Z_2^k-ACTIONS WITH A SPECIAL FIXED SET

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ABSTRACT. Let F^n be a connected, smooth and closed n-dimensional manifold satisfying the following property: if N^m is any smooth and closed m-dimensional manifold with $m > n$ and $T : N^m \to N^m$ is a smooth involution whose fixed point set is F^n, then $m = 2n$. In this paper we describe the equivariant cobordism classification of smooth actions (M^m, Φ) of the group $G = Z_2^k$ on closed smooth m-dimensional manifolds M^m for which the fixed point set of the action is a submanifold F^n with the above property. This generalizes a result of F. L. Capobianco, which obtained this classification for $F^n = RP^{2r}$ (P. E. Conner and E. E. Floyd had previously shown that RP^{2r} satisfies the property in question). In addition, we establish some properties concerning these F^n and give some new examples of these special manifolds.

1. Introduction.

Given a connected, smooth and closed n-dimensional manifold F^n, one has the twist involution $t : F^n \times F^n \to F^n \times F^n$, $t(x, y) = (y, x)$, and the identity involution $Id : F^n \to F^n$, $Id(x) = x$. The fixed point set of each of these involutions is F^n. We call F^n a manifold with *property* $\mathcal{H}$ if the only equivariant cobordism classes of involutions fixing F^n are $[F^n \times F^n, t]$ and $[F^n, Id]$.

In [1], C. Kosniowski and R. E. Stong showed that, if (W^{2n}, T) is an involution fixing F^n, then (W^{2n}, T) is equivariantly cobordant to $(F^n \times F^n, t)$. In this way, the above concept can be restated in terms of dimensions: F^n has property $\mathcal{H}$ if every involution (M^m, T) with fixed set F^n has $m = n$ or $m = 2n$. The real, complex and quaternionic even dimensional projective spaces RP^{2n}, CP^{2n} and HP^{2n} are examples of manifolds with property $\mathcal{H}$; also the Cayley projective plane QP^2 has this property (see [9]).

Now consider the group $G = Z_2^k = Z_2 \oplus ... \oplus Z_2$ (k copies). For each r with $1 \le r \le k$ one may construct a twist action of G on the product $(F^n)^{2^r} = F^n \times ... \times F^n$ (2^r factors),

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which we denote by t_r^k, in the following inductive way: considering G as the group generated by k commuting involutions $T_1, ..., T_k$, first set $t_1^1 = t : F^n \times F^n \to F^n \times F^n$. Supposing by inductive hypothesis one has constructed t_{k-1}^{k-1}, on $(F^n)^{2^k} = (F^n)^{2^{k-1}} \times (F^n)^{2^{k-1}}$ let $T_1, ..., T_{k-1}$ act as $t_{k-1}^{k-1} \times t_{k-1}^{k-1}$ and T_k act switching $(F^n)^{2^{k-1}} \times (F^n)^{2^{k-1}}$. This defines t_k^k for any $k \geq 1$. Finally, we define t_r^k letting $T_1, ..., T_r$ act as t_r^r and $T_{r+1}, ..., T_r$ act trivially; also we extend this definition to $r = 0$ letting t_0^k to be the trivial action on F^n. The fixed point set of t_r^k is the diagonal copy of F^n. On the other hand, given any G-action $(M; \Phi)$, $\Phi = (T_1, ..., T_k)$, each automorphism $\sigma : G \to G$ yields a new G-action given by $(M; \sigma(T_1), ..., \sigma(T_k))$; we denote this action by $\sigma(M; \Phi)$. In [2], F. Capobianco proved that every G-action $(M; \Phi)$ fixing RP^{2n} is cobordant to $\sigma((RP^{2n})^{2^r}; t_r^k)$ for some $\sigma : G \to G$ and $1 \leq r \leq k$. The main goal of this paper is to generalize this result showing that it is true for any manifold with property $\mathcal{H}$.

The crucial point of the Capobianco's argument was the following special property of RP^{2n}, proved by Stong in [9]: if $\eta^r \to RP^{2n}$ is an r-dimensional vector bundle over RP^{2n} which is the fixed data of an involution, then $r = 2n$ and the Stiefel-Whitney class of η^r is $W(\eta^r) = (1 + \alpha)^{2n+1}$, where $\alpha \in H^1(RP^{2n}, Z_2)$ is the generator. In particular, this fact implies that RP^{2n} has property $\mathcal{H}$ (the same type of argument works for CP^{2n}, HP^{2n} and QP^2). The subtle point of our method is that property $\mathcal{H}$, and not the above special property of RP^{2n}, CP^{2n}, HP^{2n} and QP^2, determines the desired result.

Section 2 will study manifolds with property $\mathcal{H}$ and give examples of manifolds with this property. In Section 3, the proof of the main result will be given; this result is the following

Theorem. *Suppose $(M; \Phi)$, $\Phi = (T_1, ..., T_k)$, is a G-action fixing F^n, where F^n has property $\mathcal{H}$. Then $(M; \Phi)$ is equivariantly cobordant to $\sigma((F^n)^{2^r}; t_r^k)$ for some automorphism $\sigma : G \to G$ and some $1 \leq r \leq k$.*

2. On manifolds with property $\mathcal{H}$.

We begin with some general facts concerning property $\mathcal{H}$.

