Isothermic submanifolds of Euclidean space

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Abstract: We study the problem posed by F. Burstall of developing a theory of isothermic Euclidean submanifolds of dimension greater than or equal to three. As a natural extension of the definition in the surface case, we call a Euclidean submanifold *isothermic* if it is locally the image of a conformal immersion of a Riemannian product of Riemannian manifolds whose second fundamental form is adapted to the product net of the manifold. Our main result is a complete classification of all such conformal immersions of Riemannian products of dimension greater than or equal to three. We derive several consequences of this result. We also study whether the classical characterizations of isothermic surfaces as solutions of Christoffel's problem and as envelopes of nontrivial conformal sphere congruences extend to higher dimensions.

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1 Introduction

The classical theory of isothermic surfaces has received a recent renaissance in interest mostly due to its connection with the modern theory of integrable systems (cf. [**Bu**] and the references therein). Recall that a surface in Euclidean space $\mathbb{R}^3$ is *isothermic* if, away from umbilic points, its curvature lines form an isothermic net, that is, there exist locally conformal coordinates that diagonalize the second fundamental form. Standard examples include cylinders, cones, surfaces of revolution, quadrics, surfaces of constant mean curvature and, the concept being a conformally invariant one, the images of such surfaces by an inversion in $\mathbb{R}^3$.

Isothermic surfaces have a number of characterizations from various points of view. One of them comes with the problem posed by Christoffel [**Ch**] of determining the surfaces $f\colon M^2 \to \mathbb{R}^3$ that admit locally a *dual* surface $\mathcal{F}\colon M^2 \to \mathbb{R}^3$ with parallel tangent planes to those of f which induces the same conformal structure but opposite orientation on M^2. It was shown by Christoffel himself that this property characterizes isothermic surfaces, and dual isothermic surfaces are now said to be *Christoffel transforms* one of each other.

Another characterization involves the notion of *Ribaucour transformation*. Classically, two surfaces in $\mathbb{R}^3$ are said to be related by a Ribaucour transformation when they are the enveloping surfaces of a two-parameter sphere congruence such that their curvature lines correspond. It was proved by Darboux ([**Da**$_1$]) that if two surfaces in $\mathbb{R}^3$ are related by a *conformal* Ribaucour transformation that reverts orientation then they must be a *Darboux pair* of isothermic surfaces.

A nice account of classical aspects of the subject as well as from the perspective of the modern theory of integrable systems and loop groups was given in [**Bu**]. In the concluding remarks of that paper, a list of open problems is posed that suggests interesting guides for future research. We quote here the first of them:

Is there any interesting theory of isothermic submanifolds of $\mathbb{R}^n$ of dimension greater than two? The problem here is to find a suitable definition that is not too restrictive... ...One way forward might be to study submanifolds admitting a conformal sphere congruence.

This article grew out as an attempt to give an answer to this question. The preliminary step of defining higher dimensional analogues of isothermic nets and geometrically characterizing them was considered in [**To**]. This turned out to fit into the general problem of studying local decomposability properties of *orthogonal nets* on Riemannian manifolds. Recall that an orthogonal k-net $\mathcal{E} = (E_i)_{i=1,\dots,k}$ on a Riemannian manifold M is a splitting $TM = \oplus_{i=1}^k E_i$ of the tangent bundle of M by a family of mutually orthogonal integrable subbundles. It is natural to require a higher dimensional isothermic net $\mathcal{E} = (E_i)_{i=1,\dots,k}$ on a Riemannian manifold M to have the property that each point $p \in M$ lies in an open neighborhood U such that there exists a conformal diffeomorphism of a Riemannian product $\Pi_{i=1}^k M_i$ onto U that maps the leaves of the product foliation induced by M_i onto the leaves of E_i, $1 \leq i \leq k$. A geometric characterization of such orthogonal nets was given in Theorem 4.3 of [**To**] (see Theorem 3 below), which may be regarded as a conformal version of the local decomposition theorem of de Rham. Following a more suggestive terminology used in [**MRS**] for other types of orthogonal nets, they were named *conformal product nets* in [**To**], or CP-nets for short.

The next step is to look for the appropriate extension of the property that *the curvature lines* form an isothermic net. A little thought then makes it natural to call a Euclidean submanifold *isothermic*, or more precisely, *k-isothermic*, if it carries a k-CP-net $\mathcal{E} = (E_i)_{i=1,\dots,k}$ to which its second fundamental form is adapted, in the sense that each subbundle E_i is invariant by all shape operators of the submanifold.

Summing things up, a submanifold is isothermic if it is locally given by a conformal immersion of a Riemannian product whose second fundamental form is adapted to the product net of the manifold. Thus, in studying isothermic Euclidean submanifolds of dimension $n \geq 3$ we are faced with the following problem:

Describe the conformal immersions of a Riemannian product of dimension $n \geq 3$ into Euclidean space whose second fundamental forms are adapted to its product net.

The isometric version of this problem (in any dimension $n \geq 2$) has led to a useful result due to Moore ([**Mo**]), who proved that an isometric immersion of a Riemannian product into Euclidean space whose second fundamental form is adapted to the product net of the manifold must be an extrinsic product of isometric immersions of the factors. Notice that the composition of such an extrinsic product of k isometric immersions with a conformal transformation of Euclidean space provides trivial examples of k-isothermic submanifolds for any value of k. These examples can be seen as the higher dimensional analogues of the images by an inversion of cylinders in $\mathbb{R}^3$ over plane curves. Similar examples can be produced by composing an extrinsic product of isometric immersions into a sphere with a stereographic projection of that sphere onto Euclidean space.

Further examples of isothermic submanifolds arise by means of a conformal diffeomorphism $\Theta\colon \mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) \to \mathbb{R}^N$, $1 \leq m \leq N-1$, onto the complement of an $(m-1)$-dimensional sphere (a point if $m = 1$) in $\mathbb{R}^N$ (see Section 3). Here $\mathbb{H}^m(-c) = \{X \in \mathbb{L}^{m+1} \colon \langle X, X\rangle = -1/c\}$ and $\mathbb{S}^{N-m}(c) = \{X \in \mathbb{R}^{N-m+1} \colon \langle X, X\rangle = 1/c\}$, where $\langle\ ,\ \rangle$ stands for the Lorentz inner product in Minkowski space $\mathbb{L}^{m+1}$ in the first case and the standard Euclidean inner product in the latter. Thus, if $m \geq 2$ (resp., $N - m \geq 2$) then $\mathbb{H}^m(-c)$ (resp., $\mathbb{S}^{N-m}(c)$) is the standard model of hyperbolic space (resp., the sphere) with constant sectional curvature $-c$ (resp., c) as an umbilical hypersurface of $\mathbb{L}^{m+1}$ (resp., $\mathbb{R}^{N-m+1}$). Then, for any isometric immersion $f\colon M_1 \to \mathbb{H}^m(-c)$ and any extrinsic product $g\colon M_2 \times \ldots \times M_k \to \mathbb{S}^{N-m}(c)$ of isometric immersions, the conformal immersion $\Theta \circ (f \times g)$ defines a new example of a k-isothermic submanifold. In two dimensions, these examples reduce to conformal parameterizations by curvature lines of the images of cones and surfaces of revolution by an inversion.

The abundance of isothermic surfaces in $\mathbb{R}^3$ gives a hint that their higher dimensional analogues should be equally as plentiful. The hint turns out to be surprisingly misleading: we are able to show that the examples just described actually comprise all isothermic submanifolds of dimension $n \geq 3$. We leave the precise statement for Section 4 (see Theorem 5). From the point of view of the theory of integrable systems, the result is rather disappointing: there is no such theory in connection with isothermic submanifolds of dimension $n \geq 3$. The discrepancy between the richness of the two-dimensional theory and the shortage of examples in higher dimensions should be compared to that between the classes of conformal maps between open subsets of Euclidean space in two and higher dimensions. In fact, our result contains as a special case Liouville's theorem on the classification of such conformal maps in dimension $n \geq 3$.

Having our definition of isothermic submanifolds not led to as rich a theory as in the two-dimensional case, one might think that it should be more appropriate to choose a definition based on an extension of one of the properties that characterize isothermic surfaces in Euclidean three-space. For instance, one might look for those submanifolds that admit a nontrivial conformal sphere congruence, as suggested by Burstall, or search for the solutions of some natural extension of Christoffel's problem. We carry this out and give a complete classification of both classes of submanifolds. Unfortunately, they turn out to be distinct proper subclasses of the class of isothermic submanifolds as previously defined.

Before we draw this introduction to a close, we provide a brief guide to the contents of each section and describe some applications of our main result.

We start Section 2 by reviewing from [**RS**] the notions of nets and net morphisms between netted manifolds, a suitable setting for dealing with decomposition problems for manifolds and for maps between them. Then we recall some basic facts on twisted and warped products and state a characterization of twisted and warped product metrics on product manifolds in terms of geometric properties of its product net [**MRS**]. We close the section by stating some results of [**To**], where a geometric characterization of higher dimensional isothermic nets in Riemannian manifolds was obtained, leading to a conformal version of the local de Rham theorem.

In Section 3 we give a brief description of the model of Euclidean space as an umbilical hypersurface of the light cone in Minkowski space, and summarize some facts that are needed in the sequel. In particular, we recall how to construct the aforementioned conformal diffeomorphism $\Theta\colon \mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) \to \mathbb{R}^N$ and prove a formula relating the second fundamental forms of a conformal immersion into Euclidean space and its associated isometric immersion into the light cone.

Our main results are contained in Section 4. We obtain a local classification of isothermic Euclidean submanifolds of dimension $n \geq 3$, or equivalently, we describe all conformal immersions of a Riemannian product of dimension $n \geq 3$ into Euclidean space whose second fundamental forms are adapted to the product net of the manifold. We follow with several applications. First, we specialize to surfaces in Euclidean space and generalize to arbitrary codimension a classical result of Bonnet on surfaces whose curvature lines have constant geodesic curvature (Corollary 9). Then, we classify all conformal representations of Euclidean space of dimension $n \geq 3$ as a Riemannian product, that is, we determine all conformal local diffeomorphisms of a Riemannian product of dimension $n \geq 3$ onto an open subset of Euclidean space (Corollary 10). Besides containing Liouville's theorem as a special case, this also provides a geometric proof of the known classification of conformally flat Riemannian products [**La**]. We follow with a decomposition theorem for isometric immersions of a twisted product of two Riemannian manifolds into Euclidean space (Corollary 11), which contains as a special case the main result of [**DFT**] on the classification of Euclidean submanifolds that carry a Dupin principal normal vector field with umbilical conullity. As a consequence, we provide an alternate statement (Corollary 13) of the well-known classification of the cyclides of Dupin of arbitrary dimension [**Ce**].

In Section 5, after stating some basic facts on Codazzi tensors we formulate an extension of Christoffel problem for Euclidean submanifolds of arbitrary dimension and codimension, and determine all of its solutions of dimension $n \geq 3$. We conclude the paper by classifying in the last section all Euclidean submanifolds of dimension $n \geq 3$ that admit nontrivial conformal sphere congruences.

2 Preliminaries

A suitable setting for treating decomposition results for manifolds was developed in [**RS**] by introducing the category of *netted manifolds*. A *net* $\mathcal{E} = (E_i)_{i\in I_k}$ on a connected C^∞-manifold M is a splitting $TM = \oplus_{i\in I_k} E_i$ by a family of integrable subbundles. Here and throughout the paper we denote $I_k = \{1,\ldots,k\}$. If M is a Riemannian manifold and the subbundles are mutually orthogonal then the net is said to be an *orthogonal net*. The canonical net on a product manifold $M = \Pi_{i=1}^k M_i$ is called the *product net*. A C^∞-map ψ: $M \to N$ between two *netted manifolds* $(M,\mathcal{E})$, $(N,\mathcal{F})$, that is, manifolds M, N equipped with nets $\mathcal{E} = (E_i)_{i\in I_k}$ and $\mathcal{F} = (F_i)_{i\in I_k}$, respectively, is called a *net morphism* if $d\psi(E_i(p)) \subset F_i(\psi(p))$ for all $p \in M$, $i \in I_k$, or equivalently, if for any $p \in M$ the restriction $\psi|_{L_i^{\mathcal{E}}(p)}$ of ψ to the leaf $L_i^{\mathcal{E}}(p)$ of E_i through p is a C^∞-map into the leaf $L_i^{\mathcal{F}}(\psi(p))$ of F_i through $\psi(p)$. The net morphism ψ is said to be a *net isomorphism* if in addition it is a diffeomorphism and ψ^{-1} is also a net morphism. A net $\mathcal{E}$ on M is said to be *locally decomposable* if for every $p \in M$ there exist a neighborhood U of p in M and a net isomorphism ψ from $(U, \mathcal{E}|_U)$ onto a product manifold $\Pi_{i=1}^k M_i$. The map ψ^{-1}: $\Pi_{i=1}^k M_i \to U$ is called a *product representation* of $(U, \mathcal{E}|_U)$.

