TWO COMMUTING INVOLUTIONS FIXING $F^N \cup F^{N-1}$

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1. Introduction.

In [1], C. Kosniowski and R. E. Stong proved the following result: if M^m is a closed and smooth m-dimensional manifold with a smooth involution $T : M^m \mapsto M^m$ such that the fixed point set F_T of T has constant dimension n, and if $m > 2n$, then (M^m, T) bounds equivariantly. Later, in [6], P. Pergher extended this result to Z_2^k-actions, under the restriction that the fixed set is connected: if (M^m, Φ) is a Z_2^k-action whose fixed set F_Φ has dimension n and is connected, and if $m > 2^k n$, then (M^m, Φ) bounds equivariantly. Connectedness is redundant for $k = 1$, since any involution is equivariantly cobordant to an involution with the property that the p-dimensional part of the fixed set is connected. However, this is not true for $k > 1$, since in this case different components of the p-dimensional part of the fixed set may have different normal representations.

In this paper we start the study of the case where, besides n-dimensional components, the fixed point set contains additional $(n-1)$-dimensional componentes. We will focus our attention on Z_2^2-actions; here, $G = Z_2^2$ will be considered as the group generated by two commuting involutions T_1, T_2; set $T_3 = T_1T_2$. Given a G-action (M, Φ), $\Phi = (T_1, T_2)$, the *fixed-data* of Φ is $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, a closed manifold (the fixed point set F_Φ of Φ) and a list of three vector bundles, $\varepsilon_1, \varepsilon_2$ and ε_3, where ε_i is the normal bundle of F_Φ in F_{T_i}, $i = 1, 2, 3$. We will prove the following

Theorem 1. *Let (M, Φ) be a smooth G-action on a closed smooth manifold whose fixed set F_Φ has the form $F_\Phi = F^n \cup F^{n-1}$, where F^n and F^{n-1} are connected submanifolds with dimensions n and $n-1$, respectively. Let*

$$(F^n; \varepsilon_1, \varepsilon_2, \varepsilon_3) \cup (F^{n-1}; \mu_1, \mu_2, \mu_3)$$

be the fixed-data of Φ. Suppose that at least two $\varepsilon_{i's}$, and at least two $\mu_{i's}$, have dimension greater than n. Then (M, Φ) bounds equivariantly.

As a consequence, we will see that the the Kosniowski-Stong result above mentioned is still valid when additional $(n-1)$-dimensional components occur:

Theorem 2. *If $T : M^m \mapsto M^m$ is an involution whose fixed set F_T has the form $F_T = F^n \cup F^{n-1}$, with F^n (and F^{n-1}) not necessarily connected, and with $m > 2n$, then (M^m, T) bounds equivariantly.*

It is interesting to observe that the above facts are not valid for $F^n \cup F^{n-2}$, as we will see with an example given in the end of the paper.

2. Technical Preliminaries.

Let (M, Φ), $\Phi = (T_1, T_2)$, be a G-action with fixed-data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$. Each s-dimensional component of $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ can be considered as an element of $\mathcal{N}_s(BO(n_1) \times BO(n_2) \times BO(n_3))$, the bordism of s-dimensional manifolds with a map into $BO(n_1) \times BO(n_2) \times BO(n_3)$, where n_i is the dimension of ε_i over the component and $BO(n_i)$ is the classifying space for n_i-dimensional vector bundles (this is the *simultaneous cobordism* between lists of bundles). According to [7], the equivariant cobordism class of (M, Φ) is determined by the simultaneous cobordism class of $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Now let $F_0 \subset M$ be any component of F_{T_1}. Write $l = dim(F_0)$, and denote by $F_0^i \subset F_0$, $0 \leq i < l$, the union of the i-dimensional components of F_Φ that are contained in F_0. Then, for each $0 \leq i < l$, one has that $dim(\varepsilon_2) + dim(\varepsilon_3)$ is equal to $dim(M) - l$ over F_0^i; set $m = dim(M) - l$. Consider $RP(\varepsilon_1) \mapsto F_0^i$ the real projective space bundle associated to $\varepsilon_1 \mapsto F_0^i$, and denote by $\xi \mapsto RP(\varepsilon_1)$ the line bundle of the double cover $S(\varepsilon_1) \mapsto RP(\varepsilon_1)$, $S(\varepsilon_1)$ the sphere bundle of ε_1. Then, for each $0 \leq i < l$, one has the object

$$(RP(\varepsilon_1); \xi, \varepsilon_2 \oplus (\varepsilon_3 \otimes \xi)),$$

where $\varepsilon_1, \varepsilon_2$ and ε_3 are considered over F_0^i. This object represents an element in the bordism group $\mathcal{N}_{l-1}(BO(1) \times BO(m))$.

