Gevrey regularity in time for generalized KdV type equations

Heather Hannah, A. Alexandrou Himonas and Gerson Petronilho *

Abstract

Given $s \geq 1$ we present initial data that belong to the Gevrey space G^s for which the solution to the Cauchy problem for the generalized $mk\ell$-KdV equation does not belongs to G^s in the time variable. Also, for the KdV, in the periodic case, we show that the solution to the Cauchy problem withanalytic initial data (Gevrey class G^1) belongs to G^3 in time.

1 Introduction

For $k, \ell \in \{1, 2, 3, 4, 5 \cdots\}$ and $m \in \{3, 4, 5, 6, \cdots\}$, we consider the Cauchy problem for the generalized $mk\ell$-KdV type equation

$$\partial_t u = \partial_x^m u + u^k \partial_x^\ell u, \tag{1.1}$$

$$u(x, 0) = \varphi(x), \;\; x \in \mathbb{T} \text{ or } \; x \in \mathbb{R}, \;\; t \in \mathbb{R}, \tag{1.2}$$

where φ is an appropriate function in Gevrey space G^s, $s \geq 1$. If we let $m = 3$ and $\ell = 1$, and replace t with $-t$ then we obtain the generalized KdV equation

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \tag{1.3}$$

for which it was shown in [GH1] that for appropriate analytic initial data one can construct non-analytic in time solutions. The purpose of this work is to extend to equation (1.1) the results obtained in [GH1]. Also, using the estimates obtained in [GH2], for proving analyticity in the space variable for KdV solutions, we show that these solutions belong in the Gevrey 3 space in the time variable.

*The second author was partially supported by the NSF under grant number DMS-9970857, and the third author was partially supported by CNPq.

2000 Mathematics Subject Classification. Primary 35Q53.

Key words and phrases. Generalized KdV equation, Cauchy problem, Gevrey spaces, regularity.

Analytic and Gevrey regularity properties for KdV-type equations have been studied extensively by many authors in the literature. For example, in [T], Trubowitz showed that the solution to the periodic initial value problem for the KdV with analytic initial data is anallytic in the space variable (see also [GH2] for another proof based on billinear estimates). For the non-periodic case we refer the reader to T. Kato [K], T. Kato and Masuda [KM], and K. Kato and Ogawa [KO]. For further results, we refer the reader to Bercovici, Constantin, Foias and Manley [BCFM], Bona and Grujić [BG], Bona, Grujić and Kalisch [BGK], Foias and Temam [FT], De Bouard, Hayashi and Kato [DHK], Grujić and Kukavica [GK], and Hayashi [H]. Another motivation for studying regularity properties for KdV-type equations is to contrast them with the Camassa-Holm equation (see [CH] and [FF]) which has been shown in [HM] that the solution map is analytic in time at time zero.

2 Periodic case

The main result of this section is given by the following

Theorem 2.1 *Given $s \geq 1$ the solution to the mkℓ-KdV initial value problem (1.1)-(1.2) with initial data in the Gevrey space $G^s(\mathbb{T})$ may not be in $G^s(\mathbb{R})$ in time variable t. More precisely, if*

$$\varphi(x) = i^{\frac{m-\ell}{k}} \sum_{n=1}^{\infty} \widehat{\psi}(n) e^{inx}, \tag{2.1}$$

where $\widehat{\psi}(n) = e^{-n^{1/s}}$, then the solution u to the initial value problem (1.1)-(1.2) is not in $G^s(\mathbb{R})$ in t.

Observe that the initial data $\varphi(x)$ belong in the Sobolev space $H^s(\mathbb{T})$, for any s, and therefore the Cauchy problem (1.1)-(1.2) is well-posed in $H^s(\mathbb{T})$ for s large enough when $m = 3$ and $\ell = 1$ (see Bourgain [B], Kenig, Ponce and Vega [KPV], Colliander, Keel, Staffilani, Takaoka and Tao [CKSTT1], [CKSTT2] and the references therein).

Before starting the proof of Theorem 2.1, we show the following lemma, which is crucial in estimating the higher-order derivatives of a solution with respect to t.

Lemma 2.2 *If u is a solution to (1.1)-(1.2) then for every $j \in \{1, 2, ...\}$ we have*

$$\partial_t^j u = \partial_x^{mj} u + \sum_{q=1}^{j} \sum_{|\alpha|+(m-\ell)q=mj} C_\alpha^q (\partial_x^{\alpha_1} u) \cdots (\partial_x^{\alpha_{qk+1}} u), \tag{2.2}$$

where $C_\alpha^q \geq 0$.

