Continuity of the Attractors in a Singular Problem Arising in Composite Materials

Vera Lúcia Carbone [1,1]

Departamento de Matemática, Centro de Ciências Exatas e de Tecnologia, Fundação Universidade Federal de São Carlos, Rodovia Washington Luiz km-235, Caixa Postal 676, 13565-905, São Carlos, SP, Brazil

José Gaspar Ruas-Filho

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil

Abstract

In this paper we consider the second order parabolic problem

$$\begin{cases} u^\varepsilon_t = (p_\varepsilon(x)u^\varepsilon_x)_x + c(x)u^\varepsilon + f(u^\varepsilon) \text{ in } (0,1) \\ \frac{\partial u^\varepsilon}{\partial \vec{n}} + b(x)u^\varepsilon = g(u^\varepsilon) \text{ for } x \in \{0,1\} \end{cases} \tag{P$_\varepsilon$}$$

where the diffusion coefficient becomes large in a subset which is interior to the interval $[0,1]$, as $\varepsilon \to 0$. We prove the existence of invariant manifolds for (P_ε), and using the transversality of invariant manifolds of hyperbolic points obtained in [2] we prove that equation (P_ε) and its limit problem are topologically equivalent on the attractors.

Key words: Morse-Smale systems, parabolic problems,topological equivalence.

1 Introduction

Let $\Omega = (0,1)$, ε be a positive parameter, m be a positive integer and $\Omega_0 = \cup_{i=1}^{m} \Omega_{0,i}$ be a subset of Ω, where $\Omega_{0,i} = (a_i, b_i) \subset \Omega$ with $\bar{\Omega}_{0,i} \cap \bar{\Omega}_{0,j} = \emptyset$ for $i \neq j$, and $\bar{\Omega}_0 \subset (0,1)$. We also write $\Omega_1 = \Omega \setminus \bar{\Omega}_0$.

Consider the family of semi-linear parabolic problems

$$\begin{cases} u_t = (p_\varepsilon(x) u_x)_x + c(x)u + f(u), \ x \in (0,1) \\ \frac{\partial u}{\partial \vec{n}} + b(x)u = g(u) \text{ for } x \in \{0,1\} \end{cases} \tag{P$_\varepsilon$}$$

where $\frac{\partial u}{\partial \vec{n}} = p_\varepsilon(x)\langle u_x, \vec{n}\rangle$, and $\vec{n}$ is the outward unit normal vector. The diffusion coefficients p_ε are assumed to be smooth and bounded functions, satisfying $0 < m_0 \leq p_\varepsilon(x) \leq M_\varepsilon$, for every $x \in (0,1)$ and $0 < \varepsilon \leq \varepsilon_0$. We also assume, that they become very large in Ω_0 as ε approaches zero, that is,

$$p_\varepsilon(x) \to \begin{cases} p(x) \text{ uniformly on } \Omega_1 \\ \infty \text{ uniformly on compact subsets of } \ \Omega_0 \end{cases} \tag{1}$$

as $\varepsilon \to 0$. Physically, large diffusion will imply a rapid redistribution of the spatial inhomogeneities and we expect that for small values of ε, the solution of problem (P$_\varepsilon$) will be approximately constant in each $\Omega_{0,i}$, $i = 1, \ldots, m$. Our goal is to compare the flows of equation (P$_\varepsilon$) and the "limit equation" when $\varepsilon \to 0$. To obtain the limit equation, suppose the solution u^ε of (P$_\varepsilon$) converges, in some sense, to some function u and that u takes a time dependent but spatially constant value in $\Omega_{0,i}$ which we denote by $u_{\Omega_{0,i}}(t)$, for each $i =$

Email addresses: carbone@dm.ufscar.br (Vera Lúcia Carbone),
jgrfilho@icmc.usp.br (José Gaspar Ruas-Filho).
[1] Research partially supported by FAPESP under grant 98/14545-7, Brazil.

$1, \ldots, m$. Using this information we guess that the limit equation should be

$$\begin{cases} u_t = (p(x)u_x)_x + c(x)u + f(u) \text{ in } \Omega_1 \\ \dot{u}_{\Omega_{0,i}}(t) = \frac{1}{b_i - a_i}[p(b_i)u_x(b_i^+, t) - p(a_i)u_x(a_i^-, t)] + \hat{c}_i u_{\Omega_{0,i}}(t) + f(u_{\Omega_{0,i}}(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, m \\ \frac{\partial u}{\partial \vec{n}} + b(x)u = g(u) \text{ for } x \in \{0, 1\} \end{cases} \tag{P_0}$$

where $\hat{c}_i = \frac{1}{b_i - a_i} \int_{a_i}^{b_i} c(x)dx$ for each $i = 1, \ldots, m$ and $\frac{\partial u}{\partial \vec{n}} = p(x)\langle u_x, \vec{n} \rangle$. See [2] or [6] for details.

Suppose the nonlinearities f, $g : \mathbb{R} \to \mathbb{R}$ are smooth functions with bounded derivatives, c is a smooth function in Ω, $b(0) > 0$, $b(1) > 0$ and the diffusion coefficient $p_\varepsilon \in \mathrm{C}^1([0, 1])$ satisfies (1) with $p \in \mathrm{C}^1(\Omega_1)$. From the results in [6] we know problem (P_ε) is well posed in $\mathrm{H}^1(0, 1)$ and under some mild conditions we have the existence of global attractors $\mathcal{A}_\varepsilon$. Also from [6], we know that for sufficiently small ε, there exists $\mathcal{N}$ such that

$$\mathcal{A}_\varepsilon \subset \{u \in \mathrm{C}([0, 1]) : \|u\|_\infty \leq \mathcal{N}\}.$$

The asymptotic behavior of the problem (P_ε) is then confined to the set $\{u \in \mathrm{L}^2(0, 1) : \|u\|_{\mathrm{L}^2(0,1)} \leq \mathcal{N}\}$. This allows us to cut the nonlinearities, reducing to the case where the nonlinearities are globally Lipschitz functions. We assume from now on that this cut off is made and will continue to denote the nonlinearities by f and g; see [6] and [5] for details.

Let

$$\mathrm{H}^1_{\Omega_0}(0, 1) = \{u \in \mathrm{H}^1(0, 1),\, u \text{ is constant on } \Omega_{0,i},\, i = 1, \ldots, m\}.$$

With globally Lipschitz nonlinearities, problem (P_0) is well posed in $\mathrm{H}^1_{\Omega_0}(0,1)$ and has an attractor $\mathcal{A}_0$ that lies in a bounded subset of $\mathrm{C}^\alpha([0,1])$ for some positive α, and attracts bounded sets of

$$\mathrm{L}^2_{\Omega_0}(0,1) = \{u \in \mathrm{L}^2(0,1),\, u \,\text{is constant on}\; \Omega_{0,i},\; i = 1, \dots, m\}$$

in the $\mathrm{H}^1(0,1)$ topology.

Let

$$\mathcal{D}(A_\varepsilon) = \{\phi \in \mathrm{H}^1(0,1) : (p_\varepsilon(x)\phi_x)_x \in \mathrm{L}^2(0,1) \text{ and } \frac{\partial \phi}{\partial \vec{n}} + b(x)\phi = 0, \text{ in } \{0,1\}\}$$

and $A_\varepsilon : \mathcal{D}(A_\varepsilon) \subset \mathrm{L}^2(0,1) \to \mathrm{L}^2(0,1)$ be defined by

$$A_\varepsilon \phi = -(p_\varepsilon(x)\phi_x)_x - c(x)\phi.$$

Let also $A_0 : \mathcal{D}(A_0) \subset \mathrm{L}^2_{\Omega_0}(0,1) \to \mathrm{L}^2_{\Omega_0}(0,1)$ where

$$\mathcal{D}(A_0) = \{\phi \in \mathrm{H}^1_{\Omega_0}(0,1) : (p(x)\phi_x)_x \in \mathrm{L}^2_{\Omega_0}(0,1) \text{ and } \frac{\partial \phi}{\partial \vec{n}} + b(x)\phi = 0 \text{ in } \{0,1\}\}$$

be defined by

$$A_0\phi = -[(p(x)\phi_x)_x + c(x)\phi]\chi_{\Omega_1} - \sum_{i=1}^{m}[\frac{1}{b_i - a_i}[p(b_i)\phi_x(b_i^+) - p(a_i)\phi_x(a_i^-)] + \hat{c}_i\phi]\chi_{\Omega_{0,i}}$$

where $\hat{c}_i = \frac{1}{b_i - a_i}\int_{a_i}^{b_i} c(x)dx$. We consider, as Rodríguez-Bernal, Arrieta and Carvalho in [6], (P_0) and (P_ε) as semi-linear problems written in the abstract form as

$$\begin{cases} u_t^\varepsilon = -A_\varepsilon u^\varepsilon + h(u^\varepsilon) \\ u^\varepsilon(0) = u_0^\varepsilon \end{cases} \tag{2}$$

and

$$\begin{cases} u_t = -A_0 u + h_0(u) \\ u(0) = u_0 \end{cases} \tag{3}$$

where the nonlinearities $h : \mathrm{H}^1(0,1) \to \mathrm{H}^{-1}(0,1)$ and $h_0 : \mathrm{H}^1_{\Omega_0}(0,1) \to \mathrm{H}^{-1}_{\Omega_0}(0,1)$ satisfy

$$\langle h(u), \phi \rangle = \int_0^1 f(u(x))\phi(x)dx + g(u(0))\phi(0) + g(u(1))\phi(1),$$

for any test function ϕ, sufficiently regular; see [4] for details. **Rodríguez-Bernal, Arrieta and Carvalho in [6] have also shown, comparing the dynamics of (P_ε) and (P_0), that the family of global attractors $\{\mathcal{A}_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ is upper semicontinuous at $\varepsilon = 0$ relatively to the topologies $\mathrm{H}^1((0,1))$ and $\mathrm{C}^0([0,1])$. When all equilibrium points of problem (P_0) are hyperbolic we have the following much more precise result.**

Theorem 1.1 (Main Theorem) ***Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth, bounded functions with bounded derivatives. Suppose equilibrium points of problem (P_0) are hyperbolic. For ε sufficiently small the flow of (P_ε) on the attractor $\mathcal{A}_\varepsilon$ and the flow of (P_0) on the attractor $\mathcal{A}_0$ are topologically equivalent.***

The paper is organized as follows. In section 2 we obtain the asymptotic behavior of the eigenvalues of the operator A_ε, associated with (P_ε). We then prove the existence of invariant manifolds for the problem (P_ε). Using a result of convergence of the eigenvalues of Rodríguez-Bernal [8] we show the existence of invariant manifolds for the limit problem (P_0).

