Regularity of a class of sublaplacians on the 3-dimensional torus

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Abstract

In this paper we present a necessary and sufficient condition for a family of sums of squares operators to be globally hypoelliptic on a torus. This condition says that either a Diophantine condition is satisfied or there exists a point of finite type. Also, we describe the analytic and Gevrey versions of this result. The proof is based on L^2-estimates and microlocal analysis.

1 Introduction

Given m real vector fields $X = \{X_1, \cdots, X_m\}$ on a C^∞ manifold of dimension n, $\mathcal{M}$, their sublaplacian is defined by $\Delta_X \doteq -(X_1^2 + \cdots + X_m^2)$. In general this is a degenerate elliptic operator. By Hörmander's theorem [Ho] it is locally, and therefore, globally hypoelliptic if all points of $\mathcal{M}$ are of finite type. However, it is well known that the finite type condition is not necessary for hypoellipticity. For example, the operator $-[(\partial_{t_1})^2 + (\partial_{t_2})^2 + (\exp(-1/|t_1|^p)\partial_x)^2]$, $0 < p < 1$, is hypoelliptic in $\mathbb{R}^3$ although the points on the plane $\{t_1 = 0\}$ are not of finite type (see Kusuoka and Strook [KS]).

In this work we shall focus on global hypoellipticity when the manifold is a torus $\mathbb{T}^N$. The main motivation comes from Himonas [Hi1]. There it is shown that the operator $-[\partial_{t_1}^2 + (\partial_{t_2} + a(t_1)\partial_x)^2]$ is globally hypoelliptic in $\mathbb{T}^3$ if and only if the range of $a(t_1)$ contains an irrational non-Liouville number. Our main theorem here extends this result in the case that the coefficient a depends on both variables t_1 and t_2. More precisely, we prove the following

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Theorem 1.1. *Let P be given by*

$$P = -\partial_{t_1}^2 - (\partial_{t_2} + a(t_1, t_2)\partial_x)^2 \tag{1.1}$$

where $(t_1, t_2, x) \in \mathbb{T}^3$ and $a \in C^\infty(\mathbb{T}^2)$ is real-valued. We set

$$a_0(t_1) = \frac{1}{2\pi} \int_{\mathbb{T}} a(t_1, s)ds.$$

Then, P is globallly hypoelliptic in $\mathbb{T}^3$ if and only if either the range of $a_0(t_1)$ contains an irrational non-Liouville number or there exists a point $p \in \mathbb{T}^3$ of finite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t_1, t_2)\partial_x$.

Recall that a linear partial differential operator P is locally hypoelliptic in $\mathcal{M}$ if for any $U \subset \mathcal{M}$ open set the conditions $u \in D'(U)$ and $Pu \in C^\infty(U)$ imply that $u \in C^\infty(U)$. P is said to be globally hypoelliptic in $\mathcal{M}$ if $u \in D'(\mathcal{M})$ and $Pu \in C^\infty(\mathcal{M})$ imply that $u \in C^\infty(\mathcal{M})$. Also, recall that a point $p \in \mathcal{M}$ is said to be of finite type for the vector fields $X_1, \cdots, X_m$ if among the vector fields $X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \cdots, [X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots, X_{j_k}]]], \cdots$ where $j_i = 1, \cdots, m$, there exist n (the dimension of $\mathcal{M}$) which are linearly independent at p. Finally, we recall that a number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is said to be a non-Liouville number if there exist $C > 0$ and $K > 0$ such that

$$|\alpha - \frac{p}{q}| \geq \frac{C}{|q|^K}, \ \forall \ (p, q) \in \mathbb{Z} \times \mathbb{Z}\backslash\{0\}.$$

When the coefficient $a(t_1, t_2)$ is an analytic function on $\mathbb{T}^2$ then the analytic and Gevrey version of Theorem 1.1 is true. More precisely, for any $s \geq 1$ the sublaplacian P is globallly G^s-hypoelliptic in $\mathbb{T}^3$ if and only if either the range of $a_0(t_1)$ contains an exponentially non-Liouville number with exponent s or there exists a point $p \in \mathbb{T}^3$ of finite type. This result is described in more detail in the last section.

There is an extensive literature concerning the local hypoellipticity of sublaplacians in form (1.1). For example, the case when the coefficient $a(t_1, t_2) = t_1^k$, where k is a positive integer, has been considered by Pham The Lai and Robert [PR], Helffer [H], Hanges and Himonas [HH1], Christ [C1] and Chanillo, Helffer and Laptev [CHL]. For additional results and conjectures concerning the open problem of analytic hypoellipticity (local and global) of sublaplacians we refer the reader to the works of Baouendi and Goulaouic [BG], Tartakoff [T1], [T2], Derridj and Tartakoff [DT], Treves [Tr1], [Tr2], Bernardi, Bove and Tartakoff [BBT], Bove and Treves [BT], Grigis and Sjöstrand [GS], Metivier [M], Christ [C2], [C3], [C4], [C5], Hanges and Himonas [HH2], [HH3], Cordaro and Himonas [CH1], [CH2], and the references therein.

For results about local and global C^∞ hypoellipticity of sublaplacians, in particular when the finite type condition fails, and in connection with problems arising from several complex variables, we refer the reader to the work of Oleinik and Radkevic [OR], Rothschild and Stein [RS], Kohn [K1], [K2], Christ [C6], Morimoto [Mo], Amano [A], Fujiwara and Omori [FO], Greenfield and Wallach [GW], Himonas and Petronilho [HP1], [HP2], [HP3], and the references therein.

The paper is organized as follows. In the next section we prove a reduction lemma and the necessity of the condition for global hypoellipticity. Also, we describe our approach for proving sufficiency. In Section 3 we prove global hypoellipticity when $a_0(t_1)$ contains a non-Liouville number. In Section 4 we assume that a_0 is a rational number and prove global hypoellipticity if there is a point of finite type. In Section 5 we prove global hypoellipticity when a_0 is an irrational number and there exists a point of finite type. Finally, in Section 6 we state the analytic and Gevrey versions of Theorem 1.1.

2 A reduction lemma and necessity

We begin the proof of Theorem 1.1 with the following lemma which states that the given sublaplacian can be reduced, by a conjugation, to another sublaplacian with the coefficient $a(t_1, t_2)$ replaced by $a_0(t_1)$.

Lemma 2.1. *Let P and $a_0(t_1)$ be as in Theorem 1.1 and $\tilde{P}$ be given by*

$$\tilde{P} = -(\partial_{t_1} + b(t)\partial_x)^2 - (\partial_{t_2} + a_0(t_1)\partial_x)^2, \tag{2.1}$$

where

$$b(t) = -\frac{\partial h}{\partial_{t_1}}(t_1, t_2); \;\; h(t_1, t_2) = \int_0^{t_2} a(t_1, s)ds - a_0(t_1)t_2.$$

Then, there exists an isomorphism, S, from $D'(\mathbb{T}^3)$ onto itself which is also an isomorphism from $C^\infty(\mathbb{T}^3)$ onto $C^\infty(\mathbb{T}^3)$ such that

$$SPS^{-1} = \tilde{P}$$

and therefore the operator P is globally hypoelliptic in $\mathbb{T}^3$ if and only if $\tilde{P}$ is globally hypoelliptic in $\mathbb{T}^3$.

Proof. We define

$$S : D'(\mathbb{T}^3) \to D'(\mathbb{T}^3)$$

by

$$Su = v = \sum_{\xi \in \mathbb{Z}} \hat{v}(t, \xi)e^{ix\xi} \doteq \sum_{\xi \in \mathbb{Z}} \hat{u}(t, \xi)e^{i\xi h(t)}e^{ix\xi}, \;\; \text{for all } \; u \in D'(\mathbb{T}^3).$$

We also define

$$S^{-1} : D'(\mathbb{T}^3) \to D'(\mathbb{T}^3)$$

by

$$S^{-1}v = u = \sum_{\xi\in\mathbb{Z}} \hat{u}(t,\xi)e^{ix\xi} \doteq \sum_{\xi\in\mathbb{Z}} \hat{v}(t,\xi)e^{-i\xi h(t)}e^{ix\xi}, \text{ for all } v \in D'(\mathbb{T}^3).$$

It follows from a straightforward calculation that S is an isomorphism from $D'(\mathbb{T}^3)$ onto itself which is also an isomorphism from $C^\infty(\mathbb{T}^3)$ onto itself.

Now we are going to prove the concatenation formula $SPS^{-1} = \tilde{P}$.