Proof. We prove this by induction. For $j = 1$, relation (2.2 holds since it is nothing else but equation (1.1). Next, we assume that (2.2) holds for $j \geq 1$ and we show that it holds for $j+1$. Differentiating (2.2) with respect to t and using (1.1) we obtain

$$\partial_t^{j+1} u = \partial_x^{m(j+1)} u + \partial_x^{mj}(u^k \partial_x^\ell u) + \sum_{q=1}^{j} \sum_{|\alpha|+(m-\ell)q=mj} C_\alpha^q \partial_t \left((\partial_x^{\alpha_1} u) \cdots (\partial_x^{\alpha_{qk+1}} u) \right). \tag{2.3}$$

Using Leibniz rule, the term $\partial_x^{mj}(u^k \partial_x^\ell u)$ can be written as the sum of terms of the form

$$C_\alpha (\partial_x^{\alpha_1} u)(\partial_x^{\alpha_2} u) \cdots (\partial_x^{\alpha_{k+1}} u),$$

with $C_\alpha \geq 0$, and $|\alpha| = mj + \ell$. Therefore, we have $|\alpha| + (m-\ell) \cdot 1 = m(j+1)$.

Now each term in the sum of (2.3) is of the form

$$\begin{aligned} \partial_t \left((\partial_x^{\alpha_1} u) \cdots (\partial_x^{\alpha_{qk+1}} u) \right) &= (\partial_x^{\alpha_1} \partial_t u)(\partial_x^{\alpha_2} u) \cdots (\partial_x^{\alpha_{qk+1}} u) + \cdots \\ &+ (\partial_x^{\alpha_1} u) \cdots (\partial_x^{\alpha_{qk}} u)(\partial_x^{\alpha_{qk+1}} \partial_t u). \end{aligned}$$

Substituting $\partial_t u = \partial_x^m u + u^k \partial_x^\ell u$ in each term above yields terms which order of the derivatives, $|\gamma|$, satisfies either $|\gamma| + (m-\ell)q = m(j+1)$ or $|\gamma| + (m-\ell)(q+1) = m(j+1)$. For example, the first term becomes

$$\begin{aligned} & (\partial_x^{\alpha_1} \partial_t u)(\partial_x^{\alpha_2} u) \cdots (\partial_x^{\alpha_{qk+1}} u) = (\partial_x^{\alpha_1} (\partial_x^m u + u^k \partial_x^\ell u))(\partial_x^{\alpha_2} u) \cdots (\partial_x^{\alpha_{qk+1}} u) \\ = \; & (\partial_x^{\alpha_1 + m} u)(\partial_x^{\alpha_2} u) \cdots (\partial_x^{\alpha_{qk+1}} u) + (\partial_x^{\alpha_1}(u^k \partial_x^\ell u))(\partial_x^{\alpha_2} u) \cdots (\partial_x^{\alpha_{qk+1}} u), \end{aligned}$$

where in the first term we have $qk+1$ terms of the type $\partial_x^\ell u$ and the order of derivatives satisfies $|\gamma| + (m-\ell)q = m(j+1)$, where $\gamma = (\alpha_1 + m, \alpha_2, \cdots, \alpha_{qk+1})$. In the second term, using Leibniz rule, we have $(q+1)k+1$ terms of the type $\partial_x^\ell u$ and the order of the derivatives is given by $|\gamma| = |\alpha| + \ell = mj - (m-\ell)q + \ell$, which can be written as $|\gamma| + (m-\ell)(q+1) = m(j+1)$, where we have used $\gamma = (\alpha_1 + \ell, \alpha_2, \cdots, \alpha_{qk+1})$.

This completes the proof of Lemma 2.2. □

Now, we are in the position to prove Theorem 2.1.

Proof of Theorem 2.1: We assume that the initial data is given by (2.1) and we shall prove that the solution to the $mk\ell$-KdV initial value problem (1.1)-(1.2) is not in $G^s(\mathbb{R})$ in time t.