Lemma 2.1. *The object*

$$\bigcup_{i=0}^{l-1} \big(RP(\varepsilon_1); \xi, \varepsilon_2 \oplus (\varepsilon_3 \otimes \xi)\big)$$

bounds as an element of $\mathcal{N}_{l-1}(BO(1) \times BO(m))$.

Proof. This follows from the argument outlined in [5; Section 3; pages 88, 89 and 90] (or in [6; Section 2; pages 107 and 108]), particularized to $k = 2$ and adapted to the situation in which F_Φ may have several components. ∎

Remark. We observe that if $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is fixed-data of a G-action, then the same is true for $(F_\Phi; \varepsilon_i, \varepsilon_j, \varepsilon_k)$, where (i, j, k) is any permutation of $(1, 2, 3)$. Then, in the above lemma, $\cup(RP(\varepsilon_1); \xi, \varepsilon_2 \oplus (\varepsilon_3 \otimes \xi))$ can be replaced by $\cup(RP(\varepsilon_i); \xi, \varepsilon_j \oplus (\varepsilon_k \otimes \xi))$ for any permutation (i, j, k) of $(1, 2, 3)$.

3. Proofs of the Theorems.

We first need the following

Lemma 3.1. *Let (M, Φ) be a G-action with fixed-data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, and suppose that $V \subset M$ is a n-dimensional component of F_Φ. Let P be the component of F_{T_1} that contains V. Suppose that P satisfies the following conditions:*

i) $dim(P) > 2n$;

ii) V is the unique component of F_Φ contained in P.

Then $(V; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ bounds simultaneously.

Proof. One lets

$$W(V) = 1 + w_1 + \cdots + w_n$$

be the Stiefel-Whitney class of V and

$$W(\varepsilon_i) = 1 + v_1^i + \cdots + v_{n_i}^i$$

be the Stiefel-Whitney class of $\varepsilon_i \mapsto V$, $i = 1, 2, 3$, where $n_i = dim(\varepsilon_i)$. Since $\varepsilon_1 \mapsto V$ is the normal bundle of V in P, one has $n_1 > n$. Throughout this proof we consider each ε_i as a bundle either over V or over $RP(\varepsilon_1) \mapsto V$, and in this last case we are suppressing bundles maps. Letting $c \in H^1(RP(\varepsilon_1); Z_2)$ be the first Stiefel-Whitney class of the line bundle ξ for the double cover $S(\varepsilon_1) \mapsto RP(\varepsilon_1)$, one knows that the Stiefel-Whitney class of $RP(\varepsilon_1)$ is

$$W(RP(\varepsilon_1)) = (1 + w_1 + \cdots + w_n)\{(1+c)^{n_1} + v_1^1(1+c)^{n_1-1} + \cdots + v_{n_1}^1\},$$

the Stiefel-Whitney class of ξ is

$$W(\xi) = 1 + c,$$

and the Stiefel-Whitney class of the bundle $\varepsilon_2 \oplus (\varepsilon_3 \otimes \xi)$ is

$$W(\varepsilon_2 \oplus (\varepsilon_3 \otimes \xi)) = (1 + v_1^2 + \cdots + v_{n_2}^2)\{(1+c)^{n_3} + v_1^3(1+c)^{n_3-1} + \cdots + v_{n_3}^3\}.$$

From Lemma 2.1 (with $P = F_0$ and $V = \cup_{i=0}^{l-1} F_0^i$) one has that $(RP(\varepsilon_1); \xi, \varepsilon_2 \oplus (\varepsilon_3 \otimes \xi))$ is a boundary in the bordism group $\mathcal{N}_{l-1}(BO(1) \times BO(n_2 + n_3))$, where $l = dim(P)$. Then any class of dimension $l - 1$ given by a product of the classes

$$w_i(RP(\varepsilon_1)), c, w_j(\varepsilon_2 \oplus (\varepsilon_3 \otimes \xi))$$

gives a zero characteristic number for $RP(\varepsilon_1)$. We will apply this using certain special classes, which are polynomials in the above-displayed ones, and were initially introduced in [4] and also used in [5] and in [6]. The argument is identical with that of [6; part (a) of Lemma of Section 3]; to ease the reading and mainly to establish some notations, we will rewrite it.

