

Lap Number Properties for p -Laplacian Problems by Lyapunov Methods

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Abstract

In this work we consider a one-dimensional quasilinear parabolic equation and we prove that the lap number of any solution can not increase through orbits as the time passes if the initial data is a continuous function. We deal with the lap number functional as a Lyapunov function, and apply lap number properties to reach an understanding on the asymptotic behavior of a particular problem.

1 Introduction

We consider the following reaction-diffusion problem

$$\begin{cases} u_t = \lambda(|u_x|^{p-2}u_x)_x + |u|^{q-2}u(1 - |u|^r), & (x, t) \in (a, b) \times (0, +\infty) \\ u(a, t) = u(b, t) = 0, & t \in (0, +\infty), \end{cases} \quad (1.1)$$

with initial condition $u(x, 0) = u_0(x)$, $x \in (a, b)$, where $a, b \in \mathbb{R}$, $p > 2$, $q \geq 2$, $r > 0$, and λ is a positive parameter. We can put this problem in an abstract framework by defining $D(A) = \{u \in W_0^{1,p}(a, b); \lambda(|u_x|^{p-2}u_x)_x \in L^2(a, b)\}$, and $Au = -\lambda(|u_x|^{p-2}u_x)_x$ if $u \in D(A)$. So we have a maximal monotone operator A in $H = L^2(a, b)$, in fact a subdifferential of a convex lower semi continuous (l.s.c.) function φ which is equivalent to $W_0^{1,p}(a, b)$ norm.

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Existence questions concerning this kind of problem are well-known (see [1, 2], for example), and we can associate a continuous semigroup with (1.1), defined in $D(A) = L^2(a, b)$. There have been several works in recent years that investigate the asymptotic properties of this type of evolution problems and we particularly mention [3], where the authors obtain the existence of global attractors for problems governed by monotone operators with globally Lipschitz perturbations. It is our intention in this work to start an investigation about how can be the dynamics on such attractors.

The ideas developed here have been motivated by [10] and [7]. In [10], Takeuchi and Yamada have made a complete description of the equilibrium solutions of (1.1) for each p , q and λ parameters. One important result is the existence of equilibrium solutions with flat cores which allows the appearance of a continuum equilibrium set whose connected components are determined by the number of zeros of the stationary solutions. Since the lap number of an equilibrium point ϕ , as it is defined in [7], counts its number of zeros, a study of the lap number behavior on orbits can provide more information about the connections among the stationary solutions in the attractor.

We begin by establishing the lap number properties of the solutions u of (1.1) if $r > q - 2$ in the same sense as it is done for semilinear one-dimensional problems in [7]. We show by using Lyapunov methods that the lap number $l(u(t))$ can not increase through orbits as the time passes if the initial data is continuous. This assumption allow us to deal with resolvent conditions by using elementary methods and does not represent serious restrictions once $u(t) \in D(A) \subset W_0^{1,p}(a, b) \subset C([a, b])$, $\forall t > 0$ even if $u_0 \in \overline{D(A)}$, [2]. The way we approach this issue is possible thanks to a Pazy work [9], where he exhibits a sufficient condition for a real-valued function φ to be a Lyapunov function for semigroups generated by accretive operators, without appealing to any differentiability condition neither on φ nor on the solutions of the problem.

The remaining of this paper is organized as follows: section 2 sets up the definitions and properties of the lap number function. In section 3, we establish the lap number monotonicity. Finally, in section 4, we use this result to obtain some comprehension about the connections among the equilibrium points in the attractor.

2 Notation and Preliminary Facts

In [7] we can find two definitions of the lap number of a function, and they are equivalent if we consider only continuous functions. The first one counts the least value of non-overlapping intervals where a real-valued function ω is monotone. Naturally this number is well defined if ω is a piecewise monotone function on some bounded interval of \mathbb{R} . We elect the last definition here and we describe it in details: let $I = (a, b)$, $\bar{I} = [a, b]$, $a, b \in \mathbb{R}$, and $\omega : \bar{I} \rightarrow \mathbb{R}$ a continuous piecewise monotone non-constant function. Let k be a positive integer number such that we can choose k distinct points $a = x_0 < x_1 < x_2 < \dots < x_k = b$ in a way that $\omega(x_i) \neq \omega(x_{i+1})$ and $(\omega(x_{i+1}) - \omega(x_i))(\omega(x_i) - \omega(x_{i-1})) < 0$ for $i = 1, \dots, k - 1$. The lap number $l(\omega)$ of ω is the supremum of the possible numbers k . As it is done in [7], we set $l(\omega) = 0$ when ω is

a constant function, and $l(\omega) = \infty$ when ω is not a piecewise monotone function. Also, we set $l^+(\omega) =$ the cardinal number of $\{i \in \mathbb{N}; 1 \leq i \leq k, \omega(x_i) - \omega(x_{i-1}) > 0\}$, and $l^-(\omega) =$ the cardinal number of $\{i \in \mathbb{N}; 1 \leq i \leq k, \omega(x_i) - \omega(x_{i-1}) < 0\}$. We understand $l^+(\omega) = l^-(\omega) = 0$ when ω is a constant function. It follows from the definitions that $l^-(\omega) = l^+(-\omega)$, and $l(\omega) = l^+(\omega) + l^-(\omega)$.

