Simultaneous reduction of a family of commuting real vector fields and global hypoellipticity

Gerson Petronilho *

Abstract

In this paper we consider a family of commuting real vector fields on the n-dimensional torus and show that it can be transformed into a family of constant vector fields provided that there is one of them which its transposed is globally hypoelliptic. We apply this result to prove global hypoellipticity for certain classes of sublaplacians.

1 Introduction and Preliminaries

There are few results regarding normal forms of systems of vector fields and differential operators on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Before we state some of them, we need to recall some definitions.

A linear partial differential operator $P : D'(\mathbb{T}^n) \to D'(\mathbb{T}^n)$ with coefficients in $C^\infty(\mathbb{T}^n)$ is said to be globally hypoelliptic on $\mathbb{T}^n$ if the conditions $u \in D'(\mathbb{T}^n)$ and $Pu \in C^\infty(\mathbb{T}^n)$ imply that $u \in C^\infty(\mathbb{T}^n)$. A similar definition can be given when we replace $\mathbb{T}^n$ by a compact smooth manifold without boundary. If P is defined on an open subset U of $\mathbb{R}^n$, then P is said to be locally hypoelliptic if for any open subset V of U the conditions $u \in D'(V)$ and $Pu \in C^\infty(V)$ imply that $u \in C^\infty(V)$. Note that local hypoellipticity implies global hypoellipticity.

Since in this paper we are concerned with real vector fields, we begin by recalling the real version of the well-known result:

Theorem 1.1 *(See Hounie [Hou]) Let X be a real vector field on $\mathbb{T}^2$ and suppose that X does not vanish on $\mathbb{T}^2$. If X is a globally hypoelliptic operator on $\mathbb{T}^2$ then there exists a diffeomorphism of $\mathbb{T}^2$ onto $\mathbb{T}^2$ that takes X into a non-vanishing multiple of*

$$\partial_s + A\partial_t \tag{1.1}$$

where the constant A is an irrational non-Liouville number.

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In $\mathbb{T}^n$ there exists a new reduction theorem for real vector fields due to Chen and Chi [CC]:

Theorem 1.2 *Let X be a real vector field on $\mathbb{T}^n$. Then, the transposed of X is a globally hypoelliptic operator on $\mathbb{T}^n$ if and only if there exist global coordinates y on $\mathbb{T}^n$ in which X admits the form*

$$X = \sum_{j=1}^{n} A_j \partial_{y_j} \tag{1.2}$$

with the real numbers $A_1, \cdots, A_n$ satisfying the following Diophantine condition: there exist positive constants C and K such that

$$|\sum_{j=1}^{n} \xi_j A_j| \geq \frac{C}{(1+|\xi|)^K}, \ \forall\, \xi \in \mathbb{Z}^n \backslash \{0\}. \tag{1.3}$$

Remark 1.3 Theorem 1.2 gives new results on normal forms of real vector fields on $\mathbb{T}^n$ even for $n = 2$ (cf. Theorem 1.4 in Greenfield and Wallach [GW]).

The next two examples are due to Dickinson, Gramchev and Yoshino [DGY]. In the first one they present an example of a system of overdetermined real vector fields being simultaneously transformed into constant vector fields. They consider

$$X = d_t + w(t) \wedge \partial_x, \ x \in \mathbb{T}, \ t \in \mathbb{T}^n,$$

where $w(t) = \sum_{j=1}^{n} w_j(t) dt_j$ is a real-valued smooth closed one-form on $\mathbb{T}^n$. The corresponding family of n commuting real vector fields associated with X is given by $X_j = \partial_{t_j} + w_j(t)\partial_x$, $1 \leq j \leq n$, (see Bergamasco, Cordaro and Malagutti [BCM]). It is easy to see that the family $\{X_j\}_1^n$ is transformed into the family $\{\partial_{s_j} + w_{j0}\partial_y\}_1^n$, if we define the diffeomorphism of $\mathbb{T}^{n+1}$ onto $\mathbb{T}^{n+1}$ by $y = x - h(t), s = t$, where h satisfies $\partial_{t_j} h(t) = w_j(t) - w_{j0}$, with $w_{j0} = \int_{\mathbb{T}^n} w_j(t)dt$.