Given a product $M = \Pi_{i=1}^k M_i$ of C^∞ manifolds $M_1, \ldots, M_k$, a metric $\langle\ ,\ \rangle$ on M is called a *twisted product metric* if there exist Riemannian metrics $\langle\ ,\ \rangle_i$ on M_i, $i \in I_k$, and a C^∞ *twist-function* $\rho = (\rho_1, \ldots, \rho_k)$: $M \to \mathbb{R}_+^k$ such that $\langle\ ,\ \rangle = \sum_{i=1}^k \rho_i^2 \pi_i^* \langle\ ,\ \rangle_i$. Then $(M, \langle\ ,\ \rangle)$ is said to be a *twisted product* and is denoted by ${}^\rho\Pi_{i=1}^k (M_i, \langle\ ,\ \rangle_i)$. When ρ_1 is identically 1 and $\rho_2, \ldots, \rho_k$ are independent of $M_2, \ldots, M_k$, that is, there exist $\tilde\rho_i \in C^\infty(M_1)$ such that $\rho_i = \tilde\rho_i \circ \pi_1$ for $i = 2, \ldots, k$, then $\langle\ ,\ \rangle$ is called a *warped product metric* and $(M, \langle\ ,\ \rangle) := (M_1, \langle\ ,\ \rangle_1) \times_{\tilde\rho} \Pi_{i=2}^k (M_i, \langle\ ,\ \rangle_i)$ a *warped product* with *warping function* $\tilde\rho = (\tilde\rho_2, \ldots, \tilde\rho_k)$. If ρ_i is identically 1 for all $i \in I_k$, then $\langle\ ,\ \rangle$ is a *Riemannian product metric* and $(M, \langle\ ,\ \rangle)$ a *Riemannian product*.

An orthogonal net $\mathcal{E} = (E_i)_{i\in I_k}$ on a Riemannian manifold M is called a *TP-net* if E_i is umbilical and $E_i^\perp$ is integrable for every $i \in I_k$. Recall that a subbundle E of TM is *umbilical* if there exists a vector field η in $E^\perp$ such that $(\nabla_X Y)_{E^\perp} = \langle X, Y\rangle \eta$ for all $X, Y \in \Gamma(E)$. Here and in the sequel, the space of smooth local sections of a vector bundle E over M is denoted by $\Gamma(E)$, whereas writing a vector field with a vector subbundle as a subscript indicates taking the section of that vector subbundle obtained by orthogonally projecting the vector field pointwise onto the corresponding fiber of the subbundle. The vector field η is called the *mean curvature normal* of E. If, in addition, $(\nabla_X \eta)_{E^\perp} = 0$ for all $X \in \Gamma(E)$, then E is said to be *spherical*. If E is umbilical and its mean curvature normal vanishes identically, then it is called *totally geodesic* (or *auto-parallel*). An umbilical distribution is automatically integrable, and the leaves are umbilical submanifolds of M. When E is totally geodesic or spherical, its leaves are totally geodesic or spherical submanifolds, respectively. By a *spherical submanifold* we mean an umbilical submanifold whose mean curvature vector is parallel with respect to the normal connection.

An orthogonal net $\mathcal{E} = (E_i)_{i \in I_k}$ is called a *WP-net* if E_i is spherical and $E_i^\perp$ is totally geodesic for $i = 2, \ldots, k$. This easily implies that E_1 is totally geodesic and $E_1^\perp$ is integrable, thus every WP-net is also a TP-net. The terminologies TP-net and WP-net are justified by the following result (see Proposition 4 of [**MRS**]).

Proposition 1 *On a connected product manifold $M =: \Pi_{i=1}^k M_i$ the product net $\mathcal{E} = (E_i)_{i \in I_k}$ is a TP-net (resp., WP-net) with respect to a Riemannian metric $\langle\ ,\ \rangle$ on M if and only if $\langle\ ,\ \rangle$ is a twisted product (resp., warped product) metric on M. Moreover, if $\rho = (\rho_1, \ldots, \rho_k)\colon M \to \mathbb{R}_+^k$ is the twist function and $U_i = -grad(log \circ \rho_i)$, $i \in I_k$, where the gradient is calculated with respect to $\langle\ ,\ \rangle$, then the mean curvature normal of E_i is $\eta_i = (U_i)_{E_i^\perp}$ for all $i \in I_k$.*

An orthogonal net $\mathcal{E} = (E_i)_{i \in I_k}$ on a Riemannian manifold is a *conformal product net*, or a CP-net for short, if for $i = 1, \ldots, k$ it holds that

$$E_i \text{ and } E_i^\perp \text{ are umbilical and } \langle \nabla_{X_{\perp_i}} \eta_i, X_i \rangle = \langle \nabla_{X_i} H_i, X_{\perp_i} \rangle \tag{1}$$

for all $X_i \in \Gamma(E_i)$ and $X_{\perp_i} \in \Gamma(E_i^\perp)$, where H_i and η_i are the mean curvature normals of E_i and $E_i^\perp$, respectively. We have (see Proposition 4.2 of [**To**]):

Proposition 2 *On a connected and simply connected product manifold $M = \Pi_{i=1}^k M_i$ the product net $\mathcal{E} = (E_i)_{i \in I_k}$ is a CP-net with respect to a Riemannian metric $\langle\ ,\ \rangle^\sim$ on M if and only if $\langle\ ,\ \rangle^\sim$ is conformal to a Riemannian product metric.*

By means of Proposition 2, the following conformal version of the local de Rham Theorem was obtained in [**To**]. It shows that conformal product nets are natural generalizations of isothermic nets on surfaces.

Theorem 3 *If a Riemannian manifold M carries a CP-net $\mathcal{E} = (E_i)_{i \in I_k}$, then for every $p \in M$ there exists a local product representation $\psi\colon \Pi_{i=1}^k M_i \to U$ of $\mathcal{E}$ with $p \in U \subset M$, which is conformal with respect to a Riemannian product metric on $\Pi_{i=1}^k M_i$.*

3 Möbius geometry in the light cone

We give a brief description of the model of Euclidean space as an umbilical hypersurface of the light cone of Minkowski space, a convenient setting for dealing with Möbius geometric notions. We refer the reader to [**H-J**] for further details.

Let $\mathbb{L}^{N+2}$ be the $(N+2)$–dimensional Minkowski space endowed with a Lorentz scalar product of signature $(+, \ldots, +, -)$, and let $\mathbb{V}^{N+1} = \{p \in \mathbb{L}^{N+2}\colon \langle p, p \rangle = 0\}$ denote the light cone in $\mathbb{L}^{N+2}$. Then $\mathbb{E}^N = \mathbb{E}_w^N = \{p \in \mathbb{V}^{N+1}\colon \langle p, w \rangle = 1\}$ is a model of N–dimensional Euclidean space for any $w \in \mathbb{V}^{N+1}$. Namely, choose $p_0 \in \mathbb{E}^N$ and a linear isometry $A\colon \mathbb{R}^N \to \operatorname{span}\{p_0, w\}^\perp$. Then the triple (p_0, w, A) gives rise to an isometry $\Psi = \Psi_{p_0,w,A}\colon \mathbb{R}^N \to \mathbb{E}^N \subset \mathbb{L}^{N+2}$ defined by $x \in \mathbb{R}^N \mapsto p_0 + A(x) - (1/2)\|x\|^2 w$.

Hyperspheres can be nicely described in $\mathbb{E}^N$: given a hypersphere $S \subset \mathbb{E}^N$ with (constant) mean curvature h with respect to a unit normal vector field n along S, then $v = n_p + hp \in \mathbb{L}^{N+2}$ is a constant space-like unit vector such that $\langle v, p\rangle = 0$ for all $p \in S$; thus $S = \mathbb{E}^N \cap \{v\}^\perp$. Since $h = \langle v, w\rangle$, then S is a hyperplane if and only if $\langle v, w\rangle = 0$.

The intersection angle of two (oriented) hyperspheres has also a simple description in this model: given hyperspheres $S_i = \mathbb{E}^N \cap \{v_i\}^\perp$ with unit normal vectors n_p^i, $1 \le i \le 2$, at a common point p, their intersection angle at p is given by $\langle n_p^1, n_p^2\rangle = \langle v_1, v_2\rangle$. Thus S_1 and S_2 intersect orthogonally if and only if $\langle v_1, v_2\rangle = 0$.

A hypersphere $S = \mathbb{E}^N \cap \{v\}^\perp$ has (Euclidean) center at $q_0 \in \mathbb{E}^N$ and mean curvature $h \neq 0$ if and only if $v = hq_0 + (2h)^{-1}w$. This follows from $\langle v, w\rangle = h$, $\langle v, v\rangle = 1$ and $\operatorname{span}\{q_0, w\}^\perp \subset v^\perp$, the latter being due to the fact that any hyperplane through q_0 is orthogonal to S.

Given a conformal immersion $G\colon M^n \to \mathbb{V}^{N+1}$ with conformal factor $\varphi \in C^\infty(M^n)$, for any $\mu \in C^\infty(M^n)$ the map $G_\mu\colon M^n \to \mathbb{V}^{N+1}$, $p \mapsto \mu(p)G(p)$, is also conformal with conformal factor $\mu\varphi$. Therefore, any conformal immersion $f\colon M^n \to \mathbb{R}^N$ with conformal factor $\varphi \in C^\infty(M^n)$ gives rise to an isometric immersion $\mathcal{I}(f) = \mathcal{I}_{p_0,w,A}(f) := (\Psi \circ f)_{\varphi^{-1}}\colon M^n \to \mathbb{V}^{N+1}$. Conversely, if $F\colon M^n \to \mathbb{V}^{N+1}$ is an isometric immersion with $F(M^n) \subset \mathbb{V}^{N+1} \setminus \mathbb{R}_w$, where $\mathbb{R}_w = \{tw : t > 0\}$, define $\mathcal{C}(F) = \mathcal{C}_{p_0,w,A}(F)\colon M^n \to \mathbb{R}^N$ by $\Psi \circ \mathcal{C}(F) = \Pi \circ F$, where $\Pi = \Pi_w\colon \mathbb{V}^{N+1} \setminus \mathbb{R}_w \to \mathbb{E}^N$ is the projection onto $\mathbb{E}^N$ given by $\Pi(x) = x/\langle x, w\rangle$. Since Π is conformal with conformal factor $\varphi_\Pi(x) = \langle x, w\rangle^{-1}$, then $\mathcal{C}(F)$ is also conformal with conformal factor $\varphi_\Pi \circ F = \langle F, w\rangle^{-1}$.

In particular, conformal transformations of $\mathbb{R}^N$ are linearized in this model: any $T \in \mathbb{O}_1(N+2)$ gives rise to a conformal (Möbius) transformation $\mathcal{T} = \mathcal{C}(T \circ \Psi)$ of $\mathbb{R}^N$ and, conversely, any Möbius transformation of $\mathbb{R}^N$ is given in this way by means of some $T \in \mathbb{O}_1(N+2)$. For instance, if R is the reflection $R(p) = p - 2\langle p, v\rangle v$ with respect to the hyperplane in $\mathbb{L}^{N+2}$ orthogonal to the unit space-like vector v, $\langle v, w\rangle \neq 0$, then $I = \mathcal{C}(R \circ \Psi)$ is the inversion with respect to the hypersphere $S = \mathbb{E}^N \cap \{v\}^\perp$. If $T \in \mathbb{O}_1(N+2)$ satisfies $T(w) = \lambda w$ for some $\lambda \in \mathbb{R}$ then there exists a similarity $\mathcal{H}$ of $\mathbb{R}^N$ of ratio λ such that $\mathcal{C}(T \circ \Psi) = \mathcal{H}$, i.e., $\Psi \circ \mathcal{H} = \lambda T \circ \Psi$. In particular, the isometries of $\mathbb{R}^N$ are given by those T that fix w.