Proposition 2.1. *If F^n has property $\mathcal{H}$, then F^n is nonbounding.*

Proof. This follows from the fact that if F^n bounds, then there are involutions of every dimension fixing F^n. In fact, consider W^{n+1} an $(n+1)$-dimensional manifold whose boundary $\partial(W^{n+1})$ is F^n. For any natural number $m > 0$, form $W^{n+1} \times D^m$ with the involution $(w, x) \to (w, -x)$, where D^m is the m-dimensional disc. Then $\partial(W^{n+1} \times D^m)$ is a closed $(n + m)$-dimensional manifold equipped with an involution whose fixed set is F^n. □

Proposition 2.2. *If F^n has property $\mathcal{H}$, then n is even.*

Proof. If n is odd it is known that, since the Euler characteristic of F^n is zero, the tangent bundle $\tau^n \to F^n$ has a section, that is, there is an $(n - 1)$-dimensional vector bundle $\mu^{n-1} \to F^n$ so that τ^n is equivalent to $\mu^{n-1} \oplus R \to F^n$, where $R \to F^n$ is the one-dimensional trivial bundle (see, for example, [3]). Since $\tau^n \to F^n$ is the fixed data of the twist involution $(F^n \times F^n, t)$, one has from the stability property of [4; Theorem 26.4] that also $\mu^{n-1} \to F^n$ is the fixed data of an involution. □

Remark. Obviously the above argument also serves to show that if F^n has property $\mathcal{H}$, then the tangent bundle $\tau^n \to F^n$ does not have a section. It is interesting to note that the reverse statement of this fact is not true. The essential point is that $\tau^n \to F^n$, while not having a section itself, may be cobordant to a bundle over F^n with a section. For example, consider the connected sum $RP^4 \# CP^2 = F^4$. The Euler characteristic of F^4 is $\chi(F^4) = \chi(RP^4) + \chi(CP^2) - 2 = 1 + 3 - 2 = 2$, thus the tangent bundle $\tau^4 \to F^4$ does not have a section. One has that $H^*(F^4, Z_2)$ is generated by $\alpha \in H^1(F^4, Z_2)$ and $\beta \in H^2(F^4, Z_2)$, with relations $\alpha^5 = 0$, $\beta^3 = 0$, $\alpha\beta = 0$ and $\alpha^4 = \beta^2$. The Stiefel-Whitney class of F^4 is $W(F^4) = 1 + \alpha + \beta$. Over F^4 there is a real line bundle $\lambda \to F^4$ with $W(\lambda) = 1 + \alpha$ and a complex line bundle $\xi \to F^4$ with $W(\xi) = 1 + \beta$, and one has $W(\lambda \oplus \xi) = (1 + \alpha)(1 + \beta) = 1 + \alpha + \beta$ (since $\alpha\beta = 0$). Thus $W(\lambda \oplus \xi) = W(\tau^4)$, and τ^4 is cobordant to $\lambda \oplus \xi \oplus R$. Then one has an involution (M^8, T) cobordant to $(F^4 \times F^4, t)$ with fixed data $\lambda \oplus \xi \oplus R \to F^4$. It follows that $\lambda \oplus \xi \to F^4$ is the fixed data of an involution and F^4 does not have property $\mathcal{H}$.

Proposition 2.3. *Suppose that F^n is the total space of a differentiable fibering of closed manifolds, where V^r is the base space, K^s is the fiber and $\pi : F^n \to V^r$ is the projection map, with $r, s > 0$. Then F^n does not have property $\mathcal{H}$.*

Proof. Consider $E \subset F^n \times F^n$, $E = \{(x, y) \,|\, \pi(x) = \pi(y)\}$. Then E is a closed $(n + s)$-dimensional submanifold of $F^n \times F^n$. On E one has the fiberwise twist involution fixing a diagonal copy of F^n, and since $s < n$ the result follows. □

A consequence of Proposition 2.3 is that property $\mathcal{H}$ is not a cobordism invariant. In fact, CP^2 has property $\mathcal{H}$ and is cobordant to $RP^2 \times RP^2$ which does not, since it fibers. As another example, consider the 3-dimensional vector bundle $\lambda \oplus R^2 \to RP^2$, where λ is the canonical line bundle and R^2 is the trivial 2-dimensional bundle over RP^2. Then the projective space bundle $RP(\lambda \oplus R^2)$ is cobordant to RP^4 (this can be checked by computing characteristic numbers) and does not have property $\mathcal{H}$, since it is a fibering. However, one has

Proposition 2.4. *Property $\mathcal{H}$ is a homotopy invariant.*

Proof. Suppose F^n does not have property $\mathcal{H}$ and is homotopy equivalent to V^n. Then there exists a vector bundle $\mu^r \to F^n$ which is the fixed data of an involution with $r < n$. Take $f : V^n \to F^n$ a homotopy equivalence. Then the pullback $f * (\mu^r) \to V^n$ is a vector bundle having the same Whitney numbers as μ^r, which implies that $f * (\mu^r)$ and μ^r are cobordant as elements of the bordism group of closed n-dimensional manifolds with r-dimensional vector bundles, $\mathcal{N}_n(BO(r))$. It follows that $f * (\mu^r)$ is also the fixed data of an involution and V^n does not have property $\mathcal{H}$. □

The following proposition gives the first new examples of manifolds with property $\mathcal{H}$.

Proposition 2.5. *If F^2 is nonbounding then F^2 has property $\mathcal{H}$.*

Proof. First we recall the following result of Stong and Kosniowski of [1]: if an involution (M^m, T) fixes F^n and $m > 2n$, then (M^m, T) bounds equivariantly. Now if (M^m, T) fixes the nonbounding F^2, the fixed data of (M^m, T) is nonbounding and hence (M^m, T) is nonbounding. It follows that $m \leq 4$. But from [4] one knows that an involution with codimension one fixed set bounds, hence $m = 3$ is impossible and the result is proved. □

Observe that Proposition 2.5 gives examples not homotopy equivalent to RP^2. Other new examples will be obtained with the following

Proposition 2.6. *Suppose F^{2n} is nonbounding and has $H^j(F^{2n}, Z_2) = 0$ for $0 < j < n$. Then F^{2n} has property $\mathcal{H}$.*