For any integer r, one lets

$$W[r] = \frac{W(RP(\varepsilon_1))}{(1+c)^{n_1-r}},$$

and

$$U[r] = \frac{W(\varepsilon_2 \oplus (\varepsilon_3 \otimes \xi))}{(1+c)^{n_3-r}}.$$

These classes satisfy the following special properties:

$$\begin{aligned}
W[r]_{2r-1} &= w_{r-1}c^r + \text{ terms with smaller } c \text{ powers},\\
W[r]_{2r} &= w_r c^r + \text{ terms with smaller } c \text{ powers},\\
W[r]_{2r+1} &= (w_{r+1} + v_{r+1})c^r + \text{ terms with smaller } c \text{ powers},\\
W[r]_{2r+2} &= v^1_{r+1}c^{r+1} + \text{ terms with smaller } c \text{ powers},
\end{aligned}$$

and in the same way

$$\begin{aligned}
U[r]_{2r-1} &= v^2_{r-1}c^r + \text{ terms with smaller } c \text{ powers},\\
U[r]_{2r} &= v^2_r c^r + \text{ terms with smaller } c \text{ powers},\\
U[r]_{2r+1} &= (v^2_{r+1} + v^3_{r+1})c^r + \text{ terms with smaller } c \text{ powers},\\
U[r]_{2r+2} &= v^3_{r+1}c^{r+1} + \text{ terms with smaller } c \text{ powers}.
\end{aligned}$$

We need some more notations: for a sequence of integers $\omega = (j_1, j_2, \dots, j_s)$, one lets $|\omega| = j_1 + j_2 + \cdots + j_s$, and for $V = 1 + v_1 + \cdots + v_p$, one lets $V_\omega = v_{j_1}v_{j_2}\cdots v_{j_s}$ be the product of the classes v_j. Our goal is to show that every characteristic number of $(V;\varepsilon_1,\varepsilon_2,\varepsilon_3)$ is zero. A general number of this type is obtained by taking sequences $\omega = (j_1, j_2, \dots, j_s)$ and $\omega_i = (j^i_1, j^i_2, \dots, j^i_s)$ for $i = 1, 2, 3$, with $|\omega| + \sum_{i=1}^3 |w_i| = n$; the number is then

$$W(V)_\omega \cdot \prod_{i=1}^{3} W(\varepsilon_i)_{\omega_i}[V].$$

To show that this number is zero, one forms first the class

$$X = \prod_{j\in\omega} W[j]_{2j} \cdot \prod_{j\in\omega_1} W[j-1]_{2j} \cdot \prod_{j\in\omega_2} U[j]_{2j} \cdot \prod_{j\in\omega_3} U[j-1]_{2j} \cdot c^{n_1-(n+1)},$$

which is possible since $n_1 \geq n+1$. Since $|\omega| + \sum_{i=1}^3 |w_i| = n$, X is a characteristic class of $RP(\varepsilon_1)$ of dimension $2n + n_1 - n - 1 = n + n_1 - 1 = l - 1$; thus

$$X[RP(\varepsilon_1)] = 0$$

From the properties above listed, one has

$$\begin{aligned}
\prod_{j\in\omega} W[j]_{2j} &= \prod_{j\in\omega}(w_j c^j + \text{ terms with smaller } c \text{ powers})\\
&= W(V)_\omega c^{|\omega|} + \text{ terms with smaller } c \text{ powers},
\end{aligned}$$

and

$$\begin{aligned}
\prod_{j\in\omega_1} W[j-1]_{2j} &= \prod_{j\in\omega_1} W[j-1]_{2(j-1)+2}\\
&= \prod_{j\in\omega_1} (v^1_j c^j + \text{ terms with smaller } c \text{ powers})\\
&= W(\varepsilon_1)_{\omega_1} c^{|\omega_1|} + \text{ terms with smaller } c \text{ powers}.
\end{aligned}$$