Clearly, we have $\mathcal{C}_{p_0,w,A}(\mathcal{I}_{p_0,w,A}(f)) = f$ and $\mathcal{I}_{p_0,w,A}(\mathcal{C}_{p_0,w,A}(F)) = F$ for any conformal immersion $f\colon M^n \to \mathbb{R}^N$ and for any isometric immersion $F\colon M^n \to \mathbb{V}^{N+1}$ with $F(M^n) \subset \mathbb{V}^{N+1} \setminus \mathbb{R}_w$. For distinct triples (p_0, w, A) and $(\bar{p}_0, \bar{w}, \bar{A})$, there exist an inversion I with respect to a sphere of unit radius and a similarity $\mathcal{H}$ such that

$$\mathcal{C}_{p_0,w,A}(\Psi_{\bar{p}_0,\bar{w},\bar{A}}) = I \circ \mathcal{H}. \tag{2}$$

Let us check this: consider the reflection $R(p) = p - 2\langle p, v\rangle v$ determined by the unit space-like vector $v = \langle \bar{w}, w\rangle^{-1}\bar{w} + (1/2)w$ and let $T \in \mathbb{O}_1(N+2)$ be defined by $T(w) = R(\bar{w}) = -(1/2)\langle \bar{w}, w\rangle w$, $T(p_0) = R(\bar{p}_0)$ and $T \circ A = R \circ \bar{A}$. Then $R \circ T$ takes w to $\bar{w}$, p_0

to $\bar{p}_0$ and $R \circ T \circ A = \bar{A}$, whence $\Psi_{\bar{p}_0,\bar{w},\bar{A}} = R \circ T \circ \Psi_{p_0,w,A}$. Since $\mathcal{C}_{p_0,w,A}(R \circ \Psi_{p_0,w,A}) = I$ and $\mathcal{C}_{p_0,w,A}(T \circ \Psi_{p_0,w,A}) = \mathcal{H}$ for an inversion I in $\mathbb{R}^N$ with respect to the hypersphere $S = \mathbb{E}^N_w \cap \{v\}^\perp$ of unit radius with center at $\langle \bar{w}, w\rangle^{-1}\bar{w} \in \mathbb{E}^N_w$ and a similarity $\mathcal{H}$ on $\mathbb{R}^N$ of ratio $\lambda = -(1/2)\langle \bar{w}, w\rangle$, we obtain

$$\begin{aligned}\Psi_{p_0,w,A} \circ \mathcal{C}_{p_0,w,A}(\Psi_{\bar{p}_0,\bar{w},\bar{A}}) &= \Pi_w \circ \Psi_{\bar{p}_0,\bar{w},\bar{A}} = \Pi_w \circ R \circ T \circ \Psi_{p_0,w,A}\\ &= \Pi_w \circ R \circ \Psi_{p_0,w,A} \circ \mathcal{H} = \Psi_{p_0,w,A} \circ I \circ \mathcal{H}.\end{aligned}$$

Given a time-like vector $v \in \mathbb{L}^{N+2}$ with $\langle v, v\rangle = -1/c$ and a linear isometry $B\colon \mathbb{R}^{N+1} \to \{v\}^\perp$, the isometric immersion $T_{B,v}\colon \mathbb{S}^N(c) \to \mathbb{L}^{N+2}$, $X \in \mathbb{S}^N(c) \mapsto B(X)+v$, takes values in $\mathbb{V}^{N+1}$. When $c = 1$ and (p_0, w, A) is a triple such that $\{v, u\}$ is an orthonormal basis of span$\{p_0, w\}$ with $w = v + u$ and $A = B|_{\mathbb{R}^N}$, then a direct verification shows that $\mathcal{C}_{p_0,w,A}(T_{B,v})$ is a stereographic projection of $\mathbb{S}^N$ onto $\mathbb{R}^N$. In general, there exists a similarity $\mathcal{H}$ of $\mathbb{R}^{N+1}$ and a stereographic projection $\mathcal{P}\colon \mathbb{S}^N \to \mathbb{R}^N$ such that

$$\mathcal{C}_{p_0,w,A}(T_{B,v}) = \mathcal{P} \circ \mathcal{H}. \tag{3}$$

Now we construct a conformal diffeomorphism $\Theta\colon \mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) \to \mathbb{R}^N$ onto the complement of an $(m-1)$-dimensional sphere as referred to in the introduction. Namely, given an orthogonal decomposition $\mathbb{L}^{N+2} = V \oplus W$ with V time-like and linear isometries $C\colon \mathbb{L}^{m+1} \to V$ and $D\colon \mathbb{R}^{N-m+1} \to W$, define an isometric immersion $L_{C,D}\colon \mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) \to \mathbb{V}^{N+1} \subset \mathbb{L}^{N+2}$ by $(X, Y) \mapsto C(X) + D(Y)$, and set

$$\Theta = \mathcal{C}_{p_0,w,A}(L_{C,D}) \tag{4}$$

for some triple (p_0, w, A). Observe that the image of Θ omits the $(m-1)$-dimensional sphere that is mapped onto $\mathbb{E}^N_w \cap V$ by the isometry $\Psi_{p_0,w,A}\colon\ \mathbb{R}^N \to \mathbb{E}^N$. Alternately, let $\Phi\colon\ \mathbb{R}^m_+ \times_\sigma \mathbb{S}^{N-m}(c) \to \mathbb{R}^N \setminus \mathbb{R}^{m-1}$ be the isometry given by $(X, Y) \mapsto (x_1, \ldots, x_{m-1}, \sigma(X)Y)$ for $X \in \mathbb{R}^m_+$ and $Y \in \mathbb{S}^{N-m}(c)$, where $\mathbb{R}^m_+$ (resp., $\mathbb{R}^{m-1}$) denotes the subset of points $X = (x_1, \ldots, x_m)$ of $\mathbb{R}^m$ where $x_m > 0$ (resp., $x_m = 0$), and $\sigma(X) = c^{1/2}x_m$. Endowing $\mathbb{R}^m_+$ with the metric $\sigma^{-2}\langle\ ,\ \rangle$, where $\langle\ ,\ \rangle$ is the Euclidean metric, we obtain the half-space model of $\mathbb{H}^m(-c)$, and then Φ gives rise to a conformal diffeomorphism Θ_Φ with conformal factor $\sigma \circ \pi_0$ with respect to the product metric of $\mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c)$. Any Θ just defined differs from such a Θ_Φ by an inversion in $\mathbb{R}^N$.

We conclude this section by computing the relation between the second fundamental forms of a conformal immersion $f\colon\ M^n \to \mathbb{R}^N$ with conformal factor $\varphi \in C^\infty(M^n)$ and its associated isometric immersion $F = \mathcal{I}_{p_0,w,A}(f)\colon\ M^n \to \mathbb{V}^{N+1} \subset \mathbb{L}^{N+2}$.

Lemma 4 *The normal bundle $T^\perp_F M$ of $F\colon\ M^n \to \mathbb{V}^{N+1} \subset \mathbb{L}^{N+2}$ decomposes orthogonally as $T^\perp_F M = d\Psi(T^\perp_f M) \oplus \mathbb{L}^2$, where $\mathbb{L}^2$ is the Lorentzian plane bundle spanned by the position vector F and $\eta := \varphi w - d(\Psi \circ f)(grad\, \varphi^{-1})$, and the second fundamental form of F splits accordingly as*

$$\alpha_F(X, Y) = \varphi\, Hess\ \varphi^{-1}(X, Y)F + \varphi^{-1}d\Psi(\alpha_f(X, Y)) - \langle X, Y\rangle\eta, \tag{5}$$

where Hess and grad are calculated with respect to the metric $\langle\ ,\ \rangle$ induced by F.

Proof: Differentiating $F = \varphi^{-1}(\Psi \circ f)$ gives $dF(X) = X(\varphi^{-1})(\Psi \circ f) + \varphi^{-1} d(\Psi \circ f)(X)$, which already implies that $T_F^\perp M$ splits as stated. Now we compute

$$\alpha_F(X, Y) = \bar{\nabla}_Y dF(X) - dF(\nabla_Y X), \tag{6}$$

where $\bar{\nabla}$ and ∇ stand, respectively, for the (pull-back to M^n of the) derivative in $\mathbb{L}^{N+2}$ and the Levi-Civita connection of M^n with respect to $\langle\ ,\ \rangle$. Using that

$$\bar{\nabla}_Y d(\Psi \circ f)(X) = -\varphi^2 \langle X, Y\rangle w + d\Psi(\hat{\nabla}_Y df(X)),$$

where $\hat{\nabla}$ is the (pull-back to M^n of the) derivative on $\mathbb{R}^N$, we have

$$\begin{aligned} \bar{\nabla}_Y dF(X) = {} & YX(\varphi^{-1})(\Psi \circ f) + X(\varphi^{-1}) d(\Psi \circ f)(Y) + Y(\varphi^{-1}) d(\Psi \circ f)(X) \\ & -\varphi \langle X, Y\rangle w + \varphi^{-1} d\Psi(\hat{\nabla}_Y df(X)). \end{aligned} \tag{7}$$

On the other hand, since

$$\nabla_Y X = \tilde{\nabla}_Y X + \varphi X(\varphi^{-1}) Y + \varphi Y(\varphi^{-1}) X - \varphi \langle X, Y\rangle \operatorname{grad}\varphi^{-1},$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the metric induced by f, we obtain

$$\begin{aligned} dF(\nabla_Y X) &= \nabla_Y X(\varphi^{-1})(\Psi \circ f) + \varphi^{-1} d(\Psi \circ f)(\nabla_Y X) \\ &= \nabla_Y X(\varphi^{-1})(\Psi \circ f) + \varphi^{-1} d\Psi(df(\tilde{\nabla}_Y X)) + X(\varphi^{-1}) d(\Psi \circ f)(Y) \\ &\quad + Y(\varphi^{-1}) d(\Psi \circ f)(X) - \langle X, Y\rangle d(\Psi \circ f)(\operatorname{grad}\varphi^{-1}). \end{aligned} \tag{8}$$

Equation (5) now follows from (6), (7) and (8). ∎

4 Conformal immersions of Riemannian products

In this section we prove our main result, namely, we give a complete description of all conformal immersions of a Riemannian product of dimension $n \geq 3$ into Euclidean space whose second fundamental forms are adapted to the product net of the manifold. As discussed in the introduction, this yields a local classification of all isothermic submanifolds of dimension $n \geq 3$ of Euclidean space.

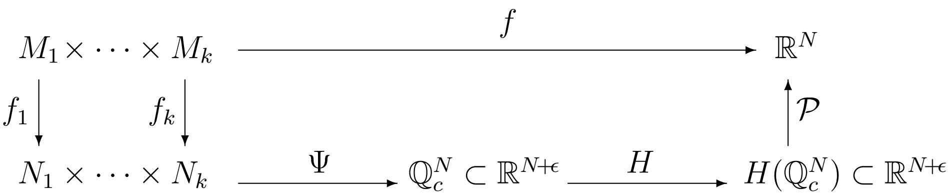
For an orthogonal decomposition $\mathbb{R}^N = \oplus_{i=1}^{k+1} \mathbb{R}^{m_i}$, with $\mathbb{R}^{m_{k+1}}$ possibly trivial, for every $i \in I_k$ let N_i denote either $\mathbb{R}^{m_i}$ or $\mathbb{S}^{m_i - 1}(c_i) = \{X_i \in \mathbb{R}^{m_i} : \langle X_i, X_i\rangle = 1/c_i\}$ (in which case $m_i \geq 2$), and define $\Psi\colon \Pi_{i=1}^k N_i \to \mathbb{R}^N$ by $\Psi(x_1, \ldots, x_k) = (i_1(x_1), \ldots, i_k(x_k), v_{k+1})$, where $i_j\colon N_j \to \mathbb{R}^{m_j}$, $1 \leq j \leq k$, is either the identity or the inclusion map, respectively, and $v_{k+1} \in \mathbb{R}^{k+1}$ is a constant vector. When $N_i = \mathbb{S}^{m_i - 1}(c_i)$ for every $i \in I_k$, then Ψ takes values in (a small sphere of) $\mathbb{S}^{N-1}(c)$, with $1/c = \sum_{i=1}^k 1/c_i + \langle v_{k+1}, v_{k+1}\rangle$. We call Ψ an *extrinsic product* of $N_1, \ldots, N_k$. Given a product manifold $M = \Pi_{i=1}^k M_i$ we denote by $T_j\colon M \to M_j \times \Pi_{i \in I_k^j} M_i$ the map $(p_1, \ldots, p_k) \mapsto (p_j, (p_1, \ldots, \hat{p}_j, \ldots, p_k))$, where $I_k^j = I_k \setminus \{j\}$ and the hat indicates that p_j is missing.

Theorem 5 *Let f: $M^n := \Pi_{i=1}^k M_i^{n_i} \to \mathbb{R}^N$, $n \geq 3$, be a conformal immersion of a Riemannian product whose second fundamental form is adapted to the product net of M^n. Then one of the following holds:*

(i) *There exist an extrinsic product Ψ: $\Pi_{i=1}^k N_i \to \mathbb{Q}_c^N \subset \mathbb{R}^{N+\epsilon}$ of complete spherical submanifolds of $\mathbb{Q}_c^N$, where $c \geq 0$, $\epsilon = 1$ if $c > 0$ and $\epsilon = 0$ if $c = 0$, and isometric immersions f_i: $M_i \to N_i$, $1 \leq i \leq k$, such that*

$$f = \mathcal{P} \circ H \circ \Psi \circ (f_1 \times \cdots \times f_k),$$

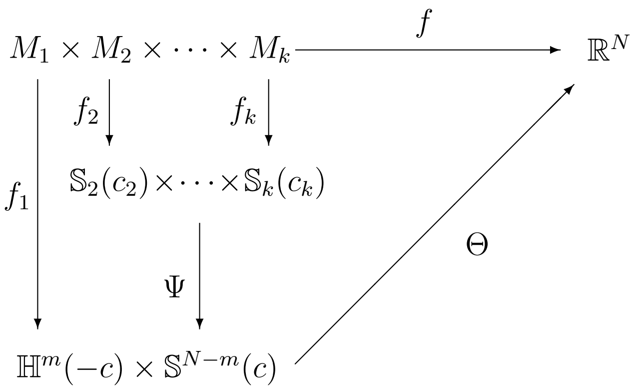
where H is a homothety of $\mathbb{R}^{N+\epsilon}$ and $\mathcal{P}$ is either a stereographic projection of $H(\mathbb{Q}_c^N) = \mathbb{S}^N$ onto $\mathbb{R}^N$ if $c > 0$ or an inversion in $\mathbb{R}^N$ with respect to a sphere of unit radius otherwise.