In the second one they consider the following family of commuting real vector fields

$$X_j = \partial_t + h_j(t,x)\partial_x, \ j = 1, \cdots, m,$$

where $h_j \in G^\sigma(\mathbb{T}^2)$, $1 \leq \sigma \leq \infty$. Let P_j and ρ_j be respectively, the Poincaré map and the rotation number of the vector field X_j. Let also R_{ρ_j} be the rotation, where $R_{\rho_j}(z) = z + \rho_j$, $z \in \mathbb{T}$, $1 \leq j \leq m$. They proved the following result

Theorem 1.4 *Let $1 \le \sigma \le +\infty$. If $m \ge 2$, assume that the Poincaré maps P_j, $1 \le j \le m$, are orientation preserving and that there exists an index $j \in \{1, \cdots, m\}$ such that $(2\pi)^{-1}\rho_j$ is irrational. Then, if a G^σ diffeomorphism u on $\mathbb{T}$ satisfying*

$$u^{-1} \circ P_j \circ u = R_{\rho_j}, \; j = 1, \cdots, m,$$

can be found then there exists a G^σ diffeomorphism on $\mathbb{T}^2$: $s = t, y = \phi(t, x)$, that transform X_k into $\partial_s + (2\pi)^{-1}\rho_k \partial_y$ for all $1 \le k \le m$.

Remark 1.5 For results on diffeomorphisms that are globally conjugated to a rotation we refer the reader to Brjuno [Br], Herman [He], Yoccoz [Y] and references therein, while for commuting diffeomorphisms that are simultaneously locally conjugated to rotations we refer the reader to Gramchev and Yoshino [GY], Moser [Mo] and references therein.

In this paper we consider a family of commuting real vector fields on $\mathbb{T}^n$, $X_j, 1 \le j \le m$, and present a sufficient condition in order to simultaneously reduce the vector fields X_j into a family of constant vector fields (see Theorem 2.1). Next, we use this result to prove global hypoellipticity for the operator P given by

$$P = -\sum_{j=1}^{m} X_j^2.$$

We also present a class of non-commuting real vector fields and we study its global hypoellipticity.

2 Simultaneous reduction

In sections 2 and 3 all the results deal with families of vector fields X_j, $1 \le j \le m$, and we shall assume that there exists $j_0 \in \{1, \cdots, m\}$ such that X_{j_0} satisfies certain conditions. Without loss of generality one may assume that $j_0 = 1$.

In this section we present a sufficient condition in order to simultaneously transform a family of commuting real vector fields with variable coefficients into a family of constant vector fields.

Theorem 2.1 *Let $X_1, \cdots, X_m$ be a family of commuting real vector fields on $\mathbb{T}^n$ such that ${}^tX_1$ is a globally hypoelliptic operator on $\mathbb{T}^n$. Then, there exist global coordinates y on $\mathbb{T}^n$ in which the family $\{X_j\}_1^m$ admits the form $\{\sum_{k=1}^n c_{jk}\partial_{y_k}\}_1^m$, where c_{jk} are real constants. Moreover, the coefficients c_{1k}, of X_1, satisfy the following Diophantine condition: there exist $K > 0, C > 0$ such that*

$$|\sum_{k=1}^{n} c_{1k}\xi_k| \ge \frac{C}{(1 + |\xi|)^K}, \; \forall \; \xi \in \mathbb{Z}^n \backslash \{0\}. \tag{2.1}$$

Remark 2.2 Before starting the proof we would like to point out that Theorem 2.1 itself also gives new results on simultaneous reduction of a system of real vector fields on $\mathbb{T}^n$ even for $n = 2$.

Proof. Since, by hypothesis, ${}^tX_1$ is a globally hypoelliptic operator on $\mathbb{T}^n$ it follows from Theorem 1.2 that there exist global coordinates y on $\mathbb{T}^n$ such that

$$X_1 = \sum_{k=1}^{n} c_{1k}\partial_{y_k} \tag{2.2}$$

with the real numbers c_{1k} satisfying the Diophantine condition (2.1).

The vector fields X_j, $2 \le j \le m$, in the coordinates y can be written as

$$X_j = \sum_{k=1}^{n} b_{jk}(y)\partial_{y_k}, \tag{2.3}$$

where the functions b_{jk} are real-valued.