In section 3, using the convergence of the linear semigroups obtained by Arrieta and Carvalho [1], we show the C^1-proximity of the invariant manifolds. From a previous work [2], we know that the stable and unstable manifolds of the hyperbolic equilibrium points of (P_ε) intersect transversally and we can conclude that the problem (P_ε) is Morse-Smale. The topological equivalence then follows from finite dimensional results.

2 Invariant Manifolds

Let X and Y be Banach spaces, $-A : \mathcal{D}(A) \subset X \to X$ be a sectorial operator such that $\Re\sigma(-A) > 0$ and $B : \mathcal{D}(B) \subset Y \to Y$ be the generator of a C^0-group of bounded linear operators $\{e^{tB} : t \geq 0\}$ on Y. Let $\{e^{-tA} : t \geq 0\}$ be the analytic semigroup of bounded linear operators generated by A and denote by $(-A)^\alpha$ the α-fractional power of $-A$ and $X^\alpha = \mathcal{D}((-A)^\alpha)$ with the graph norm. Consider the system of weakly coupled differential equations

$$\begin{cases} \dot{x} = -Ax + f(x,y) \\ \dot{y} = -By + g(x,y). \end{cases} \tag{4}$$

For the nonlinearities f and g we make the following hypotheses.

(H1) $f : X^\alpha \times Y \to X$ and $g : X^\alpha \times Y \to Y$ are continuous, locally Lipschitz functions and there are positive constants L_f, L_g, N_f and N_g, such that

$$\|f(x,y) - f(z,w)\|_X \leq L_f(\|x-z\|_{X^\alpha} + \|y-w\|_Y),$$
$$\|f(x,y)\|_X \leq N_f,$$
$$\|g(x,y) - g(z,w)\|_Y \leq L_g(\|x-z\|_{X^\alpha} + \|y-w\|_Y)$$
$$\|g(x,y)\|_X \leq N_g,$$

for every (x,y) and (z,w) in $X^\alpha \times Y$.

(H2) $f : X^\alpha \times Y \to X$ and $g : X^\alpha \times Y \to Y$ are smooth functions, satisfying

(H1), and

$$\|f_x(x,y) - f_x(z,w)\|_{\mathcal{L}(X^\alpha,X)} \leq \mathrm{L}_f(\|x-z\|_{X^\alpha} + \|y-w\|_Y),$$

$$\|f_x(x,y)\|_{\mathcal{L}(X^\alpha,X)} \leq N_f, \ \|f_y(x,y)\|_{\mathcal{L}(Y,X)} \leq N_f,$$

$$\|f_y(x,y) - f_y(z,w)\|_{\mathcal{L}(Y,X)} \leq L_f(\|x-y\|_{X^\alpha} + \|y-w\|_Y),$$

$$\|g_x(x,y) - g_x(z,w)\|_{\mathcal{L}(X^\alpha,Y)} \leq \mathrm{L}_g(\|x-z\|_{X^\alpha} + \|y-w\|_Y),$$

$$\|g_y(x,y) - g_y(z,w)\|_{\mathcal{L}(Y,Y)} \leq L_g(\|x-y\|_{X^\alpha} + \|y-w\|_Y),$$

$$\|g_x(x,y)\|_{\mathcal{L}(X^\alpha,Y)} \leq N_g, \ \|g_y(x,y)\|_{\mathcal{L}(Y,Y)} \leq N_g,$$

for every (x, y) and (z, w) in $X^\alpha \times Y$.

Definition 2.1 *A set $\mathcal{S} \subset X^\alpha \times Y$ is an invariant manifold for the differential equation* (4) *if there exists $\sigma : Y \to X^\alpha$ such that $\mathcal{S} = \{(x,y) \in X^\alpha \times Y : x = \sigma(y)\}$ and for any $(x_0, y_0) \in \mathcal{S}$ there exists a solution $(x(\cdot), y(\cdot))$ of the differential equation* (4) *with $x(0) = x_0$ and $y(0) = y_0$ such that $(x(t), y(t)) \in \mathcal{S}$ for any $t \in \mathbb{R}$. The invariant manifold $\mathcal{S}$ is exponentially attracting if there are positive constants γ and k such that*

$$\|x(t) - \sigma(y(t))\|_{X^\alpha} \leq ke^{-\gamma t}\|x(0) - \sigma(y(0))\|_{X^\alpha}$$

for $t \geq 0$, whenever $(x(t), y(t))$ is a solution of the differential equation (4).

The next theorem states the conditions to obtain the existence of an invariant manifold for system (4).

Theorem 2.2 *Let X and Y be Banach spaces, $A : \mathcal{D}(A) \subset X \to X$ be a sectorial operator and $B : \mathcal{D}(B) \subset Y \to Y$ be the generator of a C^0-groups of* *bounded linear operators. Suppose* $f : X^\alpha \times Y \to X$ *and* $g : X^\alpha \times Y \to Y$ *are functions satisfying* **(H1)**.

Assume also, that

(i) $\|e^{-At}w\|_{X^\alpha} \leq M_a e^{-\beta t}\|w\|_{X^\alpha},\ t \geq 0$

(ii) $\|e^{-At}w\|_{X^\alpha} \leq M_a t^{-\alpha} e^{-\beta t}\|w\|_X,\ t > 0$

(iii) $\|e^{-Bt}z\|_Y \leq M_b e^{-\rho t}\|z\|_Y, t \leq 0$

(iv) $\|e^{-Bt}z\|_Y \leq M_b(-t)^{-\alpha} e^{-\rho t}\|z\|_Y, t < 0$

for any $w \in X^\alpha$ *and* $z \in Y$.

If $\beta - \rho$ *is sufficiently large, there exists* $\sigma : Y \to X^\alpha$ *satisfying*

$$\|\sigma(y) - \sigma(z)\|_{X^\alpha} \leq \Delta\|y - z\|_Y,$$

for some $\Delta > 0$ *such that*

$$S = \{(x, y) \in X^\alpha \times Y : \ x = \sigma(y)\}$$

is an exponentially attracting invariant manifold for (4)*. Given* $\varepsilon > 0$*, if* $\beta - \rho$ *is large enough we can choose* $\Delta \leq \varepsilon$ *and*

$$s = \sup_{y \in Y} \|\sigma(y)\|_{X^\alpha} \leq \varepsilon.$$

If f, g *are smooth and* **(H2)** *is satisfied, then* σ *is smooth and its derivative* $D\sigma$ *satisfies*

$$\sup_{y \in Y} \|D\sigma(y)\|_{\mathcal{L}(Y, X^\alpha)} \leq \Delta$$

and

$$\|D\sigma(y) - D\sigma(y')\|_{\mathcal{L}(Y, X^\alpha)} \leq \tilde{\Delta}\|y - y'\|_Y.$$

Again, if $\beta - \rho$ *is large enough we can choose* $\tilde{\Delta}$ *small.*

PROOF. We only give here a sketch of the proof in order to fix the notation. Details of the proof can be found in [3] and [9].

For $D>0$ and $\Delta>0$ given, let $\sigma : Y \to X^\alpha$ such that

$$|||\sigma||| = \sup_{y\in Y} \|\sigma(y)\|_{X^\alpha} \le D, \quad \|\sigma(y)-\sigma(y')\|_{X^\alpha} \le \Delta \|y-y'\|_Y. \tag{5}$$

Let $\psi(t,\eta,\sigma)$ be the solution of

$$\begin{cases} \frac{dy}{dt} = -By + g(\sigma(y),y), \quad \text{for} \quad t<0, \\ y(0)=\eta, \end{cases}$$

and define

$$G(\sigma)(\eta) = \int_{-\infty}^0 e^{As} f(\sigma(\psi(s,\eta,\sigma)),\psi(s,\eta,\sigma))ds.$$

Note that

$$\|G(\sigma)(\cdot)\|_{X^\alpha} \le N_f M_a \int_{-\infty}^0 (-s)^{-\alpha} e^{\beta s} ds.$$

Suppose that σ and σ' are functions satisfying (5) and η, $\eta' \in Y$. Then,

$$\|G(\sigma)(\eta) - G(\sigma')(\eta')\|_{X^\alpha} \le I_\eta \|\eta-\eta'\|_Y + I_\sigma |||\sigma-\sigma'|||,$$

where

$$I_\sigma = M_a L_f \int_{-\infty}^0 (-s)^{-\alpha} e^{\beta s} \left(1 + c_2(1+\Delta) e^{-(\rho+c_\Gamma)s}\right) ds$$

and

$$I_\eta = c_1 M_a L_f (1+\Delta) \int_{-\infty}^{\tau} (\tau-s)^{-\alpha} e^{-[\beta-\rho-c_\Gamma](\tau-s)} ds$$

with c_1, c_2 constants and $c_\Gamma = (M_b L_g (1+\Delta)\Gamma(1-\alpha))^{1/(1-\alpha)}$.

Given $\theta < 1$, if $\beta - \rho$ is large enough then $I_\sigma \leq \theta$ and $I_\eta \leq \Delta$ so that G is a contraction map from the class of functions that satisfy (5) into itself. Therefore, it has a unique fixed point $\sigma^* = G(\sigma^*)$ in this class and $S = \{(\sigma^*(y), y) : y \in Y\}$ is the desired invariant manifold for (4).

If f and g are smooth one can also show that $\frac{d\sigma}{d\eta}$ is the unique fixed point of the application $\tilde{G} : \mathcal{L}(Y, X^\alpha) \to \mathcal{L}(Y, X^\alpha)$ given by

$$\begin{aligned}\tilde{G}(\varphi)(\eta) = &\int_{-\infty}^{0} e^{\hat{A}s} f_y(\sigma(\psi(s,\eta,\sigma)), \psi(s,\eta,\sigma)) \frac{\partial y}{\partial \eta}(s,\eta)\, ds \\ &+ \int_{-\infty}^{0} e^{\hat{A}s} f_x(\sigma(\psi(s,\eta,\sigma)), \psi(s,\eta,\sigma)) \varphi(\psi(s,\eta,\sigma)) \frac{\partial y}{\partial \eta}(s,\eta)\, ds.\end{aligned}$$

and

$$\|\frac{d\sigma}{dy}(y) - \frac{d\sigma}{dy'}(y')\|_{\mathcal{L}(Y,X^\alpha)} \leq \tilde{\Delta}\|y - y'\|_Y,$$

for some $\tilde{\Delta} > 0$. □

We can use this theorem to obtain invariant manifolds for the problems (P_ε) and (P_0) because we have truncated the nonlinearities in such way that they are now globally Lipschitz. The existence of the invariant manifolds for this truncated system ensures the existence of the invariant manifolds for the original system. Furthermore, the attractors for the truncated problems are subsets of the invariants manifolds. See Theorems 3.2 and 3.3 of [7] for details.