Differentiating (2.1) with respect to x we obtain that

$$\partial_x^q u(x,0) = i^{\frac{m-\ell}{k}} \sum_{n=1}^{\infty} \widehat{\psi}(n)(in)^q e^{inx}.$$

Therefore,

$$\partial_x^q u(0,0) = i^{q+\frac{m-\ell}{k}} A_q,$$

where

$$A_q \doteq \sum_{n=1}^{\infty} \widehat{\psi}(n) n^q > 0. \tag{2.4}$$

For $j \in \mathbb{N}$, using Lemma 2.2, we obtain

$$\begin{aligned}
\partial_t^j u(0,0) &= i^{\frac{m-\ell}{k}+mj} A_{mj} + \sum_{q=1}^{j} \sum_{|\alpha|+(m-\ell)q=mj} C_\alpha^q i^{\frac{m-\ell}{k}+\alpha_1} A_{\alpha_1} \cdots i^{\frac{m-\ell}{k}+\alpha_{qk+1}} A_{\alpha_{qk+1}} \\
&= i^{\frac{m-\ell}{k}+mj} A_{mj} + \sum_{q=1}^{j} \sum_{|\alpha|+(m-\ell)q=mj} C_\alpha^q A_{\alpha_1} \cdots A_{\alpha_{qk+1}} i^{|\alpha|+\frac{m-\ell}{k}(qk+1)} \\
&= i^{\frac{m-\ell}{k}+mj} A_{mj} + \sum_{q=1}^{j} \sum_{|\alpha|+(m-\ell)q=mj} C_\alpha^q A_{\alpha_1} \cdots A_{\alpha_{qk+1}} i^{|\alpha|+(m-\ell)q+\frac{m-\ell}{k}} \\
&= i^{\frac{m-\ell}{k}+mj} A_{mj} + \sum_{q=1}^{j} \sum_{|\alpha|+(m-\ell)q=mj} C_\alpha^q A_{\alpha_1} \cdots A_{\alpha_{qk+1}} i^{mj+\frac{m-\ell}{k}} \\
&= i^{\frac{m-\ell}{k}+mj} \left(A_{mj} + \sum_{q=1}^{j} \sum_{|\alpha|+(m-\ell)q=mj} C_\alpha^q A_{\alpha_1} \cdots A_{\alpha_{qk+1}} \right)
\end{aligned}$$

Since $\left| i^{\frac{m-\ell}{k}+mj} \right| = 1$ and $C_\alpha^q \geq 0$ it follows from the last equality and (2.4) that

$$\left| \partial_t^j u(0,0) \right| \geq A_{mj} = \sum_{n=1}^{\infty} \widehat{\psi}(n) n^{mj}. \tag{2.5}$$

We now are going to divide the proof in two cases.

First case: $1 \leq s < m$**.**

In this case we notice that

$$A_{mj} = \sum_{n=1}^{\infty} \widehat{\psi}(n) n^{mj} > \widehat{\psi}(mj)(mj)^{mj} = e^{-(mj)^{1/s}} (mj)^{mj}. \tag{2.6}$$

Thanks to the fact that $(mj)^{\frac{1}{s}} \le mj$ for all $s \ge 1$ and $j = 1, 2, \cdots$ it follows from (2.5) and (2.6) that

$$\left|\partial_t^j u(0,0)\right| \ge e^{-(mj)}(mj)^{mj}. \tag{2.7}$$

Since $(mj)^{mj} > (j!)^m$ it follows from (2.7) that

$$\left|\partial_t^j u(0,0)\right| \ge \left(\frac{1}{e^m}\right)^j (j!)^m. \tag{2.8}$$

Recall now that a function $g(t)$ is in $G^s(\mathbb{R})$ if $g(t) \in C^\infty(\mathbb{R})$ and for every compact subset K of $\mathbb{R}$ there exists a positive constant C such that

$$|g^{(j)}(t)| \le C^{j+1}(j!)^s, \; j = 0, 1, 2, \cdots \text{ and } t \in K. \tag{2.9}$$

Taking $K = \{0\}$ and using estimates (2.8) and (2.9) we conclude that $u(0, \cdot) \notin G^s(\mathbb{R})$ in the case $1 \le s < m$.

Second case: $s \ge m$**.**

In this case we shall use $[s]$ to represent the greatest integer that is less than or equal to s. We also notice that

$$A_{mj} = \sum_{n=1}^{\infty} \widehat{\psi}(n) n^{mj} > \widehat{\psi}(j^{[s]})(j^{[s]})^{mj} = e^{-(j^{[s]})^{1/s}}(j^{[s]})^{mj}. \tag{2.10}$$

Since $[s] \le s$ we have $\frac{[s]}{s} \le 1$ and therefore

$$j^{[s]/s} \le j, \text{ for all } j = 1, 2, \cdots. \tag{2.11}$$

It follows from (2.11) that

$$e^{-(j^{[s]})^{1/s}} = e^{-(j^{[s]/s})} \ge e^{-j}, \text{ for all } j = 1, 2, \cdots. \tag{2.12}$$

Since $[s] \ge s - 1$ we have $m[s] \ge ms - m$. Thanks to the fact the $s \ge m$ we can conclude that $ms - m \ge (m-1)s$. It follows from this that

$$(j^{[s]})^{mj} = j^{mj[s]} = (j^j)^{m[s]} \ge (j!)^{m[s]} \ge (j!)^{ms-m} \ge (j!)^{(m-1)s} \tag{2.13}$$

where we have used the inequality $j^j \ge j!$ and the fact that $j! \ge 1$.

