Z_2^2-ACTIONS WITH N-DIMENSIONAL FIXED POINT SET

PEDRO L. Q. PERGHER

Abstract. We describe the equivariant cobordism classification of smooth actions (M^m, Φ) of the group $G = Z_2^2$, considered as the group generated by two commuting involutions, on closed smooth m-dimensional manifolds M^m, for which the fixed point set of the action is a connected manifold of dimension n and $m = 4n-1$ or $4n-2$. For $m \geq 4n$, the classification is known.

1. Introduction

In [7], R. E. Stong determined all possible equivariant cobordism classes of smooth involutions $T : M \to M$, defined on closed smooth m-dimensional manifolds M, for which the fixed point set F has dimension n and $m = 2n-1$; specifically, he showed that any such involution pair (M, T) is equivariantly cobordant to an union of involutions of the following two types:

i) Let $\pi : N \to S^1$ be a smooth fibering, where N is a closed n-dimensional manifold and S^1 is the 1-sphere, and let $M_1 = \{(x, y) \in N \times N \ / \ \pi(x) = \pi(y)\}$, with involution $T_1(x, y) = (y, x)$.

ii) Let P and Q be closed manifolds with dimensions p and q, respectively, where $p + q = n - 1$, and let $M_2(P, Q) = \dfrac{S^1 \times P \times P \times Q \times Q}{(x, p, p', q, q') \sim (-x, p', p, q, q')}$ with the involution $T_2[x, p, p', q, q'] = [x, p', p, q', q]$.

In general, for a given (M, T), several summands of the second type, with different p values, may be needed; every such union of involutions is equivariantly cobordant to an involution with (n-dimensional) connected fixed set.

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Note. The equivariant cobordism classification for $dim(F) = n$ and $dim(M) \geq 2n$ was obtained in [1].

The purpose of this note is to extend the above result for Z_2^2-actions; here, $G = Z_2^2$ is understood as the group generated by two commuting involutions T_1, T_2. Specifically, we determine all possible equivariant cobordism classes of G-actions (M, Φ), $\Phi = (T_1, T_2)$, for which F_Φ is connected, $dim(F_\Phi) = n$ and $dim(M) = 4n-1$ or $4n-2$; here, F_Φ is the fixed point set of Φ, that is, $F_\Phi = \{x \in M \ / \ T_i(x) = x, \ i = 1, 2\}$. We remark that, for $dim(M) \geq 4n$, this type of classification was established in [3]. To state the results, we need to describe certain G-actions (M, Φ) with F_Φ connected and n-dimensional, and with $dim(M) = 4n-1$ and $4n-2$. Set $T_3 = T_1T_2$; given a G-action (M, Φ), the *fixed data* of Φ is $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, a closed manifold (the fixed set F_Φ) and a list of three vector bundles, $\varepsilon_1, \varepsilon_2$ and ε_3, where ε_i is the normal bundle of F_Φ in F_{T_i}, $i = 1, 2, 3$. If $(F; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is the fixed data of some G-action, then the same is true for $(F; \varepsilon_i, \varepsilon_j, \varepsilon_k)$, where (i, j, k) is any permutation of $(1, 2, 3)$; this is realized by new G-actions obtained from permutations of (T_1, T_2, T_3). Denote by $R \to X$ the trivial one-dimensional vector bundle over any base space X and by $\tau(F) \to F$ the tangent bundle of F, where F is a manifold. In order to obtain the above mentioned G-actions, the crucial point is the following *section theorem* of [4]: suppose (M^m, Φ) a G-action with fixed data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, and suppose that ε_1 has a section, that is, there exists a vector bundle ε_1' over F_Φ so that $\varepsilon_1' \oplus R$ is equivalent to ε_1. Then there exists a G-action (N^{n-1}, Ψ) having $(F_\Phi; \varepsilon_1', \varepsilon_2, \varepsilon_3)$ as fixed data. This will be combined with the fact that, from an involution (W, T), we can form the G-action $(W \times W, \Phi)$, $\Phi = (T_1, T_2)$, where $T_1(x, y) = (T(x), T(y))$ and $T_2(x, y) = (y, x)$; we denote this G-action by $\Gamma(W, T)$. The fixed data of $\Gamma(W, T)$ is $(F_T; \tau(F_T), \eta, \eta)$, where $\eta \to F_T$ is the normal bundle of F_T in W.