For $y \in L^2(I)$ we define the operator A_y :

$$\begin{cases} D(A_y) = \{u \in W_0^{1,p}(I); \lambda(|u_x|^{p-2}u_x)_x + |u|^{q+r-2}u \in L^2(I)\}, \\ A_y u = -\lambda(|u_x|^{p-2}u_x)_x + |u|^{q+r-2}u - y, \text{ if } u \in D(A_y). \end{cases}$$

Remark 2.1 If $y = 0$ we denote A_y by A_0 .

If $u = u(x, t)$ is a solution of

$$\begin{cases} u_t + A_y u = 0, & \text{in } (a, b) \times (0, +\infty), \\ u(x, 0) = u_0, & u_0 \in L^2(I), \end{cases} \quad (2.2)$$

for each fixed value of t we define the lap number of $u(\cdot, t)$, and then $l(u(\cdot, t))$ becomes a function of t .

A_y is a maximal monotone operator in $L^2(I)$ and the problem (2.2) has an unique solution $u(t) \in D(A_y) \subset W_0^{1,p}(I) \subset C(\bar{I})$ for $t > 0$. It is also well known [1, 5] that, if $S(t)$ denotes the semigroup generated by $-A_y$ in $\overline{D(A_y)} = L^2(I)$ and $J_\mu = (I + \mu A_y)^{-1}$, $\mu > 0$, is the resolvent operator, then

$$J_{t/n}^n u \rightarrow S(t)u \quad \text{for } u \in L^2(I) \text{ and } t > 0.$$

Thus, in order to get some information about the lap number of solutions of (1.1), we start by showing, by elementary observations, that the resolvent $J_\mu u_0 = (I + \mu A_y)^{-1} u_0$ of u_0 , $\mu > 0$, can not oscillate more than u_0 if u_0, y are continuous functions on \bar{I} and if y is chosen in a suitable way.

Lemma 2.1 If $u_0 \in C([a, b])$ and $y = |u_0|^s u_0$, $s > 0$, then $l(J_\mu(u_0)) \leq l(u_0)$.

Proof: Let $\omega = J_\mu(u_0)$. Then

$$\omega - \mu\lambda(|\omega_x|^{p-2}\omega_x)_x + \mu|\omega|^{q+r-2}\omega - \mu y = u_0. \quad (2.3)$$

We first note that, once u_0, y belong to $C([a, b])$, if we define $\phi_p(v) := |v|^{p-2}v$, then $\phi_p(\omega_x)$ belongs to $C^1([a, b])$ and so does ω . In fact, as it is done in [8], by putting

$$\mathcal{U}(x) = \int_a^x u_0(\xi) + \mu y(\xi) - \mu|\omega(\xi)|^{q+r-2}\omega(\xi) - \omega(\xi) d\xi$$

we have $\langle \mu\lambda\phi_p(\omega_x), \varphi_x \rangle = \langle -\mathcal{U}, \varphi_x \rangle$ for all $\varphi \in C_0^\infty(a, b)$. Thus we see that

$$\phi_p(\omega_x)(x) = -\mathcal{U}(x) + \text{const}, \quad \text{a.e. } x \in (a, b),$$

from where we obtain the result.

If $l(\omega) \in (0, +\infty)$, let $k = l(\omega)$. We can choose $x_i \in (a, b)$, $i = 1, 2, \dots, k-1$, $x_0 = a$, $x_k = b$, such that $x_{i-1} < x_i$ and the signs of $\omega(x_i) - \omega(x_{i-1})$ and $\omega(x_{i+1}) - \omega(x_i)$ are distinct. In fact we can pick those points so that $\omega_x(x_i) = 0$, $0 < i < k$, and ω is monotone on each sub-interval (x_{i-1}, x_i) , $0 < i < k$.