(ii) *For exactly one $j \in I_k$ there is an extrinsic product Ψ: $\Pi_{i \in I_k^j} \mathbb{S}^{m_i}(c_i) \to \mathbb{S}^{N-m}(c)$, isometric immersions f_j: $M_j \to \mathbb{H}^m(-c)$ and f_i: $M_i \to \mathbb{S}^{m_i}(c_i)$, $i \in I_k^j$, and a conformal diffeomorphism Θ: $\mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) \to \mathbb{R}^N$ such that*

$$f = \Theta \circ (f_j \times (\Psi \circ (f_0 \times \cdots \times \hat{f}_j \times \cdots \times f_k)) \circ T_j).$$

Case (ii)
for $j = 1$



Proof: Define $F = \mathcal{I}_{p_0,w,A}(f)$: $M^n \to \mathbb{V}^{N+1} \subset \mathbb{L}^{N+2}$ as in Section 3, and let $\mathcal{E} = (E_i)_{i \in I_k}$ be the product net of M^n. We first prove:

Lemma 6 *The second fundamental form of F is adapted to $\mathcal{E}$.*

Proof: It suffices to consider the case $k = 2$ and then, relabelling if necessary, we take $n_1 \geq 2$. Fixed $p = (p_1, p_2) \in M^n$ and $\hat{X} \in E_2(p)$, denote $L = M_1^{n_1} \times \{p_2\}$, let $\bar{X} = d\pi_2(p)(\hat{X}) \in T_{p_2}M_2^{n_2}$ and, for any $q \in L$, let $\hat{X}(q)$ be the unique vector in $E_2(q)$ that projects to $\bar{X}$ by $d\pi_2(q)$. Then $\hat{X}$ is a parallel vector field along L with respect to (the pull-back to L of) the Levi-Civita connection of M^n. Let $\xi = dF(\hat{X})\colon L \to \mathbb{L}^{N+2}$. Then for any $X \in TL$ we have

$$d\xi(X) = \alpha_F(X, \hat{X}) = \omega(X)F, \quad \text{with } \omega(X) = \varphi \,\mathrm{Hess}\, \varphi^{-1}(X, \hat{X}), \tag{9}$$

where the second equality follows from (5) and the assumption that the second fundamental form of f is adapted to $\mathcal{E}$. For $X, Y \in T_pL$ linearly independent, the exterior derivative of (9) gives

$$0 = d^2\xi(X, Y) = d\omega(X, Y)F - \omega(X)dF(Y) - \omega(Y)dF(X).$$

Since F, $dF(X)$ and $dF(Y)$ are linearly independent, because F is an immersion and the position vector F is a nonzero normal vector field, we conclude that ω vanishes whence $\alpha_F(X, \hat{X}) = 0$ for any $X \in TL$. ∎

Lemma 7 *Let $p, q \in M^n$, $X_i \in E_i(p)$ and $X_j \in E_j(q)$, $i \neq j$. Then $dF(X_i) \perp dF(X_j)$.*

Proof: Let $\bar{X}_i = d\pi_i(p)(X_i) \in T_{\pi_i(p)}M_i^{n_i}$ and, for any point z of the fiber $M_{\perp_i}$ of π_i through p, let $X_i(z)$ be the unique vector in $E_i(z)$ that projects to $\bar{X}_i$ by $d\pi_i(z)$. Arguing as in the beginning of the proof of Lemma 6 and using its conclusion, we obtain that $dF(X_i)\colon M_{\perp_i} \to \mathbb{L}^{N+2}$ is constant, and the statement follows. ∎

Now, for $i \in I_k$ define linear subspaces W_i of $\mathbb{L}^{N+2}$ by

$$W_i = \mathrm{span}\{dF_q(X_i) : q \in M^n, X_i \in E_i(q)\}.$$

By Lemma 7 the subspaces W_i are mutually orthogonal. We distinguish two cases.

First suppose that all the W_i inherit non-degenerate metrics from $\mathbb{L}^{N+2}$. Define $W_{k+1} = (W_1 \oplus \cdots \oplus W_k)^\perp$ and let $P_i\colon \mathbb{L}^{N+2} \to W_i$ denote orthogonal projection. Since $W_1, \ldots, W_{k+1}$ are mutually orthogonal, for any $X_j \in \Gamma(E_j)$ we have $d(P_i \circ F)(X_j) = 0$ for $i \neq j$. Thus, for $i \leq k$, $P_i \circ F$ is constant on the fibers of the projection $\pi_i\colon M^n \to M_i^{n_i}$, while $P_{k+1} \circ F$ has a constant value e_{k+1} on M^n. Fixed $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_k) \in M^n$, for $i \in I_k$ define $F_i\colon M_i^{n_i} \to W_i$ by $F_i = P_i \circ F \circ i_{\bar{p}}$, where $i_{\bar{p}}\colon M_i^{n_i} \to M^n$ denotes the isometric inclusion of $M_i^{n_i}$ into M^n given by $p_i \mapsto (\bar{p}_1, \ldots, p_i, \ldots, \bar{p}_k)$. Then F_i is an isometric immersion with $P_i \circ F = F_i \circ \pi_i$ for every $i \in I_k$, whence

$$F = \sum_{i=1}^{k+1} P_i \circ F = \sum_{i=1}^{k} F_i \circ \pi_i + e_{k+1}. \tag{10}$$

From (10) we get

$$0 = \langle F, F\rangle = \sum_{i=1}^{k} \langle F_i, F_i\rangle \circ \pi_i + \langle e_{k+1}, e_{k+1}\rangle,$$

from which we conclude that each $\langle F_i, F_i\rangle$ is a constant, say, $1/c_i$. Notice that W_j is time-like for exactly one $j \in I_{k+1}$. Assume first that $j = k+1$. Orthogonally decompose $e_{k+1} = v + u$ with $v \in W_{k+1}$ time-like, set $-1/c = \langle v, v\rangle$ and $\hat{F} = \sum_{i=1}^{k} \langle F_i, F_i\rangle \circ \pi_i + u$, whence $\langle \hat{F}, \hat{F}\rangle = 1/c$. Then there exist an extrinsic product $\Psi\colon \Pi_{i=1}^{k} \mathbb{S}^{m_i-1}(c_i) \to \mathbb{S}^N(c) \subset \mathbb{R}^{N+1} = \oplus_{i=1}^{k+1} \mathbb{R}^{m_i}$, linear isometries $B_i\colon \mathbb{R}^{m_i} \to W_i$, $i \in I_k$, $B_{k+1}\colon \mathbb{R}^{m_{k+1}} \to \hat{W}_{k+1} := W_{k+1} \cap v^\perp$ and $B = \oplus_{i=1}^{k+1} B_i\colon \mathbb{R}^{N+1} = \oplus_{i=1}^{k+1} \mathbb{R}^{N_i} \to \oplus_{i=1}^{k} W_i \oplus \hat{W}_{k+1} = v^\perp$ such that $\hat{F} = B \circ \Psi \circ (f_1 \times \cdots \times f_k)$, where $f_i\colon M_i^{n_i} \to \mathbb{R}^{m_i}$ is defined by $F_i = B_i \circ f_i$ for every $i \in I_k$. Therefore $F = T_{B,v} \circ \Psi \circ (f_1 \times \cdots \times f_k)$, and we obtain using (3) that

$$\begin{aligned} f &= \mathcal{C}_{p_0,w,A}(\mathcal{I}_{p_0,w,A}(f)) = \mathcal{C}_{p_0,w,A}(F) = \mathcal{C}_{p_0,w,A}(T_{B,v}) \circ \Psi \circ (f_1 \times \cdots \times f_k) \\ &= \mathcal{P} \circ \mathcal{H} \circ \Psi \circ (f_1 \times \cdots \times f_k), \end{aligned}$$

where $\mathcal{H}$ is a similarity of $\mathbb{R}^{N+1}$ and $\mathcal{P}$ is a stereographic projection of $\mathcal{H}(\mathbb{S}^N(c)) = \mathbb{S}^N$ onto $\mathbb{R}^N$. Writing $\mathcal{H} = H \circ S$ for a homothety H and an isometry S of $\mathbb{R}^{N+1}$, and observing that $S \circ \Psi$ is still an extrinsic product of $\Pi_{i=1}^{k} \mathbb{S}^{m_i-1}(c_i)$ into $\mathbb{S}^N(c)$, we obtain case (i) of the statement for $c > 0$.

Now suppose that $j < k+1$. Choose linear isometries $C\colon \mathbb{L}^{m+1} \to W_j$, $D_i\colon \mathbb{R}^{m_i} \to W_i$, $i \in I_{k+1}^j$, define $f_j\colon M_j^{n_j} \to \mathbb{L}^{m+1}$ by $F_j = C \circ f_j$ and $f_i\colon M_i^{n_i} \to \mathbb{R}^{m_i}$ by $F_i = D_i \circ f_i$. Set $D = \oplus_{i \in I_{k+1}^j} D_i\colon \mathbb{R}^{N-m+1} = \oplus_{i \in I_{k+1}^j} \mathbb{R}^{m_i} \to \oplus_{i \in I_{k+1}^j} W_i$. Then there exist an extrinsic product $\Psi\colon \Pi_{i \in I_k^j} \mathbb{S}^{m_i}(c_i) \to \mathbb{S}^{N-m}(c)$, $1/c = \sum_{i \in I_k^j} 1/c_i + \langle e_{k+1}, e_{k+1}\rangle$, such that $F = L_{C,D} \circ \bar{f}$, where $\bar{f} = (f_j \times (\Psi \circ (f_1 \times \cdots \times \hat{f}_j \times \cdots \times f_k)) \circ T_j)$. Therefore, defining a conformal diffeomorphism $\Theta\colon \mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) \to \mathbb{R}^N$ by (4), we obtain

$$f = \mathcal{C}_{p_0,w,A}(F) = \mathcal{C}_{p_0,w,A}(L_{C,D}) \circ \bar{f} = \Theta \circ \bar{f},$$

and we are in the situation of case (ii) of the statement.

The second case we must consider is when some of the W_i inherit a degenerate metric from $\mathbb{L}^{N+2}$. In this case, without loss of generality, we may assume that $W_1, \ldots, W_\ell$ have such a metric while $W_{\ell+1}, \ldots, W_k$ have non-degenerate (and so necessarily space-like) metric. Then there exists a 1-dimensional and light-like subspace L_0 such that $W_i \cap W_i^\perp = L_0$ for $i = 1, \ldots, \ell$. Choose a second, distinct light-like line L_1 orthogonal to $W_{\ell+1}, \ldots, W_k$. Set $\hat{W}_i = W_i \cap L_1^\perp$ (so that $\hat{W}_i = W_i$ for $i > \ell$) and finally set

$$\hat{W}_{k+1} = (L_0 \oplus \hat{W}_1 \oplus \cdots \oplus \hat{W}_k \oplus L_1)^\perp.$$

We therefore have a decomposition $\mathbb{L}^{N+2} = L_0 \oplus \hat{W}_1 \oplus \cdots \oplus \hat{W}_{k+1} \oplus L_1$ and corresponding orthogonal projections $\hat{P}_i\colon \mathbb{L}^{N+2} \to \hat{W}_i$. Arguing as in the preceding case, we obtain that, for $i \in I_k$, $\hat{P}_i \circ F$ is constant on the fibers of π_i while the components of F in L_1 and $\hat{W}_{k+1}$ are constant. Thus, there exist isometric immersions $\hat{F}_i$: $M_i^{n_i} \to \hat{W}_i$ such that

$$F = \bar{p}_0 + \sum_{i=0}^{k} \hat{F}_i \circ \pi_i + e_{k+1} + \langle F, \bar{p}_0 \rangle \bar{w}, \tag{11}$$

where $\bar{p}_0$ is a light-like constant vector in L_1, e_{k+1} is a (space-like) constant vector in $\hat{W}_{k+1}$ and $\bar{w} \in L_0$ is chosen so that $\langle \bar{p}_0, \bar{w} \rangle = 1$. From (11) and $\langle F, F \rangle = 0$ we conclude that $2\langle F, \bar{p}_0 \rangle = -\sum_{i=1}^{k} \langle \hat{F}_i \circ \pi_i, \hat{F}_i \circ \pi_i \rangle$. Identifying $(L_0 \oplus L_1)^{\perp}$ with $\mathbb{R}^N$ by means of a linear isometry $\bar{A}$, the isometric immersion $\hat{F} = \sum_{i=1}^{k} \hat{F}_i \circ \pi_i + e_{k+1}$ is an extrinsic product of $\hat{F}_1, \ldots, \hat{F}_k$ with respect to the orthogonal decomposition $\mathbb{R}^N = \oplus_{i=1}^{k+1} \hat{W}_i$, whose image lies in the affine subspace $e_{k+1} \oplus \hat{W}_{k+1}^{\perp}$, and $F = \Psi_{\bar{p}_0, \bar{w}, \bar{A}} \circ \hat{F}$. Therefore, using (2) we conclude that there exist an inversion I with respect to a sphere of unit radius and a similarity $\mathcal{H}$ on $\mathbb{R}^N$ such that

$$f = \mathcal{C}_{p_0,w,A}(F) = \mathcal{C}_{p_0,w,A}(\Psi_{\bar{p}_0,\bar{w},\bar{A}} \circ \hat{F}) = \mathcal{C}_{p_0,w,A}(\Psi_{\bar{p}_0,\bar{w},\bar{A}}) \circ \hat{F} = I \circ \mathcal{H} \circ \hat{F},$$

which gives case (i) of the statement for $c = 0$. ∎

Remarks 8. (a) An alternate statement for part (ii) is as follows:
(ii') For exactly one $j \in I_k$ there exist an isometry Φ: $\mathbb{R}_+^m \times_\sigma \mathbb{S}^{N-m}(c) \to \mathbb{R}^N$, an extrinsic product Ψ: $\Pi_{i \in I_k^j} \mathbb{S}^{m_i}(c_i) \to \mathbb{S}^{N-m}(c)$, a conformal immersion f_j: $M_j \to \mathbb{R}_+^m$ with conformal factor $(\sigma \circ f_j)$, isometric immersions f_i: $M_i \to \mathbb{S}^{m_i}(c_i)$, $i \in I_k^j$, and an inversion I in $\mathbb{R}^N$ such that $f = I \circ \Phi \circ (f_j \times (\Psi \circ (f_1 \times \cdots \times \hat{f}_j \times \cdots \times f_k))) \circ T_j$.