Proof. If (M^m, T) fixes F^{2n} and $\eta^r \to F^{2n}$ is the normal bundle of F^{2n} in M^m, then by the argument used in the proof of Proposition 2.5 one has $r \leq 2n$. Hence we must to show that $0 < r < 2n$ is impossible. We need the following fact, which follows from the Conner-Floyd's exact sequence of [4; 28.1]: if $\mu^r \to W$ is an r-dimensional vector bundle over a nonbounding and closed manifold W which is cobordant to the trivial r-dimensional bundle over W, then μ^r cannot be the fixed data of an involution. If $0 < r < n$, the hypothesis says that the Stiefel-Whitney class of η^r is $W(\eta^r) = 1$ and thus η^r is cobordant to the trivial r-dimensional bundle over F^{2n}. Hence we can assume $n \leq r < 2n$. By Poincarè duality, $H^j(F^{2n}, Z_2) = 0$ for $n < j < 2n$, thus the Stiefel-Whitney classes of η^r and F^{2n} can be written as $W(\eta^r) = 1 + u_n$, $W(F^{2n}) = 1 + w_n + w_{2n}$. Also the Wu class of F^{2n} can be written as $V(F^{2n}) = 1 + v_n$, and if Sq is the Steenrod operation one has

$$Sq(V(F^{2n})) = 1 + v_n + \sum_{i=1}^{n} Sq^i(v_n) = 1 + v_n + Sq^n(v_n) = 1 + v_n + v_n^2 \quad .$$

From $Sq(V(F^{2n})) = W(F^{2n})$ one gets then $v_n = w_n$ and $w_{2n} = v_n^2 = w_n^2$. Since F^{2n} does not bound, in particular one has $w_{2n} = w_n^2 \neq 0$. Our objective is to prove that η^r is cobordant to the r-dimensional trivial vector bundle over F^{2n}. To do this, it suffices to show that every Whitney number of η^r involving classes of η^r vanishes. But the only such a number which can be nonzero is

$$w_n u_n[F^{2n}] = v_n u_n[F^{2n}] = Sq^n(u_n)[F^{2n}] = u_n^2[F^{2n}] \quad .$$

Denote by $\lambda \to RP(\eta^r)$ the usual line bundle over the projective space bundle $RP(\eta^r)$, and write $W(\lambda) = 1 + c$. Since $\eta^r \to F^{2n}$ is the fixed data of an involution, $\lambda \to RP(\eta^r)$ bounds as an element of the bordism group of manifolds with line bundles, $\mathcal{N}_{2n+r-1}(BO(1))$. It follows that $c^{2n+r-1}[RP(\eta^r)] = 0$. Denoting by

$$\overline{W}(\eta^r) = \frac{1}{W(\eta^r)} = 1 + \overline{u_n} + \overline{u_{2n}}$$

the dual Stiefel-Whitney class of η^r, one has from [5] that $c^{2n+r-1}[RP(\eta^r)] = \overline{u_{2n}}[F^{2n}]$. But

$$\frac{1}{W(\eta^r)} = \frac{1}{1 + u_n} = 1 + u_n + u_n^2 \quad ,$$

which means that $\overline{u_{2n}} = u_n^2$ and $u_n^2 = 0$. □

For example, simply connected nonbounding 4-dimensional manifolds satisfy the hypotheses of Proposition 2.6. In particular, for every $k \geq 1$, the connected sum of CP^2 and k copies of $S^2 \times S^2$, $CP^2\#(S^2 \times S^2)\#...\#(S^2 \times S^2)$, has this property, and the same is valid for $HP^2\#(S^4 \times S^4)\#...\#(S^4 \times S^4)$ and $QP^2\#(S^8 \times S^8)\#...\#(S^8 \times S^8)$. Observe that these examples are not homotopy equivalent to the known examples CP^2, HP^2 and QP^2. For an 8-dimensional nonbounding M^8 with $H^1(M^8, Z_2) = 0$ and $H^2(M^8, Z_2) = 0$, the result is also valid. In fact, in this case any vector bundle $\nu \to M^8$ has $w_3(\nu) = 0$ and $w_5(\nu) = 0$, so the argument is the same. For example, besides $S^4 \times S^4$, we may add handles $S^3 \times S^5$ to HP^2. In the same way, a 16-dimensional nonbounding M^{16} with $H^j(M^{16}, Z_2) = 0$ for $1 \leq j \leq 4$ has property $\mathcal{H}$, which allows one to add handles $S^5 \times S^{11}$, $S^6 \times S^{10}$ and $S^7 \times S^9$ to QP^2.

Remark. Proposition 2.6 does not give examples of manifolds with property $\mathcal{H}$ in dimensions different from 2, 4, 8 and 16. This follows from the fact that if F^{2n} has $H^j(F^{2n}, Z_2) = 0$ for $0 < j < n$ and $n \neq 1, 2, 4$ and 8, then F^{2n} bounds. In fact, from the proof of Proposition 2.6 one has that F^{2n} is nonbounding if and only if $w_{2n}(F^{2n})$, which is equal to $w_n^2(F^{2n}) = Sq^n(w_n)(F^{2n})$, is different from zero. But Sq^n is decomposable through the Adem relations for $n \neq 2^s$, and is decomposable in terms of secondary operations for $n = 2^s \geq 16$.