It follows from (2.5), (2.10), (2.12) and (2.13) that

$$\begin{aligned} \left|\partial_t^j u(0,0)\right| &\ge A_{mj} = \sum_{n=1}^{\infty} \widehat{\psi}(n) n^{mj} \\ &> \widehat{\psi}(j^{[s]})(j^{[s]})^{mj} = e^{-(j^{[s]})^{1/s}}(j^{[s]})^{mj} \\ &\ge e^{-j}(j!)^{(m-1)s}, \end{aligned} \tag{2.14}$$

which implies that $u(0, \cdot) \notin G^s(\mathbb{R})$, since $m \ge 3$. This completes the proof of Theorem 2.1.

3 Non-periodic case

In the non-periodic case we consider analytic initial data and we show that the solution is not analytic in time.

Theorem 3.1 *The solution to the mkℓ-KdV initial value problem (1.1)-(1.2) with initial data an analytic function may not be analytic in the t variable. More precisely, if*

$$u(x,0) = (i-x)^{-\frac{4p+m-\ell}{k}}, \tag{3.1}$$

with $p \in \mathbb{N}$ and $k < 2m - 2\ell + 8p$, then $u(0,\cdot)$ is not analytic near $t = 0$.

Observe that for any given $s > 0$ we can choose p large enough so that the initial data $u(x,0)$ belong in the Sobolev space $H^s(\mathbb{R})$. Therefore the Cauchy problem (1.1)-(1.2) is well-posed in $H^s(\mathbb{T})$ when $m = 3$ and $\ell = 1$ (see Kenig, Ponce and Vega [KPV], Colliander, Keel, Staffilani, Takaoka and Tao [CKSTT1], [CKSTT2] and the references therein).

Proof of Theorem 3.1. We have

$$\partial_x^n u(x,0) = \frac{4p+m-\ell}{k}(\frac{4p+m-\ell}{k}+1)\cdots(\frac{4p+m-\ell}{k}+n-1)(i-x)^{-(n+\frac{4p+m-\ell}{k})}.$$

It follows from this and from Lemma 2.2 that

$$\partial_t^j u(0,0) = \frac{4p+m-\ell}{k}(\frac{4p+m-\ell}{k}+1)\cdots(\frac{4p+m-\ell}{k}+mj-1)(i)^{-(mj+\frac{4p+m-\ell}{k})}+$$

$$\sum_{q=1}^{j}\sum_{|\alpha|+(m-\ell)q=mj} C_\alpha^q\left[\frac{4p+m-\ell}{k}(\frac{4p+m-\ell}{k}+1)\cdots(\frac{4p+m-\ell}{k}+\alpha_1-1)\right]\cdots$$

$$\left[\frac{4p+m-\ell}{k}(\frac{4p+m-\ell}{k}+1)\cdots(\frac{4p+m-\ell}{k}+\alpha_{qk+1}-1)\right](i)^{-(|\alpha|+\frac{4p+m-\ell}{k}(qk+1))}.$$

Since $|\alpha| = mj-(m-\ell)q$ and $i^{4pq} = 1$ we may factor, in the last equality, the term $(i)^{-(mj+\frac{4p+m-\ell}{k})}$ and therefore we have

$$\begin{aligned} |\partial_t^j u(0,0)| &\geq \frac{4p+m-\ell}{k}(\frac{4p+m-\ell}{k}+1)\cdots(\frac{4p+m-\ell}{k}+mj-1) \\ &\geq \frac{4p+m-\ell}{k}(mj-1)! \\ &\geq (mj-1)!C_1 \end{aligned} \tag{3.2}$$

where $C_1 = \frac{4p+m-\ell}{k}$.

Since $mj-1 \geq (m-1)j$, for $j \geq 1$, we have $(mj-1)! \geq ((m-1)j)!$. By using the inequality $(\ell+n)! \geq \ell! n!$ it follows from the last inequality that $(mj-1)! \geq (j!)^{m-1}$. Thus, from this and (3.2) we obtain

$$|\partial_t^j u(,0)| \geq C_1 (j!)^{m-1}$$

which shows that $u(0,\cdot)$ cannot be analytic near $t=0$. $\square$

4 G^3 regularity in time for the KdV

Next we shall focus our attention to the periodic initial value problem for the KdV equation

$$\partial_t u = \partial_x^3 + u\partial_x u \tag{4.1}$$

$$u(x,0) = \varphi(x), \tag{4.2}$$

when $\varphi(x)$ is analytic on the torus $\mathbb{T}$. As we have mentioned before, this problem is well-posed (see, for example, [B], [KPV] and [CKSTT1]) and its solution $u(x,t)$ is analytic in the spatial variable (see [T] and [GH2]). Here we shall use the analyticity estimates obtained in [GH2] to prove the following result.