(b) Case (i) with $c > 0$ can occur only if $k \leq N-n$. This follows from the fact the codimension $N-n$ of f is greater than or equal to the codimension k of Ψ. Similarly, in case (ii) we must have $k \leq N-n+1$.

Clearly, Theorem 5 does not hold for $n = 2$. In fact, the proof of Theorem 5 yields a classification of a special class of isothermic surfaces in $\mathbb{R}^N$, which extends a theorem of Bonnet for $N = 3$ ([**Bo**], cf. [**Da**$_2$], vol. III, p. 121)).

Corollary 9 *Let f: $M^2 \to \mathbb{R}^N$ be a surface with flat normal bundle without umbilic points. Assume that the curvature lines of both families have constant geodesic curvature. Then one of the following holds:*

(i) *There exist local isothermic parameterizations by curvature lines ψ: $J_1 \times J_2 \to M^2$, an isometry Φ: $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}^N$, an inversion I in $\mathbb{R}^N$ and unit speed curves α: $J_1 \to \mathbb{R}^{N_1}$ and β: $J_2 \to \mathbb{R}^{N_2}$ such that $f \circ \psi = I \circ \Phi \circ (\alpha \times \beta)$.*

(ii) *There exist local isothermic parameterizations by curvature lines ψ: $J_1 \times J_2 \to M^2$, unit speed curves α: $J_1 \to \mathbb{H}^m(-c)$ and β: $J_2 \to \mathbb{S}^{N-m}(c)$, and a conformal diffeomorphism Θ: $\mathbb{H}^m(-c) \times \mathbb{S}^{N-m}(c) \to \mathbb{R}^N$ such that $f \circ \psi = \Theta \circ (\alpha \times \beta)$.*

Proof: Let $\mathcal{E} = (E_1, E_2)$ be the orthogonal net on M^2 determined by the principal directions of f. By assumption, E_1 and E_2 are spherical whence $\mathcal{E}$ is a CP-net by (1). By Theorem 3, there exist local isothermic parameterizations $\psi\colon J_1 \times J_2 \to M^2$ whose coordinate curves are integral curves of E_1 and E_2. Moreover, if φ denotes the conformal factor of ψ, then it is easily checked that the fact that the coordinate curves of ψ have constant geodesic curvature is equivalent to $\mathrm{Hess}\,\varphi^{-1}$ (the Hessian being computed with respect to the flat metric on $J_1 \times J_2$) being adapted to the product net of $J_1 \times J_2$, i.e., $\varphi^{-1} = \varphi_1 \circ \pi_1 + \varphi_2 \circ \pi_2$ for some $\varphi_1 \in C^\infty(J_1)$ and $\varphi_2 \in C^\infty(J_2)$. Therefore Lemma 6, whence the remaining of the proof of Theorem 5, applies for $f \circ \psi$ ∎

In the case $n = N$, Theorem 5 gives a classification of all *conformal representations* of Euclidean space as a Riemannian product, which contains as a special case Liouville's theorem on the classification of conformal mappings between open subsets of $\mathbb{R}^N$, $N \geq 3$. It also provides a geometric proof of the classification of conformally flat Riemannian products (cf. [**La**], Section D, Proposition 2).

Corollary 10 *Let $f\colon M^N := \Pi_{i=1}^k M_i^{N_i} \to \mathbb{R}^N$, $N \geq 3$, be a conformal local diffeomorphism of a Riemannian product. Then one of the following holds:*

(i) *There exist an isometry $\Phi\colon \Pi_{i=1}^k \mathbb{R}^{N_i} \to \mathbb{R}^N$, local isometries $f_i\colon M_i^{N_i} \to \mathbb{R}^{N_i}$, an inversion I and a homothety H in $\mathbb{R}^N$ such that $f = I \circ H \circ \Phi \circ (f_1 \times \cdots \times f_k)$.*

(ii) *$k = 2$ and, after relabelling the factors if necessary, there exist local isometries $f_1\colon M_1^{N_1} \to \mathbb{H}^{N_1}(-c)$ and $f_2\colon M_2^{N_2} \to \mathbb{S}^{N_2}(c)$ and a conformal diffeomorphism $\Theta\colon \mathbb{H}^{N_1}(-c) \times \mathbb{S}^{N_2}(c) \to \mathbb{R}^N$ such that $f = \Theta \circ (f_1 \times f_2)$.*

Theorem 5 has also the following consequence for isometric immersions of twisted products into Euclidean space.

Corollary 11 *Let $f\colon (M^n, \langle\ ,\ \rangle) := M^{n_1} \times M^{n_2} \to \mathbb{R}^N$, $n \geq 2$, be an isometric immersion of a simply connected twisted product whose second fundamental form is adapted to the product net $\mathcal{E} = (E_1, E_2)$ of M^n. If $n_i = 1$ for some $i \in \{1, 2\}$, suppose further that the leaves of E_i are spherical in M^n. Then $\langle\ ,\ \rangle$ is conformal to a product metric $\langle\ ,\ \rangle^\sim$ and the conclusion of Theorem 5 holds for $f\colon (M^n, \langle\ ,\ \rangle^\sim) \to \mathbb{R}^N$.*

Proof: For $n = 2$ the result reduces to Corollary 9, thus we may assume that $n \geq 3$. By Proposition 1, $\mathcal{E}$ is a TP-net. Then, Lemma 12 below and the assumption for the case when $n_i = 1$ imply that E_1 and E_2 are spherical subbundles, whence $\mathcal{E}$ is a CP-net by (1). Thus $\langle\ ,\ \rangle$ is conformal to a product metric $\langle\ ,\ \rangle^\sim$ by Proposition 2. Since the second fundamental form of f is adapted to $\mathcal{E}$, Theorem 5 applies for $f\colon (M^n, \langle\ ,\ \rangle^\sim) \to \mathbb{R}^N$.

Lemma 12. *Let $h\colon M^n \to \mathbb{R}^N$ and $g\colon L^k \to M^n$, $k \geq 2$, be isometric immersions. If g is umbilical and $\alpha_h(dg(X), Z) = 0$ for all $X \in \Gamma(TL)$, $Z \in \Gamma(T_g^\perp L)$, then g is spherical.*

Proof: Since g is umbilical, the Codazzi equation for g yields

$$\langle Y,T\rangle\nabla^{\perp}_{X}H_g - \langle X,T\rangle\nabla^{\perp}_{Y}H_g = (R(dg(X),dg(Y))dg(T))_{T_g^{\perp}L} \tag{12}$$

for all $X,Y,T\in\Gamma(TL)$, where R stands for the curvature tensor of M^n and H_g for the mean curvature vector of g. The Gauss equation of h and the assumption on α_h imply that $(R(dg(X),dg(Y))Z)_{dg(TL)} = A^h_{\alpha_h(dg(Y),Z)}dg(X) - A^h_{\alpha_h(dg(X),Z)}dg(Y) = 0$ for all $X,Y\in\Gamma(TL)$ and $Z\in\Gamma(T_g^{\perp}L)$, whence the right-hand-side of (12) vanishes. Choosing $Y=T$ orthogonal to X we conclude that H_g is parallel in the normal connection. ∎

An interesting particular case is that of a cyclide of Dupin $f\colon M^n\to\mathbb{R}^{n+1}$, in which case the eigenbundles correspondent to the two distinct principal curvatures give rise to a TP-net on M^n. Recall that f is a *cyclide of Dupin* of characteristic $(m,n-m)$ if it has everywhere two distinct principal curvatures of multiplicities m and $n-m$, respectively, which are constant along the corresponding eigenbundles. One can now easily derive the following alternate classification of the cyclides of Dupin (cf. [**Ce**]).

Corollary 13. *Let $f\colon M^n\to\mathbb{R}^{n+1}$ be a cyclide of Dupin of characteristic $(m,n-m)$. Then there exist a conformal diffeomorphism ψ of an open subset $W\subset\mathbb{Q}_c^{n-m}\times\mathbb{S}^m$, $c>-1$, onto M^n, and a conformal diffeomorphism $\Theta\colon\mathbb{H}^{n-m+1}\times\mathbb{S}^m\to\mathbb{R}^{n+1}$ such that $f\circ\psi=\Theta\circ(f_1\times i)|_W$, where $f_1\colon\mathbb{Q}_c^{n-m}\to\mathbb{H}^{n-m+1}$ is a spherical inclusion and $i\colon\mathbb{S}^m\to\mathbb{S}^m$ is the identity map. Moreover, the classes of conformally congruent cyclides of Dupin are parameterized by their characteristic and the value of c.*

Remarks 14. (i) If $c>0$ in Corollary 13 then we have an alternate description: There exist a conformal diffeomorphism ψ of an open subset $W\subset\mathbb{S}^{n-m}(c)\times\mathbb{S}^m$ onto M^n, an isometric embedding Ψ of $\mathbb{S}^{n-m}(c)\times\mathbb{S}^m$ into $\mathbb{S}^{n+1}(\tilde{c})$, $\tilde{c}=c/(c+1)$, as an extrinsic product, a homothety H in $\mathbb{R}^{n+2}$ and a stereographic projection $\mathcal{P}$ of $\mathbb{S}^{n+1}(\bar{c}) = H(\mathbb{S}^{n+1}(\tilde{c}))$ onto $\mathbb{R}^{n+1}$ such that $f\circ\psi=\mathcal{P}\circ H\circ\Psi|_W$.

(ii) Another particular case of Corollary 11 is the main result of [**DFT**], which classifies Euclidean submanifolds that carry a Dupin principal normal with umbilical conullity.

5 Christoffel problem

In this section we classify submanifolds that are solutions of a generalized Christoffel problem. We start by summarizing some facts on Codazzi tensors that will be needed in the sequel. Recall that a symmetric tensor $\mathcal{S}$ on a Riemannian manifold M is a *Codazzi tensor* if

$$(\nabla_X\mathcal{S})Y=(\nabla_Y\mathcal{S})X \ \text{ for all } \ X,Y\in\Gamma(TM),$$

where $(\nabla_X\mathcal{S})Y=\nabla_X\mathcal{S}Y-\mathcal{S}(\nabla_XY)$. The following basic result is due to Reckziegel [**Re**] (cf. Proposition 5.1 of [**To**]).

Proposition 15 *Let $\mathcal{S}$ be a Codazzi tensor on a Riemannian manifold M and let $\lambda \in C^\infty(M)$ be an eigenvalue of $\mathcal{S}$ such that $E_\lambda = \ker(\lambda I - \mathcal{S})$ has constant rank k. Then:*

(i) *If $k \geq 2$ then λ is constant along E_λ.*

(ii) *E_λ is an umbilical distribution (in fact spherical if λ is constant along E_λ) with mean curvature normal η given by $(\lambda I - \mathcal{S})\,\eta = (\nabla\,\lambda)_{E_\lambda^\perp}$.*

We now consider Codazzi tensors with exactly two distinct eigenvalues everywhere. The next result is contained in Theorem 5.2 of [**To**].

Proposition 16 *Let M be a Riemannian manifold and let $\mathcal{S}$ be a Codazzi tensor on M with exactly two distinct eigenvalues λ and μ everywhere. Let E_λ and E_μ be the corresponding eigenbundles. Then (E_λ, E_μ) is a CP-net if and only if*

$$2rX(h)Y(h) + hX(h)Y(r) + hY(h)X(r) - rh\,Hess\,h(X,Y) = 0,$$

where $\lambda = h(1-r)$ and $\mu = h(1+r)$.

The following is a special case of Corollary 5.3 of [**To**]. For a product manifold $M = M_1 \times M_2$ with product net (E_1, E_2), we denote by $\Pi_i\colon TM \to E_i$ the canonical projection onto E_i, $1 \leq i \leq 2$.