We have, for $t = (t_1, t_2) \in \mathbb{T}^2$,

$$\begin{aligned}
PS^{-1}v &= Pu = -\partial_{t_1}^2 u - (\partial_{t_2} + a(t)\partial_x)^2 u \\
&= -\partial_{t_1}^2\Big(\sum_{\xi\in\mathbb{Z}} \hat{u}(t,\xi)e^{ix\xi}\Big) - (\partial_{t_2} + a(t)\partial_x)^2\Big(\sum_{\xi\in\mathbb{Z}} \hat{u}(t,\xi)e^{ix\xi}\Big) \\
&= \sum_{\xi\in\mathbb{Z}} \left[-\partial_{t_1}^2 \hat{u}(t,\xi) - (\partial_{t_2} + i\xi a(t))^2 \hat{u}(t,\xi)\right] e^{ix\xi} \\
&= \sum_{\xi\in\mathbb{Z}} \left[-\partial_{t_1}^2 (e^{-i\xi h(t)}\hat{v}(t,\xi)) - (\partial_{t_2} + i\xi a(t))^2 (e^{-i\xi h(t)}\hat{v}(t,\xi))\right] e^{ix\xi} \\
&= \sum_{\xi\in\mathbb{Z}} \left[-(\partial_{t_1} - i\xi \frac{\partial h}{\partial_{t_1}}(t))^2 \hat{v}(t,\xi) - (\partial_{t_2} + i\xi a_0(t_1))^2 \hat{v}(t,\xi)\right] e^{-i\xi h(t)}e^{ix\xi} \\
&= \sum_{\xi\in\mathbb{Z}} \widehat{(\tilde{P}v)}(t,\xi)e^{-i\xi h(t)}e^{ix\xi} \\
&= S^{-1}\tilde{P}v.
\end{aligned}$$

Next, let $u \in D'(\mathbb{T}^3)$ be such that

$$Pu = f, \;\; f \in C^\infty(\mathbb{T}^3).$$

Letting $v \doteq Su \in D'(\mathbb{T}^3)$, $g \doteq Sf \in C^\infty(\mathbb{T}^3)$ and using concatenation $SP = \tilde{P}S$, we see that v and g satisfy the equation

$$\tilde{P}v = g, \;\; g \in C^\infty(\mathbb{T}^3).$$

If $\tilde{P}$ is globally hypoelliptic in $\mathbb{T}^3$ then $v \in C^\infty(\mathbb{T}^3)$. But $v \in C^\infty(\mathbb{T}^3)$ if and only if $u \in C^\infty(\mathbb{T}^3)$. Therefore P is globally hypoelliptic in $\mathbb{T}^3$. Similarly we show that if P is globally hypoelliptic in $\mathbb{T}^3$ then $\tilde{P}$ is globally hypoelliptic in $\mathbb{T}^3$. □

Necessity. We begin by proving the necessity of the conditions for global hypoellipticity. We do this by contradiction. If the condition in Theorem 1.1 does not hold then $a_0(t_1) \equiv \alpha$, where α is either a rational number or a Liouville number, and all points in $\mathbb{T}^3$ are of infinite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t)\partial_x$. Since $[X_1, X_2] = \frac{\partial a}{\partial t_1}(t)\partial_x$ the statement that all points of $\mathbb{T}^3$ are of infinite type reads as $\frac{\partial a}{\partial t_1}(t) \equiv 0$. Therefore, the function a depends only on t_2, i.e., $a(t) = a(t_2)$. In this case

$$h(t_1,t_2) = \int_0^{t_2} a(s)ds + \alpha t_2 = h(t_2),\ b(t) = -\frac{\partial h}{\partial_{t_1}}(t_2) = 0,$$

and the operator $\tilde{P}$ becomes

$$\tilde{P} = -\partial_{t_1}^2 - (\partial_{t_2} + \alpha\partial_x)^2.$$

Now, it is easy to show that (see [Hi1], Thm 1.2) that $\tilde{P}$ is globally hypoelliptic in $\mathbb{T}^3$ if and only if α is a non-Liouville number. Therefore P is not globally hypoelliptic in $\mathbb{T}^3$. This completes the proof of the necessity. □

Outline of sufficiency proof. Next we describe our approach for proving that the condition in Theorem 1.1 is sufficient. Assume that $u \in D'(\mathbb{T}^3)$ is such that

$$Pu = f,\ f \in C^\infty(\mathbb{T}^3).$$

Letting $v = Su \in D'(\mathbb{T}^3)$ and $g = Sf \in C^\infty(\mathbb{T}^3)$ and using concatenation $SP = \tilde{P}S$ gives

$$\tilde{P}v = g,\ g \in C^\infty(\mathbb{T}^3),$$

where $\tilde{P}$ is given by (2.1). Therefore, to show that $u \in C^\infty(\mathbb{T}^3)$ it suffices to show $v \in C^\infty(\mathbb{T}^3)$. For this we must show that for any $N \in \mathbb{N}$ there exists a positive constant C_N such that the Fourier coefficients $\hat{v}(\tau,\xi)$ of v satisfy the following inequality

$$|\hat{v}(\tau,\xi)| \leq C_N|(\tau,\xi)|^{-N},\ (\tau,\xi) \in (\mathbb{Z}^3)\backslash F, \tag{2.2}$$

where F is a finite set in $\mathbb{Z}^3$.

To prove inequality (2.2) we will use the fact that g satisfies

$$|\hat{g}(t,\xi)| \leq C_N|\xi|^{-N},\ \xi \in \mathbb{Z}\backslash\{0\},\ t \in \mathbb{T}^2, \tag{2.3}$$

and we will show that the partial Fourier transform with respect to x of g dominates the partial Fourier transform with respect to x of v in L^2-norm (like in Lemma 3.1). In addition, we will use the fact that the operator $\tilde{P}$ is elliptic in $t = (t_1, t_2)$.

3 The non-Liouville number case

Here we shall prove that if the range of $a_0(t_1)$ contains a non-Liouville number then P is globally hypoelliptic in $\mathbb{T}^3$. We begin with the following lemma, which in this case provides an L^2-estimate for the solution v of the equation $\tilde{P}v = g$.

Lemma 3.1. *If the range of $a_0(t_1)$ contains an irrational non-Liouville number then there exist positive constants C_1, C_2 and $K \geq 0$ such that*

$$||\hat{v}(\cdot,\xi)||_{L^2(\mathbb{T}^2)} \leq C_1|\xi|^{3K+1}||\hat{g}(\cdot,\xi)||_{L^2(\mathbb{T}^2)}$$

for $|\xi| > C_2$.

Before proving Lemma 3.1 we will use it to show that P is globally hypoelliptic in $\mathbb{T}^3$. It suffices to prove that $v \in C^\infty(\mathbb{T}^3)$, or equivalently that inequality (2.2) holds. By Lemma 3.1 and the fact that $g \in C^\infty(\mathbb{T}^3)$, which implies inequality (2.3), we obtain that for any $N \in \mathbb{N}$ there exists a positive constant C_N such that

$$||\hat{v}(\cdot,\xi)||^2_{L^2(\mathbb{T}^2)} \leq C_N|\xi|^{-N}, \ |\xi| > C_2.$$

From now on we will use C_N to represent a constant which depends on N and may change a finite number of times.

Since

$$\hat{v}(\tau,\xi) = \int_{\mathbb{T}^2} e^{-it\cdot\tau}\hat{v}(t,\xi)dt,$$

by the last inequality and the Cauchy-Schwarz inequality we obtain

$$|\hat{v}(\tau,\xi)| \leq C_N|\xi|^{-N}, \ (\tau,\xi) \in \mathbb{Z}^2\times\mathbb{Z}, \, |\xi| > C_2. \tag{3.1}$$

Moreover, since the operator $\tilde{P}$ is elliptic at $(t,x;\tau_0,0)$ for all $(t,x) \in \mathbb{T}^2\times\mathbb{T}$ and $\tau_0 \in \mathbb{Z}^2\backslash\{0\}$, by using the microlocal elliptic theory we obtain that there exists a conic neighborhood $\Gamma_\epsilon = \{(\tau,\xi) \in \mathbb{Z}^2\times\mathbb{Z}\colon |\xi| < \epsilon|\tau|\}$ of $(\tau_0,0)$ such that for any $N \in \mathbb{N}$ there exists a positive constant C_N such that

$$|\hat{v}(\tau,\xi)| \leq C_N|(\tau,\xi)|^{-N}, \ (\tau,\xi) \in \Gamma_\epsilon.$$

Thus, if we let $\Gamma = \{(\tau,\xi) \in \mathbb{Z}^2\times\mathbb{Z}\colon |\xi| > \frac{\epsilon}{2}|\tau|\}$, then using (3.1) we see that for $(\tau,\xi) \in \Gamma$ and $|\xi| > C_2$

$$|\hat{v}(\tau,\xi)| \leq C_N(\frac{1}{2}|\xi| + \frac{\epsilon}{4}|\tau|)^{-N} \leq C_N|(\tau,\xi)|^{-N}.$$

The last two inequalities imply that for any $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that

$$|\hat{v}(\tau,\xi)| \leq C_N|(\tau,\xi)|^{-N}, \ (\tau,\xi) \in (\mathbb{Z}^2\times\mathbb{Z})\backslash F,$$

where $F = \{(\tau,\xi) : (\tau,\xi) \notin \Gamma_\epsilon, \ |\xi| \leq C_2\}$. Hence (2.2) holds true and therefore $v \in C^\infty(\mathbb{T}^3)$.

Proof of Lemma 3.1. We start by recalling that

$$\tilde{P}v = [-(\partial_{t_1} + b(t)\partial_x)^2 - (\partial_{t_2} + a_0(t_1)\partial_x)^2]v = g.$$

In this equation, taking the partial Fourier transform with respect to x gives

$$\left[-(\partial_{t_1} + i\xi b(t))^2 - (\partial_{t_2} + i\xi a_0(t_1))^2\right] \hat{v}(t,\xi) = \hat{g}(t,\xi), \tag{3.2}$$

for all $\xi \in \mathbb{Z}$ and $t \in \mathbb{T}^2$.