From (2.3) we have

$$\begin{aligned} \omega(x_i) - \omega(x_{i-1}) &= \mu\lambda[(\phi_p(\omega_x))_x(x_i) - (\phi_p(\omega_x))_x(x_{i-1})] \\ &+ \mu(\phi_{q+r}(\omega)(x_i) - \phi_{q+r}(\omega)(x_{i-1})) \\ &= u_0(x_i) - u_0(x_{i-1}) + \mu(y(x_i) - y(x_{i-1})). \end{aligned} \quad (2.4)$$

If $\omega(x_i) - \omega(x_{i-1}) < 0$ then $\omega_x \leq 0$ on (x_{i-1}, x_i) . As $\phi_p(\omega_x)$ has the same sign of ω_x on (x_{i-1}, x_i) , and as ω_x changes successively sign at each subinterval, we must have $(\phi_p(\omega_x))_x(x_{i-1}) \leq 0$ and $(\phi_p(\omega_x))_x(x_i) \geq 0$. So, we can see that $l^-(\omega) \leq l^-(u_0 + \mu y)$ and, as $y = |u_0|^s u_0$, $l^-(\omega) \leq l^-(u_0)$. The same argument guarantees that $l^+(\omega) \leq l^+(u_0)$. Finally, if $l(\omega) = \infty$, it is enough to note that k can be taken as big as we want, so u_0 can not to be a piecewise monotone function. \blacksquare

Remark 2.2 *In fact, in order to have $l(\omega) \leq l(u_0)$ it is enough to suppose that y is a continuous function such that $l(u_0 + \mu y) \leq l(u_0)$. A special particular case is $y \equiv 0$.*

3 Lap Number as a Lyapunov Function

In this section we state our principal result, Theorem 3.2. The essential tool for proving it is the Theorem 3.1, whose proof follows closely the ideas in [9], Theorem 3.4, which give us a type of criteria for a function ϕ to be a Lyapunov function for problem (2.2) not based on the differentiability of $\phi(u(t))$. In fact, we do not even have to suppose that ϕ is a continuous function.

Lemma 3.1 *Let $t > 0$. $\{J_{t/n}^n u_0\}$ is a bounded sequence in $W_0^{1,p}(I)$ norm if $u_0, y \in W_0^{1,p}(I)$.*

Proof: As $A_y u = \partial \varphi u - y$, $u \in D(A_y)$, with

$$\varphi(u) = \begin{cases} \frac{1}{p} \int_{\Omega} |\nabla u(x)|^p dx + \frac{1}{q+r} \int_{\Omega} |u(x)|^{q+r} dx, & u \in W_0^{1,p}(I), \\ +\infty, & \text{otherwise,} \end{cases}$$

if $\mathcal{J}_\mu = (I + \mu \partial \varphi)^{-1}$, it follows from [2], Lemma 2.1, that $\varphi(\mathcal{J}_\mu u_0) \leq \varphi(u_0) \forall \mu > 0$. Thus, as $J_\mu u_0 = \mathcal{J}_\mu(u_0 + \mu y)$, we have that $\varphi(J_\mu u_0) \leq \varphi(u_0 + \mu y)$. Also, if $0 < \mu < 1$ and $x, y \in W_0^{1,p}(I)$, we can easily see that

$$\varphi(x + \mu y) \leq (1 - \mu) \varphi\left(\frac{x}{1 - \mu}\right) + \mu \varphi(y) \leq (1 - \mu)^\alpha \varphi(x) + \mu \varphi(y),$$

where $\alpha = \min\{1 - p, 1 - (q + r)\}$. Therefore, if $0 < \mu < 1$,

$$\varphi(J_\mu^n(u_0)) \leq (1 - \mu)^{n\alpha} \varphi(u_0) + \left[\sum_{j=0}^{n-1} (1 - \mu)^{j\alpha} \mu \right] \varphi(y).$$

If $\mu = t/n$ and n is big enough,

$$\begin{aligned} \|J_{t/n}^n u_0\|_{W_0^{1,p}}^p &\leq p e^{-\alpha t} \left(\frac{1}{p} \|u_0\|_{W_0^{1,p}}^p + \frac{1}{q+r} \|u_0\|_{L^{q+r}}^{q+r} \right) \\ &+ p t e^{-\alpha t} \left(\frac{1}{p} \|y\|_{W_0^{1,p}}^p + \frac{1}{q+r} \|y\|_{L^{q+r}}^{q+r} \right), \end{aligned}$$

from where we obtain the result. ■

Theorem 3.1 *If $y = |u_0|^s u_0$, $s > 0$, and u satisfies*

$$\begin{cases} u_t = -A_y u, & \text{in } (a, b) \times (0, +\infty), \\ u(\cdot, 0) = u_0, & u_0 \in D(A_y), \end{cases} \quad (3.5)$$

then $l(u(\cdot, t))$ is a non increasing function on $t \in [0, +\infty)$.