A) G-actions (M^m, Φ) with $m = 4n - 1$: let F^n be a connected and closed n-dimensional manifold such that $\tau(F^n)$ is cobordant, as a bundle, to a vector bundle $\mu^n \to F^n$ which has a section, say $\mu^n \cong \nu^{n-1} \oplus R$. From [2], one knows that there is an involution (W, T), equivariantly cobordant to the twist involution on

$F^n \times F^n$, with fixed set F^n and with μ^n being the normal bundle of F^n in W. The fixed data of $\Gamma(W,T)$ then is $(F^n; \tau(F^n), \mu^n, \mu^n)$, and by removing one section one obtains a G-action (N^{4n-1}, Ψ) with fixed data $(F^n; \tau(F^n), \mu^n, \nu^{n-1})$. Evidently, this includes the case in which $\tau(F^n)$ has itself a section; for example, when F^n is any closed manifold with n odd. In this case, these actions are obtained by removing one section from the fixed data of the action $(F^n \times F^n \times F^n \times F^n, T_1, T_2)$, $T_1(x,y,z,w) = (y,x,w,z)$, $T_2(x,y,z,w) = (z,w,x,y)$.

B) G-actions (M^m, Φ) with $m = 4n-2$: we describe two types of such actions. First, take F^n as in A), and suppose that also ν^{n-1} has a section, $\nu^{n-1} \cong \theta^{n-2} \oplus R$. By removing this additional section, one obtains a G-action (Q^{4n-2}, φ) with fixed data $(F^n; \tau(F^n), \mu^n, \theta^{n-2})$. Second, one takes the G-actions $\Gamma(W,T)$, where (W,T) is any involution with $dim(W) = 2n-1$ and with F_T connected and n-dimensional, that is, equivariantly cobordant to the ones described by Stong. This includes the case where, taking F^n as in A), we remove two sections to get a G-action with fixed data $(F^n; \tau(F^n), \nu^{n-1}, \nu^{n-1})$.

The desired classification is given by the following

Theorem. *Let (M^m, Φ) be a Z_2^2-action whose fixed point set F_Φ is connected and n-dimensional, and where either $m = 4n-1$ or $m = 4n-2$. Then, either (M^m, Φ) bounds equivariantly, or it is, up to permutations, equivariantly cobordant to an action of type A) if $m = 4n-1$ and of type B) if $m = 4n-2$.*

Note. For involutions, the connectedness of the fixed point set is redundant, since any involution is equivariantly cobordant to an involution with the property that the p-dimensional part of the fixed set is connected. However, this is not true for Z_2^2-actions, since in this case the fixed data over different components of the p-dimensional part of the fixed set may have different lists of dimensions.

2. **Proofs of the results**

In order to prove the stated result, we need some preliminaries about G-actions. Let (M, Φ), $\Phi = (T_1, T_2)$, be a G-action with fixed data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$. Each

s-dimensional component of $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ can be considered as an element of $\mathcal{N}_s(BO(n_1) \times BO(n_2) \times BO(n_3))$, the bordism of s-dimensional manifolds with a map into $BO(n_1) \times BO(n_2) \times BO(n_3)$, where n_i is the dimension of ε_i over the component and $BO(n_i)$ is the classifying space for n_i-dimensional vector bundles (this is the *simultaneous cobordism* between lists of bundles). According to [6], two G-actions are equivariantly cobordant if and only if they have fixed data simultaneously cobordant. Also, if (M, Φ) has fixed data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is simultaneously cobordant to $(F; \mu_1, \mu_2, \mu_3)$, then there exists a G-action (N, Ψ) with fixed data $(F; \mu_1, \mu_2, \mu_3)$, hence equivariantly cobordant to (M, Φ). The following result is the lemma found in [3; Section 3; page 108], particularized to Z_2^2-actions.