Remark. Proposition 2.5 says any 2-dimensional nonbounding manifold has property $\mathcal{H}$. Now every nonbounding M^2 is an RP^2 with handles, that is, $RP^2\#(S^1 \times S^1)\#...\#(S^1 \times S^1)$. This suggests considering, as in the above examples, $F^{2n} = RP^{2n}\#(S^n \times S^n)\#...\#(S^n \times S^n)$ (k copies of $S^n \times S^n$). $H^*(F^{2n}, Z_2)$ is isomorphic to $H^*(RP^{2n}, Z_2)$ except in dimension n, where we have added cohomology classes $a_i, b_i \in H^n(F^{2n}, Z_2)$, $1 \leq i \leq k$, with each $a_i b_i$ being the nonzero element of $H^{2n}(F^{2n}, Z_2)$. The Stiefel-Whitney class of F^{2n} is still $W(F^{2n}) = (1 + \alpha)^{2n+1}$, where $\alpha \in H^1(F^{2n}, Z_2)$ is the generator coming from RP^{2n}. If (M^m, T) fixes F^{2n} and $\eta^{m-2n} \to F^{2n}$ is the normal bundle, with $W(\eta^{m-2n}) = 1 + u_1 + u_2 + ...$, then one has $u_{2^s} =$ multiple of α^{2^s} except possibly in dimension n. If n is not a power of 2, then this gives $u_{2^s} =$ multiple of α^{2^s} for each $s \geq 0$ and so $W(\eta^{m-2n}) = (1 + \alpha)^t$ for some t. Summarizing, one has an involution (M^m, T) fixing $\eta^{m-2n} \to F^{2n}$ with $W(F^{2n}) = (1 + \alpha)^{2n+1}$ and $W(\eta^{m-2n}) = (1 + \alpha)^t$; this is exactly the situation used in the proof of [4; 27.7] (which established that RP^{2r} has property $\mathcal{H}$), which gives $m - 2n = 0$ or $2n$. Thus F^{2n} has property $\mathcal{H}$ when n is not a power of 2.

3. Proof of the main theorem.

First we need some preliminaries about G-actions, $G = Z_2^k$. Given a G-action $(M; \Phi)$, $\Phi = (T_1, ..., T_k)$, the fixed point set of Φ, F, is a disjoint union of closed submanifolds of M. The normal bundle of F in M, η, decomposes under the action of G into the Whitney sum of the subbundles on which G acts as one of the irreducible (nontrivial) real representations. These irreducible and nontrivial representations"of G are all one-dimensional and can be described by homomorphisms $\rho : G \to Z_2 = \{+1, -1\}$ which are onto, and G acts on the reals so that $g \in G$ acts as multiplication by $\rho(g)$. In other words,

$$\eta = \bigoplus_{\rho} \varepsilon_\rho \quad ,$$

where ε_ρ is the subbundle of η on which G acts in the fibers as ρ, that is, where each T_i acts as multiplication by $\rho(T_i)$, and where the sum excludes the trivial homomorphism

$1 \in Hom(G, Z_2)$. Alternatively, ε_ρ is the normal bundle of F in the fixed point set F_H of the subgroup $H = ker(\rho)$. Setting $\mathcal{P} = Hom(G, Z_2) - \{1\}$, the *fixed data* of $(M; \Phi)$ is $(F, \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$, a closed manifold (the fixed set) and a list of $2^k - 1$ vector bundles over it indexed by $\mathcal{P}$. Each s-dimensional component of $(F, \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$ can be considered as an element of $\mathcal{N}_s(\prod_{\rho \in \mathcal{P}} BO(n_\rho))$, the bordism of s-dimensional manifolds with a map into a product of classifying spaces $BO(n_\rho)$ for n_ρ-dimensional vector bundles, where n_ρ denotes the dimension of ε_ρ over the component (this is the *simultaneous cobordism* between lists of vector bundles: if $\mathcal{P}$ is any finite set, two lists (indexed by $\mathcal{P}$) of vector bundles over closed n-dimensional manifolds, $(F^n, \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$ and $(V^n, \{\mu_\rho\}_{\rho \in \mathcal{P}})$, are *simultaneously cobordant* if there exists a $(n+1)$-dimensional manifold W^{n+1} with boundary $\partial(W^{n+1}) = F^n \cup V^n$ (disjoint union) and a list of vector bundles over W^{n+1}, $(W^{n+1}, \{\eta_\rho\}_{\rho \in \mathcal{P}})$, so that each η_ρ restricted to $F^n \cup V^n$ is equivalent to $\varepsilon_\rho \cup \mu_\rho$). According to [8], the equivariant cobordism class of $(M; \Phi)$ is determined by the cobordism class of $(F, \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$.

For example, the fixed data of the twist G-action $((F^n)^{2^r}; t^k_r)$ described in Section 1 is $(F, \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$, where $F = F^n$ and $\{\varepsilon_\rho\}_{\rho \in \mathcal{P}}$ consists of $2^r - 1$ copies of the tangent bundle $\tau \to F^n$ and $2^k - 2^r$ copies of the zero bundle $0 \to F^n$. More precisely, $\varepsilon_\rho = \tau$ when $\rho(T_i) = 1$ for all $i \geq r+1$, and $\varepsilon_\rho = 0$ for the remaining $\rho \in \mathcal{P}$.

Remark. For every automorphism $\sigma : G \to G$, $\sigma((F^n)^{2^k}; t^k_k)$ is cobordant to $((F^n)^{2^k}; t^k_k)$. However, if $r < k$, $\sigma((F^n)^{2^r}; t^k_r)$ may or not be cobordant to $((F^n)^{2^r}; t^k_r)$. If $(F^n, \{\varepsilon_\rho\})$ and $(F^n, \{\mu_\rho\})$ are, respectively, the fixed data of $((F^n)^{2^r}; t^k_r)$ and $\sigma((F^n)^{2^r}; t^k_r)$, these actions are cobordant if and only if $\varepsilon_\rho = \mu_\rho$ for every $\rho \in \mathcal{P}$. Let $H \subset G$ be the subgroup of G generated by $T_{r+1}, ..., T_k$. By the above description of the fixed data of $((F^n)^{2^r}; t^k_r)$ and the fact that $\sigma(H) = H$ if and only if $\{\rho \in \mathcal{P} \,|\, \rho(T_i) = 1 \quad \text{for all} \quad i \geq r+1\} = \{\rho \in \mathcal{P} \,|\, \rho(\sigma(T_i)) = 1 \quad \text{for all} \quad i \geq r+1\}$, one then has that $((F^n)^{2^r}; t^k_r)$ and $\sigma((F^n)^{2^r}; t^k_r)$ are cobordant if and only if $\sigma(H) = H$. For example, write $((F^n)^4; t^4_2) = ((F^n)^4; T_1, T_2, T_3, T_4)$. Then $((F^n)^4; t^4_2)$ and $((F^n)^4; T_2T_3, T_1T_2, T_4, T_3)$ are cobordant, while $((F^n)^4; t^4_2)$ and $((F^n)^4; T_2, T_1T_3, T_3, T_1T_4)$ are not cobordant.