Similarly,

$$\prod_{j\in\omega_2} U[j]_{2j} = W(\xi_2)_{\omega_2} c^{|\omega_2|} + \text{ terms with smaller } c \text{ powers},$$

and

$$\prod_{j\in\omega_3} U[j-1]_{2j} = W(\xi_3)_{\omega_3} c^{|\omega_3|} + \text{ terms with smaller } c \text{ powers}.$$

Thus X has the form

$$X = W(V)_\omega \cdot \prod_{i=1}^{3} W(\varepsilon_i)_{\omega_i} c^{n_1-1} + \text{ terms with smaller } c \text{ powers}.$$

Now, if a term (of dimension $l-1$) has power of c less than n_1-1, it necessarily has a factor of dimension greater than n form the cohomology of V, which is zero. Also one knows that $H^*(RP(\varepsilon_1), Z_2)$ is the free $H^*(V, Z_2)$-module on $1, c, c^2, \dots, c^{n_1-1}$. It follows that

$$0 = X[RP(\varepsilon_1)] = W(V)_\omega \cdot \prod_{i=1}^{3} W(\varepsilon_i)_{\omega_i} c^{n_1-1}[RP(\varepsilon_1)] = W(V) \cdot \prod_{i=1}^{3} W(\varepsilon_i)_{\omega_i}[V],$$

which completes the proof. ■

We now proceed to prove Theorem 1. We are then considering a G-action (M, Φ) fixing $F^n \cup F^{n-1}$, where F^n and F^{n-1} are connected, with fixed-data

$$(F^n; \varepsilon_1, \varepsilon_2, \varepsilon_3) \cup (F^{n-1}; \mu_1, \mu_2, \mu_3),$$

and where at least two $\varepsilon_{i's}$ (and at least two $\mu_{i's}$) have dimension greater than n. For $i = 1, 2, 3$, denote by P_i the component of F_{T_i} containing F^n, and by Q_i the component containing F^{n-1}. Then either $P_i = Q_i$ or $P_i \cap Q_i = \emptyset$.

Lemma 3.2. *There exist at least one $i \in \{1, 2, 3\}$ for which $P_i \cap Q_i = \emptyset$.*

Proof. Suppose, by contradiction, that $P_i = Q_i$ for $i = 1, 2$ and 3. Set

$$n_i = dim(\varepsilon_i), \qquad m_i = dim(\mu_i).$$

Since, for each i, F^n and F^{n-1} belong to the same component, one has $m_i = n_i+1$. Then

$$dim(M) = n + n_1 + n_2 + n_3 = (n-1) + m_1 + m_2 + m_3 = n + n_1 + n_2 + n_3 + 2,$$

which is impossible. ■

The above lemma say us that the number of $i \in \{1, 2, 3\}$ such that $P_i \cap Q_i = \emptyset$ is 1, 2 or 3. Let us suppose first that this number is 2 or 3. Because of the hypothesis concerning the number of bundles with dimension greater than n, there exist $i, j \in \{1, 2, 3\}$ so that

$$P_i \cap Q_i = \emptyset, \ P_j \cap Q_j = \emptyset, \ dim(\varepsilon_i) > n \quad \text{and} \quad dim(\mu_j) > n.$$

By applying Lemma 3.1 on the componets $F^n \subset P_i$ and $F^{n-1} \subset Q_j$, one concludes that $(F^n; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $(F^{n-1}; \mu_1, \mu_2, \mu_3)$ bound simultaneously, which implies that (M, Φ) bounds equivariantly.

In this way, from now we can suppose that there exists an unique $i \in \{1,2,3\}$ such that $P_i \cap Q_i = \emptyset$. Set again $n_i = dim(\varepsilon_i)$, $i = 1,2,3$. By making permutations on $\{1,2,3\}$ if necessary, we can suppose with no loss that

$$P_1 = Q_1, \; P_2 \cap Q_2 = \emptyset, \; P_3 = Q_3 \quad \text{and} \quad n_1 > n.$$

Since $P_1 = Q_1$ and $P_3 = Q_3$, one has $dim(\mu_1) = n_1 + 1$ and $dim(\mu_3) = n_3 + 1$. Now

$$n + n_1 + n_2 + n_3 = (n-1) + (n_1 + 1) + dim(\mu_2) + (n_3 + 1),$$

thus $dim(\mu_2) = n_2 - 1$.