Theorem 4.1 *The solution $u(x,t)$ to the KdV initial value problem (4.1)-(4.2) belongs to G^3 in the time variable t.*

Proof of Theorem 4.1. By the work in [GH2] $u(x,t)$ is analytic in x for all t near zero. More precisely, there exist $C>0$ and $\delta>0$ such that

$$|\partial_x^k u(x,t)| \leq C^{k+1} k!,\ k=0,1,2,\cdots,\ t\in(-\delta,\delta),\ x\in\mathbb{T}. \tag{4.3}$$

In order to prove Theorem 4.1 it is enough to prove the following

Lemma 4.2 *For $k=0,1,\cdots$ and $j=0,1,2,\cdots$ the following inequality holds true*

$$\left|\partial_t^j \partial_x^k u(x,t)\right| \leq C^{k+j+1}(k+3j)!(C^2+C/2)^j, \tag{4.4}$$

for $t\in(-\delta,\delta),\ x\in\mathbb{T}$.

Proof. We will prove it by using induction on j. For $j = 0$ inequality (4.4) holds for all $k \in \{0, 1, 2, \cdots\}$ since it is nothing else but inequality (4.3). For $j = 1$ and $k \in \{0, 1, 2, \cdots\}$ it follows from (4.1) that

$$\begin{aligned}\partial_t \partial_x^k u &= \partial_x^{k+3} u + \partial_x^k (u \partial_x u) \\ &= \partial_x^{k+3} u + \sum_{p=0}^{k} \binom{k}{p} \partial_x^{k-p} u \partial_x^{p+1} u. \end{aligned} \tag{4.5}$$

First, from (4.3) we obtain that

$$|\partial_x^{k+3} u(x,t)| \leq C^{k+3+1}(k+3)! \leq C^{k+1+1}(k+3\cdot 1)! C^2, \ t \in (-\delta, \delta), \ x \in \mathbb{T}. \tag{4.6}$$

Now we notice that

$$\begin{aligned} & |\sum_{p=0}^{k} \binom{k}{p} \partial_x^{k-p} u \partial_x^{p+1} u| \leq \sum_{p=0}^{k} \frac{k!}{p!(k-p)!} C^{k-p+1}(k-p)! C^{p+1+1}(p+1)! \\ = \ & C^{k+3} k! \sum_{p=0}^{k} (p+1) = C^{k+3} k! (k+1)(k+2)/2 \\ = \ & C^{k+3}(k+2)!/2 = C^{k+1+1}(k+2)! C/2 \leq C^{k+1+1}(k+3)! C/2, \end{aligned} \tag{4.7}$$

for $t \in (-\delta, \delta), \ x \in \mathbb{T}$, where we have used the fact that

$$\sum_{p=0}^{k} (p+1) = (k+1)(k+2)/2.$$

It follows from (4.6) and (4.7) that

$$|\partial_t \partial_x^k u(x,t)| \leq C^{k+1+1}(k+3.1)!(C^2 + C/2),$$

for $t \in (-\delta, \delta), \ x \in \mathbb{T}$, which complete the proof in this case.

We now suppose that (4.4) holds for all derivatives in t of order $\leq j$ and $k \in \{0, 1, 2, \cdots\}$ and we shall prove that (4.4) holds for $j+1$ and $k \in \{0, 1, 2, \cdots\}$.

We have from (4.1) that

$$\begin{aligned}
\partial_t^{j+1}\partial_x^k u &= \partial_t^j\partial_x^{k+3}u + \partial_t^j\partial_x^k(u\cdot\partial_x u) \\
&= \partial_t^j\partial_x^{k+3}u + \partial_t^j\left(\sum_{p=0}^{k}\binom{k}{p}\partial_x^{k-p}u\partial_x^{p+1}u\right) \\
&= \partial_t^j\partial_x^{k+3}u + \sum_{p=0}^{k}\binom{k}{p}(\partial_t^j\partial_x^{k-p}u)(\partial_x^{p+1}u) \qquad (4.8)\\
&+ \sum_{p=0}^{k}\binom{k}{p}(\partial_x^{k-p}u)(\partial_t^j\partial_x^{p+1}u) \\
&+ \sum_{\ell=1}^{j-1}\sum_{p=0}^{k}\binom{j}{\ell}\binom{k}{p}(\partial_t^{j-\ell}\partial_x^{k-p}u)(\partial_t^{\ell}\partial_x^{p+1}u).
\end{aligned}$$

