  $\Psi: G \rightarrow SO(n+1)$  such that  $\Psi(G)$  acts polarly on  $\mathbb{R}^{n+1}$  with cohomogeneity  $k+1$  and  $f \circ g = \Psi(g) \circ f$  for every  $g \in G$ .*

For the proof of Theorem 6, one starts by recalling that  $f$  is equivariant with respect to a Lie-group homomorphism  $\Phi: Iso^0(M^n) \rightarrow Iso(\mathbb{R}^{n+1})$  by Theorem 5, which here must actually be an orthogonal representation of  $Iso^0(M^n)$  as a closed subgroup of  $SO(n+1)$  by the compactness and connectedness of  $Iso^0(M^n)$ . Thus, it suffices to prove that  $\Phi(G)$  acts polarly on  $\mathbb{R}^{n+1}$  and set  $\Psi = \Phi|_G$ . For that, the idea is to make use of a result of Palais-Terng [PT], according to which an orthogonal representation in  $\mathbb{R}^{n+1}$  of a compact Lie group  $G$  is polar whenever it has an orbit that is an isoparametric submanifold  $M^k \subset \mathbb{R}^{n+1}$  and, in addition, either  $M^k$  is full (i.e., not contained in any affine subspace) or, if otherwise,  $G$  acts trivially on the orthogonal complement of the linear span of  $M^k$ . Recall that a submanifold  $M^k \subset \mathbb{R}^{n+1}$  is *isoparametric* if it

has flat normal bundle and the principal curvatures with respect to every parallel normal vector field along any curve in  $M^k$  are constant (with constant multiplicities). The proof then consists of proving that these conditions are satisfied by the action of  $\Phi(G)$ , for which the main tool is to use that principal orbits of locally polar isometric actions are characterized by the fact that equivariant normal vector fields are parallel with respect to the normal connection.

The simplest examples of hypersurfaces  $f: M^n \rightarrow \mathbb{R}^{n+1}$  that are invariant by a polar action on  $\mathbb{R}^{n+1}$  of a closed subgroup of  $SO(n+1)$  are the hypersurfaces of revolution (with possibly higher dimensional profiles), which are produced by the action on a hypersurface  $L^k$  (possibly with boundary) of a half-space of  $\mathbb{R}^{k+1} \subset \mathbb{R}^{n+1}$ , disjoint to the boundary  $\mathbb{R}^k$  or orthogonal to it, of a closed subgroup  $\tilde{G}$  of  $SO(n+1)$  that fixes  $\mathbb{R}^k$  and acts transitively on the hyperspheres of the orthogonal complement  $\mathbb{R}^{n-k+1}$  of  $\mathbb{R}^k$ . The following result of [MT] gives several sufficient conditions for a compact Euclidean hypersurface as in Theorem 6 to be a hypersurface of revolution.

**Corollary 7.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 3$ , be an isometric immersion of a compact Riemannian manifold. Assume that a closed connected subgroup  $G$  of  $\text{Iso}(M^n)$  acts locally polarly on  $M^n$  with cohomogeneity  $k$ . Then any of the following additional conditions implies that  $f$  is a hypersurface of revolution:*

- (i) *there exists a totally geodesic (in  $M^n$ )  $G$ -principal orbit;*
- (ii)  *$k = n - 1$ ;*
- (iii) *the  $G$ -principal orbits are umbilical in  $M^n$ ;*
- (iv) *there exists a  $G$ -principal orbit with nonzero constant sectional curvatures;*
- (v) *there exists a  $G$ -principal orbit with positive sectional curvatures.*

*Moreover, in this case  $G$  is isomorphic to one of the closed subgroups of  $SO(n-k+1)$  that act transitively on  $\mathbb{S}^{n-k}$ .*

For the case of hypersurfaces of cohomogeneity  $k = 1$ , the same conclusion was also derived in the noncompact case in [MPST] under the assumptions of Theorem 4. Another sufficient condition obtained in [MPST] for a hypersurface of  $G$ -cohomogeneity one and dimension  $n \geq 4$  as in Theorem 4 to be a hypersurface of revolution is that some principal  $G$ -orbit be homeomorphic to a sphere. This was accomplished by combining Theorem 4 and the fact that an isoparametric hypersurface of the sphere of dimension  $n \geq 4$  can not have a sphere as its universal covering, unless it is a round sphere. The proof of the latter uses several of the known restrictions on the number  $g$  of principal curvatures of isoparametric hypersurfaces and on their multiplicities in order to show that the only possibility allowed for an isoparametric hypersurface of dimension  $n \geq 4$  covered by a sphere is that  $g = 1$ .