We will repeat the notation used in the proof of Lemma 3.1 for the suitable characteristic classes, with F^n playing the role of V. That is, we write

$$\begin{aligned} W(F^n) &= 1 + w_1 + \cdots + w_n, \\ W(\varepsilon_i) &= 1 + v_1^i + \cdots + v_{n_i}^i, \; i = 1,2,3, \\ W(\xi) &= 1 + c, \end{aligned}$$

where $\xi \mapsto RP(\varepsilon_1)$ is the usual line bundle. As before, one has

$$W(RP(\varepsilon_1)) = (1 + w_1 + \cdots + w_n)\{(1+c)^{n_1} + v_1^1(1+c)^{n_1-1} + \cdots + v_{n_1}^1\},$$

and

$$W(\varepsilon_2 \oplus (\varepsilon_3 \otimes \xi)) = (1 + v_1^2 + \cdots + v_{n_2}^2)\{(1+c)^{n_3} + v_1^3(1+c)^{n_3-1} + \cdots + v_{n_3}^3\}.$$

On the component F^{n-1},we write

$$\begin{aligned} W(F^{n-1}) &= 1 + \theta_1 + \theta_2 + \cdots + \theta_{n-1}, \\ W(\mu_1) &= 1 + u_1^1 + u_2^1 \cdots + v_{n_1+1}^1, \\ W(\mu_2) &= 1 + u_1^2 + u_2^2 \cdots + v_{n_1+1}^2, \\ W(\mu_3) &= 1 + u_1^3 + u_2^3 \cdots + v_{n_1+1}^3. \end{aligned}$$

Also we denote by $\lambda \mapsto RP(\mu_1)$ the line bundle for doubel cover $S(\mu_1) \mapsto RP(\mu_1)$, and by

$$W(\lambda) = 1 + d$$

its Stiefel-Whitney class. One has

$$W(RP(\mu_1)) = (1 + \theta_1 + \cdots + \theta_{n-1})\{(1+d)^{n_1+1} + u_1^1(1+d)^{n_1} + \cdots + u_{n_1+1}^1\}$$

and

$$W(\mu_2 \oplus (\mu_3 \otimes \lambda)) = (1 + u_1^2 + u_2^2 \cdots + v_{n_2+1}^2)\{(1+d)^{n_3+1} + u_1^3(1+d)^{n_3} + \cdots + u_{n_3+1}^3\}.$$

From Lemma 2.1 (with $F_0 = P_1$ and $\cup_{i=0}^{l-1} F_0^i = F^n \cup F^{n-1}$) one has that

$$(RP(\varepsilon_1); \xi, \varepsilon_2 \oplus (\varepsilon_3 \otimes \xi))$$

is cobordant to

$$(RP(\mu_1); \lambda, \mu_2 \oplus (\mu_3 \otimes \lambda))$$

in the bordism group

$$\mathcal{N}_{n+n_1-1}(BO(1) \times BO(n_2 + n_3)).$$

Then any class of dimension $n + n_1 - 1$ given by a product of the classes

$$w_i(RP(\varepsilon_1)), \quad c, \quad w_j(\varepsilon_2 \oplus (\varepsilon_3 \otimes \xi)),$$

evaluated on $[RP(\varepsilon_1)]$, gives the same characteristic number as the one obtained by the correspondent product of the classes

$$w_i(RP(\mu_1)), \quad d, \quad w_j(\mu_2 \oplus (\mu_3 \otimes \lambda))$$

evaluated on $[RP(\mu_1)]$. Again we will apply this using some very special classes. To do this, for any integer r, consider the polynomial

$$\overline{W}_{2r} = W[r-1]_{2r} + W[r-1]_{2r-1} \cdot W[1]_1$$

associated to $RP(\varepsilon_1)$. One has

$$W[1]_1 = c + w_1 + v_1^1,$$

$$W[r-1]_{2r} = v_r^1 c^r + \text{ terms with smaller } c \text{ powers},$$

and

$$W[r-1]_{2r-1} = (w_r + v_r^1)c^{r-1} + \text{ terms with smaller } c \text{ powers}.$$

Therefore,

$$W[r-1]_{2r-1} \cdot W[1]_1 = (w_r + v_r^1)c^r + \text{ terms with smaller } c \text{ powers},$$

and

$$\overline{W}_{2r} = w_r c^r + \text{ terms with smaller } c \text{ powers}.$$

Similarly, one considers the polynomial

$$\overline{U}_{2r} = U[r-1]_{2r} + U[r-1]_{2r-1} \cdot U[1]_1$$

associated to $\varepsilon_2 \oplus (\varepsilon_3 \otimes \xi)$, which is

$$\overline{U}_{2r} = v_r^2 c^r + \text{ terms with smaller } c \text{ powers}.$$