Proposition 17. *Let $\mathcal{S}$ be a Codazzi tensor on a Riemannian manifold M with eigenvalues λ and $-\lambda$, where $\lambda \neq 0$ everywhere. Let E_+ and E_- be the corresponding eigenbundles and assume that λ is constant along E_-. Then one of the following holds:*

(i) *λ is constant along E_+ and for every point $p \in M$ there exists a local product representation $\psi\colon M_1 \times M_2 \to U$ of (E_+, E_-) with $p \in U \subset M$, which is an isometry with respect to a Riemannian product metric $\langle\ ,\ \rangle$ on $M_1 \times M_2$. Moreover, $(d\psi)^{-1} \circ \mathcal{S} \circ d\psi = a(\Pi_2 - \Pi_1)$ for some $a \neq 0$.*

(ii) *for every point $p \in M$ there exists a local product representation $\psi\colon I \times N \to U$ of (E_+, E_-) with $p \in U \subset M$, where $I \subset \mathbb{R}$ is an open interval, which is an isometry with respect to a warped product metric $\langle\ ,\ \rangle$ on $I \times N$ with warping function ρ. Moreover, $(d\psi)^{-1} \circ \mathcal{S} \circ d\psi = a(\rho \circ \pi_1)^{-2}(\Pi_2 - \Pi_1)$ for some $a \neq 0$.*

The classical proof that isothermic surfaces are precisely the ones that admit a *dual* surface (or Christoffel transform) starts by showing that if two surfaces are mapped conformally onto each other with parallel tangent planes at corresponding points, then either they are a pair of minimal surfaces with the same conformal structures and orientations, or the correspondence between them necessarily preserves curvature lines. Then, the proof proceeds by showing that in the latter case either the surfaces differ by a homothety and a translation or they are a Christoffel pair of isothermic surfaces.

Classically, two surfaces $f\colon M^2 \to \mathbb{R}^3$ and $\mathcal{F}\colon M^2 \to \mathbb{R}^3$ that can be mapped onto each other with preservation of curvature lines and with parallel tangent planes at corresponding points are said to be related by a *Combescure transformation* (cf. [**Bi**], v.II-1, p. 108). Therefore, Christoffel's characterization of isothermic surfaces implies that a surface is isothermic if and only if it admits a nontrivial conformal Combescure transform. By nontrivial we mean that the surfaces do not differ by a composition of a homothety and a translation.

The Combescure transformation can be extended for Euclidean submanifolds of arbitrary dimension and codimension as follows (cf. [$\mathbf{DT}_2$]). Given an isometric immersion $f\colon M^n \to \mathbb{R}^N$, a map $\mathcal{F}\colon M^n \to \mathbb{R}^N$ is said to be a *Combescure transform* of f determined by a symmetric tensor $\mathcal{S}$ on M^n if $d\mathcal{F} = df \circ \mathcal{S}$. This implies (see Proposition 1 of [$\mathbf{DT}_2$]) that $\mathcal{S}$ is a *commuting* Codazzi tensor, i.e.,

$$\alpha_f(X, \mathcal{S}Y) = \alpha_f(\mathcal{S}X, Y) \text{ for all } X, Y \in \Gamma(TM^n).$$

If $\mathcal{S}$ is invertible, then $\mathcal{F}$ is an immersion with the same Gauss map as f into the Grassmann manifold of non oriented n–planes in $\mathbb{R}^N$. Moreover, the requirement that the tensor $\mathcal{S}$ be symmetric reduces in the surface case to the assumption that f and $\mathcal{F}$ have the same curvature lines, since their second fundamental forms are related by

$$\alpha_{\mathcal{F}}(X, Y) = \alpha_f(\mathcal{S}X, Y) \text{ for all } X, Y \in \Gamma(TM^n). \tag{13}$$

If M^n is simply-connected and $\mathcal{F}\colon M^n \to \mathbb{R}^N$ is any Combescure transform of f determined by a symmetric tensor $\mathcal{S}$, it was shown in Proposition 3 of [$\mathbf{DT}_2$] that there exist $\varphi \in C^\infty(M^n)$ and $\beta \in \Gamma(T_f^\perp M^n)$ satisfying

$$\alpha_f(\nabla\varphi, X) + \nabla_X^\perp \beta = 0 \text{ for all } X \in \Gamma(TM^n), \tag{14}$$

such that $\mathcal{S}$ and $\mathcal{F}$ are given by

$$\mathcal{S} = \mathcal{S}_{\varphi,\beta} := \operatorname{Hess}\varphi - A^f_\beta \quad \text{and} \quad \mathcal{F} = \mathcal{F}_{\varphi,\beta} := df(\operatorname{grad}\varphi) + \beta. \tag{15}$$

Conversely, if $\varphi \in C^\infty(M^n)$ and $\beta \in \Gamma(T_f^\perp M^n)$ satisfy (14) then $\mathcal{F}_{\varphi,\beta}$ given by (15) is a Combescure transform of f with $\mathcal{S}_{\varphi,\beta}$ as the corresponding commuting Codazzi tensor.

As in the surface case, trivial Combescure transforms of a given isometric immersion $f\colon M^n \to \mathbb{R}^N$ are compositions of f with a homothety and a translation, whose correspondent commuting Codazzi tensors are constant multiples of the identity tensor.

We say that an immersion $\mathcal{F}\colon M^n \to \mathbb{R}^N$ is a *Christoffel transform* of f if it is a nontrivial *conformal* Combescure transform of f. In the following result we classify Euclidean submanifolds of dimension $n \geq 3$ that admit Christoffel transforms.

Theorem 18. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion that admits a Christoffel transform $\mathcal{F}\colon M^n \to \mathbb{R}^N$. Then $(f, \mathcal{F})$ is a pair of 2-isothermic submanifolds. More precisely, there exists a 2-CP-net (E_+, E_-) with respect to which the second fundamental forms of both f and $\mathcal{F}$ are adapted. Moreover, if $n \geq 3$ then one of the following holds:*

(i) *for every* $p \in M^n$ *there exists a local product representation* $\psi\colon M_1 \times M_2 \to U$ *of* (E_+, E_-) *with* $p \in U \subset M^n$*, which is an isometry with respect to a Riemannian product metric* $\langle\ ,\ \rangle$ *on* $M_1 \times M_2$*, such that* $f \circ \psi = f_1 \times f_2$ *is an extrinsic product of isometric immersions with respect to an orthogonal decomposition* $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$*. Moreover,* $\mathcal{F} \circ \psi = a((-f_1) \times f_2) \circ \psi + v$ *for some* $a \neq 0$ *and* $v \in \mathbb{R}^N$*.*

(ii) *for every* $p \in M^n$ *there exist a local product representation* $\psi\colon I \times N \to U$ *of* (E_+, E_-) *with* $p \in U \subset M^n$*, which is an isometry with respect to a warped product metric* $\langle\ ,\ \rangle$ *on* $I \times N$ *with warping function* ρ*, an isometry* $\Phi\colon \mathbb{R}^m_+ \times_\sigma \mathbb{S}^{N-m} \to \mathbb{R}^N$*, a unit speed curve* $\gamma\colon I \to \mathbb{R}^m_+$ *and an isometric immersion* $g\colon N \to \mathbb{S}^{N-m}$ *such that* $\gamma_m = \rho = \sigma \circ \gamma$ *and* $f \circ \psi = \Phi \circ (\gamma \times g)$*. Moreover,* $\mathcal{F} \circ \psi = \Phi \circ (a\tilde{\gamma} \times g) + v$*, where* $a \neq 0$*,* $v \in \mathbb{R}^N$ *and* $\tilde{\gamma} = \int \gamma_m^{-2}(\tau)\gamma'(\tau)d\tau$*.*

Proof: Since the metric induced by $\mathcal{F}$ is $\langle X, Y\rangle_* = \langle d\mathcal{F}(X), d\mathcal{F}(Y)\rangle = \langle \mathcal{S}X, \mathcal{S}Y\rangle$ for all $X, Y \in \Gamma(TM^n)$, the symmetry of $\mathcal{S}$ and the assumption that f and $\mathcal{F}$ are conformal imply that $\mathcal{S}^2 = \lambda^2 I$ for some $\lambda \in C^\infty(M^n)$. Therefore, either $\mathcal{S} = \pm\lambda I$ or TM^n splits as an orthogonal direct sum $TM^n = E_+ \oplus E_-$, where E_+ and E_- are the eigenbundles of $\mathcal{S}$ correspondent to the eigenvalues λ and $-\lambda$, respectively. In the first case, λ must be a constant $a \neq 0$ by Proposition 15-(i), whence $\mathcal{F} = af + v$ for some $v \in \mathbb{R}^N$. In the latter case, it follows from Proposition 16 that (E_+, E_-) is a CP-net. Moreover, since $\mathcal{S}$ is commuting, the second fundamental form of f is adapted to (E_+, E_-) and, because of (13), the same holds for the second fundamental form of $\mathcal{F}$.

Now assume that $n \geq 3$. Then either E_+ or E_-, say, the latter, has dimension at least two, and hence λ must be constant along E_- by Proposition 15-(i). Thus Proposition 17 applies. In case (i), since the second fundamental form of f is adapted to (E_+, E_-), it follows from the main lemma in [**Mo**] that $f \circ \psi = f_1 \times f_2$ splits as an extrinsic product of isometric immersions with respect to an orthogonal decomposition $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. Moreover, integrating $d(\mathcal{F} \circ \psi) = d(f \circ \psi) \circ (d\psi)^{-1} \circ \mathcal{S} \circ d\psi$ with $(d\psi)^{-1} \circ \mathcal{S} \circ d\psi = a(\Pi_2 - \Pi_1)$ for some $a \neq 0$ yields $\mathcal{F} \circ \psi = a((-f_1) \times f_2) \circ \psi + v$ for some $v \in \mathbb{R}^N$. In case (ii), Nölker's theorem [**No**] implies that there exist an isometry $\Phi\colon \mathbb{R}^m_+ \times_\sigma \mathbb{S}^{N-m} \to \mathbb{R}^N$, a unit speed curve $\gamma\colon I \to \mathbb{R}^m_+$ and an isometric immersion $g\colon N \to \mathbb{S}^{N-m}$ such that $\gamma_m = \rho = \sigma \circ \gamma$ and $f \circ \psi = \Phi \circ (\gamma \times g)$. Finally, integrating $d(\mathcal{F} \circ \psi) = d(f \circ \psi) \circ (d\psi)^{-1} \circ \mathcal{S} \circ d\psi$ with $(d\psi)^{-1} \circ \mathcal{S} \circ d\psi = a(\rho \circ \pi_1)^{-2}(\Pi_2 - \Pi_1)$ for some $a \neq 0$ implies that $\mathcal{F} \circ \psi$ is as stated. ∎

6 Conformal sphere congruences

The Ribaucour transformation for surfaces in $\mathbb{R}^3$ was extended as follows to Euclidean submanifolds of arbitrary dimension and codimension ([**DT**$_1$], [**DT**$_2$]). Two pointwise distinct immersions $f\colon M^n \to \mathbb{R}^N$ and $\tilde{f}\colon M^n \to \mathbb{R}^N$ are said to be related by a Ribaucour transformation (or each one of them is a Ribaucour transform of the other) if there exist a vector bundle isometry $\mathcal{P}: f^*T\mathbb{R}^N \to \tilde{f}^*T\mathbb{R}^N$, a symmetric tensor D on M^n with respect to the metric induced by f and a nowhere vanishing smooth vector field $\delta \in \Gamma(f^*T\mathbb{R}^N)$ such that

(*i*) $\mathcal{P}Z - Z = \langle \delta, Z\rangle(f - \tilde{f})$ for all $Z \in \Gamma(f^*T\mathbb{R}^N)$;

(*ii*) $d\tilde{f} = \mathcal{P} \circ df \circ D$.

Geometrically, f and $\tilde{f}$ are tangent at each $p \in M^n$ to a common n-sphere $S(p)$ and $\mathcal{P}$ is the reflection in the hyperplane orthogonal to $\tilde{f}(p) - f(p)$. In classical terminology, f and $\tilde{f}$ envelope a common n-sphere congruence. If M^n is simply-connected, it was shown in [**DT**$_2$] (see Theorem 17) that there exists (φ, β) satisfying (14) such that

$$\tilde{f} = f - 2\nu\varphi\mathcal{F}, \tag{16}$$

where $\mathcal{F} = df(\operatorname{grad}\varphi) + \beta$ and $\nu^{-1} = \langle \mathcal{F}, \mathcal{F}\rangle$. Therefore $\tilde{f}$ is completely determined by (φ, β), or equivalently, by φ and $\mathcal{F}$. We denote $\tilde{f} = \mathcal{R}_{\varphi,\beta}(f)$. Moreover, $\mathcal{P}$, D and δ are given in terms of (φ, β) by

$$\mathcal{P}Z = Z - 2\nu\langle \mathcal{F}, Z\rangle\mathcal{F}, \quad D = I - 2\nu\varphi\mathcal{S}_{\varphi,\beta} \quad \text{and} \quad \delta = -\varphi^{-1}\mathcal{F}. \tag{17}$$

Conversely, given (φ, β) satisfying (14) on an open subset $U \subset M^n$ where D is invertible, then $\tilde{f}$ given by (16) defines a Ribaucour transform of $f|_U$. The induced metrics, Levi-Civita connections and second fundamental forms of f and $\tilde{f}$ are related by

$$\langle X, Y\rangle^{\sim} = \langle DX, DY\rangle, \tag{18}$$

$$D\tilde{\nabla}_X Y = \nabla_X DY + 2\nu\langle \mathcal{S}X, DY\rangle \operatorname{grad}\varphi - 2\nu\langle \operatorname{grad}\varphi, DY\rangle\mathcal{S}X, \tag{19}$$

$$\tilde{\alpha}(X, Y) = \mathcal{P}\left(\alpha(DX, Y) + 2\nu\langle \mathcal{S}X, DY\rangle\beta\right). \tag{20}$$

Equation (20) clarifies the meaning of the symmetry of the tensor D: for each $\xi \in T_f^{\perp}M$ the shape operators A^f_ξ and $\tilde{A}^{\tilde{f}}_{\mathcal{P}\xi}$ of f and $\tilde{f}$, respectively, commute.