Proof: We set

$$l(J_\mu u_0) - l(u_0) = \mu \epsilon(\mu, u_0). \quad (3.6)$$

According to Lemma 2.1, $\epsilon(\mu, u_0) \leq 0$. Also we have $J_\mu^k u_0 \in D(A_y)$ for every $\mu > 0$ and $k \geq 1$. By replacing u_0 by $J_\mu^{k-1} u_0$ in (3.6) we find

$$l(J_\mu^k u_0) - l(J_\mu^{k-1} u_0) = \mu \epsilon(\mu, J_\mu^{k-1} u_0) \leq 0,$$

and

$$l(J_\mu^n u_0) - l(u_0) = \sum_{k=1}^n \mu \epsilon(\mu, J_\mu^{k-1} u_0).$$

Let $t > 0$. If we choose $\mu = \frac{t}{n}$ we have

$$l(J_{t/n}^n u_0) - l(u_0) = \sum_{k=1}^n \frac{t}{n} \epsilon\left(\frac{t}{n}, J_{t/n}^{k-1} u_0\right) \leq 0. \quad (3.7)$$

At this point, we certainly intend to pass the limit as $n \rightarrow \infty$. We have to remark some points before doing it. It follows from [5] that $J_{t/n}^n u_0 \rightarrow S(t)u_0$ in $L^2(I)$ for $t > 0$ and $u_0 \in L^2(I)$, then there exists some subsequence $\{n_k\}$ such that $J_{t/n_k}^{n_k} u_0(x) \rightarrow S(t)u_0(x)$ a.e. in (a, b) , but this convergence is not enough to preserve the inequality (3.7) when $n \rightarrow \infty$. However, once $W_0^{1,p}(I)$ is compactly embedded in $C(\bar{I})$, it follows from Lemma

3.1 that there is a subsequence still denoted by $\{n_k\}$ such that $J_{t/n_k}^{n_k} u_0(x) \rightarrow S(t)u_0(x)$ uniformly in $[a, b]$.

As an easy consequence of the definitions, if $\omega_n(x) \rightarrow \omega(x)$ for all $x \in \bar{I}$, then $\liminf_{n \rightarrow \infty} l^+(\omega_n) \geq l^+(\omega)$ and $\liminf_{n \rightarrow \infty} l^-(\omega_n) \geq l^-(\omega)$. We are now ready to return to (3.7):

$$l(S(t)u_0) - l(u_0) \leq \liminf_{k \rightarrow \infty} l(J_{t/n_k}^{n_k} u_0) - l(u_0) \leq \limsup_{k \rightarrow \infty} \sum_{j=1}^{n_k} \frac{t}{n_k} \epsilon \left(\frac{t}{n_k}, J_{t/n_k}^{j-1} u_0 \right) \leq 0.$$

In order to get the result, it is enough to note that $D(A_y)$ is invariant, and so we can replace u_0 with $S(s)u_0$, $0 < s < t < \infty$. \blacksquare

Now we start to extend this lap number property to problem (1.1). We will do it by constructing a sequence of simple problems that approach problem (1.1). Let T be a positive real number and $u_0 \in D(A_0)$. Given $n \in \mathbb{Z}$, $n \geq 1$, we denote by u^{nk} the solution in $[(k-1)\frac{T}{n}, k\frac{T}{n}]$ of

$$\begin{cases} \frac{d}{dt} u^{nk} + A_0(u^{nk}) = |u^{n(k-1)}((k-1)\frac{T}{n})|^{q-2} u^{n(k-1)}((k-1)\frac{T}{n}), \\ u^{nk}(0) = u^{n(k-1)}((k-1)\frac{T}{n}), \end{cases}$$

for $k = 1, 2, \dots, n$, where $u^{n0} \equiv u_0$ for all integer $n \geq 1$. We define v_n in $[0, T]$ by $v_n(t) = u^{nk}(t)$ if $t \in [(k-1)\frac{T}{n}, k\frac{T}{n}]$, $k = 1, 2, \dots, n$. The function v_n is the only solution in $[0, T]$ of

$$\begin{cases} \frac{d}{dt} v_n + A_0 v_n = f_n(t) \\ v_n(0) = u_0, \end{cases} \quad (3.8)$$

where f_n is the step function in $[0, T]$ given by

$$f_n(t) = \left| u^{n(k-1)} \left((k-1)\frac{T}{n} \right) \right|^{q-2} u^{n(k-1)} \left((k-1)\frac{T}{n} \right),$$

$t \in [(k-1)\frac{T}{n}, k\frac{T}{n})$, $k = 1, 2, \dots, n$. We now briefly describe our ideas. Once A_0 is a maximal monotone operator in $H = L^2(I)$ which has compact resolvent, $-A_0$ generates a compact semigroup ([1, 2, 12]). Therefore, if we prove that the sequence f_n is uniformly integrable, we can appeal to Theorem 2.3.2, [12], to guarantee that the sequence $\{v_n\}$ is relatively compact in $C([0, T], H)$, and so there exists a subsequence v_{n_k} converging to a certain function v . By last theorem, each v_{n_k} satisfies $l(v_{n_k}(t)) \leq l(u_0)$, then we can use the lower semicontinuity of l to state the lap number non-increasing property for v . Finally we prove that this limit function v is in fact solution of problem (1.1).