Now take sequences of integers $\omega = (j_1, \ldots, j_s)$ and $\omega_i = (j_1^i, \ldots, j_{s_i}^i)$ for $i = 1, 2, 3$, with

$$|\omega| + \sum_{i=1}^{3} |\omega_i| = n.$$

We form the class

$$X = \prod_{j\in\omega} \overline{W}_{2j} \cdot \prod_{j\in\omega_1} W[j-1]_{2j} \cdot \prod_{j\in\omega_2} \overline{U}_{2j} \cdot \prod_{j\in\omega_3} U[j-1]_{2j} \cdot c^{n_1-n-1},$$

which is possible since $n_1 \geq n+1$. Since $|\omega| + \sum_{i=1}^{3} |\omega_i| = n$, X is a characteristic class of $RP(\varepsilon_1)$ of dimension $2n + n_1 - n - 1 = n + n_1 - 1$, and because of our previous computations of $\overline{W}_{2r}$ and $\overline{U}_{2r}$, X has the form

$$X = W(F^n)_\omega \cdot W(\varepsilon_1)_{\omega_1} \cdot W(\varepsilon_2)_{\omega_2} \cdot W(\varepsilon_3)_{\omega_3} c^{n_1-1} + \text{ terms with smaller } c \text{ powers.}$$

Again using the fact that if a term (with dimension $n + n_1 - 1$) has power of c less than $n_1 - 1$, it necessarily has a factor of dimension greater than n from the cohomology of F^n, one has

$$X = W(F^n)_\omega \cdot \prod_{i=1}^{3} W(\varepsilon_i)_{\omega_i} \cdot c^{n_1-1}.$$

Thus

$$X[RP(\varepsilon_1)] = W(F^n)_\omega \cdot \prod_{i=1}^{3} W(\varepsilon_i)_{\omega_i}[F^n]$$

yields a general characteristic number for $(F^n; \varepsilon_1, \varepsilon_2, \varepsilon_3)$. Our nest task is to analyze the class Y over F^{n-1} which corresponds to the previous X. The class over F^{n-1} which corresponds to $\overline{W}_{2r}$ is

$$\widetilde{W}_{2r} = W[r]_{2r} + W[r]_{2r-1} \cdot W[2]_1$$

assocated to $RP(\mu_1)$. In this case, one has

$$W[2]_1 = \theta_1 + u_1^1,$$
$$W[r]_{2r-1} = \theta_{r-1} d^r + \text{ terms with smaller } d \text{ powers,}$$

and

$$W[r]_{2r} = \theta_r d^r + \text{ terms with smaller } d \text{ powers.}$$

It follows that

$$W[r]_{2r-1} \cdot W[2]_1 = \theta_{r-1}(\theta_1 + u_1^1) d^r + \text{ terms with smaller } d \text{ powers}$$

and

$$\widetilde{W}_{2r} = (\theta_r + \theta_{r-1}(\theta_1 + u_1^1)) d^r + \text{ terms with smaller } d \text{ powers.}$$

Similarly, one considers

$$\widetilde{U}_{2r} = U[r]_{2r} + U[r]_{2r-1} \cdot U[2]_1,$$

associated to $\mu_2 \oplus (\mu_3 \otimes \lambda)$, which is

$$\widetilde{U}_{2r} = (u_r^2 + u_{r-1}^2(u_1^2 + u_1^3)) d^r + \text{ terms with smaller } d \text{ powers.}$$

Then the wished class Y must be

$$Y = \prod_{j\in\omega} \widetilde{W}_{2j} \cdot \prod_{j\in\omega_1} W[j]_{2j} \cdot \prod_{j\in\omega_2} \widetilde{U}_{2j} \cdot \prod_{j\in\omega_3} U[j]_{2j} \cdot d^{n_1-n-1}.$$