Thanks to the fact that the coefficients of X_1 are constants we obtain, for $2 \le j \le m$,

$$\begin{aligned} X_1X_j &= (X_1b_{j1})\,\partial_{y_1} + \cdots + (X_1b_{jn})\,\partial_{y_n} \\ &+ b_{j1}X_1\partial_{y_1} + \cdots + b_{jn}X_1\partial_{y_n} \\ &= (X_1b_{j1})\,\partial_{y_1} + \cdots + (X_1b_{jn})\,\partial_{y_n} \\ &+ b_{j1}\partial_{y_1}X_1 + \cdots + b_{jn}\partial_{y_n}X_1 \\ &= (X_1b_{j1})\,\partial_{y_1} + \cdots + (X_1b_{jn})\,\partial_{y_n} + X_jX_1. \end{aligned}$$

Thus, we have

$$[X_1, X_j] = X_1X_j - X_jX_1 = (X_1b_{j1})\,\partial_{y_1} + \cdots + (X_1b_{jn})\,\partial_{y_n}. \tag{2.4}$$

It follows from (2.4) and from the commutativity hypothesis that

$$X_1b_{jk} = 0,\ 1 \le k \le n,\ 2 \le j \le m. \tag{2.5}$$

Taking Fourier series in equations (2.5) we obtain

$$i\left(\sum_{\ell=1}^{n} c_{1\ell}\xi_\ell\right)\widehat{b_{jk}}(\xi) = 0,\ \xi \in \mathbb{Z}^n.$$

Since (2.1) implies that $\sum_{\ell=1}^{n} c_{1\ell}\xi_\ell \neq 0$ for all $\xi \in \mathbb{Z}\backslash\{0\}$, it follows from the last equality that

$$\widehat{b_{jk}}(\xi) = 0,\ \xi \in \mathbb{Z}^n\backslash\{0\}.$$

Thanks to this fact we have

$$b_{jk}(y) = \widehat{b_{jk}}(0),\ \text{for all}\, y \in \mathbb{T}^n, \tag{2.6}$$

since $b_{jk}(y) = \sum_{\xi\in\mathbb{Z}^n} \widehat{b_{jk}}(\xi)e^{iy\cdot\xi}$.

By setting $c_{jk} = \widehat{b_{jk}}(0)$, which are real numbers, it follows from (2.3) and (2.6) that

$$X_j = \sum_{k=1}^{n} c_{jk}\partial_{y_k},\ 2 \le j \le m. \tag{2.7}$$

The proof of Theorem 2.1 is complete. □

3 Global hypoellipticity

If $X = \{X_1, \cdots, X_m\}$ is a family of real vector fields on a C^∞ manifold $\mathcal{M}$, then the formulation of necessary and sufficient conditions for the global or local hypoellipticity of their *sublaplacian* $\Delta_X = -(X_1^2 + \cdots + X_m^2)$, is an open problem.

It is well-known that the bracket condition (see the famous theorem of Hörmander [Hö2]) implies local and therefore global hypoellipticity for Δ_X.

When $\mathcal{M} = \mathbb{T}^n$ and the bracket condition may fail we prove global hypoellipticity for two classes of sublaplacians. In the first one, we consider a family of commuting real vector fields, while in the second one, this property may not be satisfied.

For some results on global hypoellipticity, when the bracket condition fails, we refer the reader to [FO], [HP], [OK] and references therein.

While no satisfactory characterization of global hypoellipticity exists in the literature, it is hoped that our results provide some insight into this open problem.

We begin by proving global hypoellipticity for the sublaplacian of the family of commuting real vector fields X_j given in Theorem 2.1.

Theorem 3.1 *Let $X_1, \cdots, X_m$ be a family of real vector fields satifying the same conditions as in Theorem 2.1. Then the operator*

$$P = -\sum_{j=1}^{m} X_j^2 \tag{3.1}$$

is globally hypoelliptic on $\mathbb{T}^n$.

Proof. Since, by hypothesis, ${}^tX_1$ is a globally hypoelliptic operator on $\mathbb{T}^n$, it follows from Theorem 2.1 that there exist global coordinates y on $\mathbb{T}^n$ in which the vector fields X_j are constant, i.e.,

$$X_j = \sum_{k=1}^{n} c_{1k}\partial_{y_k},\ \ 1 \le j \le m, \tag{3.2}$$

where c_{jk} are real numbers and c_{1k}, $1 \leq k \leq n$, satisfy the Diophantine condition (2.1).