Lemma 3.2 *The sequence $\{f_n\}$ is uniformly bounded in H norm if $r > q - 2$.*

Proof: We need only of two basic inequalities we state bellow. If u satisfies the relation

$$\frac{d}{dt}u + \partial\varphi u = |y|^{q-2}y,$$

in some interval $[t_1, t_2]$, then

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 \leq \| |y|^{q-2}y \|_H \|u\|_H,$$

and so

$$\|u(t)\|_H \leq \|u(t_1)\|_H + (t_2 - t_1) \| |y|^{q-2}y \|_H^{q-1}, \quad t \in [t_1, t_2]. \quad (3.9)$$

On the other hand, we know that

$$\left\| \frac{d}{dt}u(t) \right\|_H^2 + \frac{d}{dt}\varphi(u(t)) = \left\langle |y|^{q-2}y, \frac{d}{dt}u(t) \right\rangle,$$

from where we get

$$\frac{1}{p} \|u(t)\|_{W_0^{1,p}}^p + \frac{1}{q+r} \|u(t)\|_{L^{q+r}}^{q+r} \leq \frac{1}{p} \|u(t_1)\|_{W_0^{1,p}}^p + \frac{1}{q+r} \|u(t_1)\|_{L^{q+r}}^{q+r} + \frac{1}{2} (t_2 - t_1) \| |y|^{2q-2} \|_{L^{2q-2}},$$

and finally, for $t \in [t_1, t_2]$,

$$\|u(t)\|_{W_0^{1,p}}^p + \|u(t)\|_{L^{q+r}}^{q+r} \leq \rho \left[\|u(t_1)\|_{W_0^{1,p}}^p + \|u(t_1)\|_{L^{q+r}}^{q+r} + (t_2 - t_1) \| |y|^{2q-2} \|_{L^{2q-2}} \right], \quad (3.10)$$

where $\rho = \max\{\frac{p}{2}, \frac{q+r}{2}\}$. Let n be a positive integer number and u^{nk} as defined above, $k = 1, 2, \dots, n$. By using repeatedly the inequalities (3.9) and (3.10), once we are assuming $r > q - 2$, we have $q + r > 2q - 2$ and then,

$$\begin{aligned} \left\| u^{n1} \left(\frac{T}{n} \right) \right\|_H &\leq \|u_0\|_H + \frac{T}{n} \|u_0\|_{L^{2q-2}}^{q-1} \\ \left\| u^{n2} \left(\frac{2T}{n} \right) \right\|_H &\leq \left\| u^{n1} \left(\frac{T}{n} \right) \right\|_H + \frac{T}{n} \left\| u^{n1} \left(\frac{T}{n} \right) \right\|_{L^{2q-2}}^{q-1} \\ &\leq \|u_0\|_H + \frac{T}{n} \|u_0\|_{L^{2q-2}}^{q-1} + \frac{T}{n} \mathcal{C} \left(\|u_0\|_{W_0^{1,p}}^p + \|u_0\|_{L^{q+r}}^{q+r} + \frac{T}{n} \|u_0\|_{L^{2q-2}}^{2q-2} \right)^{\frac{q-1}{q+r}} \end{aligned}$$

where \mathcal{C} is the immersion constant of $L^{q+r}(I) \subset L^{2q-2}(I)$ multiplied by $\rho^{\frac{q-1}{q+r}}$.