For the proof of Corollary 7, since one already knows from Theorem 6 that there exists an orthogonal representation  $\Psi: G \rightarrow SO(n+1)$  such that  $\tilde{G} = \Psi(G)$  acts polarly on  $\mathbb{R}^{n+1}$  and  $f \circ g = \Psi(g) \circ f$  for every  $g \in G$ , it suffices to show



that any of the conditions in the statement implies that

(2)  $f(Gp)$  is a round sphere (circle, if  $k = n - 1$ ) for some principal orbit  $Gp$ .

Indeed, since  $\tilde{G}$  acts polarly on  $\mathbb{R}^{n+1}$  it fixes the subspace orthogonal to the linear span of  $f(Gp)$ , and hence  $f$  is a hypersurface of revolution. Moreover, a standard covering map argument shows that  $f|_{Gp}$  must be an embedding if  $k \neq n - 1$ . Thus, if  $k \neq n - 1$  and  $\Psi(g)$  is the identity for some  $g \in G$  then it follows from  $f \circ g = \Psi(g) \circ f$  and the injectivity of  $f|_{Gp}$  that  $g$  must fix any point of  $Gp$ . Since  $Gp$  is a principal orbit, this easily implies that  $g = \text{id}$ , and hence that  $\Psi$  is an isomorphism of  $G$  onto  $\tilde{G}$ .

Now, to prove (2) one uses the fact that  $f$  immerses  $Gp$  as an isoparametric submanifold, since  $f$  is equivariant with respect to  $\Psi$  and principal orbits of polar representations are isoparametric submanifolds [PT]. If some principal  $G$ -orbit is totally geodesic, then the first normal spaces of  $f|_{Gp}$  in  $\mathbb{R}^{n+1}$ , that is, the subspaces of the normal spaces spanned by the image of its second fundamental form, are one-dimensional. Since for isoparametric submanifolds the first normal spaces always form a parallel subbundle of the normal bundle with respect to the normal connection, a standard reduction of codimension result implies that  $f(Gp) = \tilde{G}(f(p))$  is contained as a hypersurface in some affine hyperplane  $\mathcal{H} \subset \mathbb{R}^{n+1}$ , and thus is a round hypersphere of  $\mathcal{H}$  (a circle if  $k = n - 1$ ). Conditions (ii) and (iii) both imply (i). In fact, if  $k = n - 1$  then the principal orbit of maximal length must be a geodesic. Similarly, if all principal orbits are umbilical in  $M^n$  then the one of maximal volume is both minimal and umbilical, whence totally geodesic. Finally, one can show that an isoparametric submanifold that has either constant and nonzero or positive sectional curvatures must be a round sphere, which yields the statement under the remaining conditions (iv) and (v), respectively.

Corollary 7 and the results of [MPST] for cohomogeneity one hypersurfaces mentioned after it generalize and give simpler proofs of Podestá-Spiro's theorem and some of its extensions described in Section 3.1. Further extensions have been recently obtained in [Mou], where the case of complete Euclidean hypersurfaces of dimension  $n \geq 4$  acted on locally polarly with umbilical principal orbits of codimension  $k$  by a closed connected subgroup  $G$  of its isometry group was considered. In case  $G$  is compact and  $1 \leq k \leq n - 3$  the hypersurface was shown to be of revolution if the connected components of the flat part of the hypersurface are bounded. If  $G$  is not compact, the hypersurface is not everywhere flat and, for a given section  $\Sigma$ , the connected components of the subset of  $\Sigma$  formed by flat points of the hypersurface are assumed to be bounded, it was shown that the hypersurface is either an extrinsic product  $\Sigma^k \times \mathbb{R}^{n-k}$ , the principal orbits being the fibers  $\{y\} \times \mathbb{R}^{n-k}$ ,  $y \in \Sigma^k$ , or an extrinsic product  $\mathbb{R}^k \times \mathbb{S}^\ell \times \mathbb{R}^{n-k-\ell}$ ,  $\ell \geq 2$ , in which case the principal orbits are the fibers  $\{x\} \times \mathbb{S}^\ell \times \mathbb{R}^{n-k-\ell}$ ,  $x \in \mathbb{R}^k$ . For  $k = 1$  these results reduce to the ones in [MS] referred to at the end of Section 3.1.