Thus, we have

$$P = -\sum_{j=1}^{m} X_j^2 = -\sum_{j=1}^{m} \left(\sum_{k=1}^{n} c_{jk} \partial_{y_k} \right)^2 . \tag{3.3}$$

Since the property "globally hypoelliptic" does not change under diffeomorphisms we will prove that P is globally hypoelliptic on $\mathbb{T}^n$ considering its representation (3.3). For this let $u \in D'(\mathbb{T}^n)$ be such that

$$Pu = f \in C^\infty(\mathbb{T}^n). \tag{3.4}$$

By taking Fourier series in (3.4) we obtain

$$\left[\sum_{j=2}^{m} \left(\sum_{k=1}^{n} c_{jk} \eta_k \right)^2 + \left(\sum_{k=1}^{n} c_{1k} \eta_k \right)^2 \right] \hat{u}(\eta) = \hat{f}(\eta). \tag{3.5}$$

For $\eta \in \mathbb{Z}^n \backslash \{0\}$ it follows from (2.1) and (3.5) that there exist $C > 0$ and $K > 0$ such that

$$|\hat{u}(\eta)| \leq \frac{|\hat{f}(\eta)|}{\left| \sum_{k=1}^{n} c_{1k} \eta_k \right|^2} \leq \frac{1}{C^2} (1 + |\eta|)^{2K} |\hat{f}(\eta)|. \tag{3.6}$$

Since $f \in C^\infty(\mathbb{T}^n)$ it follows easily from (3.6) that $u \in C^\infty(\mathbb{T}^n)$. Hence, P is globally hypoelliptic on $\mathbb{T}^n$. □

Corollary 3.2 *Let $X_1, \cdots, X_m$ be a family of commuting real vector fields on $\mathbb{T}^n$. Suppose also that there exist global coordinates y on $\mathbb{T}^n$ in which X_1 admits the form $\sum_{k=1}^{n} c_{1k} \partial_{y_k}$ with the real numbers c_{1k} satisfying the Diophantine condition (2.1). Then the operator $P = -\sum_{j=1}^{m} X_j^2$ is globally hypoelliptic on $\mathbb{T}^n$.*

Proof. Thanks to the hypotheses it follows from Theorem 1.2 that the transposed, ${}^tX_1$, of the vector field X_1 is a globally hypoelliptic operator on $\mathbb{T}^n$. Thus, it follows from Theorem 3.1 that P is globally hypoelliptic on $\mathbb{T}^n$. □

In the next result we present a family of non-commuting real vector fields, X_j, and we present a sufficient condition for the global hypoellipticity of their sublaplacian $P = -\sum_{j=1}^{n} X_j^2$.

From now on we shall use the letters C and C_N to represent constants, which may change a finite number of times.

Theorem 3.3 *Let $X_j, 1 \leq j \leq m$, be a family of real vector fields on $\mathbb{T}^m \times \mathbb{T}^n$, where one can choose coordinates x, y on $\mathbb{T}^m$ and $\mathbb{T}^n$ respectively, in which the above vector fields admits the form $X_j = \partial_{x_j} + \sum_{k=1}^n a_{jk}(x)\partial_{y_k}$, $1 \leq j \leq m$, where $a_{1k}(x) = \lambda_k$ and the vector $(\lambda_1, \ldots, \lambda_n)$ is non-Liouville. Then the operator $P = -\sum_{j=1}^m X_j^2$ is globally hypoelliptic on $\mathbb{T}^m \times \mathbb{T}^n$.*

Proof. In order to prove that P is a globally hypoelliptic operator on $\mathbb{T}^m \times \mathbb{T}^n$ let $u \in D'(\mathbb{T}^m \times \mathbb{T}^n)$ be such that

$$Pu = f, \ \ f \in C^\infty(\mathbb{T}^m \times \mathbb{T}^n). \tag{3.7}$$

To complete the proof of Theorem 3.3 we must show that $u \in C^\infty(\mathbb{T}^m \times \mathbb{T}^n)$. For this it suffices to show that for any $N \in \mathbb{N}$ there exists a positive constant C_N such that the Fourier coefficients $\hat{u}(\xi, \eta)$ of u on $\mathbb{T}^m \times \mathbb{T}^n$ satisfy the following inequality

$$|\hat{u}(\xi, \eta)| \leq C_N(|\xi| + |\eta|)^{-N}, \ \ (\xi, \eta) \in (\mathbb{Z}^m \times \mathbb{Z}^n)\backslash\{0\}. \tag{3.8}$$

To show (3.8) we will use the fact that f satisfies such an inequality, i.e.,

$$|\hat{f}(\xi, \eta)| \leq C_N(|\xi| + |\eta|)^{-N}, \ \ (\xi, \eta) \in (\mathbb{Z}^m \times \mathbb{Z}^n)\backslash\{0\}. \tag{3.9}$$

We will show that the partial Fourier transform with respect to y of f dominates the partial Fourier transform with respect to y of u in L^2-norm, and we will also use the fact that the operator P is elliptic in x.