By using the induction assumption we obtain

$$\begin{aligned}
|\partial_t^j\partial_x^{k+3}u| &\leq C^{k+3+j+1}(k+3+3j)!(C^2+C/2)^j \\
&= C^{k+(j+1)+1}(k+3(j+1))!(C^2+C/2)^jC^2, \qquad (4.9)
\end{aligned}$$

for $t\in(-\delta,\delta),\ x\in\mathbb{T}$.

For the second term in the formula (4.8), by using the induction assumption, we obatin

$$\begin{aligned}
&\left|\sum_{p=0}^{k}\binom{k}{p}(\partial_t^j\partial_x^{k-p}u)(\partial_x^{p+1}u)\right| \\
\leq\ & \sum_{p=0}^{k}\frac{k!}{p!(k-p)!}C^{k-p+j+1}(k-p+3j)!(C^2+C/2)^jC^{p+1+1}(p+1)! \\
=\ & C^{k+j+3}(C^2+C/2)^jk!\sum_{p=0}^{k}(p+1)(k-p+1)(k-p+2).\cdots.(k-p+3j) \\
\leq\ & C^{k+j+3}(C^2+C/2)^jk!\sum_{p=0}^{k}(p+1)(k+1)(k+2).\cdots.(k+3j).
\end{aligned}$$

By using again the fact that $\sum_{p=0}^{k}(p+1) = (k+1)(k+2)/2$ it follows from the last inequality that

$$\begin{aligned}
&\left|\sum_{p=0}^{k}\binom{k}{p}(\partial_t^j\partial_x^{k-p}u)(\partial_x^{p+1}u)\right| \\
\leq\ & C^{k+j+3}(C^2+C/2)^j(k+3j)!(k+1)(k+2)/2 \qquad (4.10)\\
\leq\ & C^{k+j+3}(C^2+C/2)^j(k+3j)!(k+3j+1)(k+3j+2)/2 \\
\leq\ & C^{k+j+3}(C^2+C/2)^j(k+3(j+1))!\frac{1}{2(k+3(j+1))} \\
\leq\ & \frac{1}{3}C^{k+(j+1)+1}(C^2+C/2)^j(k+3(j+1))!C/2.
\end{aligned}$$

For the third term in the formula (4.8) we have

$$\begin{aligned}
&\left|\sum_{p=0}^{k}\binom{k}{p}(\partial_x^{k-p}u)(\partial_t^j\partial_x^{p+1}u)\right| \\
\leq\ & \sum_{p=0}^{k}\frac{k!}{p!(k-p)!}C^{k-p+1}(k-p)!C^{p+1+j+1}(p+1+3j)!(C^2+C/2)^j \\
=\ & C^{k+j+3}(C^2+C/2)^j k!\sum_{p=0}^{k}(p+1)(p+2)\cdots(p+1+3j) \\
\leq\ & C^{k+j+3}(C^2+C/2)^j k!\sum_{p=0}^{k}(p+1)(k+2)\cdots(k+1+3j).
\end{aligned}$$

As in (4.10) we have

$$\begin{aligned}
&\left|\sum_{p=0}^{k}\binom{k}{p}(\partial_x^{k-p}u)(\partial_t^j\partial_x^{p+1}u)\right| \\
\leq\ & C^{k+j+3}(C^2+C/2)^j k!(k+2)\cdots(k+1+3j)(k+1)(k+2)/2 \\
\leq\ & C^{k+j+3}(C^2+C/2)^j(k+3j+1)!(k+2)/2 \qquad (4.11)\\
\leq\ & C^{k+j+3}(C^2+C/2)^j(k+3j+1)!(k+3j+2)/2 \\
\leq\ & C^{k+(j+1)+1}(C^2+C/2)^j(k+3(j+1))!C/2\frac{1}{k+3(j+1)} \\
\leq\ & \frac{1}{3}C^{k+(j+1)+1}(C^2+C/2)^j(k+3(j+1))!C/2.
\end{aligned}$$

In order to estimate the last term in (4.8) we shall recall that for $\ell \leq j$ and $p \leq k$ we have the following inequality

$$\binom{j}{\ell}\binom{k}{p} \leq \binom{j+k}{\ell+p},$$

(see [DHK, Lemma 2.8]).