If $n \geq 3$ we can continue this process, and we obtain

$$\begin{aligned}
\left\| u^{n3} \left(\frac{3T}{n} \right) \right\|_H &\leq \left\| u^{n2} \left(\frac{2T}{n} \right) \right\|_H + \frac{T}{n} \left\| u^{n2} \left(\frac{2T}{n} \right) \right\|_{L^{2q-2}}^{q-1} \\
&\leq \|u_0\|_H + \frac{T}{n} \|u_0\|_{L^{2q-2}}^{q-1} + \frac{T}{n} \mathcal{C} \left(\|u_0\|_{W_0^{1,p}}^p + \|u_0\|_{L^{q+r}}^{q+r} + \frac{T}{n} \|u_0\|_{L^{2q-2}}^{2q-2} \right)^{\frac{q-1}{q+r}} \\
&\quad + \frac{T}{n} \mathcal{C} \left(\left\| u^{n1} \left(\frac{T}{n} \right) \right\|_{W_0^{1,p}}^p + \left\| u^{n1} \left(\frac{T}{n} \right) \right\|_{L^{q+r}}^{q+r} + \frac{T}{n} \left\| u^{n1} \left(\frac{T}{n} \right) \right\|_{L^{2q-2}}^{2q-2} \right)^{\frac{q-1}{q+r}} \\
&\leq \|u_0\|_H + \frac{T}{n} \|u_0\|_{L^{2q-2}}^{q-1} + \frac{T}{n} \mathcal{C} \left(\|u_0\|_{W_0^{1,p}}^p + \|u_0\|_{L^{q+r}}^{q+r} + \frac{T}{n} \|u_0\|_{L^{2q-2}}^{2q-2} \right)^{\frac{q-1}{q+r}} \\
&\quad + \frac{T}{n} \mathcal{C} \left(\rho \left(\|u_0\|_{W_0^{1,p}}^p + \|u_0\|_{L^{q+r}}^{q+r} + \frac{T}{n} \|u_0\|_{L^{2q-2}}^{2q-2} \right) \right) \\
&\quad + \frac{T}{n} \mathcal{C} \left(\rho \left(\|u_0\|_{W_0^{1,p}}^p + \|u_0\|_{L^{q+r}}^{q+r} + \frac{T}{n} \|u_0\|_{L^{2q-2}}^{2q-2} \right) \right)^{\frac{2q-2}{q+r}} \frac{q-1}{q+r}.
\end{aligned}$$

If we set $\mathcal{K} = \max \left\{ 1, \rho \left(\|u_0\|_{W_0^{1,p}}^p + \|u_0\|_{L^{q+r}}^{q+r} + T \|u_0\|_{L^{2q-2}}^{2q-2} \right) \right\}$, $\alpha = \frac{q-1}{q+r}$, $\beta = \frac{2q-2}{q+r}$, and $\gamma = T\mathcal{C}$, we need to assure that we can estimate the summation with $n-1$ terms

$$\begin{aligned}
&\frac{\gamma}{n} \mathcal{K}^\alpha + \frac{\gamma}{n} (\mathcal{K} + \frac{\gamma}{n} \mathcal{K}^\beta)^\alpha + \frac{\gamma}{n} (\mathcal{K} + \frac{\gamma}{n} (\mathcal{K} + \frac{\gamma}{n} \mathcal{K}^\beta)^\beta)^\alpha \\
&\quad + \dots + \frac{\gamma}{n} \left(\mathcal{K} + \frac{\gamma}{n} \left(\mathcal{K} + \frac{\gamma}{n} \left(\mathcal{K} + \dots + \frac{\gamma}{n} (\mathcal{K} + \frac{\gamma}{n} \mathcal{K}^\beta)^\beta \right)^\beta \right)^\beta \right)^\alpha \quad (3.11)
\end{aligned}$$

when n goes to infinity. We can choose T in such way that $\gamma < 1$ and, as $\alpha < \beta < 1$, we can see that (3.11) is less than

$$\sum_{j=1}^{n-1} \frac{n-j}{n^j} \mathcal{K} \leq \mathcal{K} \sum_{j=1}^{n-1} \frac{1}{n^{j-1}} \leq \mathcal{K} \sum_{j=1}^{n-1} \left(\frac{1}{2} \right)^{j-1} \uparrow 2\mathcal{K}, \quad \text{if } n \geq 2.$$

Now, we note that

$$\left\| u^{nk} \left(\frac{kT}{n} \right) \right\|_H \leq \|u_0\|_H + \frac{T}{n} \|u_0\|_{L^{2q-2}}^{q-1} + 2\mathcal{K},$$

$\forall n \geq 2$ and $k = 0, 1, \dots, n$. Therefore, the sequence $\{f_n\}$ is uniformly bounded in H . \blacksquare

From the last lemma we can conclude that the sequence $\{f_n\}$ is uniformly integrable in $L^1(0, T; H)$, which means that for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\int_E \|f_n(t)\|_H dt \leq \epsilon$ for each measurable subset $E \subset [0, T]$ whose Lebesgue measure is less than $\delta(\epsilon)$, uniformly for $n \geq 1$. It follows from Theorem 2.3.2, [12], that $\{v_n\}$ is relatively compact in $C([0, T], H)$.