The solution $\hat{v}(\cdot,\xi)$ to equation (3.2) is in $C^\infty(\mathbb{T}^2)$ for all $\xi \in \mathbb{Z}$ since it is elliptic in $t \in \mathbb{T}^2$. Therefore, if we multiply (3.2) by $\bar{\hat{v}}$, and integrate by parts with respect to $t \in \mathbb{T}^2$, then we obtain

$$\begin{aligned}\int_{\mathbb{T}^2} |\partial_{t_1}\hat{v}(t,\xi) + i\xi b(t)\hat{v}(t,\xi)|^2 dt \quad &+ \quad \int_{\mathbb{T}^2} |\partial_{t_2}\hat{v}(t,\xi) + i\xi a_0(t_1)\hat{v}(t,\xi)|^2 dt \\ &= \quad \int_{\mathbb{T}^2} \hat{g}(t,\xi)\bar{\hat{v}}(t,\xi)dt. \end{aligned} \tag{3.3}$$

Since by hypothesis the range of $a_0(t_1)$ contains an irrational non-Liouville number we expect that the second term of the left-hand side in (3.3) to be the "dominant" term.

Next, we shall analyze this second term. For $\varphi_\xi(t) \in C^\infty(\mathbb{T}^2)$, with $\xi \in \mathbb{Z}$, we define

$$\partial_{t_2}\varphi_\xi(t) + i\xi a_0(t_1)\varphi_\xi(t) \doteq \psi_\xi(t). \tag{3.4}$$

The following lemma provides a formula for expressing $\varphi_\xi(t)$ in terms of $\psi_\xi(t)$.

Lemma 3.2. *If $e^{i2\pi\xi a_0(t_1)} - 1 \neq 0$ then the equation (3.4) has a unique periodic solution which is given by*

$$\varphi_\xi(t) = \frac{1}{e^{i2\pi\xi a_0(t_1)} - 1} \int_{\mathbb{T}} e^{i\xi a_0(t_1)s}\psi_\xi(t_1, t_2 + s)ds. \tag{3.5}$$

Proof of Lemma 3.2. Multiplying (3.4) by $e^{i\xi a_0(t_1)t_2}$ we obtain

$$\partial_{t_2}(e^{i\xi a_0(t_1)t_2}\varphi_\xi) = e^{i\xi a_0(t_1)t_2}\psi_\xi.$$

Then integrating and solving for $\varphi_\xi(t)$ gives

$$\varphi_\xi(t) = e^{-i\xi a_0(t_1)t_2} \int_0^{t_2} e^{i\xi a_0(t_1)s}\psi_\xi(t_1, s)ds + e^{-i\xi a_0(t_1)t_2}C(t_1). \tag{3.6}$$

Imposing the periodicity condition $\varphi_\xi(t_1, 0) = \varphi_\xi(t_1, 2\pi)$ we obtain

$$C(t_1) = \frac{1}{e^{i2\pi\xi a_0(t_1)} - 1} \int_0^{2\pi} e^{i\xi a_0(t_1)s}\psi_\xi(t_1, s)ds.$$

Substituting this into (3.6) and making the change of variables $r = s - t_2$ gives, after some manipulation,

$$\begin{aligned}\varphi_\xi(t) \quad &= \quad \frac{1}{e^{i2\pi\xi a_0(t_1)} - 1} e^{i2\pi\xi a_0(t_1)} \int_{-t_2}^{0} e^{i\xi a_0(t_1)r}\psi_\xi(t_1, r + t_2)dr \\ &+ \quad \frac{1}{e^{i2\pi\xi a_0(t_1)} - 1} \int_0^{2\pi - t_2} e^{i\xi a_0(t_1)r}\psi_\xi(t_1, r + t_2)dr.\end{aligned}$$

Using the change of variables $\theta = r + 2\pi$ in the first integral and combining the resulting terms gives the desired formula. $\square$

Thus it follows from Lemma 3.2 that if $e^{i2\pi\xi a_0(t_1)} - 1 \neq 0$ then

$$\begin{aligned} |\varphi_\xi(t)| &\leq \frac{1}{|e^{i2\pi\xi a_0(t_1)} - 1|} \int_{\mathbb{T}} |\psi_\xi(t_1, t_2 + s)|\, ds \\ &= \frac{1}{|e^{i2\pi\xi a_0(t_1)} - 1|} \int_{\mathbb{T}} |\psi_\xi(t_1, t_2)|\, dt_2 \\ &= \frac{1}{|e^{i2\pi\xi a_0(t_1)} - 1|} \int_{\mathbb{T}} |\partial_{t_2}\varphi_\xi(t) + i\xi a_0(t_1)\varphi_\xi(t)|\, dt_2. \end{aligned}$$

Let $\xi \in \mathbb{Z}\backslash\{0\}$ be fixed. If we apply the last inequality with $\varphi_\xi(t) = \hat{v}(t, \xi)$ then we obtain

$$|\hat{v}(t,\xi)| \leq \frac{1}{|e^{i2\pi\xi a_0(t_1)} - 1|} \int_{\mathbb{T}} |\partial_{t_2}\hat{v}(t,\xi) + i\xi a_0(t_1)\hat{v}(t,\xi)|\, dt_2, \tag{3.7}$$

provided $e^{i2\pi\xi a_0(t_1)} - 1 \neq 0$.

We now split the proof in two cases.

Case 1. $a_0(t_1) \equiv \alpha$ with α being a non-Liouville number.

In this case we need the following elementary

Lemma 3.3. *Suppose that α is a non-Liouville number. Then there exist $C > 0$ and $K \geq 0$ such that*

$$\left|e^{i2\pi\xi\alpha} - 1\right| \geq C|\xi|^{-K}, \ \forall\, \xi \in \mathbb{Z}\backslash\{0\}. \tag{3.8}$$

Proof of Lemma 3.3. Observe that

$$|e^{i2\pi\xi\alpha} - 1| = 2|\sin(\pi\xi\alpha)|,$$

and that for each $\xi \in \mathbb{Z}\backslash\{0\}$ fixed one of the following conditions occurs:

(i) There exists $k \in \mathbb{Z}$ such that $|\pi\xi\alpha - k\pi| < \frac{\pi}{3}$.

(ii) $|\pi\xi\alpha - k\pi| \geq \frac{\pi}{3}$ for all $k \in \mathbb{Z}$.

Suppose that the condition (i) holds. Then there exists θ_0 between $\pi\xi\alpha$ and $k\pi$ such that

$$\begin{aligned} |\sin(\pi\xi\alpha) - \sin(k\pi)| &= |\cos(\theta_0)||\pi\xi\alpha - k\pi| \\ &\geq \frac{1}{2}|\pi\xi\alpha - k\pi| = \frac{\pi}{2}|\xi\alpha - k|. \end{aligned}$$

Since α is a non-Liouville number there exist constants $C > 0$ and $K > 0$ such that the last inequality is bounded from below by $C|\xi|^{-K}$. Thus, in this case, inequality (3.8) holds. On the other hand if condition (ii) holds then $2|sin(\pi\xi\alpha)| \geq \sqrt{3}$ which is also greater than $C|\xi|^{-K}$, by redefining C if necessary. This completes the proof of this lemma. □

From now on we shall use the letter C to represent a constant, which may change a finite number of times.

Thus, it follows from Lemma 3.2 and (3.8) that equation (3.4) has a unique periodic solution and the following estimate follows from (3.7) and (3.8):

$$|\hat{v}(t,\xi)| \leq C|\xi|^{K} \int_{\mathbb{T}} |\partial_{t_2}\hat{v}(t,\xi) + i\xi\alpha\hat{v}(t,\xi)|dt_2, \ \forall\ t_1 \in \mathbb{T},\ \xi \in \mathbb{Z}\backslash\{0\}. \quad (3.9)$$

In (3.9), applying Cauchy-Schwarz inequality first, and then integrating with respect to $t = (t_1, t_2) \in \mathbb{T}^2$ gives

$$||\hat{v}(\cdot,\xi)||^2_{L^2(\mathbb{T}^2)} \leq C|\xi|^{2K} \int_{\mathbb{T}^2} |\partial_{t_2}\hat{v}(t,\xi) + i\xi a_0(t_1)\hat{v}(t,\xi)|^2 dt.$$

Using this together with (3.3) we obtain

$$||\hat{v}(\cdot,\xi)||_{L^2(\mathbb{T}^2)} \leq C|\xi|^{2K}||\hat{g}(\cdot,\xi)||_{L^2(\mathbb{T}^2)}$$

for $\xi \in \mathbb{Z}\backslash\{0\}$. Thus, the proof of Lemma 3.1 in case 1 is complete.

Case 2. The function $a_0(t_1)$ is non-constant.

Since non-Liouville numbers are dense in $\mathbb{R}$ and $a_0(t_1)$ is non-constant we can find a $t_1^0 \in (0, 2\pi)$ such that $a_0(t_1^0) = \alpha$ with α being a non-Liouville number. It follows from (3.8) that there exist constants $C > 0$ and $K \geq 0$ such that

$$|\sin(\pi\xi a_0(t_1^0)| = |\sin(\pi\xi\alpha)| \geq C|\xi|^{-K},\ \forall\ \xi \in \mathbb{Z}\backslash\{0\}. \quad (3.10)$$

Using these values of K and C and the new constants

$$\delta \doteq \min\{2\pi - t_1^0, t_1^0\} \text{ and } \beta \doteq ||a_0'||_{L^\infty} > 0$$

we state the following key result which is the "variable" version of Lemma 3.3.

Lemma 3.4. *Suppose that $a_0(t_1)$ is non-constant. Then for $|\xi|^{K+1} > \frac{C}{\pi\beta\delta}$, with $\xi \in \mathbb{Z}$, there exists an interval I_ξ such that*

$$|e^{i2\pi\xi a_0(t_1)} - 1| \geq C|\xi|^{-K},\ \forall\ t_1 \in I_\xi,$$

with $|I_\xi| = \frac{C}{\pi\beta|\xi|^{K+1}}$ and $t_1^0 \in I_\xi$.