Lemma 2.1. *Let (M, Φ) be a G-action with fixed data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, where F_Φ is connected and n-dimensional. Then,*

a) if $dim(\varepsilon_1) > n$, $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ bounds simultaneously;

b) if $dim(\varepsilon_1) = n$, $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is simultaneously cobordant to $(F_\Phi; \tau(F_\Phi), \varepsilon_2, \varepsilon_3)$.

Now consider a n-dimensional vector bundle $\varepsilon \to F$, where F is a connected, closed and n-dimensional manifold, and let $RP(\varepsilon) \to F$ be the real projective space bundle associated to ε. Denote by $\lambda \to RP(\varepsilon)$ the line bundle of the double cover $S(\varepsilon) \to RP(\varepsilon)$, $S(\varepsilon)$ the sphere bundle of ε. Suppose that μ and ν are two additional vector bundles over F, with $dim(\mu) = p$, $dim(\nu) = q$ and $p \geq q$. The crucial point of the argument to be used is the following

Lemma 2.2. *Suppose that the list $(RP(\varepsilon); \lambda, \mu \oplus (\nu \otimes \lambda))$ bounds as an element of $\mathcal{N}_{2n-1}(BO(1) \times BO(p+q))$. Then the list $(F; \varepsilon, \mu, \nu)$ is simultaneously cobordant to the list $(F; \varepsilon, \nu \oplus (p-q)R, \nu)$. Here, $(p-q)R$ means the Whitney sum of $p-q$ copies of R, and the bundles μ and ν are considered over $RP(\varepsilon)$ via pullbacks by the projection.*

Proof. One lets $W(F) = 1 + w_1 + \cdots + w_n$, $W(\varepsilon) = 1 + v_1 + \cdots + v_n$, $W(\mu) = 1 + u_1 + \cdots + u_p$, $W(\nu) = 1 + \theta_1 + \cdots + \theta_q$ and $W(\lambda) = 1 + c$ be the Stiefel-Whitney classes of F, ε, μ, ν and λ. One knows that the Stiefel-Whitney class of $RP(\varepsilon)$ is

$$W(RP(\varepsilon)) = (1 + w_1 + ... + w_n).\{(1 + c)^n + v_1(1 + c)^{n-1} + ... + v_{n-1}(1 + c) + v_n\}$$

and the Stiefel-Whitney class of the bundle $\mu \oplus (\nu \otimes \lambda)$ is

$$W(\mu \oplus (\nu \otimes \lambda)) = (1 + u_1 + \cdots + u_p).\{(1 + c)^q + \theta_1(1 + c)^{q-1} + ... + \theta_{q-1}(1 + c) + \theta_q\} \ .$$

Because the list $(RP(\varepsilon); \lambda, \mu \oplus (\nu \otimes \lambda))$ is a simultaneous boundary, any class of dimension $2n - 1$ given by a product of classes $w_i(RP(\varepsilon))$, c, $w_j(\mu \oplus (\nu \otimes \lambda))$, evaluated on the fundamental homology class $[RP(\varepsilon)]$, gives a zero characteristic number. For any r, one lets

$$W[r] = \frac{W(RP(\varepsilon))}{(1 + c)^{n-r}} \quad \text{and} \quad V[r] = \frac{W(\mu \oplus (\nu \otimes \lambda))}{(1 + c)^{q-r}} \ .$$

That is,

$$W[r] = (1+w_1+\cdots+w_n)\{(1+c)^r+v_1(1+c)^{r-1}+...+v_r+\frac{v_{r+1}}{1+c}+...+\frac{v_n}{(1+c)^{n-r}}\}$$

and

$$V[r] = (1+u_1+\cdots+u_p)\{(1+c)^r+\theta_1(1+c)^{r-1}+...+\theta_r+\frac{\theta_{r+1}}{1+c}+...+\frac{\theta_q}{(1+c)^{q-r}}\} \ .$$