Remark 19. For later use we observe the following invariance property of the Ribaucour transformation. If $\tilde{f}: M^n \to \mathbb{R}^N$ is a Ribaucour transform of $f: M^n \to \mathbb{R}^N$ with data $(\mathcal{P}, D, \delta)$ and $\psi: \tilde{M}^n \to M^n$ is a diffeomorphism, then the tensor $\bar{D}$ on $\tilde{M}^n$ defined by $d\psi \circ \bar{D} = D \circ d\psi$ is symmetric with respect to the metric induced by $f \circ \psi$, and $\tilde{f} \circ \psi$ is a Ribaucour transform of $f \circ \psi$ with data $(\bar{\mathcal{P}}, \bar{D}, \bar{\delta})$, where $\bar{\mathcal{P}} = \mathcal{P} \circ \psi$ and $\bar{\delta} = \delta \circ \psi$. Moreover, if $\tilde{f} = \mathcal{R}_{\varphi,\beta}(f)$ then $\tilde{f} \circ \psi = \mathcal{R}_{\bar{\varphi},\bar{\beta}}(f \circ \psi)$ for $\bar{\varphi} = \varphi \circ \psi$ and $\bar{\beta} = \beta \circ \psi$.

It follows from (18) and the symmetry of D that if f and $\tilde{f}$ induce conformal metrics on M^n then $D^2 = r^2 I$ for some $r \in C^\infty(M^n)$. Therefore, either $D = \pm rI$ or TM^n splits orthogonally as $TM^n = E_+ \oplus E_-$, where E_+ and E_- are the eigenbundles of D correspondent to the eigenvalues r and $-r$, respectively. Since $D = I - 2\nu\varphi\mathcal{S}$ by (17), in the first case $\mathcal{S}$ must be a constant multiple of the identity tensor by Proposition 15 -(i), in which case the proof of Corollary 32 of [$\mathbf{DT}_2$] implies that there exists an inversion I in $\mathbb{R}^N$ such that $L'(\tilde{f}) = I(L(f))$, where L and L' are compositions of a homothety and a translation. We say that $\tilde{f}$ is a *Darboux transform* of f if the second possibility holds, in which case E_+ and E_- are also the eigenbundles of $\mathcal{S}$ correspondent to its distinct eigenvalues $\lambda = h(1-r)$ and $\mu = h(1+r)$, respectively, where $h = (2\nu\varphi)^{-1}$. Thus, $\tilde{f}$ is a Darboux transform of f if and only if the associated Codazzi tensor $\mathcal{S}$ has exactly two distinct eigenvalues λ, μ everywhere satisfying

$$(\lambda + \mu)\varphi = \nu^{-1} = \langle \mathcal{F}, \mathcal{F} \rangle. \tag{21}$$

We now classify Euclidean submanifolds of dimension $n \geq 3$ that admit Darboux transforms.

Theorem 20. *Let $f\colon M^n \to \mathbb{R}^N$ be an isometric immersion admitting a Darboux transform $\tilde{f} = \mathcal{R}_{\varphi,\beta}(f)\colon M^n \to \mathbb{R}^N$. Then $(f, \tilde{f})$ is a pair of 2-isothermic submanifolds. More precisely, there exists a 2-CP-net (E_+, E_-) with respect to which the second fundamental forms of both f and $\tilde{f}$ are adapted. Moreover, if $n \geq 3$ then for every $p \in M^n$ there exist a local product representation $\psi\colon M_1 \times M_2 \to U$ of (E_+, E_-) with $p \in U \subset M^n$, a homothety H and an inversion I in $\mathbb{R}^N$ such that one of the following holds:*

- (i) *ψ is a conformal diffeomorphism with respect to a Riemannian product metric on $M_1 \times M_2$ and $f \circ \psi = H \circ I \circ g$, where $g = g_1 \times g_2\colon M_1 \times M_2 \to \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} = \mathbb{R}^N$ is an extrinsic product of isometric immersions. Moreover, there exists $i \in \{1, 2\}$ such that either M_i is one-dimensional or $g_i(M_i)$ is contained in some sphere $\mathbb{S}^{N_i - 1}(P_i; r_i) \subset \mathbb{R}^{N_i}$.*
- (ii) *ψ is a conformal diffeomorphism with respect to a warped product metric on $M_1 \times M_2$ and $f \circ \psi = H \circ I \circ \Phi \circ (g_1 \times g_2)$, where $\Phi\colon \mathbb{R}^m_+ \times_\sigma \mathbb{S}^{N-m} \to \mathbb{R}^N$ is an isometry and $g_1\colon M_1 \to \mathbb{R}^m_+$ and $g_2\colon M_2 \to \mathbb{S}^{N-m}$ are isometric immersions.*

Conversely, any such isometric immersion admits a Darboux transform.

Proof: Let $TM = E_+ \oplus E_-$ be the orthogonal splitting of TM by the eigenbundles of D correspondent to the eigenvalues r and $-r$, respectively, where $r \in C^\infty(M)$. Since E_+ and E_- are also the eigenbundles of $\mathcal{S}$ correspondent to the eigenvalues $\lambda = h(1-r)$ and $\mu = h(1+r)$, respectively, where $h = (2\nu\varphi)^{-1}$, it follows from Proposition 15 that E_+ and E_- are umbilical distributions with mean curvature normals

$$\eta^+ = (\lambda - \mu)^{-1}(\operatorname{grad}\lambda)_{E_-} \quad \text{and} \quad \eta^- = (\mu - \lambda)^{-1}(\operatorname{grad}\mu)_{E_+}.$$

We claim that (E_+, E_-) is a CP-net. By Proposition 16, this is the case if and only if

$$2rX_+(h)X_-(h) + hX_+(h)X_-(r) + hX_-(h)X_+(r) - rh\operatorname{Hess} h(X_+, X_-) = 0. \tag{22}$$

Here and in the sequel, X_+ and X_- will always denote sections of E_+ and E_-, respectively. We now compute $\operatorname{Hess} h(X_+, X_-)$. Using (15) we have

$$X(\nu^{-1}) = X\langle \mathcal{F}, \mathcal{F}\rangle = 2\langle d\mathcal{F}(X), \mathcal{F}\rangle = 2\langle df(\mathcal{S}X), \mathcal{F}\rangle = 2\langle \mathcal{S}X, \operatorname{grad}\varphi\rangle,$$

whence $\operatorname{grad}\nu = -2\nu^2\mathcal{S}\operatorname{grad}\varphi$ and

$$\operatorname{grad} h^{-1} = 2(\varphi\operatorname{grad}\nu + \nu\operatorname{grad}\varphi) = 2(-2\nu^2\varphi\,\mathcal{S}\operatorname{grad}\varphi + \nu\operatorname{grad}\varphi) = 2\nu D\operatorname{grad}\varphi.$$

It follows that

$$\operatorname{grad} h = -2h^2\nu D\operatorname{grad}\varphi, \tag{23}$$

and thus

$$\nabla_X \operatorname{grad} h = -4hX(h)\nu D\operatorname{grad}\varphi - 2h^2X(\nu)D\operatorname{grad}\varphi - 2h^2\nu\nabla_X D\operatorname{grad}\varphi.$$

Therefore,

$$\begin{aligned}\operatorname{Hess} h(X_+, X_-) = \langle\nabla_{X_-}\operatorname{grad} h, X_+\rangle = -4hX_-(h)\nu\langle D\operatorname{grad}\varphi, X_+\rangle - \\ 2h^2X_-(\nu)\langle D\operatorname{grad}\varphi, X_+\rangle - 2h^2\nu\langle\nabla_{X_-} D\operatorname{grad}\varphi, X_+\rangle.\end{aligned} \tag{24}$$

We compute each of the three terms in the right-hand-side of (24). By (23), we have that $X_-(h) = 2h^2\nu r X_-(\varphi)$ and $X_+(h) = -2h^2\nu r X_+(\varphi)$. Setting $r = -\varphi/\tau$, these equations can be rewritten as $X_-(h) = -(h/\tau)X_-(\varphi)$ and $X_+(h) = (h/\tau)X_+(\varphi)$. Using that $\langle D\operatorname{grad}\varphi, X_+\rangle = rX_+(\varphi) = (r\tau/h)X_+(h)$, we obtain

$$-4hX_-(h)\nu\langle D\operatorname{grad}\varphi, X_+\rangle = -4r\tau\nu X_-(h)X_+(h) = 2h^{-1}X_-(h)X_+(h). \tag{25}$$

Moreover, $X_-(\nu) = -2\nu^2\langle\mathcal{S}\operatorname{grad}\varphi, X_-\rangle = -2\nu^2h(1+r)X_-(\varphi) = 2\nu^2(1+r)\tau X_-(h)$ gives

$$-2h^2X_-(\nu)\langle D\operatorname{grad}\varphi, X_+\rangle = -4h\nu^2 r(1+r)\tau^2X_-(h)X_+(h) = -\frac{1+r}{rh}X_-(h)X_+(h). \tag{26}$$

It remains to compute $\langle\nabla_{X_-} D\operatorname{grad}\varphi, X_+\rangle$. By (19) we have

$$D\,\tilde{\nabla}_X Y = \nabla_X DY + 2\nu\langle\mathcal{S}X, DY\rangle\operatorname{grad}\varphi - 2\nu\langle\operatorname{grad}\varphi, DY\rangle\mathcal{S}X.$$

On the other hand, since $\langle\ ,\ \rangle^\sim = r^2\langle\ ,\ \rangle$, the connections ∇ and $\tilde{\nabla}$ are also related by

$$\tilde{\nabla}_X Y = \nabla_X Y + r^{-1}(\langle\operatorname{grad} r, X\rangle Y + \langle\operatorname{grad} r, Y\rangle X - \langle X, Y\rangle\operatorname{grad} r).$$

Thus,

$$\begin{aligned}\nabla_X D\operatorname{grad}\varphi = D\,\nabla_X\operatorname{grad}\varphi + r^{-1}\langle\operatorname{grad} r, X\rangle D\operatorname{grad}\varphi + r^{-1}\langle\operatorname{grad} r, \operatorname{grad}\varphi\rangle DX - \\ r^{-1}\langle X, \operatorname{grad}\varphi\rangle D\operatorname{grad} r - 2\nu\langle\mathcal{S}X, D\operatorname{grad}\varphi\rangle\operatorname{grad}\varphi + 2\nu\langle\operatorname{grad}\varphi, D\operatorname{grad}\varphi\rangle\mathcal{S}X.\end{aligned}$$

We obtain

$$\begin{aligned}\langle \nabla_{X_-} D\,\mathrm{grad}\,\varphi, X_+\rangle &= X_-(r)X_+(\varphi) - X_-(\varphi)X_+(r) + 2\nu h(1+r)rX_-(\varphi)X_+(\varphi)\\ &= \frac{\tau}{h}(X_-(r)X_+(h) + X_+(r)X_-(h)) + \tau(1+r)X_-(h)X_+(h).\end{aligned}$$

Hence,

$$-2h^2\nu\langle \nabla_{X_-} D\,\mathrm{grad}\,\varphi, X_+\rangle = r^{-1}(X_-(r)X_+(h) + X_+(r)X_-(h)) + \frac{1+r}{rh}X_-(h)X_+(h). \tag{27}$$

Then (22) follows by computing from (24), (25), (26) and (27) that

$$\mathrm{Hess}\,h(X_-, X_+) = 2h^{-1}X_-(h)X_+(h) + r^{-1}(X_-(r)X_+(h) + X_+(r)X_-(h)).$$

Now, since $\mathcal{S}$ is commuting, the second fundamental form of f is adapted to (E_+, E_-). Because of (20), the same holds for the second fundamental form of $\tilde{f}$.