Let $(M; \Phi)$ be a G-action with fixed data $(F, \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$, and let Ω be a subgroup of $Hom(G, Z_2)$. Our first step will be to show that the part of the fixed data of $(M; \Phi)$ given by $(F, \{\varepsilon_\rho\}_{\rho \in \Omega \cap \mathcal{P}})$ can be realized as the fixed data of some subgroup $H \subset G$ acting (by restriction) on the fixed point set of the restriction of Φ to some appropriate subgroup $K \subset G$. First note that there exists a subgroup $H \subset G$ so that the restriction $Hom(G, Z_2) \to Hom(H, Z_2)$ maps Ω isomorphically onto $Hom(H, Z_2)$. In fact, consider the natural isomorphism $G \to Hom(Hom(G, Z_2), Z_2)$ given by $T \to \varphi_T$, where $\varphi_T(\rho) = \rho(T)$ for any $\rho \in Hom(G, Z_2)$. Choose a basis $(\rho_1, \rho_2, ..., \rho_t, \xi_1, \xi_2, ..., \xi_{k-t})$ for $Hom(G, Z_2)$ so that $(\rho_1, \rho_2, ..., \rho_t)$ is a basis for Ω, and consider $(T_1, T_2, ..., T_t, S_1, S_2, ..., S_{k-t})$ the basis for G which corresponds to the dual basis $(\rho^*_1, \rho^*_2, ..., \rho^*_t, \xi^*_1, \xi^*_2, ..., \xi^*_{k-t})$ of $Hom(Hom(G, Z_2), Z_2)$ under the above isomorphism. Evidently, $(\rho_1, \rho_2, ..., \rho_t, \xi_1, \xi_2, ..., \xi_{k-t})$ is the dual basis of $(T_1, T_2, ..., T_t, S_1, S_2, ..., S_{k-t})$. Set $H = < T_1, T_2, ..., T_t >$. Since $\rho_i(T_j) = -1$ if $i = j$ and $\rho_i(T_j) = 1$ if $i \neq j$, one has that $(\rho_{1|H}, \rho_{2|H}, ..., \rho_{t|H})$ is a basis for $Hom(H, Z_2)$, thus the restriction maps Ω isomorphically onto $Hom(H, Z_2)$. Now set $K = < S_1, S_2, ..., S_{k-t} >$, $F_K =$ the fixed point set of K and $\Psi=$ the restriction of Φ to $H \times F_K$. One then has the following

Lemma 3.1. *The fixed data of the H-action $(F_K;\Psi)$ is $(F,\{\mu_{\rho'}\}_{\rho'\in\mathcal{P}'})$, where for each $\rho'\in\mathcal{P}'=Hom(H,Z_2)-\{1\}$ one has $\mu_{\rho'}=\varepsilon_\rho$, where ρ is the unique element of $\Omega\cap\mathcal{P}$ with $\rho_{|H}=\rho'$. In other words, the fixed data of H acting on the fixed set of K is F with the subbundles ε_ρ, $\rho\in\Omega\cap\mathcal{P}$, and in terms of $\mathcal{P}'=Hom(H,Z_2)-\{1\}$, these subbundles are indexed under the restriction $\Omega\cap\mathcal{P}\to\mathcal{P}'$.*

Proof. The inverse of the restriction $\Omega\to Hom(H,Z_2)$ is the isomorphism $Hom(H,Z_2)\to\Omega$ given by $\rho'\to\rho$, where $\rho_{|H}=\rho'$ and ρ is the trivial homomorphism on K. The fixed set of $(F_K;\Psi)$ is F, and if $(F,\{\mu_{\rho'}\}_{\rho'\in\mathcal{P}'})$ is the fixed data, each $\mu_{\rho'}$ is equal to ε_ρ for some $\rho\in\mathcal{P}$. Thus, to complete the argument, it suffices to show that, when $\mu_{\rho'}=\varepsilon_\rho$, then $\rho_{|H}=\rho'$ and ρ is the trivial homomorphism on K. Take $T\in H$. Then T acts on $\mu_{\rho'}$ as $\rho'(T)$, and since $\mu_{\rho'}=\varepsilon_\rho$ also T acts on $\mu_{\rho'}$ as $\rho(T)$. Hence $\rho(T)=\rho'(T)$ and $\rho_{|H}=\rho'$. Now take $T\in K$. Note that $\mu_{\rho'}$ is a subbundle of the normal bundle of F in F_K, which is the fixed set of K. Thus T acts on $\mu_{\rho'}$ identically. Since $\mu_{\rho'}=\varepsilon_\rho$ and T acts on ε_ρ as $\rho(T)$, we conclude that $\rho(T)=1$, which gives the result. □

Remark. Suppose $(F,\{\varepsilon_\rho\}_{\rho\in\mathcal{P}})$ is the fixed data of a G-action $(M;\Phi)$. Denote by $\mathcal{A}$ the set of vector bundles over F which lie in $\{\varepsilon_\rho\}$. Then $(F,\{\varepsilon_\rho\})$ gives a map $\theta:\mathcal{P}\to\mathcal{A}$, and if $\sigma:G\to G$ is an automorphism, $\sigma(M;\Phi)$ gives rise to a new map $\mathcal{P}\to\mathcal{A}$ which is θ composed with some bijection $\mathcal{P}\to\mathcal{P}$. We note that not every bijection $\mathcal{P}\to\mathcal{P}$ gives a map $\mathcal{P}\to\mathcal{A}$ which necessarily is derived from some automorphism $G\to G$, since the number of such bijections may be greater than the number of bases of G. In particular, we cannot in principle guarantee that all such maps $\mathcal{P}\to\mathcal{A}$ come from G-actions. This is not the case, however, when $G=Z_2^2$; if $(F;\{\varepsilon_{\rho_1},\varepsilon_{\rho_2},\varepsilon_{\rho_3}\})$ is a fixed data with map $\mathcal{P}=\{\rho_1,\rho_2,\rho_3\}\to\mathcal{A}$, then all of the other possible maps $\mathcal{P}\to\mathcal{A}$ come from Z_2^2-actions, since they are derived from automorphisms $Z_2^2\to Z_2^2$. Therefore the next lemma is independent of the map $\mathcal{P}\to\mathcal{A}$.