In part (v) of Corollary 7, it was shown in [MT] that weakening the assumption to non-negativity of the sectional curvatures of some principal  $G$ -orbit implies  $f$  to be a *multi-rotational hypersurface* in the sense of [DN]:

**Corollary 8.** *Under the assumptions of Theorem 6, suppose further that there exists a  $G$ -principal orbit  $Gp$  with nonnegative sectional curvatures. Then there exist an orthogonal decomposition  $\mathbb{R}^{n+1} = \bigoplus_{i=0}^k \mathbb{R}^{n_i}$  into  $\tilde{G}$ -invariant subspaces, where  $\tilde{G} = \Psi(G)$ , and connected Lie subgroups  $G_1, \dots, G_k$  of  $\tilde{G}$  such that  $G_i$  acts on  $\mathbb{R}^{n_i}$ , the action being transitive on  $\mathbb{S}^{n_i-1} \subset \mathbb{R}^{n_i}$ , and the action of  $\tilde{G} = G_1 \times \dots \times G_k$  on  $\mathbb{R}^{n+1}$  given by*

$$(g_1 \dots g_k)(v_0, v_1, \dots, v_k) = (v_0, g_1 v_1, \dots, g_k v_k)$$

*is orbit equivalent to the action of  $\tilde{G}$ . In particular, if  $Gp$  is flat then  $n_i = 2$  and  $G_i$  is isomorphic to  $SO(2)$  for  $i = 1, \dots, k$ .*

Corollary 7 was used in [MT] to study a problem that is seemingly unrelated to isometric actions. Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$  be a hypersurface of revolution as described in the paragraph preceding Corollary 7. Then the open and dense subset of  $M^n$  that is mapped by  $f$  onto the complement of the axis  $\mathbb{R}^k$  is isometric to the warped product  $L^k \times_\rho N^{n-k}$ , where  $N^{n-k}$  is the orbit of some fixed point  $f(p) \in L^k$  under the action of  $\tilde{G}$ , and the warping function  $\rho: L^k \rightarrow \mathbb{R}_+$  is a constant multiple of the distance to  $\mathbb{R}^k$ . Recall that a *warped product*  $N_1 \times_\rho N_2$  of Riemannian manifolds  $(N_1, \langle \cdot, \cdot \rangle_{N_1})$  and  $(N_2, \langle \cdot, \cdot \rangle_{N_2})$  with *warping function*  $\rho: N_1 \rightarrow \mathbb{R}_+$  is the product manifold  $N_1 \times N_2$  endowed with the metric  $\langle \cdot, \cdot \rangle = \pi_1^* \langle \cdot, \cdot \rangle_{N_1} + (\rho \circ \pi_1)^2 \pi_2^* \langle \cdot, \cdot \rangle_{N_2}$ , where  $\pi_i: N_1 \times N_2 \rightarrow N_i$ ,  $1 \leq i \leq 2$ , denote the canonical projections. The problem is then to determine whether hypersurfaces of revolution are characterized by their intrinsic warped product structure. For compact Euclidean hypersurfaces of dimension  $n \geq 3$  this was answered affirmatively in [MT]:

**Theorem 9.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 3$ , be a compact hypersurface. If there exists an isometry onto an open and dense subset  $U \subset M^n$  of a warped product  $L^k \times_\rho N^{n-k}$  with  $N^{n-k}$  connected and complete (in particular if  $M^n$  is isometric to a warped product  $L^k \times_\rho N^{n-k}$  with  $N^{n-k}$  connected) then  $f$  is a hypersurface of revolution.*

Theorem 9 can be seen as a global version in the hypersurface case of the local classification in [DT] of isometric immersions in codimension  $\ell \leq 2$  of warped products  $L^k \times_\rho N^{n-k}$ ,  $n-k \geq 2$ , into Euclidean space. It follows from the results of [DT] that an isometric immersion  $f: L^k \times_\rho N^{n-k} \rightarrow \mathbb{R}^{n+1}$ ,  $n-k \geq 2$ , of a warped product with no flat points is either a hypersurface of revolution or an extrinsic product  $\mathbb{R}^{k-1} \times CN^{n-k}$ , where  $CN^{n-k} \subset \mathbb{R}^{n-k+2}$  denotes the cone over a hypersurface  $N^{n-k}$  of  $\mathbb{S}^{n-k+1}$ .

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