Proof of Lemma 3.4. We begin with the identity

$$\left|e^{i2\pi\xi a_0(t_1)}-1\right| = 2|\sin(\pi\xi a_0(t_1))|. \tag{3.11}$$

Then, for $\xi\in\mathbb{Z},\ |\xi|^{K+1}>\frac{C}{\pi\beta\delta}$, we define the following interval

$$I_\xi = \left\{t_1\in[0,2\pi] : |t_1-t_1^0|\le \frac{C}{2\pi\beta|\xi|^{K+1}}\right\}.$$

Since $|\xi|^{K+1}>\frac{C}{\pi\beta\delta}$ we have $\frac{C}{\pi\beta|\xi|^{K+1}}<\delta$ and therefore $I_\xi\subset(0,2\pi)$. Hence, $|I_\xi|=\frac{C}{\pi\beta|\xi|^{K+1}}$.

Applying the fundamental theorem of calculus we obtain

$$\sin(\pi\xi a_0(t_1)) = \sin(\pi\xi a_0(t_1^0)) + \pi\xi\int_{t_1^0}^{t_1}\cos(\pi\xi a_0(r))a_0'(r)dr.$$

Then estimating from below gives

$$|\sin(\pi\xi a_0(t_1))| \ge |\sin(\pi\xi\alpha)| - \pi|\xi|\left|\int_{t_1^0}^{t_1}\cos(\pi\xi a_0(r))a_0'(r)dr\right|. \tag{3.12}$$

Since, for $t_1\in I_\xi$ we have

$$\begin{aligned}\pi|\xi|\left|\int_{t_1^0}^{t_1}\cos(\pi\xi a_0(r))a_0'(r)dr\right| &\le \pi|\xi|\beta|t_1-t_1^0|\\ &\le \pi|\xi|\beta\frac{C}{2\pi\beta|\xi|^{K+1}} = \frac{C}{2}|\xi|^{-K},\end{aligned}$$

it follows from (3.11), (3.12) and (3.10) that

$$\left|e^{i2\pi\xi a_0(t_1)}-1\right| \ge 2C|\xi|^{-K} - C|\xi|^{-K} = C|\xi|^{-K},\ \forall\ t_1\in I_\xi.$$

The proof of Lemma 3.4 is now complete. □

Now we return to the proof of Lemma 3.1 in case 2. From inequality (3.7), using Lemma 3.4, we obtain

$$|\hat{v}(t,\xi)| \le C|\xi|^K\int_{\mathbb{T}}|\partial_{t_2}\hat{v}(t,\xi)+i\xi a_0(t_1)\hat{v}(t,\xi)|\,dt_2,\ \forall\ t_1\in I_\xi,\ |\xi|^{K+1}>\frac{C}{\pi\beta\delta}.$$

From the last inequality, using the Cauchy-Schwarz inequality first and then integrating with respect to $t_1\in I_\xi$ and $t_2\in\mathbb{T}$, we obtain

$$\begin{aligned}\int_{\mathbb{T}}\int_{I_\xi}|\hat{v}(t,\xi)|^2dt_1dt_2 &\le C|\xi|^{2K}\int_{\mathbb{T}}\int_{I_\xi}|\partial_{t_2}\hat{v}(t,\xi)+i\xi a_0(t_1)\hat{v}(t,\xi)|^2dt_1dt_2\\ &\le C|\xi|^{2K}\int_{\mathbb{T}^2}|\partial_{t_2}\hat{v}(t,\xi)+i\xi a_0(t_1)\hat{v}(t,\xi)|^2dt,\end{aligned} \tag{3.13}$$

for $|\xi|^{K+1} > \frac{C}{\pi\beta\delta}$.

Next we show how to obtain the desired estimates about $||\hat{v}(\cdot,\xi)||_{L^2(\mathbb{T}^2)}$ from inequality (3.13). We start by noting that for any $\varphi \in C^\infty(\mathbb{T}^2)$ we have

$$\varphi(t_1,t_2) = \varphi(s_1,t_2) + \int_{s_1}^{t_1} \partial_r \varphi(r,t_2)dr$$

which implies that

$$|\varphi(t_1,t_2)|^2 \leq 2|\varphi(s_1,t_2)|^2 + 4\pi||\partial_r\varphi(\cdot,t_2)||^2_{L^2(\mathbb{T})}.$$

Integrating the last inequality with respect to $t \in \mathbb{T}^2$ and $s_1 \in I_\xi$ we obtain

$$|I_\xi|||\varphi||^2_{L^2(\mathbb{T}^2)} \leq C\left(\int_{\mathbb{T}}\int_{I_\xi} |\varphi(s_1,t_2)|^2 ds_1 dt_2 + |I_\xi|||\varphi_{t_1}||^2_{L^2(\mathbb{T}^2)}\right). \qquad (3.14)$$

Then, applying (3.14) with $\varphi(t) = e^{-i\xi h(t)}\hat{v}(t,\xi)$ gives

$$\begin{aligned} |I_\xi|||\hat{v}(\cdot,\xi)||^2_{L^2(\mathbb{T}^2)} &\leq C\int_{\mathbb{T}}\int_{I_\xi} |\hat{v}(s_1,t_2,\xi)|^2 ds_1 dt_2 \\ &+ C|I_\xi|\int_{\mathbb{T}^2} |\partial_{t_1}\hat{v}(t,\xi) + i\xi b(t)\hat{v}(t,\xi)|^2 dt. \qquad (3.15) \end{aligned}$$

Now, using the fact that $|I_\xi| = C|\xi|^{-(K+1)}$ it follows from (3.15), (3.13), (3.3) and Cauchy-Schwarz inequality that for $|\xi|^{K+1} > \frac{C}{\pi\beta\delta}$ we have

$$\begin{aligned} ||\hat{v}(\cdot,\xi)||^2_{L^2(\mathbb{T}^2)} &\leq C\int_{\mathbb{T}^2} |\partial_{t_1}\hat{v}(t,\xi) + i\xi b(t)\hat{v}(t,\xi)|^2 dt \\ &+ C|\xi|^{3K+1}\int_{\mathbb{T}^2} |\partial_{t_2}\hat{v}(t,\xi) + i\xi a_0(t_1)\hat{v}(t,\xi)|^2 dt \\ &\leq C|\xi|^{3K+1}\int_{\mathbb{T}^2} \hat{g}(t,\xi)\bar{\hat{v}}(t,\xi)dt \\ &\leq C|\xi|^{3K+1}||\hat{g}(\cdot,\xi)||_{L^2(\mathbb{T}^2)}||\hat{v}(\cdot,\xi)||_{L^2(\mathbb{T}^2)}. \end{aligned}$$

Thus, for $\xi \in \mathbb{Z}$, $|\xi|^{K+1} > \frac{C}{\pi\beta\delta}$, we can conclude that

$$||\hat{v}(\cdot,\xi)||_{L^2(\mathbb{T}^2)} \leq C|\xi|^{3K+1}||\hat{g}(\cdot,\xi)||_{L^2(\mathbb{T}^2)}.$$

This completes the proof of Lemma 3.1. □

4 Finite type points and rational a_0

What is left now is to show that P is globally hypoelliptic in $\mathbb{T}^3$ when $a_0(t_1)$ is a constant, which is a rational or a Liouville number, and there exists a point of finite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t)\partial_x$. Here, we shall consider the case where a_0 is a rational number. The case where a_0 is a Liouville number will be considered in the next section.

Throughout this section we shall assume that $a_0(t_1) = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$ and p and q do not have a common factor which is larger than one.

In this case we shall need the following two inequalities.

Lemma 4.1. *Let I_1 and I_2 be two open intervals in $(0, 2\pi)$. Then there exists a positive constant C depending on the length of each interval such that for any $\varphi \in C^\infty(\mathbb{T}^2)$ we have*

$$\int_{I_2}\int_{\mathbb{T}} |\varphi(t_1,t_2)|^2 dt_1 dt_2 \leq C\left(\int_{I_2}\int_{I_1} |\varphi(t_1,t_2)|^2 dt_1 dt_2 + ||\varphi_{t_1}||^2_{L^2(\mathbb{T}^2)}\right) \tag{4.1}$$

$$||\varphi||^2_{L^2(\mathbb{T}^2)} \leq C\left(\int_{I_2}\int_{\mathbb{T}} |\varphi(t_1,t_2)|^2 dt_1 dt_2 + ||\varphi_{t_2}||^2_{L^2(\mathbb{T}^2)}\right). \tag{4.2}$$

Proof of Lemma 4.1. Using the fundamental theorem of calculus we obtain

$$\varphi(t_1,t_2) = \varphi(s,t_2) + \int_s^{t_1} \varphi_r(r,t_2)dr.$$

Then, using the Cauchy-Schwarz inequality we have

$$|\varphi(t_1,t_2)|^2 \leq C\left(|\varphi(s,t_2)|^2 + ||\varphi_{t_1}(\cdot,t_2)||^2\right).$$