Lemma 3.2. *Suppose F^n is a nonbounding, connected and closed n-dimensional manifold. Let η be any p-dimensional vector bundle over F^n $(p>0)$ and μ a n-dimensional vector bundle over F^n cobordant to the tangent bundle $\tau\to F^n$. Denote by 0 the zero bundle over F^n. Then $(F^n;\{\eta,\mu,0\})$ cannot be the fixed data of a Z_2^2-action.*

Proof. If ε_1, ε_2, ε_3 are three vector bundles over F^n and $(F^n;\{\varepsilon_1,\varepsilon_2,\varepsilon_3\})$ is the fixed data of a Z_2^2-action, then the argument outlined in [6; Section 3; pages 88, 89 and 90] (or in [7; Section 2; pages 107 and 108]) shows that $(RP(\varepsilon_1);\lambda,\varepsilon_2\oplus(\varepsilon_3\otimes\lambda))$, the projective space bundle of ε_1 with its standard line bundle $\lambda\to RP(\varepsilon_1)$ and the bundle $\varepsilon_2\oplus(\varepsilon_3\otimes\lambda)\to RP(\varepsilon_1)$, bounds as an element of the bordism group $\mathcal{N}_{n+s_1-1}(BO(1)\times BO(s_2+s_3))$; here, $s_i=dim(\varepsilon_i)$ and we are suppressing bundle maps. In particular, if we suppose by contradiction that $(F^n;\{\eta,\mu,0\})$ is the fixed data of a Z_2^2-action, by taking $\varepsilon_1=\eta$, $\varepsilon_2=\mu$ and $\varepsilon_3=0$, one has that $(RP(\eta);\lambda,\mu)$ bounds as an element of $\mathcal{N}_{n+p-1}(BO(1)\times BO(n))$. Since F^n is nonbounding, there is a Stiefel-Whitney number $w_{i_1}(\tau)...w_{i_r}(\tau)[F^n]$ which is nonzero. Since μ is cobordant to τ, $w_{i_1}(\mu)...w_{i_r}(\mu)[F^n]$ is also nonzero. Denoting $W(\lambda)=1+c$, one knows that $H^*(RP(\eta),Z_2)$ is the free $H^*(F^n,Z_2)$-module on $1,c,c^2,...,c^{p-1}$. Therefore $c^{p-1}w_{i_1}(\mu)...w_{i_r}(\mu)$ is the nonzero class of $H^{n+p-1}(RP(\eta),Z_2)$. Since $c^{p-1}w_{i_1}(\mu)...w_{i_r}(\mu)[RP(\eta)]$ is a characteristic number of $(RP(\eta);\lambda,\mu)$, this gives the contradiction. □

Lemma 3.3. *Let (M^m, Φ) be a G-action ($G = Z_2^k$) with fixed set F^n being a connected n-dimensional submanifold, and where $m = 2^k n$. Then (M^m, Φ) is equivariantly cobordant to $((F^n)^{2^k}; t_k^k)$.*

Proof. This is the main result of [7]. □

Now we have on hand the necessary tools to prove our main result. Suppose then (M^m, Φ), $\Phi = (T_1, ..., T_k)$, a G-action fixing F^n, where F^n has property $\mathcal{H}$. As stated in the introduction, our aim is to prove that $(M^m; \Phi)$ is equivariantly cobordant to $\sigma((F^n)^{2^r}; t_r^k)$ for some automorphism $\sigma : G \to G$ and some $1 \leq r \leq k$. The essential point is that Lemma 3.1 and Lemma 3.2 allow to find a special subgroup $H \subset G$ so that Lemma 3.3 can be applied to the restriction of Φ to $H \times M^m$. Let $(F^n, \{\varepsilon_\rho\}_{\rho \in \mathcal{P}})$ be the fixed data of Φ.

Lemma 3.4. *For each $\rho \in \mathcal{P}$, ε_ρ is either cobordant to the tangent bundle $\tau \to F^n$ or cobordant to the zero bundle $0 \to F^n$; in particular, $m = pn$ for some $1 \leq p \leq 2^k$ (note that $p - 1$ is the number of bundles cobordant to $\tau \to F^n$).*

Proof. For each $\rho \in \mathcal{P}$, let V_ρ be the component of the fixed set of the subgroup $ker(\rho)$ containing F^n, and set $n_\rho = dim(V_\rho)$. Taking $T \in G - ker(\rho)$, one has that (V_ρ, T) is an involution fixing F^n. Since F^n has property $\mathcal{H}$, $n_\rho = 0$ or $2n$. If $n_\rho = 0$, $(V_\rho, T) = (F^n, Id)$ and ε_ρ is the zero bundle $0 \to F^n$. If $n_\rho = 2n$, by the result of C. Kosniowski and R. E. Stong of [1] cited in the introduction, (V_ρ, T) is cobordant to $(F^n \times F^n, t)$ and $\varepsilon_\rho \to F^n$ is cobordant to the tangent bundle $\tau \to F^n$. □

If $p = 1$, Φ is the trivial G-action, that is, $\Phi = t_0^k$; in this case one has only zero bundles. If $p = 2^k$, $m = 2^k n$ and Lemma 3.3 says in this case that (M^m, Φ) is cobordant to $((F^n)^{2^k}; t_k^k)$ (which is cobordant to $\sigma((F^n)^{2^k}; t_k^k)$ for any automorphism $\sigma : G \to G$); in this case, one has only bundles cobordant to the tangent bundle $\tau \to F^n$. Therefore we can assume $1 < p < 2^k$, which means that the two possible cobordism types of bundles occur. To ease the notation, write $\varepsilon_\rho \equiv \tau$ when ε_ρ is cobordant to τ.