The following result from [8] gives the convergence of the eigenfunctions and eigenvalues of the operator A_ε to the eigenfunctions and eigenvalues of the operator A_0, as ε goes to 0.

Theorem 2.3 *Assume the spectrum of A_ε is given by $\lambda_0^\varepsilon < \lambda_1^\varepsilon < \cdots < \lambda_n^\varepsilon < \ldots$, counting multiplicities and, for each n, let ϕ_n^ε be an eigenfunction of λ_n^ε such that $\|\phi_n^\varepsilon\|_{\mathrm{L}^2(0,1)} = 1$ and such that $\{\phi_n^\varepsilon\}_n$ is a Hilbert basis of $\mathrm{L}^2(0,1)$.*

Also, assume that A_0 *has spectrum* $\lambda_0^0 < \lambda_1^0 < \cdots < \lambda_n^0 < \ldots$, *counting multiplicities. Then the following conditions hold.*

(1) For each $n \in \mathbb{N}$,

$$\lambda_n^\varepsilon \to \lambda_n^0, \ \text{as } \varepsilon \to 0.$$

(2) For each $n \in \mathbb{N}$, *and for any sequence converging to zero that we still denote* $\varepsilon \to 0$, *there exists a subsequence* ε_j *such that*

$$\phi_n^{\varepsilon_j} \to \phi_n^0 \in \mathrm{H}^1_{\Omega_0}(0,1), \ \text{as } j \to \infty,$$

strongly in $\mathrm{H}^1(0,1)$ *and* $\lim_{j\to\infty} \int_{\Omega_0} p_{\varepsilon_j} |\phi_n^{\varepsilon_j}|^2 = 0$, *where* ϕ_n^0 *is an eigenfunction of* A_0 *corresponding to* λ_n^0 *and* $\{\phi_n^0\}_n$ *is a Hilbert basis of* $\mathrm{L}^2_{\Omega_0}(0,1)$.

If A_ε and A_0 have spectra as in Theorem 2.3, let

$$W_\varepsilon = [\phi_0^\varepsilon, \ldots, \phi_{n-1}^\varepsilon] \quad \text{and} \quad {W_\varepsilon}^\perp = \{\phi \in \mathrm{L}^2(0,1) : \langle \phi, w \rangle = 0, \ w \in W_\varepsilon\},$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathrm{L}^2(0,1)$. We put

$$A_\varepsilon = \begin{bmatrix} \hat{A}_n^\varepsilon & 0 \\ & \\ 0 & B_n^\varepsilon \end{bmatrix}$$

where $B_n^\varepsilon = A_\varepsilon\big|_{W^\varepsilon}$ and $\hat{A}_n^\varepsilon = A_\varepsilon\big|_{\mathcal{D}(A_\varepsilon)\cap {W_\varepsilon}^\perp}$. Since W_ε is finite dimensional, we can write B_n^ε as a diagonal matrix

$$B_n^\varepsilon = \begin{bmatrix} \lambda_0^\varepsilon & 0 & \cdots & 0 \\ & & & \\ 0 & \lambda_1^\varepsilon & \cdots & 0 \\ & & & \\ \cdots & \cdots & \cdots & \cdots \\ & & & \\ 0 & 0 & \cdots & \lambda_{n-1}^\varepsilon \end{bmatrix}.$$

If $u(x,t)$ a solution of (2), then for each t, we can write

$$u(x,t) = v_0(t)\phi_0^\varepsilon(x) + v_1(t)\phi_1^\varepsilon(x) + \cdots + v_{n-1}(t)\phi_{n-1}^\varepsilon(x) + w(x,t)$$

where $w(x,t) = u(x,t) - \sum_{i=0}^{n-1} v_i(t)\phi_i^\varepsilon(x)$ and $v_i(t) = \int_0^1 u(x,t)\phi_i^\varepsilon(x)dx$, for $i = 0,\ldots,n-1$.

Thus

$$\dot{v}_i(t) = \lambda_i^\varepsilon v_i(t) + \langle h(u^\varepsilon), \phi_i^\varepsilon\rangle,\ \ i = 0,\ldots,n-1$$

and

$$w_t + \hat{A}_n^\varepsilon w = h(u) - \sum_{i=0}^{n-1}\langle h(u), \phi_i^\varepsilon\rangle\phi_i^\varepsilon.$$

Setting $u = (v,w)$ with $v = (v_0,\ldots,v_{n-1})$, equation (2) can be written as

$$\begin{aligned} \dot{v} + B_n^\varepsilon v &= g_n^\varepsilon(v,w) \\ w_t + \hat{A}_n^\varepsilon w &= f_n^\varepsilon(v,w) \end{aligned} \tag{6}$$

with

$$g_n^\varepsilon(v,w) = \left(\langle h(u), \phi_0^\varepsilon\rangle, \ldots, \langle h(u), \phi_{n-1}^\varepsilon\rangle\right)^\top$$

and

$$f_n^\varepsilon(v,w) = h(u) - \sum_{i=0}^{n-1}\langle h(u), \phi_i^\varepsilon\rangle\phi_i^\varepsilon.$$

Let $W_0 = [\phi_0^0,\ldots,\phi_{n-1}^0]$ and, as above, consider the operators $B_n^0 = A_0\big|_{W_0}$ and $\hat{A}_n^0 = A_0\big|_{\mathcal{D}(A_0)\cap {W_0}^\perp}$. Writing $u = (v,w)$ where $v = (v_0,\ldots,v_{n-1})$, and $B_n^0 = \mathrm{diag}(\lambda_0^0, \lambda_1^0, \ldots, \lambda_{n-1}^0)$, equation (3) can be written as the system

$$\begin{aligned} \dot{v} + B_n^0 v &= g_n^0(v,z) \\ z_t + \hat{A}_n^0 w &= f_n^0(v,z) \end{aligned} \tag{7}$$

where

$$g_n^0(v,w) = (\langle h_0(u), \phi_0^0\rangle, \ldots, \langle h(u), \phi_{n-1}^0\rangle)^\top$$

and

$$f_n^0(v,w) = h_0(u) - \sum_{i=0}^{n-1} \langle h(u), \phi_i^0\rangle \phi_i^0.$$

Observe that if the functions $f, g : \mathbb{R} \to \mathbb{R}$ are bounded and Lipschitzian then the functions f_n^ε, g_n^ε, f_n^0 and g_n^0 satisfy the hypothesis **(H1)** with $X^\alpha = {W_\varepsilon}^\perp$, $X^\alpha = {W_0}^\perp$ and $Y = \mathbb{R}^n$ respectively. If f and g are bounded and smooth with bounded derivatives then f_n^ε, g_n^ε, f_n^0 and g_n^0 satisfy **(H2)**.

2.1 Asymptotic Behavior of the Eigenvalues of A_ε

To show that the hypotheses of Theorem 2.2 above are satisfied by system (6) we study the asymptotic behavior of the eigenvalues of the operator A_ε.

Proposition 2.4 *The eigenvalues λ_n^ε of the operator A_ε satisfy*

$$\lambda_n^\varepsilon = \frac{1}{l^2} n^2 \pi^2 + \mathrm{o}(n),$$

as $n \to +\infty$, where $l = \int_0^1 p_\varepsilon(s)^{-1/2} ds$.

PROOF. If ϱ^ε is an eigenvalue of $-A_\varepsilon$ then

$$\begin{cases} (p_\varepsilon(x) u_x(x))_x + c(x)u(x) - \varrho^\varepsilon u(x) = 0, \ x \in (0,1) \\ \frac{\partial u}{\partial \vec{n}} + b(x) u = 0, \ x \in \{0,1\}. \end{cases} \tag{8}$$

If u is a solution of the (8) and

$$v(t) = p_\varepsilon(x)^{1/4} u(x), \quad t = \int_0^x p_\varepsilon(s)^{-1/2} ds, \quad l = \int_0^1 p_\varepsilon(s)^{-1/2} ds,$$

then

$$\begin{cases} v''(t) - r_\varepsilon(t)v(t) - \varrho^\varepsilon v(t) = 0, \quad 0 < t < l \\ v'(t) + \tilde{b}_\varepsilon(t)v(t) = 0, \quad t \in \{0, l\} \end{cases}$$

where $r_\varepsilon(t) = \frac{\eta_\varepsilon''(t)}{\eta_\varepsilon(t)} - c(t)$, $\eta_\varepsilon(t) = p_\varepsilon(t)^{1/4}$, $\tilde{b}_1 = \tilde{b}_\varepsilon(l) = [b(1) - \frac{1}{4}p_\varepsilon'(1)]p_\varepsilon(1)^{-1/2}$ and $\tilde{b}_0 = -\tilde{b}_\varepsilon(0) = [\frac{1}{4}p_\varepsilon'(0) + b(0)]p_\varepsilon(0)^{-1/2}$.

A characterization of the eigenvalues ϱ^ε is obtained considering the bilinear form $\tau : \mathrm{L}^2(0,l) \times \mathrm{L}^2(0,l) \to \mathbb{R}$,

$$\tau(u,v) = -\left[\int_0^l u_x v_x + \int_0^l r_\varepsilon uv + \tilde{b}_0 u(0)v(0) + \tilde{b}_1 u(l)v(l)\right].$$

If $v \in \mathrm{L}^2(0,1)$ and $\|v\|_{\mathrm{L}^2(0,1)} = 1$, then

$$\left|\tau(v,v) + \int_0^l v_x^2 dx + b(0)v(0)^2 + b(l)v(l)^2\right| \leq \max_{0\leq x\leq l} |r_\varepsilon(x)| + |b(0) - \tilde{b}_0| + |b(l) - \tilde{b}_1|.$$

Therefore $\varrho_n^\varepsilon = \mu_n^\varepsilon + \mathrm{O}(1)$, as $n \to +\infty$, where μ_n^ε are the eigenvalues of the problem

$$\phi'' = \mu\phi \quad \text{such that} \quad \phi'(0) = b_0\phi(0), \quad \phi'(l) = -b_1\phi(l)$$

with $b_0 = b(0) > 0$ and $b_1 = b(l) > 0$.