Integrating the last inequality with respect to $t_1 \in \mathbb{T}$, $t_2 \in I_2$ and $s \in I_1$ gives

$$\int_{I_2}\int_{\mathbb{T}} |\varphi(t_1,t_2)|^2 dt_1 dt_2 \leq C\left(\int_{I_2}\int_{I_1} |\varphi(s,t_2)|^2 ds dt_2 + ||\varphi_{t_1}||^2_{L^2(\mathbb{T}^2)}\right),$$

which is (4.1). To prove (4.2) we write $\varphi(t_1,t_2) = \varphi(t_1,s) + \int_s^{t_2} \varphi_r(t_1,r)dr$ and so $|\varphi(t_1,t_2)|^2 \leq C\left(|\varphi(t_1,s)|^2 + ||\varphi_{t_2}(t_1,\cdot)||^2\right)$. Finally, integranting this inequality with respect to $s \in I_2$ and $t = (t_1,t_2) \in \mathbb{T}^2$ gives inequality (4.2). Therefore the proof of Lemma 4.1 is complete. □

Now, we begin the proof that P is globally hypoelliptic in $\mathbb{T}^3$. Let $u \in D'(\mathbb{T}^3)$ be such that $Pu = f \in C^\infty(\mathbb{T}^3)$. Then, defining $v = Su$ and $g = Sf$ we have that v and g satisfy the equation $\tilde{P}v = g$, where, in this case,

$$\tilde{P} = -(\partial_{t_1} + b(t)\partial_x)^2 - (\partial_{t_2} + \frac{p}{q}\partial_x)^2, \text{ with } b(t) = -\frac{\partial h}{\partial t_1} = -\int_0^{t_2} a_{t_1}(t_1,s)ds.$$

As before, for each $\xi \in \mathbb{Z}$ fixed we have $\hat{v}(\cdot, \xi) \in C^\infty(\mathbb{T}^2)$ and $\hat{v}(\cdot, \xi)$ satisfies

$$\int_{\mathbb{T}^2} |\partial_{t_1}\hat{v}(t,\xi) + i\xi b(t)\hat{v}(t,\xi)|^2 dt \;+\; \int_{\mathbb{T}^2} |\partial_{t_2}\hat{v}(t,\xi) + i\xi\frac{p}{q}\hat{v}(t,\xi)|^2 dt = \int_{\mathbb{T}^2} \hat{g}(t,\xi)\bar{\hat{v}}(t,\xi)dt. \tag{4.3}$$

As we have seen in section 3, in order to show that P is globally hypoelliptic in $\mathbb{T}^3$ it suffices to prove following result.

Lemma 4.2. *Suppose that $a_0(t_1) = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$ and p and q do not have a common factor which is larger than one. Also, suppose that there exists a point of finite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t)\partial_x$. Then, there exists a constant $C > 0$ such that*

$$||\hat{v}(\cdot,\xi)||_{L^2(\mathbb{T}^2)} \leq C||\hat{g}(\cdot,\xi)||_{L^2(\mathbb{T}^2)},\ \forall\ \xi \in \mathbb{Z}\backslash\{0\}. \tag{4.4}$$

Proof of Lemma 4.2. We define the sets $\mathcal{A} = q\mathbb{Z}$ and $\mathcal{B} = \mathbb{Z}\backslash\mathcal{A}$ and split the proof in two cases. Namely, Case 1: $\xi \in \mathcal{A}$, and Case 2: $\xi \in \mathcal{B}$.

Case 1. $\xi \in \mathcal{A}$.

Thus, there exists $\ell \in \mathbb{Z}$ such that $\xi = \ell q$.

Since there exists a point $(t_1^\circ, t_2^\circ, x^\circ) \in \mathbb{T}^3$ of finite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t)\partial_x$ we can find $\delta > 0$ and $k = j + p \in \mathbb{N}$ such that

$$\frac{\partial^k a}{\partial^j_{t_1}\partial^p_{t_2}}(t_1,t_2) \neq 0,\ \forall (t_1,t_2) \in [-\delta,\delta]^2$$

and

$$\partial_x = \frac{1}{\frac{\partial^k a}{\partial^j_{t_1}\partial^p_{t_2}}(t_1,t_2)}[\underbrace{X_2,\cdots[X_2}_{p},\underbrace{[X_1,\cdots[X_1}_{j-1},[X_1,X_2]]\cdots]]\cdots], \tag{4.5}$$

$\forall\ (t_1,t_2) \in [-\delta,\delta]^2$. Setting

$$Y_1 = \partial_{t_1},\ Y_2 = \partial_{t_2} + i\xi a(t),$$

and taking the partial Fourier transform with respect to x in (4.5) gives

$$i\xi = \frac{1}{\frac{\partial^k a}{\partial^j_{t_1}\partial^p_{t_2}}(t_1,t_2)}Y_k, \quad \forall\ (t_1,t_2) \in [-\delta,\delta]^2, \tag{4.6}$$

where

$$Y_k = [\underbrace{Y_2,\cdots[Y_2}_{p},\underbrace{[Y_1,\cdots[Y_1}_{j-1},[Y_1,Y_2]]\cdots]]\cdots], \quad \text{with } p+j=k.$$

Now, let χ be a function such that $\chi \in C^\infty(\mathbb{T}^2)$, $\chi \geq 0$, $\chi \equiv 1$ on $[-\frac{\delta}{2}, \frac{\delta}{2}]^2$ and supp $\chi \subset [-\delta, \delta]^2$. For $\xi \in \mathbb{Z}\backslash\{0\}$ and $\varphi \in C^\infty(\mathbb{T}^2)$ we have

$$
\begin{aligned}
\int_{[-\frac{\delta}{2},\frac{\delta}{2}]^2} |\varphi(t)|^2 dt &= \int_{[-\frac{\delta}{2},\frac{\delta}{2}]^2} \left(\frac{1}{i\xi \frac{\partial^k a}{\partial^j_{t_1} \partial^p_{t_2}}(t_1,t_2)} Y_k \right) |\varphi(t)|^2 dt \\
&= \int_{[-\frac{\delta}{2},\frac{\delta}{2}]^2} \chi(t) \left(\frac{1}{i\xi \frac{\partial^k a}{\partial^j_{t_1} \partial^p_{t_2}}(t_1,t_2)} Y_k \right) |\varphi(t)|^2 dt \\
&\leq \int_{[-\delta,\delta]^2} \chi(t) \left(\frac{1}{i\xi \frac{\partial^k a}{\partial^j_{t_1} \partial^p_{t_2}}(t_1,t_2)} Y_k \right) |\varphi(t)|^2 dt \qquad (4.7) \\
&= \int_{\mathbb{T}^2} \left(\chi(t) \frac{1}{i\xi \frac{\partial^k a}{\partial^j_{t_1} \partial^p_{t_2}}(t_1,t_2)} \right) Y_k |\varphi(t)|^2 dt \\
&= \left| \int_{\mathbb{T}^2} \left(\chi(t) \frac{1}{i\xi \frac{\partial^k a}{\partial^j_{t_1} \partial^p_{t_2}}(t_1,t_2)} \right) Y_k |\varphi(t)|^2 dt \right| \\
&= \frac{1}{|\xi|} |(A(t) Y_k \varphi, \varphi)|,
\end{aligned}
$$

where

$$
A(t) = \chi(t) \frac{1}{\frac{\partial^k a}{\partial^j_{t_1} \partial^p_{t_2}}(t_1,t_2)}.
$$

It follows from Cordaro and Himonas [CH1] that there exists a constant $C > 0$ such that

$$
|(A(t) Y_k \varphi, \varphi)| \leq C |\xi| \sum_{j=1}^{2} \left(||Y_j \varphi||^2_{L^2(\mathbb{T}^2)} + ||\varphi||_{L^2(\mathbb{T}^2)} ||Y_j \varphi||_{L^2(\mathbb{T}^2)} \right), \qquad (4.8)
$$

where $\xi \in \mathbb{Z}\backslash\{0\}$.

Now we apply inequalities (4.7) and (4.8) with $\xi = \ell q$ and $\varphi(t) = \hat{u}(t, \ell q)$

and we obtain

$$\begin{aligned}
\int_{[-\frac{\delta}{2},\frac{\delta}{2}]^2} |\hat{u}(t,\ell q)|^2 dt &\leq C\sum_{j=1}^{2} ||Y_j\hat{u}(t,\ell q)||^2_{L^2(\mathbb{T}^2)} \\
&+ C\sum_{j=1}^{2} ||\hat{u}(t,\ell q)||_{L^2(\mathbb{T}^2)}||Y_j\hat{u}(t,\ell q)||_{L^2(\mathbb{T}^2)} \\
&\leq C\sum_{j=1}^{2} ||Y_j\hat{u}(t,\ell q)||^2_{L^2(\mathbb{T}^2)} \qquad (4.9)\\
&+ C\sum_{j=1}^{2}\left(\frac{\epsilon^2}{2}||\hat{u}(t,\ell q)||^2_{L^2(\mathbb{T}^2)} + \frac{1}{2\epsilon^2}||Y_j\hat{u}(t,\ell q)||^2_{L^2(\mathbb{T}^2)}\right) \\
&= C\epsilon^2||\hat{u}(t,\ell q)||^2_{L^2(\mathbb{T}^2)} \\
&+ C(1+\frac{1}{2\epsilon^2})\sum_{j=1}^{2}||Y_j\hat{u}(t,\ell q)||^2_{L^2(\mathbb{T}^2)}.
\end{aligned}$$