By using it and the induction assumption we obtain

$$\begin{aligned}
& \left|\sum_{\ell=1}^{j-1}\sum_{p=0}^{k}\binom{j}{\ell}\binom{k}{p}(\partial_t^{j-\ell}\partial_x^{k-p}u)(\partial_t^{\ell}\partial_x^{p+1}u)\right| \\
\leq & \sum_{\ell=1}^{j-1}\sum_{p=0}^{k}\binom{j+k}{\ell+p}C^{k-p+j-\ell+1}[(k-p+3(j-\ell)]!(C^2+C/2)^{j-\ell} \\
\times & C^{p+1+\ell+1}(p+1+3\ell)!(C^2+C/2)^{\ell} \\
= & C^{k+j+3}(C^2+C/2)^j\sum_{\ell=1}^{j-1}\sum_{p=0}^{k}\frac{(k+j)!}{(\ell+p)!(k+j-\ell-p)!} \qquad (4.12)\\
\times & [k-p+3(j-\ell)]!(p+1+3\ell)! \\
= & C^{k+j+3}(C^2+C/2)^j(k+j)!\sum_{\ell=1}^{j-1}\sum_{p=0}^{k}(p+\ell+1)(p+\ell+2).\cdots.(p+\ell+1+2\ell) \\
\times & (k+j-\ell-p+1)(k+j-\ell-p+2).\cdots.(k+j-\ell-p+2j-2\ell).
\end{aligned}$$

We now notice that for any $\nu \in \mathbb{N}$ we have

$$(p+\ell+\nu) \leq (k+j+\nu-1)$$

since $p \leq k$ and $\ell \leq j-1$, and

$$(k+j-\ell-p+\nu) \leq (k+j+\nu-1)$$

since the maximum value is given when $p=0$ and $\ell=1$.

It follows from these inequalities and from (4.12) that

$$\begin{aligned}
& \left|\sum_{\ell=1}^{j-1}\sum_{p=0}^{k}\binom{j}{\ell}\binom{k}{p}(\partial_t^{j-\ell}\partial_x^{k-p}u)(\partial_t^{\ell}\partial_x^{p+1}u)\right| \\
\leq & C^{k+j+3}(C^2+C/2)^j(k+j)!\sum_{\ell=1}^{j-1}\sum_{p=0}^{k}(p+\ell+1)(k+j+1).\cdots.(k+j+2\ell) \\
\times & (k+j)(k+j+1).\cdots.(k+j+2j-2\ell-1). \qquad (4.13)
\end{aligned}$$

Since $k+j+\nu \leq k+j+\nu+2\ell+1$, for any $\nu \in \mathbb{N}$, it follows from this and (4.13) that

$$\begin{aligned}
&\left|\sum_{\ell=1}^{j-1}\sum_{p=0}^{k}\binom{j}{\ell}\binom{k}{p}(\partial_t^{j-\ell}\partial_x^{k-p}u)(\partial_t^{\ell}\partial_x^{p+1}u)\right| \\
\leq\ & C^{k+j+3}(C^2+C/2)^j(k+j)!\sum_{\ell=1}^{j-1}\sum_{p=0}^{k}(p+\ell+1)(k+j+1).\cdots.(k+j+2\ell) \\
\times\ & (k+j+2\ell+1)(k+j+2\ell+2).\cdots.(k+3j) \qquad (4.14)\\
=\ & C^{k+j+3}(C^2+C/2)^j(k+3j)!\sum_{\ell=1}^{j-1}\sum_{p=0}^{k}(p+\ell+1).
\end{aligned}$$

Since

$$\sum_{\ell=1}^{j-1}\sum_{p=0}^{k}(p+\ell+1) = (k+1)(j-1)(k+j+2)/2,$$

it follows from this and (4.14) that

$$\begin{aligned}
&\left|\sum_{\ell=1}^{j-1}\sum_{p=0}^{k}\binom{j}{\ell}\binom{k}{p}(\partial_t^{j-\ell}\partial_x^{k-p}u)(\partial_t^{\ell}\partial_x^{p+1}u)\right| \\
\leq\ & C^{k+j+3}(C^2+C/2)^j(k+3j)!(k+j+2)(k+1)(j-1)/2 \\
\leq\ & C^{k+j+3}(C^2+C/2)^j(k+3j)!(k+3j+1)(k+1)(j-1)/2 \\
\leq\ & C^{k+j+3}(C^2+C/2)^j(k+3j+1)!(k+1)(j-1)/2 \qquad (4.15)\\
\leq\ & C^{k+(j+1)+1}(C^2+C/2)^j(k+3j+1)!C/2(k+1)(j-1) \\
\leq\ & C^{k+(j+1)+1}(C^2+C/2)^j(k+3(j+1))!C/2\frac{(k+1)(j-1)}{(k+3j+2)(k+3j+3)} \\
\leq\ & \frac{1}{3}C^{k+(j+1)+1}(C^2+C/2)^j(k+3(j+1))!C/2,
\end{aligned}$$