Letting $\mu = -\omega^2 < 0$, from the conditions in 0 and l we get

$$(b_0 b_1 - \omega^2)\sin(\omega l) = -\omega(b_0 + b_1)\cos(\omega l).$$

Since b_0 and b_1 are both positive we can not have $\omega l = n\pi$, and if $\omega l \neq n\pi$, $n \in \mathbb{Z}^*$, we put $\xi = \omega l$, $\alpha = \frac{1}{(b_0+b_1)l}$ and $\beta = \frac{b_0 b_1 l}{b_0 + b_1}$. Then

$$\cot(\xi) = \alpha\xi - \beta\frac{1}{\xi}.$$

For each $n = 0, 1, 2, \dots$ there exists $\xi_n \in (n\pi, (n+1)\pi)$, with $\xi_{n+1} < \xi_n + \pi$, such that $\cot(\xi_n) = \alpha\xi_n - \beta\frac{1}{\xi_n}$. We let $\theta_n = \xi_n - n\pi$ so that $\theta_n \in (0, \pi)$ and $\theta_{n+1} = \xi_{n+1} - (n+1)\pi < \xi_n - n\pi = \theta_n$. Let $\bar{\theta} = \lim_{n\to+\infty}\theta_n$. From

$$\cot(\theta_n) = \cot(n\pi + \theta_n) = \alpha(n\pi + \theta_n) - \beta\frac{1}{n\pi + \theta_n}$$

we obtain

$$\lim_{n\to+\infty} \frac{1}{n\pi} \cot(\theta_n) = \alpha \neq 0.$$

Then $\bar{\theta} = 0$ and $\xi_n = n\pi + \mathrm{o}(1)$, as $n \to +\infty$; that is, $\varrho_n^\varepsilon = -\frac{1}{l^2}n^2\pi^2 + \mathrm{o}(n)$, as $n \to +\infty$. $\square$

We can now use this proposition and Theorem 2.2 to get the following result for systems (6) and (7).

Theorem 2.5 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be bounded and smooth with bounded derivatives. If n is sufficiently large there exist smooth exponentially attracting invariant manifolds*

$$S_n^\varepsilon = \{(v, y);\, v = \sigma_n^\varepsilon(y),\, y \in \mathbb{R}^n\}$$

for (6) *and*

$$S_n^0 = \{(v, y);\, v = \sigma_n^0(y),\, y \in \mathbb{R}^n\}$$

for (7)*. The flow on S_n^ε is given by $(\sigma_n^\varepsilon(y(t)), y(t))$ where y is the solution of*

$$\dot{y} + B_n^\varepsilon y = g_n^\varepsilon(\sigma_n^\varepsilon(y), y)$$

and the flow on S_n^0 is given by $(\sigma_n^0(y(t)), y(t))$ where y is the solution of

$$\dot{y} + B_n^0 y = g_n^0(\sigma_n^0(y), y).$$

Furthermore, if $s_\varepsilon(n) = \sup_{v\in W_\varepsilon} \|\sigma_n^\varepsilon(v)\|$, $s_0(n) = \sup_{v\in W_0} \|\sigma_n^0(v)\|$ *and* $\Delta_0(n)$ *and* $\Delta_\varepsilon(n)$ *are the Lipschitz constants of* σ_n^0 *and* σ_n^ε*, respectively, then* $s_\varepsilon(n)$*,* $s_0(n)$*,* $\Delta_0(n)$ *and* $\Delta_\varepsilon(n)$ *all go to* 0 *as* $n \to +\infty$*.*

PROOF. If we let $\beta^\varepsilon(n) = \lambda_n^\varepsilon$ and $\rho^\varepsilon(n) = \lambda_{n-1}^\varepsilon$ and let $L_f = L_g = L_h$ where L_h is a Lipschitz constant of h and $N_f = N_g = N_h$ where N_h is an upper bound for $|h|$ then from Theorem 2.2 there exists K_0 such that if

$$\beta^\varepsilon(n) - \rho^\varepsilon(n) \geq K_0$$

then

$$S_n^\varepsilon = \{(v,y);\, v = \sigma_n^\varepsilon(y),\, y \in \mathbb{R}^n\}$$

is a smooth exponentially attracting invariant manifold for (6).

Now, if we let $\beta^0(n) = -\lambda_n^0$ and $\rho^0(n) = -\lambda_{n-1}^0$ and $L_f = L_g = L_{h_0}$ where L_{h_0} is a Lipschitz constant of h_0 and $N_f = N_g = N_{h_0}$ where N_{h_0} is an upper bound for $|h_0|$ then, again from Theorem 2.2, there exists K_1 such that if

$$\beta^0(n) - \rho^0(n) \geq K_1$$

then

$$S_n^0 = \{(v,y);\, v = \sigma_n^0(y),\, y \in \mathbb{R}^n\}$$

is a smooth exponentially attracting invariant manifold for (7).

From Proposition 2.4 we have

$$\beta^\varepsilon(n) - \rho^\varepsilon(n) = 2n\frac{\pi^2}{l^2} + \mathrm{o}(n) \to +\infty \text{ as } n \to +\infty.$$

Using Theorem 2.3 choose ε_0 such that $|\lambda_n^\varepsilon - \lambda_n^0| \leq 1$ for $\varepsilon \leq \varepsilon_0$. Choose N_0 such that $\beta^\varepsilon(n) - \rho^\varepsilon(n) \geq \max\{K_0, K_1\} + 2$ for $n \geq N_0$, and the result follows from Theorem 2.2. □

3 Proximity of the Invariant Manifolds

In this section we fix $X = W_\varepsilon{}^\perp$ and n sufficiently large so that the conclusions of Theorem 2.5 are valid. We show that the invariant manifolds S_n^ε and S_n^0 obtained in that theorem are C^1-close. To simplify the notation, from now on, we omit the indices n. We first show they are C^0-close.

3.1 C^0*-Proximity of the Invariant Manifolds*

From the proof of Theorem 2.2 it follows that the invariant manifolds S^ε of (6) and S^0 of (7) are graphs of C^1-functions $\sigma^\varepsilon : \mathbb{R}^n \to X$ and $\sigma^0 : \mathbb{R}^n \to X$, fixed points of

$$G_\varepsilon(\sigma)(\eta) = \int_{-\infty}^{0} e^{\hat{A}_\varepsilon s} f^\varepsilon(\sigma(y_\varepsilon(s,\eta)), y_\varepsilon(s,\eta))ds$$

and

$$G_0(\sigma)(\eta) = \int_{-\infty}^{0} e^{\hat{A}_0 s} f^0(\sigma(y_0(s,\eta)), y_0(s,\eta))ds,$$

respectively.

If $L_\varepsilon < 1$ is the contraction constant of G_ε, then it is easy see that

$$\sup_{\eta\in\mathbb{R}^n} \|\sigma^\varepsilon(\eta) - \sigma^0(\eta)\|_X \leq \frac{1}{1-L_\varepsilon} \sup_{\eta\in\mathbb{R}^n} \|G_\varepsilon(\sigma^0)(\eta) - G_0(\sigma^0)(\eta)\|_{\mathrm{L}^2},$$

and to obtain the C^0-proximity of the invariant manifolds it is then sufficient to estimate $\|G_\varepsilon(\sigma^0)(\eta) - G_0(\sigma^0)(\eta)\|_{\mathrm{L}^2}$.

The following result from [1] is used to obtain such estimate.

Lemma 3.1 *Define the operators* $\hat{A}_\varepsilon = A_\varepsilon\Big|_{\mathcal{D}(A_\varepsilon)\cap W_\varepsilon{}^\perp}$ *and* $\hat{A}_0 = A_0\Big|_{\mathcal{D}(A_0)\cap W_0{}^\perp}$. *If* $\beta \in (0,1)$ *and* $\lambda_n^0 < -a < 0$ *then*

$$\|e^{-\hat{A}_\varepsilon t}u - e^{-\hat{A}_0 t}u\|_{\mathrm{L}^2} \leq \theta(\varepsilon)t^{-\beta}e^{-at}\|u\|_{\mathrm{L}^2},\ t>0,\ u \in \mathrm{L}^2(0,1)$$

where $\theta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

PROOF. Choose a constant c, independent of ε, such that

$$\|e^{-\hat{A}_\varepsilon t}u\|_{\mathrm{L}^2} \leq ce^{-at}\|u\|_{\mathrm{L}^2}, \quad \text{and} \quad \|e^{-\hat{A}_0 t}u\|_{\mathrm{L}^2} \leq ce^{-at}\|u\|_{\mathrm{L}^2},\ t \geq 0.$$

Let $\eta > 0$ be a small parameter. If $t \in (0,\eta]$,

$$\begin{aligned}\|e^{-\hat{A}_\varepsilon t}u - e^{-\hat{A}_0 t}u\|_{\mathrm{L}^2} &\leq \|e^{-\hat{A}_\varepsilon t}u\|_{\mathrm{L}^2} + \|e^{-\hat{A}_0 t}u\|_{\mathrm{L}^2} \\ &\leq 2c\eta^\beta t^{-\beta}e^{-at}\|u\|_{\mathrm{L}^2}.\end{aligned}$$

If $t > \eta$ we can choose a $l = l(\eta) > 0$ such that if $z \geq l$ then $e^{-2zt} \leq \eta t^{-\beta}e^{-at}$ for all $t \geq \eta$. Since $\lambda_n^\varepsilon \to \lambda_n^0$ as $\varepsilon \to 0$ and $-\lambda_n^0 \to +\infty$ as $n \to +\infty$, there exists $N = N(\eta)$ such that $-\lambda_n^\varepsilon \geq l(\eta)$, for $\varepsilon \in (0,\varepsilon_0)$ and $n \geq N(\eta)$.