The classes $W[r]_t$ and $V[r]_l$ are polynomials in $w_i(RP(\varepsilon))$, c, $w_j(\mu\oplus(\nu\otimes\lambda))$, hence they can be used to give characteristic numbers; also, for these classes, one has the following special properties (see [5]):

$$W[r]_{2r} = w_r c^r + \text{terms with smaller } c \text{ powers},$$
$$W[r]_{2r+1} = (w_{r+1} + v_{r+1})c^r + \text{terms with smaller } c \text{ powers},$$
$$W[r]_{2r+2} = v_{r+1}c^{r+1} + \text{terms with smaller } c \text{ powers},$$

and in the same way,

$$V[r]_{2r} = u_r c^r + \text{terms with smaller } c \text{ powers},$$
$$V[r]_{2r+1} = (u_{r+1} + \theta_{r+1})c^r + \text{terms with smaller } c \text{ powers},$$
$$V[r]_{2r+2} = \theta_{r+1}c^{r+1} + \text{terms with smaller } c \text{ powers}.$$

For a sequence $\omega = (i_1, ..., i_s)$ of natural numbers, one lets $|\omega| = i_1 + ... + i_s$, and for $w = 1 + w_1 + ... + w_p$, one lets $w_\omega = w_{i_1}...w_{i_s}$ be the product of the classes w_i. Then given sequences $\omega = (i_1, ..., i_s)$, $\omega' = (j_1, ..., j_t)$, $\beta = (a_1, ..., a_k)$ and $\beta' = (b_1, ..., b_l)$, and a natural number $1 \leq r \leq p$ with $|\omega| + |\omega'| + |\beta| + |\beta'| + r = n$, one may form the class

$$X = (\prod_{i\in\omega} W[i]_{2i}).(\prod_{i\in\omega'} W[i-1]_{2i}).(\prod_{i\in\beta} V[i]_{2i}).(\prod_{i\in\beta'} V[i-1]_{2i}).V[r-1]_{2r-1} \ .$$

Since X has dimension $2n-1$, it gives the zero characteristic number $X[RP(\varepsilon)]$. From the properties above listed, one has

$\prod_{i\in\omega} W[i]_{2i} = W(F)_\omega.c^{|\omega|}$ + terms with smaller c powers,
$\prod_{i\in\omega'} W[i-1]_{2i} = W(\varepsilon)_{\omega'}.c^{|\omega'|}$ + terms with smaller c powers,
$\prod_{i\in\beta} V[i]_{2i} = W(\mu)_\beta.c^{|\beta|}$ + terms with smaller c powers,
$\prod_{i\in\beta'} V[i-1]_{2i} = W(\nu)_{\beta'}.c^{|\beta'|}$ + terms with smaller c powers and
$V[r-1]_{2r-1} = (u_r+\theta_r).c^r$ + terms with smaller c powers.

It follows that X has the form

$$X = W(F)_\omega.W(\varepsilon)_{\omega'}.W(\mu)_\beta.W(\nu)_{\beta'}.(u_r+\theta_r).c^{n-1} + \text{terms with smaller } c \text{ powers.}$$

Now if a term of dimension $2n-1$ involves a power of c less than $n-1$, it necessarily has a factor of dimension greater than n coming from the cohomology of F, which is zero. Also one knows that $H^*(RP(\varepsilon);Z_2)$ is the free $H^*(F;Z_2)$ module on $1, c, c^2, ..., c^{n-1}$. Therefore

$$0 = X[RP(\varepsilon)] = W(F)_\omega.W(\varepsilon)_{\omega'}.W(\mu)_\beta.W(\nu)_{\beta'}.(u_r+\theta_r).c^{n-1}[RP(\varepsilon)] = W(F)_\omega.W(\varepsilon)_{\omega'}.W(\mu)_\beta.W(\nu)_{\beta'}.(u_r+\theta_r)[F]$$

and thus

$$W(F)_\omega.W(\varepsilon)_{\omega'}.W(\mu)_\beta.W(\nu)_{\beta'}.u_r[F] = W(F)_\omega.W(\varepsilon)_{\omega'}.W(\mu)_\beta.W(\nu)_{\beta'}.\theta_r[F]\ .$$

This says that any class u_r in a characteristic number of $(F;\varepsilon,\mu,\nu)$ can be replaced by θ_r without changing the value of the characteristic number. In particular, if $q < r \leq p$, the class u_r can be replaced by the zero class, and thus $(F;\varepsilon,\mu,\nu)$ and $(F;\varepsilon,\nu\oplus(p-q)R,\nu)$ have the same characteristic numbers, which gives the result. □