Lemma 3.5. *Let Ω be the subset of $Hom(G, Z_2)$ given by $\Omega = \{1\} \cup \{\rho \in \mathcal{P} \,|\, \varepsilon_\rho \equiv \tau\}$. Then Ω is a subgroup of $Hom(G, Z_2)$. In particular, the number of bundles ε_ρ with $\varepsilon_\rho \equiv \tau$ is $2^r - 1$ for some $1 \leq r \leq k - 1$ (r is the dimension of Ω as Z_2-vector space); that is, $p = 2^r$ and $m = 2^r n$.*

Proof. Take $\rho_1, \rho_2 \in \Omega$. Then $\{1, \rho_1, \rho_2, \rho_1\rho_2\}$ is a subgroup of $Hom(G, Z_2)$. By Lemma 3.1, there exists subgroups $H, K \subset G$ with H isomorphic to Z_2^2 and $G = H \oplus K$, so that the fixed data of the Z_2^2-action obtained by letting H act on the fixed set of K is $(F^n; \{\varepsilon_{\rho_1}, \varepsilon_{\rho_2}, \varepsilon_{\rho_1\rho_2}\})$. Since $\varepsilon_{\rho_1} \equiv \tau$ and $\varepsilon_{\rho_2} \equiv \tau$, one has from Lemma 3.2 that $\varepsilon_{\rho_1\rho_2} \equiv \tau$, and Ω is a subgroup of $Hom(G, Z_2)$. □

Now we can complete the argument. The desired special subgroup $H \subset G$ on which Lemma 3.3 will be applied is any subgroup of G which corresponds to $\Omega = \{1\} \cup \{\rho \in \mathcal{P} \,|\, \varepsilon_\rho \equiv \tau\}$ through Lemma 3.1. First note that Lemma 3.5 indicates a similarity between the fixed data of Φ and an action of type $\sigma((F^n)^{2^r}; t_r^k)$. We call attention, however, to the following subtle point: one has a list $\{\varepsilon_\rho\}_{\rho \in \Omega \cap \mathcal{P}}$ with each ε_ρ being individually cobordant to τ, but this list might not be simultaneously cobordant to the list $\{\mu_\rho\}_{\rho \in \Omega \cap \mathcal{P}}$ with each μ_ρ being equal to τ, and the desired result requires simultaneous cobordism (this obstacle does not appear when $F^n = RP^{2s}$, since $W(\varepsilon_\rho) = (1 + \alpha)^{2s+1}$ automatically yields the simultaneous cobordism in this case). In the final remark of the paper we present a example illustrating this situation. Fortunately, the essence of this point is bypassed already by Lemma 3.3. In fact, by Lemma 3.1, there exist subgroups $H, K \subset G$, with H isomorphic to Z_2^r and $G = H \oplus K$, so that the fixed data of H acting on the fixed set F_K of K is $(F^n, \{\varepsilon_\rho\}_{\rho \in \Omega \cap \mathcal{P}})$. More precisely, and in terms of $\mathcal{P}' = Hom(H, Z_2) - \{1\}$, this fixed data is $(F^n, \{\mu_{\rho'}\}_{\rho' \in \mathcal{P}'})$, where for each $\rho' \in \mathcal{P}'$ one has $\mu_{\rho'} = \varepsilon_\rho$, with $\rho_{|H} = \rho'$ and $\rho_{|K}$= the trivial homomorphism. In particular, $dim(F_K) = 2^r n = dim(M^m)$, and thus F_K is the component of M^m containing F^n (note that each $T \in K$ acts on M^m trivially). Since $(M^m - F_K, \Phi)$ is a G-action without fixed points, the main result of [8] says that $(M^m - F_K, \Phi)$ bounds as a manifold with G-action. Thus we can suppose, without loss of generality, that $F_K = M^m$. Then one has an action $(M^m; \Phi_{|H \times M^m})$ with H isomorphic to Z_2^r and $m = 2^r n$; by Lemma 3.3, this action is equivariantly cobordant to $((F^n)^{2^r}; t_r^r)$. In particular, the list $\{\varepsilon_\rho\}_{\rho \in \Omega \cap \mathcal{P}}$ is simultaneously cobordant to the list $\{\mu_\rho\}_{\rho \in \Omega \cap \mathcal{P}}$ with each μ_ρ being equal to τ. Now choose a basis $(T'_1, ..., T'_r, T''_{r+1}, ..., T''_k)$ for G so that $(T'_1, ..., T'_r)$ is a basis for H and $(T''_{r+1}, ..., T''_k)$ is a basis for K. Consider the automorphism $\varphi : G \to G$ where $\varphi(T_i) = T'_i$ if $1 \leq i \leq r$ and $\varphi(T_i) = T''_i$ if $r < i \leq k$, and the G-action $\varphi(M^m, \Phi)$. To describe the fixed data of this action, note that if $\rho \in \mathcal{P}$ is the trivial homomorphism on K, then $\rho \in \Omega$ and thus $\varepsilon_\rho \equiv \tau$; otherwise, $\rho \notin \Omega$, which means that ε_ρ is the zero bundle. Since the list $\{\varepsilon_\rho\}_{\rho \in \Omega \cap \mathcal{P}}$ is simultaneously cobordant to the list $\{\mu_\rho\}_{\rho \in \Omega \cap \mathcal{P}}$, the fixed data of $\varphi(M^m, \Phi)$ then is simultaneously cobordant to the list $\{\varepsilon_\rho\}_{\rho \in \mathcal{P}}$ given by $\varepsilon_\rho = \tau$ when ρ is the trivial homomorphism on K and $\varepsilon_\rho = 0$ otherwise. But then $\varphi(M^m, \Phi)$ is equivariantly cobordant to $((F^n)^{2^r}; t_r^k)$ and (M^m, Φ) is equivariantly cobordant to $\sigma((F^n)^{2^r}; t_r^k)$, where $\sigma = \varphi^{-1}$.