From the spectral decomposition of the linear semigroups, we obtain

$$\begin{aligned}\|e^{-\hat{A}_\varepsilon t}u - e^{-\hat{A}_0 t}u\|_{\mathrm{L}^2} &\leq \|\sum_{k=1}^{N(\eta)} e^{\lambda_k^\varepsilon t}\langle u,\phi_k^\varepsilon\rangle\phi_k^\varepsilon - \sum_{k=1}^{N(\eta)} e^{\lambda_k^0 t}\langle u,\phi_k^0\rangle\phi_k^0\|_{\mathrm{L}^2} \\ &+ \|\sum_{k=N(\eta)+1}^{\infty} e^{\lambda_k^\varepsilon t}\langle u,\phi_k^\varepsilon\rangle\phi_k^\varepsilon\|_{\mathrm{L}^2} + \|\sum_{k=N(\eta)+1}^{\infty} e^{\lambda_k^0 t}\langle u,\phi_k^0\rangle\phi_k^0\|_{\mathrm{L}^2} \\ &= S_1 + S_2 + S_3.\end{aligned}$$

We have,

$$S_2 \leq \sum_{k=N(\eta)+1}^{\infty} e^{2\lambda_k^\varepsilon t}|\langle u,\phi_k^\varepsilon\rangle|^2 \leq \eta t^{-\beta}e^{-at}\|u\|_{\mathrm{L}^2}, \tag{9}$$

$$S_3 \leq \sum_{k=N(\eta)+1}^{\infty} e^{2\lambda_k^0 t}|\langle u,\phi_k^0\rangle|^2 \leq \eta t^{-\beta}e^{-at}\|u\|_{\mathrm{L}^2}, \tag{10}$$

and,

$$S_1 \leq \sum_{k=1}^{N(\eta)} |e^{\lambda_k^\varepsilon t} - e^{\lambda_k^0 t}|\,|\langle u, \phi_k^\varepsilon \rangle| + \sum_{k=1}^{N(\eta)} e^{\lambda_k^0 t} \|\langle u, \phi_k^\varepsilon \rangle \phi_k^\varepsilon - \langle u, \phi_k^0 \rangle \phi_k^0 \|.$$

For each $k = 1, \ldots, N(\eta)$ there exists $\xi_k \in (\lambda_k^\varepsilon, \lambda_k^0)$ such that $|e^{\lambda_k^\varepsilon t} - e^{\lambda_k^0 t}| \leq te^{\xi_k t}|\lambda_k^\varepsilon - \lambda_k^0|$. Then

$$\begin{aligned} S_1 &\leq \sum_{k=1}^{N(\eta)} te^{\lambda_k^0 t}|\lambda_k^\varepsilon - \lambda_k^0|\, \|u\|_{\mathrm{L}^2} + \sum_{k=1}^{N(\eta)} 2e^{\lambda_k^0 t} \|\phi_k^\varepsilon - \phi_k^0\|\, \|u\|_{\mathrm{L}^2} \\ &\leq \eta t^{-\beta} e^{-at} \left(\sum_{k=1}^{N(\eta)} |\lambda_k^\varepsilon - \lambda_k^0| + \sum_{k=1}^{N(\eta)} \|\phi_k^\varepsilon - \phi_k^0\| \right) \|u\|_{\mathrm{L}^2}. \end{aligned} \tag{11}$$

From (9), (10) and (11)

$$\|e^{\hat{A}_\varepsilon t}u - e^{\hat{A}_0 t}u\|_{\mathrm{L}^2} \leq \eta t^{-\beta} e^{-at} \tilde{\theta}(\varepsilon)\|u\|_{\mathrm{L}^2} + 2\eta t^{-\beta} e^{-at}\|u\|_{\mathrm{L}^2},\ t > \eta$$

where $\tilde{\theta}(\varepsilon) = \sum_{k=1}^{N(\eta)} |\lambda_k^\varepsilon - \lambda_k^0| + \sum_{k=1}^{N(\eta)} \|\phi_k^\varepsilon - \phi_k^0\|$. Thus,

$$\|e^{-\hat{A}_\varepsilon t}u - e^{-\hat{A}_0 t}u\|_{\mathrm{L}^2} \leq \theta(\varepsilon) t^{-\beta} e^{-at}\|u\|_{\mathrm{L}^2},\ t > 0$$

where $\theta(\varepsilon) = \max\{2c\eta^\beta, 2\eta + \tilde{\theta}(\varepsilon)\} \to 0$ as $\varepsilon \to 0$. $\square$

To simplify the notation, define

$$\lambda(\varepsilon) = \sum_{j=0}^{n-1} |\lambda_j^\varepsilon - \lambda_j^0|$$

and

$$\varphi(\varepsilon) = \sum_{j=0}^{n-1} \|\phi_j^\varepsilon - \phi_j^0\|.$$

Observe that by Theorem 2.3 we have $\lambda(\varepsilon) \to 0$ and $\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Theorem 3.2 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be bounded and smooth with bounded derivatives.*

Then

$$\sup_{\eta\in\mathbb{R}^n}\|\sigma^\varepsilon(\eta)-\sigma^0(\eta)\|\to 0 \quad as \quad \varepsilon\to 0.$$

PROOF. From the proof of the Theorem 2.2 we know that

$$\|G_\varepsilon(\sigma^0)(\eta)-G_0(\sigma^0)(\eta)\|_{\mathrm{L}^2}\leq N_f\int_{-\infty}^0\|e^{\hat{A}_\varepsilon s}-e^{\hat{A}_0 s}\|_{\mathcal{L}(L^2)}ds$$
$$+\int_{-\infty}^0\|e^{\hat{A}_0 s}\|_{\mathcal{L}(\mathrm{L}^2)}\|f^\varepsilon(\sigma^0(y_\varepsilon(s)),y_\varepsilon(s))-f^0(\sigma^0(y_0(s)),y_0(s))\|_{\mathrm{L}^2}\,ds. \quad (12)$$

From Lemma 3.1, it follows that

$$\int_{-\infty}^0\|e^{\hat{A}_\varepsilon s}-e^{\hat{A}_0 s}\|_{\mathcal{L}(L^2)}ds\leq\theta(\varepsilon)\int_{-\infty}^0(-s)^{-\beta}e^{as}ds=a^{\beta-1}\Gamma(1-\beta)\theta(\varepsilon), \quad (13)$$

where $\beta\in(0,1)$, $\lambda_n^0<-a<0$ and $\theta(\varepsilon)\to 0$ as $\varepsilon\to 0$. The second integral in (12) satisfies

$$\int_{-\infty}^0\|e^{\hat{A}_0 s}\|_{\mathcal{L}(\mathrm{L}^2)}\,\|f^\varepsilon(\sigma^0(y_\varepsilon(s)),y_\varepsilon(s))-f^0(\sigma^0(y_0(s)),y_0(s))\|_{\mathrm{L}^2}\,ds$$
$$\leq c_0L_f\int_{-\infty}^0 e^{\lambda_n^0 s}[\|\sigma^0(y_\varepsilon(s))-\sigma^0(y_0(s))\|_X+\|y_\varepsilon(s)-y_0(s)\|_{\mathbb{R}^n}]ds$$
$$+c_0\int_{-\infty}^0 e^{\lambda_n^0 s}\|f^\varepsilon(\sigma^0(y_0(s)),y_0(s))-f^0(\sigma^0(y_0(s)),y_0(s))\|_{\mathrm{L}^2}\,ds$$
$$\leq c_0L_f(1+\Delta_0)\int_{-\infty}^0 e^{\lambda_n^0 s}\|y_\varepsilon(s)-y_0(s)\|_{\mathbb{R}^n}ds+\frac{2N_hc_0}{-\lambda_n^0}\varphi(\varepsilon) \quad (14)$$

where c_0 is a constant and N_h is a upper bounded of h.

The variation of constants formula imply

$$\|y_\varepsilon(t)-y_0(t)\|_{\mathbb{R}^n}\leq$$
$$M\sum_{j=0}^{n-1}|e^{-\lambda_j^\varepsilon t}-e^{-\lambda_j^0 t}|\,\|\eta\|_{\mathbb{R}^n}+MN_g\int_t^0\sum_{j=0}^{n-1}|e^{-\lambda_j^\varepsilon(t-s)}-e^{-\lambda_j^0(t-s)}|ds$$
$$+M\int_t^0 e^{-\lambda_{n-1}^0(t-s)}\|g^\varepsilon(\sigma^0(y_0(s)),y_0(s))-g^0(\sigma^0(y_0(s)),y_0(s))\|_{\mathrm{L}^2}\,ds$$
$$+ML_g(1+\Delta_0)\int_t^0 e^{-\lambda_{n-1}^0(t-s)}\|y_\varepsilon(s)-y_0(s)\|_{\mathbb{R}^n}ds,$$

where $M>0$ is constant.

For each $j = 0, \ldots, n-1$ there exists ξ_j^ε between $-\lambda_j^\varepsilon$ and $-\lambda_j^0$ such that

$$|e^{-\lambda_j^\varepsilon t} - e^{-\lambda_j^0 t}| \leq |t| e^{\xi_j^\varepsilon t} |\lambda_j^\varepsilon - \lambda_j^0|.$$

Let $\delta > 0$ such that $-\lambda_{n-1}^0 - \delta < 0 < -\lambda_{n-1}^0 + \delta/2$. We know from Theorem 2.3, that $-\lambda_j^\varepsilon \to -\lambda_j^0$ as $\varepsilon \to 0$ and thus there exists $\varepsilon_1 > 0$ such that

$$\xi_j^\varepsilon \in (-\lambda_j^0 - \delta/2, -\lambda_j^0 + \delta/2),$$

for each $j = 0, \ldots, n-1$ and $0 < \varepsilon < \varepsilon_1$. If $\varepsilon \in (0, \varepsilon_1)$ and $t < 0$ then

$$\sum_{j=0}^{n-1} |e^{-\lambda_j^\varepsilon t} - e^{-\lambda_j^0 t}| \leq c_1 e^{(-\lambda_{n-1}^0 - \delta)t} \lambda(\varepsilon), \tag{15}$$

where c_1 is a constant.Therefore

$$\|y_\varepsilon(t) - y_0(t)\|_{\mathbb{R}^n} \leq c_1 M \|\eta\|_{\mathbb{R}^n} \lambda(\varepsilon) e^{(-\lambda_{n-1}^0 - \delta)t} + \frac{c_1 M N_g}{\delta + \lambda_{n-1}^0} \lambda(\varepsilon) e^{(-\lambda_{n-1}^0 - \delta)t}$$
$$+ \frac{M N_h}{\delta + \lambda_{n-1}^0} \varphi(\varepsilon) e^{(-\lambda_{n-1}^0 - \delta)t} + M L_g (1 + \Delta_0) \int_t^0 e^{-\lambda_{n-1}^0 (t-s)} \|y_\varepsilon(s) - y_0(s)\| ds.$$

From the generalized Gronwall's inequality (see [9], for example) we obtain

$$\|y_\varepsilon(t, \eta) - y_0(t, \eta)\| \leq a(\varepsilon) e^{(-\lambda_{n-1}^0 - \delta - M L_g (1 + \Delta_0))t}, \quad t < 0 \tag{16}$$

where $a(\varepsilon) = c_1 M \left(\|\eta\|_{\mathbb{R}^n} + \frac{N_g}{\delta + \lambda_{n-1}^0} \right) \lambda(\varepsilon) + \frac{M N_h}{\delta + \lambda_{n-1}^0} \varphi(\varepsilon)$.