Lemma 2.3. *Let (M,Φ) be a G-action with fixed data $(F;\varepsilon,\mu,\nu)$, where F is connected, dim(F) = dim$(\varepsilon) = n$, dim$(\mu) = p$ and dim$(\nu) = q$, with $p \geq q$. Then (M,Φ) is equivariantly cobordant to an action obtained by removing $p-q$ sections from the third bundle of the fixed data of the action $\Gamma(W,T)$, where (W,T) is an involution fixing F and with the normal bundle of F in W being $\nu\oplus(p-q)R$ (if $p=q$, (M,Φ) is equivariantly cobordant to the action $\Gamma(W,T)$, where (W,T) is an involution with fixed set and normal bundle $\nu \to F$).*

Proof. From the argument outlined in [3; Section 2; pages 107 and 108], particularized to Z_2^2-actions, one has that the list $(RP(\varepsilon);\lambda,\mu\oplus(\nu\otimes\lambda))$ bounds as an element of $\mathcal{N}_{2n-1}(BO(1) \times BO(p+q))$. First using Lemma 2.2 for $(F; \varepsilon, \mu, \nu)$, and next using part b) of Lemma 2.1 for $(F; \varepsilon, \nu \oplus (p-q)R, \nu)$, one concludes that (M, Φ) is equivariantly cobordant to a G-action (N, Ψ), $\Psi = (T_1, T_2)$, with fixed data $(F; \tau(F), \nu \oplus (p-q)R, \nu)$. Let $W \subset N$ be the component of F_{T_2} that contains F. Then the involution (W, T_1) fixes F with normal bundle $\nu \oplus (p-q)R \to F$, and thus (N, Ψ) (hence (M, Φ)) is equivariantly cobordant to a G-action obtained by removing $p-q$ sections from the third bundle of the fixed data of $\Gamma(W, T_1)$. $\square$

Now we prove the result. Let $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ denote the fixed data of the G-action (M^m, Φ). If some ε_i has dimension greater than n, then part a) of Lemma 2.1, used up to permutation, says that $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ bounds simultaneously, and thus (M^m, Φ) bounds equivariantly. Therefore we can suppose that $dim(\varepsilon_i) \leq n$ for $i = 1, 2, 3$. By making permutations if necessary, we can suppose that $(dim(\varepsilon_1), dim(\varepsilon_2), dim(\varepsilon_3))$ then is $(n, n, n-1)$ if $m = 4n-1$, and either $(n, n, n-2)$ or $(n, n-1, n-1)$ if $m = 4n-2$. In the case $(n, n, n-1)$ $((n, n, n-2))$, Lemma 2.3 says that (M^m, Φ) is equivariantly cobordant to an action obtained by removing one section (two sections) from the third bundle of the fixed data of the action $\Gamma(W, T)$, where (W, T) is an involution with fixed set and normal bundle $\varepsilon_3 \oplus R \to F_\Phi$ $(\varepsilon_3 \oplus 2R \to F_\Phi)$. Since $dim(F_\Phi) = n$ and $dim(W) = 2n$, one has from [1] that (W, T) is equivariantly cobordant to $(F_\Phi \times F_\Phi, twist)$, and thus $\varepsilon_3 \oplus R \to F_\Phi$ $(\varepsilon_3 \oplus 2R \to F_\Phi)$ is cobordant to $\tau(F_\Phi)$. This means that (M^m, Φ) is equivariantly cobordant to an action of type A) (of type B)).

In the case $(n, n-1, n-1)$, Lemma 2.3 says that (M^m, Φ) is equivariantly cobordant to the action $\Gamma(W, T)$, where (W, T) is an involution with fixed set and normal bundle $\varepsilon_3 \to F_\Phi$. Since $dim(F_\Phi) = n$ and $dim(W) = 2n-1$, (W, T) is of the Stong's type, and thus (M^m, Φ) is equivariantly cobordant to an action of type B). This ends the proof.

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Pedro L. Q. Pergher
Departamento de Matemática
Universidade Federal de São Carlos
Caixa Postal 676
São Carlos, SP 13565-905, Brazil
E-mail: *pergher@dm.ufscar.br*