We start by taking the partial Fourier transform in (3.7) with respect to y. Then we obtain the equation

$$-\sum_{j=1}^m Y_j^2 \hat{u}(x, \eta) = \hat{f}(x, \eta), \ \ \text{for all} \ \ \eta \in \mathbb{Z}^n \tag{3.10}$$

where $Y_j = \partial_{x_j} + i\sum_{k=1}^n a_{jk}(x)\eta_k$.

For any $\eta \in \mathbb{Z}^n$ fixed, $\hat{u}(\cdot, \eta)$ is in $C^\infty(\mathbb{T}^m)$ since (3.10) is elliptic in x. Therefore, if we multiply (3.10) with $\bar{\hat{u}}$ and integrate by parts with respect to $x \in \mathbb{T}^m$, then we obtain

$$\sum_{j=1}^m \|Y_j \hat{u}(\cdot, \eta)\|^2_{L^2(\mathbb{T}^m)} = \int_{\mathbb{T}^m} \hat{f}(x, \eta)\bar{\hat{u}}(x, \eta)dx. \tag{3.11}$$

We shall need the following lemma:

Lemma 3.4 *There exist positive constants C and K such that*

$$||\hat{u}(\cdot, \eta)||^2_{L^2(\mathbb{T}^m)} \leq C|\eta|^{2K}||Y_1\hat{u}(\cdot, \eta)||^2_{L^2(\mathbb{T}^m)}.$$

Proof. For $\eta \in \mathbb{Z}^n$ fixed let $\varphi_\eta \in C^\infty(\mathbb{T}^m)$. Thus,

$$Y_1\varphi_\eta(x) = \partial_{x_1}\varphi_\eta(x) + \left(i\sum_{k=1}^{n}\lambda_k\eta_k\right)\varphi_\eta(x) \doteq \psi_\eta(x). \tag{3.12}$$

By taking the partial Fourier transform with respect to x_1 we obtain

$$i\left(\xi_1 + \sum_{k=1}^{n}\lambda_k\eta_k\right)\hat{\varphi}_\eta(\hat{x},\xi_1) = \hat{\psi}_\eta(\hat{x},\xi_1) \tag{3.13}$$

where $\hat{x} = (x_2,\ldots,x_n)$.

Since $(\lambda_1,\ldots,\lambda_n)$ is a non-Liouville vector (see definition, e.g., in Himonas and Petronilho [HP]) there exist $C > 0, K > 0$ such that

$$|\xi_1 + \sum_{k=1}^{n}\lambda_k\eta_k| \geq \frac{C}{|\eta|^K},\ \ \forall\ \eta \in \mathbb{Z}^n\backslash\{0\},\ \ \forall\ \xi_1 \in \mathbb{Z}. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$|\hat{\varphi}_\eta(\hat{x},\xi_1)|^2 \leq C|\eta|^{2K}|\hat{\psi}_\eta(\hat{x},\xi_1)|^2,\ \forall\ \xi_1 \in \mathbb{Z},\ \forall\ \eta \in \mathbb{Z}^n\backslash\{0\},\ \forall\ \hat{x} \in \mathbb{T}^{m-1}.$$

By using the last inequality and Parseval identity we obtain

$$\begin{aligned}\int_{\mathbb{T}_{x_1}}|\varphi_\eta(x)|^2dx_1 &= \sum_{\xi_1\in\mathbb{Z}}|\hat{\varphi}_\eta(\hat{x},\xi_1)|^2\\ &\leq C|\eta|^{2K}\sum_{\xi_1\in\mathbb{Z}}|\hat{\psi}_\eta(\hat{x},\xi_1)|^2\\ &= C|\eta|^{2K}\int_{\mathbb{T}_{x_1}}|\psi_\eta(x)|^2dx_1.\end{aligned}$$