Substituting (16) in (14) we obtain

$$\int_{-\infty}^0 \|e^{\hat{A}_0 s}\|_{\mathcal{L}(\mathrm{L}^2)} \|f^\varepsilon(\sigma^0(y_\varepsilon(s)), y_\varepsilon(s)) - f^0(\sigma^0(y_0(s)), y_0(s))\|_{\mathrm{L}^2} \, ds \leq$$
$$\frac{c_0 L_f (1 + \Delta_0)}{\lambda_n^0 - \lambda_{n-1}^0 - \delta - M L_g (1 + \Delta_0)} a(\varepsilon) + \frac{2 N_h c_0}{-\lambda_n^0} \varphi(\varepsilon). \tag{17}$$

Putting together the estimates in (13) and (17) we obtain

$$\|G_\varepsilon(\sigma^0)(\eta) - G_0(\sigma^0)(\eta)\| \leq N_f a^{\beta - 1} \Gamma(1 - \beta) \theta(\varepsilon)$$
$$+ \frac{L_f (1 + \Delta_0) c_0}{\lambda_n^0 - \lambda_{n-1}^0 - \delta - M L_g (1 + \Delta_0)} a(\varepsilon) + \frac{2 N_h c_0}{-\lambda_n^0} \varphi(\varepsilon).$$

Then $\|G_\varepsilon(\sigma^0)(\eta) - G_0(\sigma^0)(\eta)\| \to 0$ as $\varepsilon \to 0$ and the proof is complete. $\square$

3.2 C^1*-Proximity of the Invariant Manifolds*

From the proof of Theorem 2.2 we know that $\frac{d\sigma^\varepsilon}{d\eta}(\eta)$ is the unique fixed point of the operator $\tilde{G}_\varepsilon : \mathcal{L}(\mathbb{R}^n, X) \to \mathcal{L}(\mathbb{R}^n, X)$ given by

$$\begin{aligned}\tilde{G}_\varepsilon(\psi)(\eta) = \int_{-\infty}^0 & e^{\hat{A}_\varepsilon s} f_y^\varepsilon(\sigma^\varepsilon(y_\varepsilon(s,\eta)), y_\varepsilon(s,\eta)) \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta) ds \\ & + \int_{-\infty}^0 e^{\hat{A}_\varepsilon s} f_x^\varepsilon(\sigma^\varepsilon(y_\varepsilon(s,\eta)), y_\varepsilon(s,\eta)) \psi(y_\varepsilon(s,\eta)) \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\, ds,\end{aligned}$$

and $\frac{d\sigma^0}{d\eta}(\eta)$ is the unique fixed point of the operator $\tilde{G}_0 : \mathcal{L}(\mathbb{R}^n, X) \to \mathcal{L}(\mathbb{R}^n, X)$ given by

$$\begin{aligned}\tilde{G}_0(\psi)(\eta) = \int_{-\infty}^0 & e^{\hat{A}_0 s} f_y^0(\sigma^0(y_0(s,\eta)), y_0(s,\eta)) \frac{\partial y_0}{\partial \eta}(s,\eta) ds \\ & + \int_{-\infty}^0 e^{\hat{A}_0 s} f_x^0(\sigma^0(y_0(s,\eta)), y_0(s,\eta)) \psi(y_0(s,\eta)) \frac{\partial y_0}{\partial \eta}(s,\eta)\, ds.\end{aligned}$$

Let $\tilde{\Delta}$ and $\tilde{\Delta}_0$ be the Lipschitz constants of $\frac{d\sigma^\varepsilon}{d\eta}(\eta)$and $\frac{d\sigma^0}{d\eta}(\eta)$, respectively.

Theorem 3.3 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be bounded and smooth with bounded derivatives. Then,*

$$\sup_{\eta \in \mathbb{R}^n} \| \frac{d\sigma^\varepsilon}{d\eta}(\eta) - \frac{d\sigma^0}{d\eta}(\eta) \|_{\mathcal{L}(\mathbb{R}^n, X)} \to 0 \quad as \quad \varepsilon \to 0.$$

PROOF. As in the case of C^0-proximity, to obtain the C^1-proximity of the invariant manifolds it is enough to compare $\tilde{G}_\varepsilon(\frac{d\sigma^0}{d\eta})$ and $\tilde{G}_0(\frac{d\sigma^0}{d\eta})$. We have

$$\begin{aligned}
&\|\tilde{G}_\varepsilon(\frac{d\sigma^0}{d\eta})(\eta) - \tilde{G}_0(\frac{d\sigma^0}{d\eta})(\eta)\|_{\mathcal{L}(\mathbb{R}^n,X)} \leq \\
&\qquad \int_{-\infty}^0 \Big\| e^{\hat{A}_\varepsilon s} f_y^\varepsilon(\sigma^\varepsilon(y_\varepsilon(s,\eta)), y_\varepsilon(s,\eta)) \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta) \\
&\qquad\qquad - e^{\hat{A}_0 s} f_y^0(\sigma^0(y_0(s,\eta)), y_0(s,\eta)) \frac{\partial y_0}{\partial \eta}(s,\eta)\Big\| ds \\
&\quad + \int_{-\infty}^0 \| e^{\hat{A}_\varepsilon s} f_x^\varepsilon(\sigma^\varepsilon(y_\varepsilon(s,\eta)), y_\varepsilon(s,\eta)) \frac{d\sigma^0}{d\eta}(y_\varepsilon(s,\eta)) \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta) \\
&\qquad\qquad - e^{\hat{A}_0 s} f_x^0(\sigma^0(y_0(s,\eta)), y_0(s,\eta)) \frac{d\sigma^0}{d\eta}(y_0(s,\eta)) \frac{\partial y_0}{\partial \eta}(s,\eta) \| ds \\
&= I_1 + I_2. \qquad (18)
\end{aligned}$$

We first estimate I_1.

$$\begin{aligned}
I_1 \leq & \int_{-\infty}^0 \| e^{\hat{A}_\varepsilon s} - e^{\hat{A}_0 s}\| \, \| f_y^\varepsilon(\sigma^\varepsilon(y_\varepsilon(s,\eta)), y_\varepsilon(s,\eta)) \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\| ds \\
&+ \int_{-\infty}^0 \| e^{\hat{A}_0 s}\| \| f_y^\varepsilon(\sigma^\varepsilon(y_\varepsilon(s,\eta)), y_\varepsilon(s,\eta)) \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta) \\
&\qquad - f_y^0(\sigma^0(y_0(s,\eta)), y_0(s,\eta)) \frac{\partial y_0}{\partial \eta}(s,\eta)\| ds
\end{aligned}$$

$$\begin{aligned}
&\leq N_f \theta(\varepsilon) \int_{-\infty}^0 (-s)^{-\beta} e^{as} \| \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\| ds \\
&\quad + c_0 L_f \int_{-\infty}^0 e^{\lambda_n^0 s} \Big[(1+\Delta)\| y_\varepsilon(s,\eta) - y_0(s,\eta)\| \\
&\qquad\qquad + \|(\sigma^\varepsilon - \sigma^0)(y_0(s,\eta))\| \Big] \| \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\| ds \\
&\quad + c_0 \int_{-\infty}^0 e^{\lambda_n^0 s} \| f_y^\varepsilon(\sigma^0(y_0(s,\eta)), y_0(s,\eta)) \\
&\qquad\qquad - f_y^0(\sigma^0(y_0(s,\eta)), y_0(s,\eta))\| \, \| \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\| ds \\
&\quad + c_0 N_f \int_{-\infty}^0 e^{\lambda_n^0 s} \| \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta) - \frac{\partial y_0}{\partial \eta}(s,\eta)\| ds.
\end{aligned}$$

Since $\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)$ satisfies

$$\begin{cases} \dot{z} = -B^\varepsilon z + \frac{\partial g^\varepsilon}{\partial y}(\sigma^\varepsilon(y(t,\eta)), y(t,\eta))z \\ \frac{\partial y_\varepsilon}{\partial \eta}(0,\eta) = I, \end{cases}$$

we have

$$\|\frac{\partial y_\varepsilon}{\partial \eta}(t,\eta)\| \leq Me^{-\lambda_{n-1}^\varepsilon t} + MN_g(1+\Delta)\int_t^0 e^{-\lambda_{n-1}^\varepsilon (t-s)}\|\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\|ds$$

for $t \leq 0$. By the generalized Gronwall's inequality,

$$\|\frac{\partial y_\varepsilon}{\partial \eta}(t,\eta)\| \leq Me^{(-\lambda_{n-1}^\varepsilon - MN_g(1+\Delta))t} \tag{19}$$

for $t \leq 0$. Similarly we obtain

$$\|\frac{\partial y_0}{\partial \eta}(t,\eta)\| \leq Me^{(-\lambda_{n-1}^0 - MN_g(1+\Delta_0))t} \tag{20}$$

for $t \leq 0$. Then,

$$\begin{aligned}
\|\frac{\partial y_\varepsilon}{\partial \eta}(t,\eta) - \frac{\partial y_0}{\partial \eta}(t,\eta)\| &\leq \|e^{-B^\varepsilon t} - e^{-B^0 t}\| \\
&\quad + \int_t^0 \|e^{-B^\varepsilon (t-s)} g_x^\varepsilon(\sigma^\varepsilon(y_\varepsilon(s,\eta)), y_\varepsilon(s,\eta))\frac{d\sigma^\varepsilon}{d\eta}(s,\eta)\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta) \\
&\qquad - e^{-B^0(t-s)} g_x^0(\sigma^0(y_0(s,\eta)), y_0(s,\eta))\frac{d\sigma^0}{d\eta}(y_0(s,\eta))\frac{\partial y_0}{\partial \eta}(s,\eta)\|ds \\
&\quad + \int_t^0 \|e^{-B^\varepsilon (t-s)} g_y^\varepsilon(\sigma^\varepsilon(y_\varepsilon(s,\eta)), y_\varepsilon(s,\eta))\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta) \\
&\qquad - e^{-B^0(t-s)} g_y^0(\sigma^0(y_0(s,\eta)), y_0(s,\eta))\frac{\partial y_0}{\partial \eta}(s,\eta)\|ds \\
&= \|e^{-B^\varepsilon t} - e^{-B^0 t}\| + I_3 + I_4.
\end{aligned}$$