Note that

$$\prod_{j\in\omega} \widetilde{W}_{2j} = \left(\prod_{j\in\omega}(\theta_j + \theta_{j-1}(\theta_1 + u_1^1))\right) d^{|\omega|} + \text{ terms with smaller } d \text{ powers},$$

$$\prod_{j\in\omega_1} W[j]_{2j} = \left(\prod_{j\in\omega_1} \theta_j\right) d^{|\omega_1|} + \text{ terms with smaller } d \text{ powers},$$

$$\prod_{j\in\omega_2} \widetilde{U}_{2j} = \left(\prod_{j\in\omega_2} (u_j^2 + u_{j-1}^2(u_1^2 + u_1^3))\right) d^{|\omega_2|} + \text{ terms with smaller } d \text{ powers},$$

and

$$\prod_{j\in\omega_3} U[j]_{2j} = \left(\prod_{j\in\omega_3} u_j^2\right) d^{|\omega_3|} + \text{ terms with smaller } d \text{ powers}.$$

It follows that

$$Y = \left(\prod_{j\in\omega}(\theta_j + \theta_{j-1}(\theta_1 + u_1^1))\right) \cdot \left(\prod_{j\in\omega_1} \theta_j\right) \cdot \left(\prod_{j\in\omega_2} (u_j^2 + u_{j-1}^2(u_1^2 + u_1^3))\right) \cdot$$

$$\cdot \left(\prod_{j\in\omega_3} u_j^2\right) \cdot d^{n_1-1} + \text{ terms with smaller } d \text{ powers}.$$

Note that, in Y, the factor which is multiplied by d^{n_1-1} has dimension

$$|\omega| + \sum_{i=1}^{3} |\omega_i| = n$$

and comes from the cohomology of F^{n-1}, thus it is zero. The other terms have factors of dimension greater than n from the cohomology of F^{n-1}, thus they are also zero. Hence $Y = 0$. Putting together with the calculations on F^n, we conclude that every characteristic number of $(F^n; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is zero. Thus $(F^n; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ bounds simultaneously, which means that it can be equivariantly removed to give a G-action (N, Ψ), $\Psi = (S_1, S_2)$, equivariantly cobordant to (M, Φ) and with fixed-data $(F^{n-1}; \mu_1, \mu_2, \mu_3)$. Since, by hypothesis, one has at least one (in fact, two) μ_i with $dim(\mu_i) > n-1$, we use again Lemma 3.1 to conclude that $(F^{n-1}; \mu_1, \mu_2, \mu_3)$ bounds simultaneously. It follows that (N, Ψ) (and thus (M, Φ)) bounds equivariantly, and Theorem 1 is proved.

Now we prove Theorem 2. One are considering an involution $T : M^m \mapsto M^m$ with fixed set $F_T = F^n \cup F^{n-1}$, with $m > 2n$, and wants to show that (M^m, T) bounds equivariantly. As preveiously remarked, we can assume with no loss that F^n and F^{n-1} are connected. Suppose that $\varepsilon^{m-n} \mapsto F^n$ and $\mu^{m-n+1} \mapsto F^{n-1}$ are the normal bundles. From [3], it is suffice to show that ε^{m-n} and μ^{m-n+1} bound. In $M^m \times M^m$, consider the Z_2^2-action $\Phi = (T \times T, S)$ where $S(x, y) = (y, x)$. Then $F_\Phi = F^n \cup F^{n-1}$, and the fixed-data of Φ is

$$(F^n; \varepsilon^{m-n}, \varepsilon^{m-n}, \tau(F^n)) \cup (F^{n-1}; \mu^{m-n+1}, \mu^{m-n+1}, \tau(F^{n-1})),$$

where $\tau()$ means the tangent bundle. It follows that $(M^m \times M^m, \Phi)$ satisfies the hypotheses of Thorem 1, and thus

$$(F^n; \varepsilon^{m-n}, \varepsilon^{m-n}, \tau(F^n)) \quad \text{and} \quad (F^{n-1}; \mu^{m-n+1}, \mu^{m-n+1}, \tau(F^{n-1}))$$

bound simultaneously. In particular, ε^{m-n} and μ^{m-n+1} bound and Theorem 2 is proved.