From now on we assume that $n \geq 3$. By Theorem 3, for every $p \in M^n$ there exists a local product representation $\psi\colon M_1 \times M_2 \to U$ of (E_+, E_-) with $p \in U \subset M^n$, which is a conformal diffeomorphism with respect to a Riemannian product metric $\langle\ ,\ \rangle = \pi_1^*\langle\ ,\ \rangle_1 + \pi_2^*\langle\ ,\ \rangle_2$ on $M_1 \times M_2$. Therefore, Theorem 5 can be applied to $f \circ \psi$. We obtain (see Remark 8-(a)) that there exist a homothety H and an inversion I in $\mathbb{R}^N$ with respect to a sphere $\mathbb{S}^{N-1}(P_0)$ of unit radius such that one of the following holds :

(a) $f \circ \psi = H \circ I \circ (g_1 \times g_2)$, where $g_i\colon M_i \to \mathbb{R}^{N_i}$, $1 \leq i \leq 2$, are isometric immersions and $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is an orthogonal decomposition.

(b) after relabelling if necessary, $f\circ\psi = H\circ I\circ\Phi\circ(f_1\times f_2)$, where $\Phi\colon \mathbb{R}^m_+ \times_\sigma \mathbb{S}^{N-m} \to \mathbb{R}^N$ is an isometry, $f_1\colon M_1 \to \mathbb{R}^m_+$ is a conformal immersion with conformal factor $\rho := (\sigma \circ f_1)$ and $f_2\colon M_2 \to \mathbb{S}^{N-m}$ is an isometric immersion.

In the latter case, define $g_1 := f_1\colon (M_1, \langle\ ,\ \rangle_1^\sim) \to \mathbb{R}^m_+$, where $\langle\ ,\ \rangle_1^\sim = \rho^2\langle\ ,\ \rangle_1$, and set $g_2 = f_2$. Then g_1 and g_2 are isometric immersions and $\psi\colon (M_1 \times M_2, \langle\ ,\ \rangle^\sim) \to U$ is still a conformal diffeomorphism with respect to the warped product metric $\langle\ ,\ \rangle^\sim = \pi_1^*\langle\ ,\ \rangle_1^\sim + (\rho\circ\pi_1)^2\pi_2^*\langle\ ,\ \rangle_2$. This gives case (ii) of the statement.

In order to complete the proof of the direct statement, it remains to show that the restriction in case (i) holds. First notice that, by Remark 19 and the fact that ψ is conformal, we have that $\tilde{f} \circ \psi$ is a Darboux transform $\tilde{f} \circ \psi = \mathcal{R}_{\bar{\varphi},\bar{\beta}}(f \circ \psi)$ of $f \circ \psi$, where $\bar{\varphi} = \varphi \circ \psi$ and $\bar{\beta} = \beta \circ \psi$. Thus $\mathcal{S}_{\bar{\varphi},\bar{\beta}}$ is given by $d\psi \circ \mathcal{S}_{\bar{\varphi},\bar{\beta}} = \mathcal{S}_{\varphi,\beta} \circ d\psi$, and therefore the eigenbundle net of $\mathcal{S}_{\bar{\varphi},\bar{\beta}}$ is the product net of $M_1 \times M_2$. It now follows from Proposition 31 and equation (45) in [**DT**$_2$] that $\tilde{f} \circ \psi = H \circ I \circ \tilde{g}$, where $\tilde{g} = \mathcal{R}_{\hat{\varphi},\hat{\beta}}(g)$ is a Darboux transform of $g = g_1 \times g_2$ such that $\mathcal{S}_{\hat{\varphi},\hat{\beta}} = \tau I + \|g - P_0\|^2\mathcal{S}_{\bar{\varphi},\bar{\beta}}$, with $\tau = 2(\bar{\varphi} - \langle g - P_0, \mathcal{F}_{\bar{\varphi},\bar{\beta}}\rangle)$. In particular, the eigenbundle net of $\mathcal{S}_{\hat{\varphi},\hat{\beta}}$ is also the product net of $M_1 \times M_2$. It now follows easily from Proposition 15 that either one of the factors is one-dimensional or there exist $a_1, a_2 \in \mathbb{R}$ with $a_1 \neq a_2$ such that $\mathcal{S}_{\hat{\varphi},\hat{\beta}} = a_1\Pi_1 + a_2\Pi_2$. Thus, it suffices to prove that in the latter case there must exist $i \in \{1, 2\}$ such that $g_i(M_i)$ is contained in some sphere $\mathbb{S}^{N_i-1}(P_i; r_i) \subset \mathbb{R}^{N_i}$. Integrating $d\mathcal{F}_{\hat{\varphi},\hat{\beta}} = dg \circ \mathcal{S}_{\hat{\varphi},\hat{\beta}}$ and $d\hat{\varphi} = \langle \mathcal{F}_{\hat{\varphi},\hat{\beta}}, dg\rangle$ gives

$$\mathcal{F}_{\hat{\varphi},\hat{\beta}} = a_1(g_1 - P_1) \times a_2(g_2 - P_2) \quad \text{and} \quad 2\hat{\varphi} = a_1\|g_1 \circ \pi_1 - P_1\|^2 + a_2\|g_2 \circ \pi_2 - P_2\|^2 + C,$$

for some $P_1 \in \mathbb{R}^{N_1}$, $P_2 \in \mathbb{R}^{N_2}$ and $C \in \mathbb{R}$. Using that $\hat{\varphi}(a_1 + a_2) = \hat{\nu}^{-1} = \langle \mathcal{F}_{\hat{\varphi},\hat{\beta}}, \mathcal{F}_{\hat{\varphi},\hat{\beta}}\rangle$, as follows from (21), we obtain

$$a_1(a_2 - a_1)\|g_1 \circ \pi_1 - P_1\|^2 + a_2(a_2 - a_1)\|g_2 \circ \pi_2 - P_2\|^2 + C(a_1 + a_2) = 0,$$

which implies that $a_1\|g_1 - P_1\|^2$ and $a_2\|g_2 - P_2\|^2$ must be constants $a_1r_1^2$ and $a_2r_2^2$, respectively. Since a_1 and a_2 can not be both zero, we conclude that $\|g_i - P_i\|^2 = r_i^2$ for at least one $i \in \{1, 2\}$.

We now prove the converse. It suffices to show that if g is either an extrinsic product of isometric immersions $g = g_1 \times g_2$: $M^n = M_1 \times M_2 \to \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} = \mathbb{R}^N$ as in (i), or it is given by $g = \Phi \circ (g_1 \times g_2)$: $M^n = M_1 \times_\rho M_2 \to \mathbb{R}^N$, where Φ: $\mathbb{R}^m_+ \times_\sigma \mathbb{S}^{N-m} \to \mathbb{R}^N$ is an isometry and g_1: $M_1 \to \mathbb{R}^m_+$ and g_2: $M_2 \to \mathbb{S}^{N-m}$ are isometric immersions, then g admits a Darboux transform $\tilde{g}$. For if this is the case and $f = H \circ I \circ g$ for such a g, then $H \circ I \circ \tilde{g}$ is a Darboux transform of f by Proposition 31 in [$\mathbf{DT}_2$].

Assume first that $g = g_1 \times g_2$: $M^n = M_1 \times M_2 \to \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} = \mathbb{R}^N$ is an extrinsic product of isometric immersions with $\|g_2 - P_2\|^2 = r_2^2$ for some $r_2 > 0$, $P_2 \in \mathbb{R}^{N_2}$. Define $\mathcal{F} = g_2 \circ \pi_2 - P_2$ and $\varphi = r_2^2$. Then $d\varphi = 0 = \langle \mathcal{F}, dg\rangle$ and $d\mathcal{F} = dg \circ \Pi_2$. Therefore φ and $\mathcal{F}$ determine a Ribaucour transform $\tilde{g}$ of g whose associated commuting Codazzi tensor is $\mathcal{S} = \Pi_2$. Moreover, (21) is satisfied, since the eigenvalues of $\mathcal{S}$ are $\lambda = 0$ and $\mu = 1$. Hence $\tilde{g}$ is in fact a Darboux transform of g.

Now suppose that, say, M_1 is an open interval $I \subset \mathbb{R}$, so that $\alpha := g_1$: $I \to \mathbb{R}^{N_1}$ is a unit speed curve. We may assume that the Frenet curvatures $k_1, \ldots, k_{N_1-1}$ of α are nowhere vanishing and consider the linear first order system of ODE's

$$\begin{cases} (i)\ \lambda' = \beta, \quad (ii)\ \beta' = \lambda + k_1V_2, \\ (iii)\ V_2' = -k_1\beta + k_2V_3, \quad (iv)\ V_{N_1}' = -k_{N_1-1}V_{N_1-1}, \\ (v)\ V_j' = -k_{j-1}V_{j-1} + k_jV_{j+1}, \ \ 3 \leq j \leq N_1 - 1, \end{cases} \tag{28}$$

which has the first integral

$$\lambda^2 - \lambda'^2 - \sum_{j=2}^{N_1} V_j^2 = K \in \mathbb{R}. \tag{29}$$

Let $(\lambda, \beta = \lambda', V_2, \ldots, V_{N_1})$ be a solution of (28) with initial conditions chosen so that the constant K in the right-hand-side of (29) vanishes. Let γ: $I \to \mathbb{R}^{N_1}$ be defined by $\gamma = \lambda'\alpha' + \sum_{j=2}^{N_1} V_j e_j$, where $e_1 = \alpha', e_2, \ldots, e_{N_1}$ is the Frenet frame of α. Using (28) we obtain that $\gamma' = \lambda\alpha'$. Now set $\mathcal{F} = \gamma \circ \pi_1$ and $\varphi = \lambda \circ \pi_1$. Since $\lambda' = \langle \gamma, \alpha' \rangle$, it follows that $d\varphi = \langle \mathcal{F}, dg \rangle$. Moreover, $d\mathcal{F} = dg \circ ((\lambda \circ \pi_1)\Pi_1)$ whence φ and $\mathcal{F}$ determine a Ribaucour transform $\tilde{g}$ of g whose associated commuting Codazzi tensor is $\mathcal{S} = (\lambda \circ \pi_1)\Pi_1$. Furthermore, since (29) holds with $K = 0$ and $\mathcal{S}$ has eigenvalues $\lambda \circ \pi_1$ and 0, it follows that (21) is satisfied, for

$$\varphi(\lambda \circ \pi_1) = (\lambda \circ \pi_1)^2 = \|\gamma \circ \pi_1\|^2 = \langle \mathcal{F}, \mathcal{F} \rangle.$$

We conclude that $\tilde{g}$ is a Darboux transform of g.

Finally, we prove that $g = \Phi \circ (g_1 \times g_2)\colon M^n = M_1 \times_\rho M_2 \to \mathbb{R}^N$, where $\Phi\colon \mathbb{R}^m_+ \times_\sigma \mathbb{S}^{N-m} \to \mathbb{R}^N$ is an isometry and $g_1\colon M_1 \to \mathbb{R}^m_+$ and $g_2\colon M_2 \to \mathbb{S}^{N-m} \subset \mathbb{R}^{N-m+1}$ are isometric immersions, also admits a Darboux transform. Let g be parameterized by $g = (h_1, \ldots, h_{m-1}, h_m g_2)$, where $g_1 = (h_1, \ldots, h_{m-1}, h_m)$, $\rho = h_m$ and $g_2 = (k_1, \ldots, k_{N-m+1})$ has unit length. Define

$$\mathcal{F} = (0, \ldots, 0, k_1 \circ \pi_2, \ldots, k_{N-m+1} \circ \pi_2) \quad \text{and} \quad \varphi = h_m \circ \pi_1.$$

Then $d\varphi = d(h_m \circ \pi_1) = \langle \mathcal{F}, dg \rangle$ and $d\mathcal{F} = dg \circ \mathcal{S}$, where $\mathcal{S} = (h_m \circ \pi_1)^{-1}\Pi_2$. It follows that φ and $\mathcal{F}$ determine a Ribaucour transform $\tilde{g}$ of g whose associated commuting Codazzi tensor is $\mathcal{S}$. Moreover, since the eigenvalues of $\mathcal{S}$ are $\lambda = 0$ and $\mu = (h_m \circ \pi_1)^{-1}$, we have that $\varphi(\lambda + \mu) = 1 = \langle \mathcal{F}, \mathcal{F} \rangle$. Thus (21) is satisfied, whence $\tilde{g}$ is a Darboux transform of g. ∎

Remark 21. Our definition of an immersion $\tilde{f}\colon M^n \to \mathbb{R}^N$ being a Darboux transform of a given isometric immersion $f\colon M^n \to \mathbb{R}^N$ does not rule out the possibility that $\tilde{f}$ and f differ by a rigid motion of $\mathbb{R}^N$ that induces a vector bundle isometry $\mathcal{P}$ between the normal bundles of f and $\tilde{f}$ and a tensor D on M^n having two constant eigenvalues with the same absolute values and opposite signs such that $d\tilde{f} = \mathcal{P} \circ df \circ D$. In fact, the Darboux transforms of f constructed in the proof of the converse in Theorem 20 are of this type, except for those in the sub-case of case (i) in which one of the factors is one-dimensional. Had we excluded this possibility in the definition, it is not difficult to check that in order for the converse statement of Theorem 20 to remain true one should add the requirement that in both cases either M_1 or M_2 be one-dimensional.

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