We now obtain the estimates for I_3 and I_4. We have

$$
\begin{aligned}
I_3 \le{}& MN_g\Delta \int_t^0 \sum_{j=0}^{n-1} |e^{-\lambda_j^\varepsilon(t-s)} - e^{-\lambda_j^0(t-s)}| \, \| \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\| ds \\
&+ ML_g\Delta \int_t^0 e^{-\lambda_{n-1}^0(t-s)} \Big[(1+\Delta)\|y_\varepsilon(s,\eta) - y_0(s,\eta)\| \\
&\qquad\qquad + \|(\sigma^\varepsilon - \sigma^0)(y_0(s,\eta))\|\Big] \, \|\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\| \, ds \\
&+ MN_g\Delta \int_t^0 e^{-\lambda_{n-1}^0(t-s)} \|\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta) - \frac{\partial y_0}{\partial \eta}(s,\eta)\| \, ds \\
&+ MN_g\tilde{\Delta} \int_t^0 e^{-\lambda_{n-1}^0(t-s)} \|y_\varepsilon(s,\eta) - y_0(s,\eta)\| \, \|\frac{\partial y_0}{\partial \eta}(s,\eta)\| ds \\
&+ MN_g \int_t^0 e^{-\lambda_{n-1}^0(t-s)} \|\frac{d\sigma^\varepsilon}{d\eta}(y_0(s,\eta)) - \frac{d\sigma^0}{d\eta}(y_0(s,\eta))\| \, \|\frac{\partial y_0}{\partial \eta}(s,\eta)\| \, ds \\
&+ M\Delta_0 \int_t^0 e^{-\lambda_{n-1}^0(t-s)} \|g_x^\varepsilon(\sigma^0(y_0(s,\eta)), y_0(s,\eta)) \\
&\qquad\qquad - g_x^0(\sigma^0(y_0(s,\eta)), y_0(s,\eta))\| \, \|\frac{\partial y_0}{\partial \eta}(s,\eta)\| \, ds,
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 \le{}& MN_g \int_t^0 \sum_{j=0}^{n-1} |e^{-\lambda_j^\varepsilon(t-s)} - e^{-\lambda_j^0(t-s)}| \, \| \frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\| ds \\
&+ ML_g \int_t^0 e^{-\lambda_{n-1}^0(t-s)} \Big[(1+\Delta)\|y_\varepsilon(s,\eta) - y_0(s,\eta)\| \\
&\qquad\qquad + \|(\sigma^\varepsilon - \sigma^0)(y_0(s,\eta))\|\Big] \, \|\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\| ds \\
&+ MN_g \int_t^0 e^{-\lambda_{n-1}^0(t-s)} \|\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta) - \frac{\partial y_0}{\partial \eta}(s,\eta)\| ds \\
&+ M \int_t^0 e^{-\lambda_{n-1}^0(t-s)} \|g_y^\varepsilon(\sigma^0(y_0(s,\eta)), y_0(s,\eta)) \\
&\qquad\qquad - g_y^0(\sigma^0(y_0(s,\eta)), y_0(s,\eta))\| \, \|\frac{\partial y_0}{\partial \eta}(s,\eta)\| ds.
\end{aligned}
$$

Substituting the estimates (15), (16), (19) and (20) in the inequalities for I_3 and I_4 we obtain

$$
\begin{aligned}
&\|\frac{\partial y_\varepsilon}{\partial \eta}(t,\eta) - \frac{\partial y_0}{\partial \eta}(t,\eta)\| \leq c_1 M e^{(-\lambda^0_{n-1}-\delta)t}\lambda(\varepsilon)\\
&+ M^2 N_g(1+\Delta)c_1\lambda(\varepsilon)\int_t^0 e^{(-\lambda^0_{n-1}-\delta)(t-s)}e^{(-\lambda^\varepsilon_{n-1}-MN_g(1+\Delta))s}\,ds\\
&+ M^2 L_g(1+\Delta)^2 a(\varepsilon)\int_t^0 e^{-\lambda^0_{n-1}(t-s)}e^{(-\lambda^0_{n-1}-\delta-ML_g(1+\Delta))s}e^{(-\lambda^\varepsilon_{n-1}-MN_g(1+\Delta))s}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ M^2 L_g(1+\Delta)\sup_{\psi\in\mathbb{R}^n}\|\sigma^\varepsilon(\psi)-\sigma^0(\psi)\|\int_t^0 e^{-\lambda^0_{n-1}(t-s)}e^{(-\lambda^\varepsilon_{n-1}-MN_g(1+\Delta))s}ds\\
&+ M^2 N_g\tilde{\Delta}a(\varepsilon)\int_t^0 e^{-\lambda^0_{n-1}(t-s)}e^{(-\lambda^0_{n-1}-\delta-ML_g(1+\Delta))s}e^{(-\lambda^0_{n-1}-MN_g(1+\Delta_0))s}ds\\
&+ M^2 N_g\sup_{\psi\in\mathbb{R}^n}\|\frac{d\sigma^\varepsilon}{d\eta}(\psi)-\frac{d\sigma^0}{d\eta}(\psi)\|\int_t^0 e^{-\lambda^0_{n-1}(t-s)}e^{(-\lambda^0_{n-1}-MN_g(1+\Delta_0))s}ds\\
&+ M^2\left[\Delta_0\sup_{\psi\in\mathbb{R}^n}\|(g^\varepsilon_x-g^0_x)(\sigma^0(\psi),\psi)\| + \sup_{\psi\in\mathbb{R}^n}\|(g^\varepsilon_y-g^0_y)(\sigma^0(\psi),\psi)\|\right]\times\\
&\qquad\qquad\int_t^0 e^{-\lambda^0_{n-1}(t-s)}e^{(-\lambda^0_{n-1}-MN_g(1+\Delta_0))s}ds\\
&+ MN_g(1+\Delta)\int_t^0 e^{-\lambda^0_{n-1}(t-s)}\|\frac{\partial y_\varepsilon}{\partial\eta}(s,\eta)-\frac{\partial y_0}{\partial\eta}(s,\eta)\|ds.
\end{aligned}
$$

Since $\lambda^0_{n-1}-\delta<0$, we have

$$
\begin{aligned}
&e^{\lambda^0_{n-1}t}\|\frac{\partial y_\varepsilon}{\partial\eta}(t,\eta)-\frac{\partial y_0}{\partial\eta}(t,\eta)\|\\
&\qquad\leq \tilde{\alpha}(\varepsilon)e^{-M_\varepsilon t} + MN_g(1+\Delta)\int_t^0 e^{\lambda^0_{n-1}s}\|\frac{\partial y_\varepsilon}{\partial\eta}(s,\eta)-\frac{\partial y_0}{\partial\eta}(s,\eta)\|ds,
\end{aligned}
$$

for small ε, where

$$
\tilde{\alpha}(\varepsilon) = b(\varepsilon) + \frac{M}{(1+\Delta_0)}\sup_{\eta\in\mathbb{R}^n}\|\frac{d\sigma^\varepsilon}{d\eta}(\eta)-\frac{d\sigma^0}{d\eta}(\eta)\|,
$$

$$
M_\varepsilon = \frac{3\delta}{2} + ML_g(1+\Delta) + MN_g\max\{1+\Delta, 1+\Delta_0)\},
$$

and

$$
\begin{aligned}
b(\varepsilon) = {} & Mc_1\lambda(\varepsilon) + \frac{M^2N_g(1+\Delta)c_1}{\lambda_{n-1}^{\varepsilon}+\delta+MN_g(1+\Delta)}\lambda(\varepsilon) \\
& + \frac{M^2L_g(1+\Delta)}{\lambda_{n-1}^{\varepsilon}-\lambda_{n-1}^{0}+MN_g(1+\Delta)} \sup_{\psi\in\mathrm{R}^n} \|(\sigma^{\varepsilon}-\sigma^0)\psi\| \\
& + \Bigg(\frac{M^2N_g\tilde{\Delta}}{\lambda_{n-1}^{0}+\delta+M(L_g(1+\Delta)+N_g(1+\Delta_0))} \\
& \qquad\qquad + \frac{M^2L_g(1+\Delta)^2}{\lambda_{n-1}^{\varepsilon}+\delta+M(L_g+N_g)(1+\Delta)}\Bigg) a(\varepsilon) \\
& + \frac{M}{N_g(1+\Delta_0)}\Bigg[\tilde{\Delta}_0 \sup_{\psi\in\mathrm{R}^n} \|(g_x^{\varepsilon}-g_x^0)(\sigma^0(\psi),\psi)\| \\
& \qquad\qquad + \sup_{\psi\in\mathrm{R}^n} \|(g_y^{\varepsilon}-g_y^0)(\sigma^0(\psi),\psi)\|\Bigg] \to 0, \text{ as } \varepsilon\to 0.
\end{aligned}
$$

From the generalized Gronwall's inequality,

$$
\|\frac{\partial y_{\varepsilon}}{\partial \eta}(t,\eta) - \frac{\partial y_0}{\partial \eta}(t,\eta)\| \le \tilde{\alpha}(\varepsilon) e^{(-\lambda_{n-1}^0-(M_{\varepsilon}+MN_g(1+\Delta)))t}. \tag{21}
$$

Using estimates (19) and (21), we obtain

$$
\begin{aligned}
I_1 \le {} & MN_f\theta(\varepsilon)\int_{-\infty}^{0}(-s)^{-\beta}e^{(-\lambda_{n-1}^{\varepsilon}+a-MN_g(1+\Delta))s}ds \\
& + Mc_0L_f(1+\Delta)a(\varepsilon)\int_{-\infty}^{0} e^{(\lambda_n^0-\lambda_{n-1}^0-\lambda_{n-1}^{\varepsilon}-\delta-M(L_g(1+\Delta_0)+N_g(1+\Delta)))s}ds \\
& + Mc_0\Bigg[L_f \sup_{\psi\in\mathbb{R}^n}\|(\sigma^{\varepsilon}-\sigma^0)\psi\| + \sup_{\psi\in\mathbb{R}^n}\|(f_y^{\varepsilon}-f_y^0)(\sigma^0(\psi),\psi)\|\Bigg]\times \\
& \qquad\qquad \int_{-\infty}^{0} e^{(\lambda_n^0-\lambda_{n-1}^{\varepsilon}-MN_g(1+\Delta))s}ds \\
& + c_0N_f\Bigg[b(\varepsilon)+\frac{1}{1+\Delta_0}\sup_{\eta\in\mathbb{R}^n}\|\frac{d\sigma^{\varepsilon}}{d\eta}(\eta)-\frac{d\sigma^0}{d\eta}(\eta)\|\Bigg]\times \\
& \qquad\qquad \int_{-\infty}^{0} e^{(\lambda_n^0-\lambda_{n-1}^0-(M_{\varepsilon}+MN_g(1+\Delta)))s}ds.
\end{aligned}
$$

Increasing n, if necessary, so that $\lambda_n^0-\lambda_{n-1}^0$ is sufficiently large to make the above integrals convergent, we finally get an estimate for I_1.