Let $\{v_{n_k}\}$ be a subsequence of $\{v_n\}$ converging to v in $C([0, T], H)$. We claim that f_n goes to f given by $f(t) = |v(t)|^{q-2}v(t)$ in $L^1(0, T; H)$ as $n \rightarrow \infty$. In fact, given $\varepsilon \geq 0$, let N be large enough to

$$\|f(t_1) - f(t_2)\|_H < \frac{\varepsilon}{2} \quad \text{if } |t_2 - t_1| \leq \frac{T}{N},$$

and also

$$\sup_{t \in [0, T]} \|\phi_q(v(t)) - \phi_q(v_n(t))\|_H < \frac{\varepsilon}{2} \quad \text{for } n \geq N - 1.$$

If $t \in [(k-1)\frac{T}{N}, k\frac{T}{N}]$, $1 \leq k \leq N$,

$$\begin{aligned} \|f(t) - f_N(t)\|_H &\leq \left\| f(t) - f\left(\frac{(k-1)T}{N}\right) \right\|_H + \left\| f\left(\frac{(k-1)T}{N}\right) - f_N(t) \right\|_H \\ &= \left\| f(t) - f\left(\frac{(k-1)T}{N}\right) \right\|_H \\ &\quad + \left\| \phi_q\left(v\left(\frac{(k-1)T}{N}\right)\right) - \phi_q\left(u^{N(k-1)}\left(\frac{(k-1)T}{N}\right)\right) \right\|_H \\ &= \left\| f(t) - f\left(\frac{(k-1)T}{N}\right) \right\|_H \\ &\quad + \left\| \phi_q\left(v\left(\frac{(k-1)T}{N}\right)\right) - \phi_q\left(v_{N-1}\left(\frac{(k-1)T}{N}\right)\right) \right\|_H \\ &\leq \varepsilon. \end{aligned}$$

It follows from Proposition 3.6, [2], that v is in fact solution of problem (1.1). Therefore we can proof the following result

Theorem 3.2 *Let v be the solution of*

$$\begin{cases} v_t - \lambda(|v_x|^{p-2}v_x)_x = |v|^{q-2}v - |v|^{q+r-2}v & (x, t) \in (a, b) \times (0, +\infty), \\ v(a, t) = v(b, t) = 0, & t \in (0, +\infty), \\ v(x, 0) = v_0, & x \in (a, b), \end{cases} \quad (3.12)$$

where $p > 2$, $q \geq 2$, $r > q - 2$ and $u_0 \in D(A_0)$. Then $l(v(\cdot))$ is a nonincreasing function in $[0, +\infty)$.

Proof: It is enough to note that, for each $t \in [0, T]$ and once $A_0 = \partial\varphi$, we have from (3.8) that

$$\frac{1}{2} \left\| \frac{d}{dt} v_n(t) \right\|_H^2 + \frac{d}{dt} \varphi(v_n(t)) \leq \frac{1}{2} \|f_n(t)\|_H^2.$$

Then, it follows from Lemma 3.2 that $\{v_n(t)\}$ is bounded in $W_0^{1,p}(I)$ and so there is a subsequence $\{v_{n_k}(t)\}$ which converges to $v(t)$ in $C([a, b])$. Thus

$$l(v(t)) \leq \liminf l(v_n(t)) \leq l(v_0).$$

As the choice of T in (3.11) was not dependent on u_0 , the same argument works on $t \in [kT, (k+1)T]$, $k \geq 1$. \blacksquare

4 Applications

Now, we apply the previous result to the equation:

$$\begin{cases} u_t = \lambda(|u_x|^{p-2}u_x)_x + |u|^{p-2}u(1 - |u|^r), & (x, t) \in (0, 1) \times (0, +\infty) \\ u(0, t) = u(1, t) = 0, & t \in (0, +\infty), \end{cases} \quad (4.13)$$

with initial condition $u(x, 0) = u_0(x)$, $x \in (0, 1)$, and $r > p - 2$. In this case, Takeuchi and Yamada in [10] proved that there are two sequences λ_k and $\lambda_k(1)$, both going to zero when $k \rightarrow \infty$, such that for $\lambda \in [\lambda_{k+1}, \lambda_k)$, the equilibrium set E_λ is given by

$$E_\lambda = \{0\} \cup \bigcup_{l=0}^k \{E_{\lambda,l}\} \quad (4.14)$$

where $E_{\lambda,l}$ contains equilibrium points ϕ_λ^l with l zeros in the interval $(0, 1)$. $E_{\lambda,0} = \{\pm\phi_\lambda^0\}$ for all $\lambda > 0$ and, if $l > 0$, $E_{\lambda,l} = \{\pm\phi_\lambda^l\}$ for $\lambda \geq \lambda_l(1)$ and $E_{\lambda,l}$ is diffeomorphic to $[0, 1]^l$ for $\lambda < \lambda_l(1)$. The sign \pm indicates positive or negative initial derivative.