Since $\hat{v}(t,\ell q) = e^{i\ell q h(t)}\hat{u}(t,\ell q)$, $\hat{g}(t,\ell q) = e^{i\ell q h(t)}\hat{f}(t,\ell q)$ and

$$\sum_{j=1}^{2}||Y_j\hat{u}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)} = \int_{\mathbb{T}^2}\hat{f}(t,\ell q)\bar{\hat{u}}(t,\ell q)dt,$$

it follows from the last inequality and from Cauchy-Schwarz inequality that

$$\begin{aligned}
\int_{[-\frac{\delta}{2},\frac{\delta}{2}]^2} |\hat{u}(t,\ell q)|^2 dt &\leq C\epsilon^2||\hat{v}(t,\ell q)||^2_{L^2(\mathbb{T}^2)} \\
&+ C(1+\frac{1}{2\epsilon^2})\left(\frac{\epsilon_1^2}{2}||\hat{v}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)} + \frac{1}{2\epsilon_1^2}||\hat{g}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)}\right) \\
&= \left(C\epsilon^2 + C(1+\frac{1}{2\epsilon^2})\frac{\epsilon_1^2}{2}\right)||\hat{v}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)} \qquad (4.10)\\
&+ C(1+\frac{1}{2\epsilon^2})\frac{1}{2\epsilon_1^2}||\hat{g}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)}.
\end{aligned}$$

Applying (4.2) with $\varphi = e^{i\ell p t_2}\hat{v}(t,\ell q)$ and $I_2 = [-\delta,\delta]$ we can conclude that there exists a positive constant C which depends only on δ such that

$$\begin{aligned}
||\hat{v}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)} &\leq C\int_{[-\delta,\delta]}\int_{\mathbb{T}}|\hat{u}(t_1,t_2,\ell q)|^2 dt_1 dt_2 \\
&+ C||(\partial_{t_2}+i\ell p)\hat{v}(t,\ell q)||^2_{L^2(\mathbb{T}^2)}.
\end{aligned}$$

Estimating the first integral in the right-hand side of the last inequality by using (4.1) with $\varphi = \hat{u}(t,\ell q)$ and $I_1 = I_2 = [-\delta,\delta]$ we obtain

$$\begin{aligned}||\hat{v}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)} &\leq C\left(\int_{[-\delta,\delta]}\int_{[-\delta,\delta]}|\hat{u}(t_1,t_2,\ell q)|^2dt_1dt_2 + ||\hat{u}_{t_1}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)}\right)\\ &+ C||(\partial_{t_2}+i\ell p)\hat{v}(t,\ell q)||^2_{L^2(\mathbb{T}^2)}, \qquad (4.11)\end{aligned}$$

where the constant C depends only on δ.

Since $\hat{u}_{t_1}(t,\ell q) = e^{-i\ell q h(t)}(\partial_{t_1}+i\ell q b(t))\hat{v}(t,\ell q))$ it follows from (4.10) and (4.11) that there exists a constant $C>0$ such that

$$\begin{aligned}||\hat{v}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)} &\leq \left(C\epsilon^2 + C(1+\frac{1}{2\epsilon^2})\frac{\epsilon_1^2}{2}\right)||\hat{v}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)}\\ &+ C(1+\frac{1}{2\epsilon^2})\frac{1}{2\epsilon_1^2}||\hat{g}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)}\\ &+ C||(\partial_{t_1}+i\ell q b(t))\hat{v}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)}\\ &+ C||(\partial_{t_2}+i\ell p)\hat{v}(t,\ell q)||^2_{L^2(\mathbb{T}^2)}.\end{aligned}$$

Choosing ϵ and ϵ_1 so that

$$C\epsilon^2 + C(1+\frac{1}{2\epsilon^2})\frac{\epsilon_1^2}{2} = \frac{1}{2}$$

the last inequality gives

$$\begin{aligned}||\hat{v}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)} &\leq C||\hat{g}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)}\\ &+ C||(\partial_{t_1}+i\ell q b(t))\hat{v}(\cdot,\ell q)||^2_{L^2(\mathbb{T}^2)} \qquad (4.12)\\ &+ C||(\partial_{t_2}+i\ell p)\hat{v}(t,\ell q)||^2_{L^2(\mathbb{T}^2)}.\end{aligned}$$

Finally, from (4.12), using (4.3) and the Cauchy-Schwarz inequality, we conclude that there exists a constant $C>0$ such that

$$||\hat{v}(\cdot,\ell q)||_{L^2(\mathbb{T}^2)} \leq C||\hat{g}(\cdot,\ell q)||_{L^2(\mathbb{T}^2)}),\ \forall\ \ell\in\mathbb{Z}\backslash\{0\}. \qquad (4.13)$$

This completes the proof of Lemma 4.2 in the Case 1. □

Remark 4.3 When $q=1$ then $\mathcal{A}=\mathbb{Z}$ and therefore $\mathcal{B}=\emptyset$. In this case (4.13) implies that

$$||\hat{v}(\cdot,\xi)||_{L^2(\mathbb{T}^2)} \leq C_N|\xi|^{-N},\ \forall\ \xi\in\mathbb{Z}\backslash\{0\} \qquad (4.14)$$

since $g\in C^\infty(\mathbb{T}^3)$.

The last inequality implies, as before, that $u\in C^\infty(\mathbb{T}^3)$ and therefore P is globally hypoelliptic in $\mathbb{T}^3$. Thus, when $q=1$ the proof is complete.

Remark 4.4 We would like to point out that when $p = 0$ we have $q = 1$.

From now on we assume that $q \geq 2$ and $p \neq 0$.

Case 2. $\xi \in \mathcal{B}$.

By definition $\xi \in \mathcal{B}$ means that ξ is not a multiple of q, in particulat $\xi \neq 0$. Therefore we have $|\xi p - kq| \geq 1$ for any $k \in \mathbb{Z}$. Thus,

$$\left| \frac{p}{q} - \frac{k}{\xi} \right| = \frac{1}{|q||\xi|} |\xi p - qk| \geq \frac{1}{q|\xi|}, \ \forall \ k \in \mathbb{Z}. \tag{4.15}$$

The last inequality is saying that p/q is a non-Liouville number with respect to denominators ξ belonging to $\mathcal{B}$.

By using inequality (4.15) one can easily prove, like in Lemma 3.3 that there exists a postive constant C such that

$$\left| e^{i2\pi\xi\frac{p}{q}} - 1 \right| \geq C. \tag{4.16}$$

Then, using inequality (4.16) one can prove (following the proof of Lemma 3.1 in the case where a_0 is a non-Liouville number) that

$$||\hat{v}(\cdot, \xi)||_{L^2(\mathbb{T}^2)} \leq C||\hat{g}(\cdot, \xi)||_{L^2(\mathbb{T}^2)}, \ \forall \ \xi \in \mathcal{B}. \tag{4.17}$$

It follows from (4.13) and (4.17) that there exists a constant $C > 0$ such that

$$||\hat{v}(\cdot, \xi)||_{L^2(\mathbb{T}^2)} \leq C||\hat{g}(\cdot, \xi)||_{L^2(\mathbb{T}^2)}, \ \forall \ \xi \in \mathbb{Z}\backslash\{0\}. \tag{4.18}$$

This completes the proof of Lemma 4.2. □

5 Finite type points and irrational a_0

To complete the proof of Theorem 1.1 we must show that if $a_0(t_1) \equiv \alpha$, where α is a Liouville number, and if there exists a point of finite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t)\partial_x$ then the sublaplacian P is globally hypoelliptic in $\mathbb{T}^3$.

In fact here we shall prove a more general result by allowing α to be any irrational number.

For this we shall use the fact that every real number can be approximated by rational numbers. In particular, the continued fraction representation provides convinient rational approximations which allows us to obtain good estimates for $||\hat{v}(\cdot, \xi)||_{L^2(\mathbb{T}^2)}$.

Recall that for a given $x \in \mathbb{R}$ its integer part $[x]$ is defined to be the unique integer number such that $[x] \leq x < [x] + 1$. Then, define recursively $\alpha_0 =$

x, $a_n = [\alpha_n]$ and if $\alpha_n \notin \mathbb{Z}$, $\alpha_{n+1} = \frac{1}{\alpha_n - a_n}$, for all $n \in \mathbb{N}$. If, for some n, $\alpha_n = a_n$ then we have

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}} \doteq [a_0; a_1, a_2, \cdots, a_n].$$

Otherwise, we denote

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \doteq [a_0; a_1, a_2, \cdots].$$

Let $\alpha = [a_0; a_1, a_2, \cdots]$ and $\frac{p_n}{q_n} = [a_0; a_1, a_2, \cdots, a_n]$, $n \geq 0$ be the nth-order convergent. Now we shall write down some important facts about continued fractions that will be fundamental in the sequence.

For all $n \geq 0$, p_n and q_n satisfy the recurrent formulas $p_{n+2} = a_{n+2}p_{n+1} + p_n$ and $q_{n+2} = a_{n+2}q_{n+1} + q_n$ and $\{q_n\}$ is an increasing sequence of positive integers for $n > 0$, what implies that

$$q_n \to \infty \text{ as } n \to \infty. \tag{5.1}$$

Furthermore, we also have

$$\left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}, \quad \forall n \in \mathbb{N}. \tag{5.2}$$

Now we state the following (more general than we need) result.

Proposition 5.1. *If $a_0(t_1) \equiv \alpha$ with α being an irrational number and there is a point $p \in \mathbb{T}^3$ of finite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t)\partial_x$ then, P is globally hypoelliptic on $\mathbb{T}^3$.*

In order to show, in this case, that P is globally hypoelliptic in $\mathbb{T}^3$ it suffices to prove the following lemma.

Lemma 5.2. *There exists a constant $C > 0$ and a finite subset F of $\mathbb{Z}$ such that*

$$||\hat{v}(\cdot, \xi)||_{L^2(\mathbb{T}^2)} \leq C|\xi|^4 ||\hat{g}(\cdot, \xi)||_{L^2(\mathbb{T}^2)}, \ \forall \ \xi \in \mathbb{Z} \backslash F, \tag{5.3}$$

provided the hypotheses of Proposition 5.1 are fulfilled.