$$
\begin{aligned}
I_1 \le H_1(\varepsilon) + {} & \frac{c_0N_f}{(1+\Delta_0)(\lambda_n^0-\lambda_{n-1}^0-(M_{\varepsilon}+MN_g(1+\Delta)))}\times \\
& \qquad \sup_{\eta\in\mathbb{R}^n}\|\frac{d\sigma^{\varepsilon}}{d\eta}(\eta)-\frac{d\sigma^0}{d\eta}(\eta)\|
\end{aligned} \tag{22}
$$

where

$$H_1(\varepsilon) = MN_f\Gamma(1-\beta)(\lambda_n^0 - \lambda_{n-1}^\varepsilon - (MN_g(1+\Delta) - (a+\lambda_n^0)))^{\beta-1}\theta(\varepsilon)$$
$$+ \frac{c_0 ML_f(1+\Delta)}{\lambda_n^0 - \lambda_{n-1}^0 - \delta/2 - M(L_g(1+\Delta_0) + N_g(1+\Delta))} a(\varepsilon)$$
$$+ Mc_0\Big[L_f \sup_{\psi\in \mathrm{R}^n} \|(\sigma^\varepsilon - \sigma^0)(\psi)\| + \sup_{\psi\in \mathrm{R}^n} \|(f_y^\varepsilon - f_y^0)(\sigma^0(\psi),\psi)\|\Big] \times$$
$$\frac{1}{\lambda_n^0 - \lambda_{n-1}^\varepsilon - MN_g(1+\Delta)}$$
$$+ \frac{c_0 N_f}{\lambda_n^0 - \lambda_{n-1}^0 - (M_\varepsilon + MN_g(1+\Delta))} b(\varepsilon).$$

We now obtain an estimate for I_2.

$$I_2 \le N_f\theta(\varepsilon)\Delta_0 \int_{-\infty}^0 (-s)^{-\beta} e^{as} \,\|\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\|\, ds$$

$$+ c_0 L_f \Delta_0 \int_{-\infty}^0 e^{\lambda_n^0 s}\big[(1+\Delta)\|y_\varepsilon(s,\eta) - y_0(s,\eta)\|$$
$$+ \|(\sigma^\varepsilon - \sigma^0)(y_0(s,\eta))\|\big] \,\|\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\|\, ds$$
$$+ c_0\Delta_0 \int_{-\infty}^0 e^{\lambda_n^0 s} \|(f_x^\varepsilon - f_x^0)(\sigma^0(y_0(s,\eta)), y_0(s,\eta))\| \,\|\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta)\|\, ds$$
$$+ c_0 N_f \Delta_0 \int_{-\infty}^0 e^{\lambda_n^0 s} \|\frac{\partial y_\varepsilon}{\partial \eta}(s,\eta) - \frac{\partial y_0}{\partial \eta}(s,\eta)\|\, ds$$
$$+ c_0 N_f \tilde{\Delta}_0 \int_{-\infty}^0 e^{\lambda_n^0 s} \|y_\varepsilon(s,\eta) - y_0(s,\eta)\| \,\|\frac{\partial y_0}{\partial \eta}(s,\eta)\|\, ds.$$

Using estimates (16), (19), (20) and (21) it follows that

$$
\begin{aligned}
I_2 \le{}& MN_f\Delta_0\theta(\varepsilon)\int_{-\infty}^{0}(-s)^{-\beta}e^{(\lambda_n^0-\lambda_{n-1}^{\varepsilon}-(MN_g(1+\Delta)+\lambda_n^0-a))s}\,ds \\
&+c_0ML_f\Delta_0(1+\Delta)a(\varepsilon)\int_{-\infty}^{0}e^{(\lambda_n^0-\lambda_{n-1}^0-(\delta+\lambda_{n-1}^{\varepsilon}+M(L_g(1+\Delta_0)+N_g(1+\Delta))))s}\,ds \\
&+c_0M\Delta_0\Big[L_f\sup_{\psi\in\mathrm{R}^n}\|(\sigma^{\varepsilon}-\sigma^0)(\psi)\|+\sup_{\psi\in\mathrm{R}^n}\|(f_x^{\varepsilon}-f_x^0)(\sigma^0(\psi),\psi)\|\Big]\times \\
&\qquad\int_{-\infty}^{0}e^{(\lambda_n^0-\lambda_{n-1}^{\varepsilon}-MN_g(1+\Delta))s}ds \\
&+c_0N_f\Delta_0\int_{-\infty}^{0}\tilde{\alpha}(\varepsilon)e^{(\lambda_n^0-\lambda_{n-1}^0-(M_\varepsilon+MN_g(1+\Delta)))s}ds \\
&+c_0MN_f\tilde{\Delta}_0a(\varepsilon)\int_{-\infty}^{0}e^{(\lambda_n^0-\lambda_{n-1}^0-(\lambda_{n-1}^0+\delta+M(N_g(1+\Delta)+L_g(1+\Delta_0)))s}ds.
\end{aligned}
$$

Evaluating the integrals we obtain

$$
\begin{aligned}
I_2 \le H_2(\varepsilon)+&\frac{c_0\Delta_0N_fM}{(1+\Delta_0)(\lambda_n^0-\lambda_{n-1}^0-(M_\varepsilon+MN_g(1+\Delta)))}\times \\
&\sup_{\eta\in\mathbb{R}^n}\|\frac{d\sigma^{\varepsilon}}{d\eta}(\eta)-\frac{d\sigma^0}{d\eta}(\eta)\|,
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
H_2(\varepsilon) ={}& M\Delta_0N_f\Gamma(1-\beta)(\lambda_n^0-\lambda_{n-1}^{\varepsilon}-(MN_g(1+\Delta)+\lambda_n^0-a)))^{\beta-1}\theta(\varepsilon) \\
&+\frac{c_0M\Delta_0L_f(1+\Delta)}{\lambda_n^0-\lambda_{n-1}^0-(3\delta/2+M(L_g(1+\Delta_0)+N_g(1+\Delta)))}a(\varepsilon) \\
&+c_0M\Delta_0\Big[L_f\sup_{\psi\in\mathbb{R}^n}\|(\sigma^{\varepsilon}-\sigma^0)(\psi)\|+\sup_{\psi\in\mathbb{R}^n}\|(f_x^{\varepsilon}-f_x^0)(\sigma^0(\psi),\psi)\|\Big]\times \\
&\qquad\frac{1}{\lambda_n^0-\lambda_{n-1}^{\varepsilon}-MN_g(1+\Delta)} \\
&+\frac{c_0\Delta_0N_f}{\lambda_n^0-\lambda_{n-1}^0-(M_\varepsilon+MN_g(1+\Delta))}b(\varepsilon) \\
&+\frac{c_0M\tilde{\Delta}_0N_f}{\lambda_n^0-\lambda_{n-1}^0-(3\delta/2+M(N_g(1+\Delta_0)+L_g(1+\Delta)))}a(\varepsilon).
\end{aligned}
$$

Substituting (22) and (23) in (18) we have

$$\|\tilde{G}_\varepsilon(\frac{d\sigma^0}{d\eta})(\eta) - \tilde{G}_0(\frac{d\sigma^0}{d\eta})(\eta)\|_{\mathcal{L}(\mathbb{R}^n,X)} \quad \leq \quad H_1(\varepsilon) + H_2(\varepsilon)$$
$$+ \frac{c_0 N_f(1+\Delta_0 M)}{(1+\Delta_0)(\lambda_n^0 - \lambda_{n-1}^0 - (M_\varepsilon + MN_g(1+\Delta)))} \sup_{\eta\in\mathbb{R}^n} \|\frac{d\sigma^\varepsilon}{d\eta}(\eta) - \frac{d\sigma^0}{d\eta}(\eta)\|.$$

Therefore

$$\sup_{\eta\in\mathbb{R}^n} \|\frac{d\sigma^\varepsilon}{d\eta}(\eta) - \frac{d\sigma^0}{d\eta}(\eta)\|$$
$$\leq \left[1 - \frac{c_0 N_f(1+\Delta_0)M}{(1-\tilde{L}_\varepsilon)(\lambda_n^0 - \lambda_{n-1}^0 - (M_\varepsilon + MN_g(1+\Delta)))(1+\Delta_0)}\right]^{-1} \times$$
$$\frac{1}{1-\tilde{L}_\varepsilon}(H_1(\varepsilon) + H_2(\varepsilon)) \to 0, \text{ as } \varepsilon \to 0,$$

where $\tilde{L}_\varepsilon < 1$ is the contraction constant of $\tilde{G}_\varepsilon$. □

3.3 Topological Equivalence of Problems (P_ε) and (P_0)

Using the results above we can easily obtain the topological equivalence of the original problem (P_ε) and the limit problem (P_0) on the attractors.

From Theorems 3.2 and 3.3 we conclude that the vector fields of equations (2) and (3) over the invariant manifolds are C^1-close.

Lemma 3.4 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth bounded functions with bounded derivatives. For ε sufficiently small, the vector fields of* (2) *and* (3)*, on S^ε and S^0, respectively, are* C^1*-close.*

Theorem 3.5 (Main Theorem) *Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth, bounded functions with bounded derivatives. Suppose that all equilibrium points of problem* (P_0) *are hyperbolic. For ε sufficiently small the flow from* (P_ε) *in the attractor $\mathcal{A}_\varepsilon$ and the flow from* (P_0) *in the attractor $\mathcal{A}_0$ are topologically equivalent.*

PROOF. From the existence of the invariant manifolds it follows that the flows in the attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ are determined, respectively, by the ordinary differential equations

$$\dot{v} + B_n^\varepsilon v = g_n^\varepsilon(\sigma^\varepsilon(v), v) \tag{24}$$

and

$$\dot{v} + B_n^0 v = g_n^0(\sigma^0(v), v). \tag{25}$$

As a consequence of Lemma 3.4 the flows generated by equations (24) and (25) are C^1-close for small ε. We proved in [2], Lemma 6.1, that the stable and unstable manifolds of the hyperbolic equilibria are transverse. Then (3) is Morse-Smale and the topological equivalence follows from finite dimensional results. □

References

[1] J. M. Arrieta and A. N. Carvalho, *Continuity of attractors of reaction-diffusion equations under perturbations of the domain.* Submitted for publication.

[2] V. L. Carbone and J. G. Ruas-Filho, *Transversality of stable and unstable manifolds for parabolic problems arising in composite materials.* Accepted for publication in Journal of Mathematical Analysis and Applications.

[3] D. B. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, 1980.

[4] A. Rodríguez-Bernal, J. M. Arrieta and A. N. Carvalho, *Parabolic problems with nonlinear boundary conditions and critical nonlinearities*, J. Differential Equations, (1999).

[5] ——, *Attractors of parabolic problems with nonlinear: Boundary conditions. uniform bounds*, Commun. in Partial Differential Equations, (2000).

[6] ——, *Upper semicontinuity for attractors of parabolic problems with localized large diffusion and nonlinear boundary conditions*, Journal Differential Equations, 168 (2000).

[7] A. Rodríguez-Bernal, *Inertial manifolds for dissipative semiflows in banch spaces*, Appl. Anal., (1990).

[8] ——, *Localized spatial homogenizations and large diffusion*, SIAM J. Math. Anal., (1998).

[9] J. G. Ruas-Filho, S. M. Bruschi, A. N. Carvalho, *The dynamics of a one-dimensional parabolic problem versus the dynamics of its discretization*, J. Differential Equations, (2000).