A Z_p-INDEX HOMOMORPHISM FOR Z_p-SPACES

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ABSTRACT. Let (X, T) be a Z_p-*space* , that is, a topological space X equipped with a free action of the cyclic group Z_p, generated by a periodic homeomorphism $T : X \to X$ of period p. The goal of this paper is to construct a Z_p-index graded homomorphism $J : H_r(X, T) \to Z_p$ associated with (X, T), where $H_r(X, T)$ is the *rth equivariant homology* Z_p-*module* of (X, T). Using this Z_p-index we prove that, if (X, T) and (Y, S) are Z_p-spaces and $p = 2q$ with q odd, then, under certain homological conditions on X and Y, there is no equivariant map $f : (X, T) \to (Y, S)$. This result includes the situation in which (Y, S) is the odd dimensional sphere S^{2n+1} equipped with the standard free periodic homeomorphism of period $p = 2q$ with q odd. This is a special case of a result of T. Kobayashi **[TK]** .

1. INTRODUCTION

Let G be a compact Lie group. In **[FH]** , E. Fadell and S. Husseini introduced a nonnumerical index associated to pairs (X, ϕ), where X is a Hausdorff and paracompact space and ϕ is a continuous action of G on X. If G is a cyclic group of order p and the action ϕ is free, this index is obtained by using the homomorphism $g^* : \check{H}^*(L_p^\infty) \to \check{H}^*(X/G)$ induced in Cech cohomology mod p by a classifying map $g : X/G \to L_p^\infty$ for the principal G-bundle $X \to X/G$, where L_p^∞ is the infinite lens space and X/G is the orbit space of X by ϕ; specifically, the index of (X, ϕ) is defined in this case as the kernel of g^*, and alternatively has a numerical version given by $dim(\frac{\check{H}^*(L_p^\infty)}{index(X,\phi)})$. In this way, this index is collected "outside X", since its construction requires classifying maps, which in turn require the Hausdorff and paracompactness properties on X. In fact, paracompactness can be removed by assuming $X \to X/G$ to be trivialized over some partition of unity of X/G (this means that there is an indexed open covering $\{U_\alpha\}$ of X/Z_p having an associated partition of unity $\{p_\alpha\}$, such that the restriction of $X \to X/Z_p$ to U_α is a trivial bundle for every α).

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Let (X, T) be a Z_p-*space* , that is, a topological space X equipped with a free action of the cyclic group Z_p, generated by a periodic homeomorphism $T : X \to X$ of period p. The classic example of a Z_p-space is given by the odd dimensional standard sphere S^{2n+1} in complex $(n+1)$-space C^{n+1}, equipped with the map $T_p : S^{2n+1} \to S^{2n+1}$ given by

$$T_p(z_0, z_1, ..., z_n) = (e^{2\pi i/p} z_0, e^{2\pi i/p} z_1, ..., e^{2\pi i/p} z_n),$$

where p is a natural number and $z_0, z_1, ..., z_n$ are complex numbers with $\sum_{i=0}^{n} |z_i|^2 = 1$. Evidently T_2 is the antipodal map. The main objective of this paper is to construct a Z_p-index associated with (X, T) so that it is collected "inside X", and with no topological requirement on X. In particular, our method covers the case where simultaneously X is not paracompact and $X \to X/Z_p$ cannot be trivialized over some partition of unity of X/Z_p. Specifically, our index will be given by a graded Z_p-homomorphism $J : H_r(X, T) \to Z_p$ invariant under the effect of homomorphisms induced by equivariant maps $f : (X, T) \to (Y, S)$. This construction was inspired by the approach of C. T. Yang in **[CY]** to define his Z_2-index homomorphism $\nu : H_r(X, T) \to Z_2$. In fact, for $p = 2$, our Z_p-homomorphism J reduces to ν, but we will see that this extension for $p > 2$ of the Yang's Z_2-index is not so automatic.

In addition, we will prove that, if (X, T) is a Z_p-space where $p = 2q$ with q odd, and X is pathwise connected with singular Z_p-homology $H_r(X, Z_p) = 0$ for $1 \le r \le n$, then $J(H_r(X, T)) \ne 0$ for $1 \le r \le n + 1$. This result will be used to establish a generalization of the Borsuk-Ulam Theorem, a generalization of a theorem of J. Walker in **[JW]** , and a generalization of a special case of a theorem of T. Kobayashi in **[TK]** .

2. A $\mathbf{Z_p}$-INDEX HOMOMORPHISM

Let (X, T) be any Z_p-space and $S_r(X, Z_p)$ the singular chain Z_p-module of X, and consider the induced chain map $T_\# : S_r(X, Z_p) \to S_r(X, Z_p)$. An r-chain $c \in S_r(X, Z_p)$ is called a (T, r)-*chain* if $T_\#(c) = c$. All the (T, r)-chains form a Z_p-submodule $S_r(X, T) \subset S_r(X, Z_p)$, and the boundary operator $\partial : S_r(X, Z_p) \to S_{r-1}(X, Z_p)$ maps $S_r(X, T)$ into $S_{r-1}(X, T)$. Hence one has the *equivariant homology Z_p-modules*

$$H_r(X, T) = \frac{Z_r(X, T)}{B_r(X, T)},$$

where $Z_r(X, T) = \{c \in S_r(X, T)/\partial(c) = 0\}$ and $B_r(X, T) = \partial(S_{r+1}(X, T))$.