Proof of Lemma 5.2. Let α be the given irrational number. Take its approximation by continued fractions and let $\frac{p_n}{q_n}$ denote the nth-order convergent satisfying (5.1) and (5.2). Also, Let $K > 0$ be a fixed constant to be chosen later. Thanks to (5.1) there is $n_0 \in \mathbb{N}$ such that $q_{n_0} > 8K + 1$. Let $\xi \in \mathbb{Z} \backslash \{0\}$ be such that $|\xi| \geq \sqrt{q_{n_0}}$. Since $\{q_j\}$ increases as j goes to ∞ we have that $\{\sqrt{q_j}\}$ increases too. Thus, there exists some $j_0 \geq n_0$ such that

$$\sqrt{q_{j_0}} \leq |\xi| \leq \sqrt{q_{j_0+1}}. \tag{5.4}$$

Taking partial Fourier transform with respect to x in equation $\tilde{P}v = g$ we obtain

$$-(\partial_{t_1} + ib(t)\xi)^2\hat{v}(t,\xi) - (\partial_{t_2} + i\alpha\xi)^2\hat{v}(t,\xi) = \hat{g}(t,\xi). \tag{5.5}$$

Using the identity

$$(\partial_{t_2} + i\xi\alpha)^2 = (\partial_{t_2} + \frac{p_{j_0}}{q_{j_0}}i\xi)^2 + 2(\alpha - \frac{p_{j_0}}{q_{j_0}})i\xi(\partial_{t_2} + \frac{p_{j_0}}{q_{j_0}}i\xi) - \xi^2(\alpha - \frac{p_{j_0}}{q_{j_0}})^2$$

(5.5) takes the following form

$$\begin{aligned} - & \quad (\partial_{t_1} + ib(t)\xi)^2\hat{v}(t,\xi) - (\partial_{t_2} + \frac{p_{j_0}}{q_{j_0}}i\xi)^2\hat{v}(t,\xi) = \hat{g}(t,\xi) \\ + & \quad 2(\alpha - \frac{p_{j_0}}{q_{j_0}})i\xi(\partial_{t_2} + \frac{p_{j_0}}{q_{j_0}}i\xi)\hat{v}(t,\xi) - \xi^2(\alpha - \frac{p_{j_0}}{q_{j_0}})^2\hat{v}(t,\xi). \end{aligned} \tag{5.6}$$

Therefore, if we multiply (5.6) by $\bar{\hat{v}}$, and integrate by parts with respect to $t \in \mathbb{T}^2$, then we obtain

$$\begin{aligned} & \int_{\mathbb{T}^2} |(\partial_{t_1} + ib(t)\xi)\hat{v}(t,\xi)|^2 dt + \int_{\mathbb{T}^2} |(\partial_{t_2} + i\frac{p_{j_0}}{q_{j_0}}\xi)\hat{v}(t,\xi)|^2 dt \\ = & \int_{\mathbb{T}^2} \hat{g}(t,\xi)\bar{\hat{v}}(t,\xi)dt + 2(\alpha - \frac{p_{j_0}}{q_{j_0}})i\xi \int_{\mathbb{T}^2} (\partial_{t_2}\hat{v}(t,\xi) + \frac{p_{j_0}}{q_{j_0}}i\xi\hat{v}(t,\xi))\bar{\hat{v}}(t,\xi)dt \\ - & \ \xi^2(\alpha - \frac{p_{j_0}}{q_{j_0}})^2 \int_{\mathbb{T}^2} |\hat{v}(t,\xi)|^2 dt. \end{aligned} \tag{5.7}$$

Since both sides of (5.7) are real and $-\xi^2(\alpha - \frac{p_{j_0}}{q_{j_0}})^2 \int_{\mathbb{T}^2} |\hat{v}(t,\xi)|^2 dt \le 0$ it follows from (5.7) that

$$\begin{aligned} & \int_{\mathbb{T}^2} |(\partial_{t_1} + ib(t)\xi)\hat{v}(t,\xi)|^2 dt + \int_{\mathbb{T}^2} |(\partial_{t_2} + i\frac{p_{j_0}}{q_{j_0}}\xi)\hat{v}(t,\xi)|^2 dt \\ \le & \ \|\hat{g}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} \\ +2 & \left|\alpha - \frac{p_{j_0}}{q_{j_0}}\right||\xi| \int_{\mathbb{T}^2} |\partial_{t_2}\hat{v}(t,\xi) + \frac{p_{j_0}}{q_{j_0}}i\xi\hat{v}(t,\xi)||\hat{v}(t,\xi)|dt. \end{aligned}$$

Using (5.2) and (5.4), last inequality gives

$$\begin{aligned}
&\int_{\mathbb{T}^2} |(\partial_{t_1} + ib(t)\xi)\hat{v}(t,\xi)|^2 dt + \int_{\mathbb{T}^2} |(\partial_{t_2} + i\frac{p_{j_0}}{q_{j_0}}\xi)\hat{v}(t,\xi)|^2 dt \\
\leq\ & \|\hat{g}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} \\
+\ & 2\frac{1}{q_{j_0}q_{j_0+1}} q_{j_0+1}^{\frac{1}{2}} \|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}\|(\partial_{t_2} + \frac{p_{j_0}}{q_{j_0}} i\xi)\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} \\
=\ & \|\hat{g}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} \\
+\ & 2\left(\frac{\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}}{\sqrt{4K}q_{j_0}}\right)\left(\frac{1}{q_{j_0+1}^{\frac{1}{2}}}\|(\partial_{t_2} + \frac{p_{j_0}}{q_{j_0}} i\xi)\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}\sqrt{4K}\right) \\
\leq\ & \|\hat{g}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} \\
+\ & \frac{1}{4Kq_{j_0}^2}\|\hat{v}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^2)} + \frac{4K}{q_{j_0+1}}\|(\partial_{t_2} + \frac{p_{j_0}}{q_{j_0}} i\xi)\hat{v}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^2)},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_{\mathbb{T}^2} |(\partial_{t_1} + ib(t)\xi)\hat{v}(t,\xi)|^2 dt + \left(1 - \frac{4K}{q_{j_0+1}}\right)\int_{\mathbb{T}^2} |(\partial_{t_2} + i\frac{p_{j_0}}{q_{j_0}}\xi)\hat{v}(t,\xi)|^2 dt \\
\leq\ & \|\hat{g}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} + \frac{1}{4Kq_{j_0}^2}\|\hat{v}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^2)}.
\end{aligned}$$

Since $\frac{1}{q_{j_0+1}} < \frac{1}{q_{j_0}} < \frac{1}{8K}$ and therefore $1 - \frac{4K}{q_{j_0}+1} > \frac{1}{2}$ it follows from the last inequality that

$$\begin{aligned}
&\int_{\mathbb{T}^2} |(\partial_{t_1} + ib(t)\xi)\hat{v}(t,\xi)|^2 dt + \int_{\mathbb{T}^2} |(\partial_{t_2} + i\frac{p_{j_0}}{q_{j_0}}\xi)\hat{v}(t,\xi)|^2 dt \qquad (5.8)\\
\leq\ & 2\|\hat{g}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} + \frac{1}{2Kq_{j_0}^2}\|\hat{v}(\cdot,\xi)\|^2_{L^2(\mathbb{T}^2)},\ \forall\ \xi \in \mathbb{Z}\backslash\{0\}.
\end{aligned}$$

In order to obtain estimates for $||\hat{v}(\cdot,\xi)||_{L^2(\mathbb{T}^2)}$ we define the following sets: $\mathcal{A}_0 = q_{j_0}\mathbb{Z}$, $\mathcal{B}_0 = \mathbb{Z}\backslash\mathcal{A}_0$ and $\mathcal{C}_0 = \{\xi \in \mathbb{Z} : \sqrt{q_{j_0}} \leq |\xi| \leq \sqrt{q_{j_0+1}}\}$. We split the proof in two cases.

Case 1. $\xi \in \mathcal{A}_0 \cap \mathcal{C}_0$.

Let $\xi \in \mathcal{A}_0 \cap \mathcal{C}_0$. Thus, there exists $\ell \in \mathbb{Z}$ such that $\xi = \ell q_{j_0}$ with $\sqrt{q_{j_0}} \leq |\xi| = |\ell q_{j_0}| \leq \sqrt{q_{j_0+1}}$. Proceeding in a way similar to proving inequality (4.12) in Case 1 of Lemma 4.2, we can show that there exists a contant $C > 0$ independent of ℓ such that (see (4.12)

$$\begin{aligned}
||\hat{v}(\cdot,\ell q_{j_0})||^2_{L^2(\mathbb{T}^2)} \ \leq\ & C||\hat{g}(\cdot,\ell q_{j_0})||^2_{L^2(\mathbb{T}^2)} + C||(\partial_{t_1} + i\ell q_{j_0}b(t))\hat{v}(t,\ell q_{j_0})||^2_{L^2(\mathbb{T}^2)} \\
+\ & C||(\partial_{t_2} + i\ell p_{j_0})\hat{v}(t,\ell q_{j_0})||^2_{L^2(\mathbb{T}^2)}.
\end{aligned}$$