The semigroup associated with (4.13) is a gradient system in $W_0^{1,p}(I)$, and we have a Lyapunov functional $V : W_0^{1,p}(I) \rightarrow \mathbb{R}$ given by

$$V(u) = \frac{\lambda}{p} \|u'\|_{L^p}^p - \int_0^1 F(u(x)) dx$$

where $F(s) = \frac{|s|^p}{p} - \frac{|s|^{p+r}}{p+r}$. We have that

1. $V(u)$ is continuous and bounded below;
2. $V(u) \rightarrow \infty$, as $\|u\|_{W_0^{1,p}} \rightarrow \infty$;
3. $\frac{dV}{dt}(u(t, u_0)) = -\|u_t\|_2^2$ thus $V(u(t, u_0))$ is nonincreasing in t for each $u_0 \in W_0^{1,p}(I)$;
4. if $u(t, u_0)$ is defined for all t and $V(u(t, u_0)) \equiv V(u_0)$ then u_0 is a equilibrium point.

Therefore all ω -limit set contains only equilibrium points. As an easy consequence of Theorem 3.2, we get the following

Theorem 4.1 *Let u_0 be an initial condition in the attractor \mathcal{A}_λ and w_0 an equilibrium point in $\omega(u_0)$, the ω -limit of u_0 . Then $l(u_0) \geq l(w_0)$.*

Remark 4.1 *In other words w_0 has $l(u_0)$ or less zeros in $[0, 1]$ since $w_0 \subset E_\lambda$ and the lap number of an equilibrium solution ϕ is related with its number k of zeros in $(0, 1)$ by the following expression*

$$l(\phi) = k + 2.$$

It is known, see [3, 4], that there exists a global attractor \mathcal{A}_λ in $W_0^{1,p}(I)$ to (4.13). Thus, there are at least one complete orbit through each u_0 in the attractor and every negative orbit $\phi^-(u_0) = \{u(t, u_0), t < 0\}$ is precompact. Therefore, the set $\gamma = \bigcap_{s < 0} \overline{\{u(t, u_0), t < s\}}$ which is part of the α -limit set of u_0 , $\alpha(u_0)$, satisfies $\gamma \subset E_\lambda$ for $u_0 \in \mathcal{A}_\lambda$. Furthermore, the attractor is described by the unstable set W^u of the equilibrium set, [6, 11],

$$\mathcal{A}_\lambda = W^u(E_\lambda).$$

Since E_λ is given by (4.14) as an union of a finite number of connected components $E_{\lambda,l}$, then

$$\mathcal{A}_\lambda = W^u(\{0\}) \cup \bigcup_{l=0}^k W^u(E_{\lambda,l}).$$

So we know that \mathcal{A}_λ is an union of the components $E_{\lambda,l}$ and complete orbits going from one component to another. There is a pattern for equilibrium solutions in $E_{\lambda,l}$. Let $X(\lambda, \phi'(0))$ be the x -time that the solution ϕ spends going from 0 to 1. By symmetry, this x -time is equal to the time the same solution spends going from 1 to 0, from 0 to -1 and from -1 to 0. Also, each $\phi \in E_{\lambda,l}$ has exactly l zeros in $(0, 1)$, thus $1 = (2 + 2l)X(\lambda, \phi'(0)) + Y(\lambda, \phi'(0))$ where $Y(\lambda, \phi'(0))$ is the remaining x -time. $Y(\lambda, \phi'(0)) \geq 0$ and when $Y(\lambda, \phi'(0)) > 0$ it can be distributed in $l + 1$ flat-cores. Using the function V and this pattern we have that all equilibrium solutions in the component $E_{\lambda,l}$ have the same energy level, that means, $V(\phi) = m(l, \lambda)$ for all $\phi \in E_{\lambda,l}$. Therefore there is no orbit with α -limit and ω -limit in the same set $E_{\lambda,l}$ but the stationary solutions.

We would like to say that for every $u_0 \in \mathcal{A}_\lambda$ the α -limit set $\alpha(u_0)$ contains only functions with lap number greater or equal than $l(u_0)$ but, unfortunately, we can only guarantee it when $\lambda \geq \lambda_1(1)$, because the value $\lambda = \lambda_k(1)$ is the point where ϕ_λ^k starts to be allowed to have flat cores. In fact, if ϕ is constant in some interval, then as close to ϕ as one wishes there exists a function ψ whose lap number is greater than $l(\phi)$. So, each flat core of an equilibrium point in this context presents the same kind of problem that occurs with the trivial solution in the case of laplacian, [7], where the only situation in which the lap number of the α -limit is less than the lap number of the initial condition occurs when the α -limit is the trivial solution $\phi = 0$.

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