If we integrate the last inequality with respect to $\hat{x} \in \mathbb{T}^{m-1}$ then we obtain

$$\begin{aligned}\|\varphi_\eta\|^2_{L^2(\mathbb{T}^m)} &\leq C|\eta|^{2K}\int_{\mathbb{T}^m}|\psi_\eta(x)|^2dx\\ &= C|\eta|^{2K}\int_{\mathbb{T}^m}|\partial_{x_1}\varphi_\eta(x) + \left(i\sum_{k=1}^{n}\lambda_k\eta_k\right)\varphi_\eta(x)|^2dx. \end{aligned}\tag{3.15}$$

If we apply (3.15) with $\varphi_\eta(x) = \hat{u}(x,\eta)$ we obtain

$$\begin{aligned}\|\hat{u}(\cdot,\eta)\|^2_{L^2(\mathbb{T}^m)} &\leq C|\eta|^{2K}\int_{\mathbb{T}^m}|\partial_{x_1}\hat{u}(x,\eta) + \left(i\sum_{k=1}^{n}\lambda_k\eta_k\right)\hat{u}(x,\eta)|^2dx\\ &= C|\eta|^{2K}\|Y_1\hat{u}(\cdot,\eta)\|^2_{L^2(\mathbb{T}^m)}.\end{aligned}$$

The proof of Lemma 3.4 is complete. $\square$

We now use Lemma 3.4 in order to show that $||\hat{f}(\cdot,\eta)||_{L^2(\mathbb{T}^m)}$ dominates $||\hat{u}(\cdot,\eta)||_{L^2(\mathbb{T}^m)}$.

It follows from Lemma 3.4 and (3.11) that

$$\begin{aligned} ||\hat{u}(\cdot,\eta)||^2_{L^2(\mathbb{T}^m)} &\leq C|\eta|^{2K}||Y_1\hat{u}(\cdot,\eta)||^2_{L^2(\mathbb{T}^m)} \\ &\leq C|\eta|^{2K}\left(\sum_{j=1}^{m}||Y_j\hat{u}(\cdot,\eta)||^2_{L^2(\mathbb{T}^m)}\right) \\ &= C|\eta|^{2K}\int_{\mathbb{T}^m}\hat{f}(x,\eta)\bar{\hat{u}}(x,\eta)dx. \end{aligned}$$

Thus, the Cauchy-Schwarz inequality implies that

$$\|\hat{u}(\cdot,\eta)\|^2_{L^2(\mathbb{T}^m)} \leq C|\eta|^{2K}\|\hat{f}(\cdot,\eta)\|_{L^2(\mathbb{T}^m)}\|\hat{u}(\cdot,\eta)\|_{L^2(\mathbb{T}^m)}$$

and therefore

$$\|\hat{u}(\cdot,\eta)\|_{L^2(\mathbb{T}^m)} \leq C|\eta|^{2K}\|\hat{f}(\cdot,\eta)\|_{L^2(\mathbb{T}^m)}. \tag{3.16}$$

Next we will use (3.16) to prove that $u \in C^\infty(\mathbb{T}^m \times \mathbb{T}^n)$, or equivalently that inequality (3.8) holds. By (3.16) and the fact that $f \in C^\infty(\mathbb{T}^m \times \mathbb{T}^n)$ we obtain that for any $N \in \mathbb{N}$ there exists a positive constant C_N such that

$$||\hat{u}(\cdot,\eta)||^2_{L^2(\mathbb{T}^m)} \leq C_N|\eta|^{-N},\ \ \eta \in \mathbb{Z}^n\backslash\{0\}.$$

Since

$$\hat{u}(\xi,\eta) = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} e^{-ix\cdot\xi}\hat{u}(x,\eta)dx,$$

by the last inequality and the Cauchy-Schwarz inequality we obtain

$$|\hat{u}(\xi,\eta)| \leq C_N|\eta|^{-N},\ \ (\xi,\eta) \in \mathbb{Z}^m \times \mathbb{Z}^n,\ \eta \neq 0. \tag{3.17}$$