Using (5.8), with $\xi = \ell q_{j_0}$, it follows from the last inequality that there exists a contant $C > 0$ independent of ℓ such that

$$\begin{aligned}
||\hat{v}(\cdot, \ell q_{j_0})||^2_{L^2(\mathbb{T}^2)} &\leq C||\hat{g}(\cdot, \ell q_{j_0})||^2_{L^2(\mathbb{T}^2)} + 2C\|\hat{g}(\cdot, \ell q_{j_0})\|_{L^2(\mathbb{T}^2)}\|\hat{v}(\cdot, \ell q_{j_0})\|_{L^2(\mathbb{T}^2)} \\
&+ \frac{C}{2Kq_{j_0}^2}\|\hat{v}(\cdot, \ell q_{j_0})\|^2_{L^2(\mathbb{T}^2)} \\
&\leq C||\hat{g}(\cdot, \ell q_{j_0})||^2_{L^2(\mathbb{T}^2)} + \frac{C^2}{\epsilon^2}\|\hat{g}(\cdot, \ell q_{j_0})\|^2_{L^2(\mathbb{T}^2)} \\
&+ \epsilon^2\|\hat{v}(\cdot, \ell q_{j_0})\|^2_{L^2(\mathbb{T}^2)} + \frac{C}{2Kq_{j_0}^2}\|\hat{v}(\cdot, \ell q_{j_0})\|^2_{L^2(\mathbb{T}^2)}.
\end{aligned}$$

Taking $\epsilon = \frac{1}{2}$, $K > 4C$ and recalling that $2q_{j_0}^2 \geq 1$ last inequality gives

$$\begin{aligned}
||\hat{v}(\cdot, \ell q_{j_0})||^2_{L^2(\mathbb{T}^2)} &\leq C||\hat{g}(\cdot, \ell q_{j_0})||^2_{L^2(\mathbb{T}^2)} + 4C^2\|\hat{g}(\cdot, \ell q_{j_0})\|^2_{L^2(\mathbb{T}^2)} \\
&+ \frac{1}{2}\|\hat{v}(\cdot, \ell q_{j_0})\|^2_{L^2(\mathbb{T}^2)},
\end{aligned}$$

which implies that

$$||\hat{v}(\cdot, \ell q_{j_0})||_{L^2(\mathbb{T}^2)} \leq C\|\hat{g}(\cdot, \ell q_{j_0})\|_{L^2(\mathbb{T}^2)}. \tag{5.9}$$

Hence Lemma 5.2 is proved in this case.

Case 2. $\xi \in \mathcal{B}_0 \cap \mathcal{C}_0$.

Let $\xi \in \mathcal{B}_0 \cap \mathcal{C}_0$. Since ξ is not a multiple of q_{j_0} we have $|\xi p_{j_0} - kq_{j_0}| \geq 1$ for any $k \in \mathbb{Z}$. Hence,

$$\left|\frac{p_{j_0}}{q_{j_0}} - \frac{k}{\xi}\right| = \frac{1}{|q_{j_0}||\xi|}|\xi p_{j_0} - q_{j_0}k| \geq \frac{1}{|q_{j_0}||\xi|}, \ \ \forall\ k \in \mathbb{Z}. \tag{5.10}$$

Inequality (5.10) says that $\frac{p_{j_0}}{q_{j_0}}$ is a non-Liouville number with respect to denominators belonging to $\mathcal{B}_0$. Therefore we have

$$\left|e^{i2\pi\xi\frac{p_{j_0}}{q_{j_0}}} - 1\right| \geq \frac{1}{q_{j_0}}. \tag{5.11}$$

Using (3.7) with $a_0(t_1)$ replaced by $\frac{p_{j_0}}{q_{j_0}}$ gives

$$|\hat{v}(t, \xi)| \leq \frac{1}{\left|e^{i2\pi\xi\frac{p_{j_0}}{q_{j_0}}} - 1\right|}\int_{\mathbb{T}}\left|\partial_{t_2}\hat{v}(t, \xi) + i\xi\frac{p_{j_0}}{q_{j_0}}\hat{v}(t, \xi)\right| dt_2.$$

From this inequality, using (5.11) and the Cauchy-Schwarz inequality, we obtain

$$|\hat{v}(t, \xi)| \leq \sqrt{2\pi}q_{j_0}\left(\int_{\mathbb{T}}|\partial_{t_2}\hat{v}(t, \xi) + i\xi\frac{p_{j_0}}{q_{j_0}}\hat{v}(t, \xi)|^2 dt_2\right)^{\frac{1}{2}},$$

which implies that

$$|\hat{v}(t,\xi)|^2 \leq 2\pi q_{j_0}^2 \int_{\mathbb{T}} |\partial_{t_2}\hat{v}(t,\xi) + i\xi\frac{p_{j_0}}{q_{j_0}}\hat{v}(t,\xi)|^2 dt_2. \tag{5.12}$$

Integranting (5.12) with respect to $t \in \mathbb{T}^2$ gives

$$||\hat{v}(\cdot,\xi)||_{L^2(\mathbb{T}^2)}^2 \leq 4\pi^2 q_{j_0}^2 \int_{\mathbb{T}^2} |\partial_{t_2}\hat{v}(t,\xi) + i\xi\frac{p_{j_0}}{q_{j_0}}\hat{v}(t,\xi)|^2 dt.$$

Choosing $K > \max\{4C, 4\pi^2\}$ it follows from the last inequality that

$$\begin{aligned} \|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}^2 &\leq K q_{j_0}^2 \int_{\mathbb{T}^2} |(\partial_{t_2} + i\frac{p_{j_0}}{q_{j_0}}\xi)\hat{v}(\cdot,\xi)|^2 dt \\ &\leq K q_{j_0}^2 \int_{\mathbb{T}^2} |(\partial_{t_1} + ib(t)\xi)\hat{v}(t,\xi)|^2 dt \\ &+ K q_{j_0}^2 \int_{\mathbb{T}^2} |(\partial_{t_2} + i\frac{p_{j_0}}{q_{j_0}}\xi)\hat{v}(t,\xi)|^2 dt. \end{aligned} \tag{5.13}$$

From (5.13), using (5.8) and recalling that $\xi \in \mathcal{C}_0$, we obtain

$$\begin{aligned} \|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}^2 &\leq K q_{j_0}^2 \left(2\|\hat{g}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} \|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} + \frac{1}{2Kq_{j_0}^2}\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}^2 \right) \\ &\leq 2K|\xi|^4 \|\hat{g}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} \|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} + \frac{1}{2}\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}^2. \end{aligned}$$

Thus, we obtain

$$\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}^2 \leq 4K|\xi|^4 \|\hat{g}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} \|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)},$$

which gives

$$\|\hat{v}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)} \leq 4K|\xi|^4 \|\hat{g}(\cdot,\xi)\|_{L^2(\mathbb{T}^2)}. \tag{5.14}$$

By taking $F = \{\xi \in \mathbb{Z} : |\xi| \leq \sqrt{q_{n_0}}\}$ where $q_{n_0} > 8K + 1$ with $K > \max\{4C, 4\pi^2\}$ the proof of Lemma 5.2 is now complete. □

6 Analytic and Gevrey regularity

Next we shall describe the analytic and Gevrey version of Theorem 1.1. But first, we recall the needed definitions and facts (also, see [Hi2]).

Let $s \geq 1$. We say that a function $f(x) \in C^\infty(\mathbb{T}^N)$ is in the Gevrey class $G^s(\mathbb{T}^N)$ if there exists a contant $C > 0$ such that $|\partial_x^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s$, for all $\alpha \in \mathbb{Z}_+^N$, $x \in \mathbb{T}^N$. In particular, $G^1(\mathbb{T}^N)$ is the space of all periodic analytic functions, denoted by $C^\omega(\mathbb{T}^N)$. One can prove that $u \in D'(\mathbb{T}^N)$ is in $G^s(\mathbb{T}^N)$ if and only if there exist positive constants ϵ and C such that

$$|\hat{u}(\xi)| \leq Ce^{-\epsilon|\xi|^{1/s}}, \ \forall\, \xi \in \mathbb{Z}^N \backslash \{0\}.$$

A linear partial differential operator P defined on $\mathbb{T}^N$ with coefficients in $C^\omega(\mathbb{T}^N)$ is said to be globally G^s- hypoelliptic in $\mathbb{T}^N$ if the conditions $u \in D'(\mathbb{T}^N)$ and $Pu \in G^s(\mathbb{T}^N)$ imply that $u \in G^s(\mathbb{T}^N)$.

Finally we recall that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is said to be an exponentially non-Liouville number with exponent s if for any $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$|\eta - \alpha\xi| \geq C_\epsilon e^{-\epsilon|\xi|^{1/s}}, \ \forall \ (\eta, \xi) \in \mathbb{Z} \times \mathbb{Z}\backslash\{0\}.$$

Next, we state the analytic and Gevrey version of Theorem 1.1.

Theorem 6.1. *Let $s \geq 1$, and P be given by*

$$P = -\partial_{t_1}^2 - (\partial_{t_2} + a(t_1, t_2)\partial_x)^2,$$

where $(t_1, t_2, x) \in \mathbb{T}^3$ with $a \in C^\omega(\mathbb{T}^2)$ is real-valued. We set

$$a_0(t_1) = \frac{1}{2\pi}\int_{\mathbb{T}} a(t_1, s)ds.$$

Then, P is globallly G^s-hypoelliptic in $\mathbb{T}^3$ if and only if either the range of $a_0(t_1)$ contains an exponentially non-Liouville number with exponent s or there exists a point $p \in \mathbb{T}^3$ of finite type for the vector fields $X_1 = \partial_{t_1}$ and $X_2 = \partial_{t_2} + a(t)\partial_x$.

The proof of Theorem 6.1 is similar to that of Theorem 1.1.

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