Remark. Let (M, T) be an involution. For each r with $1 \leq r \leq k$, one may form a Z_2^k-action $\Phi = (T_1, T_2, ..., T_k)$ on the produt $M^{2^{r-1}}$ by letting $T_1 = T$ and letting $(T_2, ..., T_k)$ be the twist Z_2^{k-1}-action t_{r-1}^{k-1}. Denote this action by $\Gamma_r^k(M, T)$. Note that if $(F \times F, t)$ is the twist involution, then $\Gamma_r^k(F \times F, t) = (F^{2^r}; t_r^k)$. Also if (M, T) and (V, S) are Z_2-cobordant, then $\Gamma_r^k(M, T)$ and $\Gamma_r^k(V, S)$ are Z_2^k-cobordant (it suffices to look at the fixed data of $\Gamma_r^k(M, T)$: if $\eta \to F$ is the fixed data of (M, T), then the fixed point set of $\Gamma_r^k(M, T)$ also is F and the fixed data consists of 2^{r-1} copies of η, $2^{r-1} - 1$ copies of the tangent bundle over F and $2^k - 2^r$ zero bundles).

Now suppose that F^n has property $\mathcal{H}$, and let (M^{2n}, T) be any involution fixing F^n. Then (M^{2n}, T) is equivariantly cobordant to the twist involution $(F^n \times F^n, t)$, and thus $\sigma\Gamma_r^k(M^{2n}, T)$ is equivariantly cobordant to $\sigma\Gamma_r^k(F^n \times F^n, t) = \sigma((F^n)^{2^r}; t_r^k)$ for every automorphism $\sigma : Z_2^k \to Z_2^k$, $k \geq 1$ and $1 \leq r \leq k$. In this way, every Z_2^k-action fixing F^n is equivariantly cobordant to $\sigma\Gamma_r^k(M^{2n}, T)$ for some automorphism $\sigma : G \to G$ and some $1 \leq r \leq k$. For example, every Z_2^k-action fixing RP^{2n} is equivariantly cobordant to $\sigma\Gamma_r^k(CP^{2n}, c)$ for some automorphism $\sigma : G \to G$ and some $1 \leq r \leq k$, where c means complex conjugation on homogeneous coordinates.

Still in this context, let $F^n \subset R^m$ be a smooth and closed n-dimensional manifold F^n which is a real algebraic variety, and where R^m is an adequate euclidean real m-dimensional space. Let $M^{2n} \subset C^m$ be the corresponding complex algebraic variety, with real dimension $2n$. One has that M^{2n} is invariant under the complex conjugation c on C^m, and the fixed point set of the involution (M^{2n}, c) is F^n. Thus if F^n has property $\mathcal{H}$ every Z_2^k-action fixing F^n is equivariantly cobordant to $\sigma\Gamma_r^k(M^{2n}, c)$ for some automorphism $\sigma : G \to G$ and some $1 \leq r \leq k$.

We thank the referee for having inspired this remark.

Remark. Consider $F^4 = CP^2\#(S^2 \times S^2) = CP^2\#(CP^1 \times CP^1)$. One knows that F^4 has property $\mathcal{H}$. $H^*(F^4, Z_2)$ is generated by α, β, $\gamma \in H^2(F^4, Z_2)$, with $\alpha^2 = \beta\gamma$ being the nonzero element of $H^4(F^4, Z_2)$ and where $\alpha\beta = \alpha\gamma = \beta^2 = \gamma^2 = 0$. The Stiefel-Whitney class of F^4 is $W(F^4) = 1 + \alpha + \alpha^2$. Denote by $\xi_1 \to CP^1 \times CP^1$ the pullback of the usual complex line bundle over CP^1 through the first projection, and by $R^2 \to CP^1 \times CP^1$ the trivial 2-dimensional bundle over $CP^1 \times CP^1$. Then over F^4 one has a 4-dimensional bundle $\mu^4 \to F^4$ given by forming the connected sum of the tangent bundle $\tau(CP^2) \to CP^2$ and $\xi_1 \oplus R^2 \to CP^1 \times CP^1$. One has $W(\mu^4) = 1 + (\alpha + \beta) + \alpha^2$, and computing characteristic numbers it is easy to see that μ^4 is cobordant to the tangent bundle $\tau(F^4) \to F^4$.

Similarly one has a 4-dimensional bundle $\eta^4 \to F^4$ given by forming the connected sum of $\tau(CP^2)$ and $R^2 \oplus \xi_2 \to CP^1 \times CP^1$, where ξ_2 is the pullback of the usual complex line bundle over CP^1 through the second projection. One has $W(\eta^4) = 1 + (\alpha + \gamma) + \alpha^2$, and in the same way η^4 is cobordant to $\tau(F^4)$. However, there is no simultaneous cobordism of $(F^4; \mu^4, \eta^4)$ with $(F^4; \tau(F^4), \tau(F^4))$. In fact, one has $w_2(\mu^4)w_2(\eta^4) = (\alpha + \beta)(\alpha + \gamma) = \alpha^2 + \beta\gamma = 0$ and $w_2(\tau(F^4))w_2(\tau(F^4)) = \alpha\alpha = \alpha^2$. This obviously extends to $CP^2\#(CP^1 \times CP^1)\#...\#(CP^1 \times CP^1)$ to get any number of bundles. It also works for $RP^2\#(RP^1 \times RP^1)\#...\#(RP^1 \times RP^1)$, $HP^2\#(HP^1 \times HP^1)\#...\#(HP^1 \times HP^1)$ and $QP^2\#(QP^1 \times QP^1)\#...\#(QP^1 \times QP^1)$.

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