An Example. First we recall the following construction from [3]: starting with an involution (M^n, T) with fixed-data $\eta \mapsto F$, we can form the closed $(n+1)$-dimensional manifold

$$\frac{M^n \times S^1}{T \times A},$$

where S^1 is the 1-sphere and $A : S^1 \mapsto S^1$ is the antipodal involution. On $\dfrac{M^n \times S^1}{T \times A}$ one has the involution $t[m, z] = [m, \overline{z}]$, where $\overline{z}$ means complex conjugation. The fixed-data of t is

$$(\eta \oplus R \mapsto F) \cup (R \mapsto M^n),$$

where R denotes the trivial one-dimensional bundle over the manifold. If M^n bounds, $R \mapsto M^n$ also bounds, thus

$$\left(\frac{M^n \times S^1}{T \times A}, t\right)$$

is equivariantly cobordant to an involution with fixed-data $(\eta \oplus R \mapsto F)$ in this case. Call

$$\left(\frac{M^n \times S^1}{T \times A}, t\right) = \Gamma(M^n, T).$$

Now take n even, and begin with the involution $T_1 : RP^{2n-1} \mapsto RP^{2n-1}$ on the $(2n-1)$-dimensional real projective space RP^{2n-1} given by

$$T_1[x_0, x_1, \ldots, x_{2n-1}] = [-x_0, -x_1, \ldots, -x_n, x_{n+1}, \ldots, x_{2n-1}].$$

T_1 fixes the disjoint union $RP^n \cup RP^{n-2}$, and since RP^{2n-1} bounds, $\Gamma(RP^{2n-1}, T_1)$ is equivariantly cobordant to an involution (V^{2n}, T_2) also fixing $RP^n \cup RP^{n-2}$. We claim that V^{2n} bounds. It is suffice to see that

$$\frac{RP^{n-2} \times S^1}{T \times A}$$

bounds. But we can see that $\dfrac{RP^{n-2} \times S^1}{T \times A}$ is homeomorphic to the projective space bundle associated to the vector bundle $(n+1)\mu \oplus (n-1)R \mapsto RP^1$, where μ is the canonical line bundle over RP^1. From [2; Lemma 2.2], every projective space bundle over RP^1 bounds, which completes the argument.

In this way, $\Gamma(V^{2n}, T_2)$ is equivariantly cobordant to an involution (W^{2n+1}, T_3) fixing $RP^n \cup RP^{n-2}$; call $(\eta^{n+1} \mapsto RP^n) \cup (\mu^{n+3} \mapsto RP^{n-2})$ the fixed-data of T_3. In $W^{2n+1} \times W^{2n+1}$, consider the Z_2^2-action $\Phi = (T_3 \times T_3, S)$, where $S(x,y) = (y,x)$. As we remarked before, the fixed-data of Φ is

$$(RP^n; \eta^{n+1}, \eta^{n+1}, \tau(RP^n)) \cup (RP^{n-2}; \mu^{n+3}, \mu^{n+3}, \tau(RP^{n-2})).$$

Since n is even, RP^n (and also RP^{n-2}) does not bound, and thus the fixed-data of Φ (and also the fixed-data of T_3) does not bound. Therefore $(W^{2n+1} \times W^{2n+1}, \Phi)$ and (W^{2n+1}, T_3) are examples showing that Theorems 1 and 2 have not analogues for $F^n \cup F^{n-2}$.

Remark. A natural generalization for Theorem 2 (and Theorem 1) to Z_2^k-actions would be the following result: let (M, Φ), $\Phi = (T_1, \dots, T_k)$, be a smooth Z_2^k-action on a closed smooth manifold whose fixed set F_Φ has the form $F_\Phi = F^n \cup F^{n-1}$, where F^n and F^{n-1} are connected submanifolds with dimensions n and $n-1$, respectively. Let $(F^n; \{\varepsilon_\rho\}_\rho) \cup (F^{n-1}; \{\mu_\rho\}_\rho)$ be the fixed-data of Φ, where ρ runs over the $2^k - 1$ irreducible and nontrivial (real) representations of Z_2^k. Suppose that at least 2^{k-1} bundles ε_ρ have dimension greater than n, and at least 2^{k-1} bundles μ_ρ have dimension greater than $n-1$. Then (M, Φ) bounds equivariantly.

Unfortunately, we are not able to hanle this generalization, even in the case $k = 3$.

Another way to generalize Theorem 2 would be an extension of the Pergher's result of [6] to $F^n \cup F^{n-1}$: if (M^m, Φ) is a Z_2^k-action whose fixed-set F_Φ has the form $F_\Phi = F^n \cup F^{n-1}$, where F^n and F^{n-1} are connedted, and if $m > 2^k n$, then (M^m, Φ) bounds equivariantly. Again we are not able to handle this stuation.