Since the operator P is elliptic at $(x,y;\xi_0,0)$ for all $(x,y) \in \mathbb{T}^m \times \mathbb{T}^n$ and $\xi_0 \in \mathbb{Z}^m\backslash\{0\}$, by using the microlocal elliptic theory we obtain that there exists a cone $\Gamma_\epsilon = \{(\xi,\eta) \in \mathbb{Z}^m \times \mathbb{Z}^n : |\eta| < \epsilon|\xi|\}$ containing $(\xi_0,0)$ such that for any $N \in \mathbb{N}$ there exists a positive constant C_N such that

$$|\hat{u}(\xi,\eta)| \leq C_N(|\xi| + |\eta|)^{-N},\ \ (\xi,\eta) \in \Gamma_\epsilon.$$

Now let $\Gamma = \{(\xi,\eta) \in \mathbb{Z}^m \times \mathbb{Z}^n : |\eta| > \frac{\epsilon}{2}|\xi|\}$. We notice that if $(\xi,\eta) \in \Gamma$ then $\eta \neq 0$. Therefore, if $(\xi,\eta) \in \Gamma$ then it follows from (3.17) that

$$|\hat{u}(\xi,\eta)| \leq C_N(\frac{1}{2}|\eta| + \frac{\epsilon}{4}|\xi|)^{-N} \leq C_N(|\xi| + |\eta|)^{-N}.$$

The last two inequalities imply that for any $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that

$$|\hat{u}(\xi, \eta)| \leq C_N(|\xi| + |\eta|)^{-N}, \;\; (\xi, \eta) \in (\mathbb{Z}^m \times \mathbb{Z}^n)\backslash\{0\}.$$

Hence (3.8) holds true and therefore $u \in C^\infty(\mathbb{T}^m \times \mathbb{T}^n)$. The proof of Theorem 3.3 is complete. □

In the next resul we present an application of Theorem 3.3.

Corollary 3.5 *Let $X_j, 1 \leq j \leq m$, be a family of real vector fields on $\mathbb{T}^m \times \mathbb{T}^n$, where one can choose coordinates x, y on $\mathbb{T}^m$ and $\mathbb{T}^n$ respectively, in which the above vector fields admits the form $X_j = \partial_{x_j} + \sum_{k=1}^n a_{jk}(x)\partial_{y_k}$, $1 \leq j \leq m$, where $a_{1k}(x) = a_{1k}(x_1)$, $1 \leq k \leq n$, and the vector $(a_{11}^0, \cdots, a_{1m}^0)$, where $a_{1k}^0 = \int_{\mathbb{T}} a_{1k}(x_1)dx_1$, $1 \leq k \leq n$, is non-Liouville. Then the operator $P = -\sum_{j=1}^m X_j^2$ is globally hypoelliptic on $\mathbb{T}^m \times \mathbb{T}^n$.*

Proof. If we define $t_j = x_j$, $1 \leq j \leq m$, and $s_k = y_k - \int_0^{x_1} a_{1k}(r)dr + a_{1k}^0 x_1$, $1 \leq k \leq n$, then in the new coordinates the family $\{X_j\}_1^m$ is transformed into $\{\partial_{t_j} + \sum_{k=1}^n b_{jk}(t)\partial_{s_k}\}_1^m$ with $b_{1k}(t) = a_{1k}^0$, $1 \leq k \leq n$. Since the vector $(a_{11}^0, \ldots, a_{1n}^0)$ is non-Liouville it follows from Theorem 3.3 that P is globally hypoelliptic on $\mathbb{T}^m \times \mathbb{T}^n$. □

Corollary 3.2 and Theorem 3.3 lead us to make the following

CONJECTURE: Let $X_1, \cdots, X_m$ be a family of real vector fields on $\mathbb{T}^n$. If there exist global coordinates y on $\mathbb{T}^n$ in which the vector field X_1 admits the form $X_1 = \sum_{k=1}^n \lambda_k \partial_{y_k}$ with the numbers $\lambda_1, \ldots, \lambda_n$ satisfying the following condition: there exist $C > 0, K > 0$ such that

$$|\lambda \cdot \eta| = \left| \sum_{k=1}^n \lambda_k \eta_k \right| \geq \frac{C}{|\eta|^K}, \;\; \eta \in \mathbb{Z}^n\backslash\{0\},$$

then the operator $P = -\sum_{j=1}^m X_j^2$ is globally hypoelliptic on $\mathbb{T}^n$.

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Gerson Petronilho
Universidade Federal de São Carlos
Departamento de Matemática
São Carlos, SP 13565-905, Brazil
E-mail: *gerson@dm.ufscar.br*
E-mail: *gersonp@power.ufscar.br*