where we have used that $k+j+2 \leq k+j+2j+1 = k+3j+1$ since $1 \leq j$ implies that $2 \leq 2j \leq 2j+1$ and we also have used that $k+3j+2 \geq k+1$ and $k+3j+3 \geq 3j+3 = 3(j+1) \geq 3(j-1)$.

Finally by using (4.8), (4.9), (4.10), (4.11) and (4.15) we obtain

$$|\partial_t^{j+1}\partial_x^k u(x,t)| \leq C^{k+(j+1)+1}[k+3(j+1)]!(C^2+C/2)^{j+1}.$$

This completes the proof of Lemma 4.2, and therefore it also completes the proof of Theorem 4.1. □

References

[BCFM] H. Bercovici, P. Constantin, C. Foias and O. Manley, Exponential decay of the power spectrum of turbulence, J. Statist. Phys. **80** (1995) 579–602.

[BG] J. Bona and Z. Grujić, Spatial analyticity properties of nonlinear waves, Dedicated to Jim Douglas, Jr. on the occasion of his 75th birthday, *Math. Models Methods Appl. Sci.* **13**, (2003), 345–360.

[BGK] J. Bona, Z. Grujić and H. Kalisch, Algebraic lower bounds for the uniform radius of spacial analyticity for the generalized KdV equation, *preprint.*

[B] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part 2: KdV equation*, Geom. Funct. Anal. **3**, (1993), 209–262.

[CH] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), 1661–1664.

[CKSTT1] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc. **16**, No. 3, (2003), 705–749.

[CKSTT2] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Multilinear estimates for periodic KdV equations, and applications*, J. Funct. Anal. **211**, (2004), no. 1, 173–218.

[DHK] A. De Bouard, N. Hayashi and K. Kato, Gevrey regularizing effect for the (generalized) Korteweg-de Vries equation and nonlinear Schrdinger equations, *Ann. Inst. H. Poincar Anal. Non Linaire* **12** (1995) 673–725.

[FT] C. Foias and R. Temam, Approximation of attractors by algebraic or analytic sets, *SIAM J. Math. Anal.* **25** (1994) 1269–1302.

[FF] A. Fokas and B. Fuchssteiner, Symplectic structures, their Bäklund transformations and hereditary symmetries, *Phys. D* **4** (1981/82) 47–66.

[GH1] J. Gorsky and A. Himonas, *Construction of non-analytic solutions for the generalized KdV equation*, to appear in JMAA.

[GH2] J. Gorsky and A. Himonas, *On analyticity in space variable of solutions to the KdV equation*, Contemporary Mathematics of AMS **368**, (2005), 233–247.

[GK] Z. Grujić and I. Kukavica, A remark on time-analyticity for the Kuramoto-Sivashinsky equation, *Nonlinear Anal.* **52** (2003) 69–78.

[H] N. Hayashi, Analyticity of solutions of the Korteweg-de Vries equation, *SIAM J. Math. Anal.* **22**, (1991), 1738–1743.

[HM] A. Himonas and G. Misiołek, Analyticity of the Cauchy problem for an integrable evolution equation, *Math. Ann.* **327** (2003) 575–584.

[K] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Adv. Math. Suppl. Studies **8** (1983), 93–128.

[KM] T. Kato and K. Masuda, *Nonlinear evolution equations. I*, Ann. Inst. Henri Poincare. Anal. non-lineaire, **3**, (1986), 455–467.

[KO] K. Kato and T. Ogawa, *Analyticity and smoothing effect for the Korteweg-de Vries equation with a single point singularity*, Math. Ann. **316**, (2000), 577–608.

[KPV] C. Kenig, G. Ponce and L. Vega, *A Bilinear Estimate with applications to the KdV Equation*, J. Amer. Math. Soc. **9** (1996), 573–603.

[T] E. Trubowitz, *The inverse problem for periodic potentials*, Comm. Pure Appl. Math. **30** (1977), 321–337.

Heather Hannah, Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-5683, E-mail: hhannah@nd.edu

A. Alexandrou Himonas, Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-5683, E-mail: himonas.1@nd.edu

Gerson Petronilho, Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP 13565-905, Brasil, E-mail: gerson@dm.ufscar.br