We say that a map $f : (X, T) \to (Y, S)$ of Z_p-spaces is *equivariant* if $Sf = fT$. In this case, $f : (X, T) \to (Y, S)$ induces a Z_p-homomorphism $f_* : H_r(X, T) \to H_r(Y, S)$.

Consider the chain map

$$\theta_T = Id + T_\# + T_\#^2 + ... + T_\#^{p-1} : S_r(X, Z_p) \to S_r(X, Z_p),$$

where Id denotes the identity homomorphism.

Proposition 1. *The homomorphism θ_T has the property that $S_r(X,T) = \theta_T(S_r(X, Z_p))$ for all $r \geq 0$.*

PROOF. It is easy to check that image$(\theta_T) \subset S_r(X, T)$. To prove the opposite inclusion, suppose $c \in S_r(X, Z_p)$ is an r-chain such that $T_\#(c) = c$. Then c can be written as $c = a_1c_1 + a_2c_2 + ... + a_sc_s$, where $a_i \in Z_p$ and the $c_{i's}$ are singular r-simplexes of X. The condition $T_\#(c) = c$ implies that $T_\#$ determines a free Z_p-action on the set $A = \{c_1, c_2, ..., c_s\}$, and we have then l Z_p-orbits $\beta_1, ..., \beta_l$ of this action where $s = pl$. Moreover, $T_\#(c) = c$ yields $T_\#^j(c) = c$ for $1 \leq j \leq p-1$, hence if c_{i_u} and c_{i_v} belong to the same orbit one has $a_{i_u} = a_{i_v}$. Pick then one element in each orbit, say $c_{i_1} \in \beta_1$,...,$c_{i_l} \in \beta_l$. Writing $d = a_{i_1}c_{i_1} + a_{i_2}c_{i_2} + ... + a_{i_l}c_{i_l}$, one clearly has

$$c = d + T_\#(d) + T_\#^2(d) + ... + T_\#^{p-1}(d) = \theta_T(d).$$

□

Remark. The representation of $c \in S_r(X,T)$ as $c = \theta_T(d)$ is not unique. A consequence of the above argument is that the inverse image of c, $\theta_T^{-1}(c)$, is in one-to-one correspondence with the cartesian product of l copies of the set $\{0, 1, 2, ..., p-1\}$. In fact, using the terminology developed in the proof of Proposition 1,

$$\theta_T^{-1}(c) = \{a_{i_1}T_\#^{v_1}(c_{i_1}) + a_{i_2}T_\#^{v_2}(c_{i_2}) + ... + a_{i_l}T_\#^{v_l}(c_{i_l}) \mid 0 \leq v_i \leq p-1\}.$$

The Z_p-index graded homomorphism $J : H_r(X,T) \to Z_p$ will first be constructed at the (T, r)-cycle level, using recurrence on r. Let $c = \theta_T(d) \in Z_0(X,T)$, and suppose $d = a_1d_1 + a_2d_2 + ... + a_sd_s$, $a_i \in Z_p$, $d_i \in X$. Set $J(c) = a_1 + a_2 + ... + a_s$. By the above remark, this definition is independent of the choice of d. Further, $J : Z_0(X,T) \to Z_p$ is clearly a Z_p-homomorphism that annihilates $B_0(X,T)$. In fact, if $c \in Z_0(X,T)$ satisfies $c = \partial(w)$, $w \in S_1(X,T)$, and $w = \theta_T(b)$, then $c = \theta_T(\partial(b))$. Write $b = a_1b_1 + a_2b_2 + ... + a_sb_s$, where b_i is a path joining b_i^0 to b_i^1. Then $\partial(b) = \sum_{i=1}^s a_i(b_i^1 - b_i^0)$ and $J(c) = \sum_{i=1}^s (a_i - a_i) = 0$.

Now we proceed by induction and assume that J is a well-defined Z_p-homomorphism $J : Z_{r-1}(X,T) \to Z_p$ and annihilates $B_{r-1}(X,T)$ $(r > 0)$. We introduce the chain maps $\Lambda_T = Id - T_\#$ and

$$\Psi_T = T_\# + 2T_\#^2 + 3T_\#^3 + ... + (p-1)T_\#^{p-1} = \sum_{j=1}^{p-1} jT_\#^j.$$

Hereafter, the subscripts of θ_T, Λ_T and Ψ_T will not be used when T is clear from context. Note that

$$\Lambda\Psi = \Psi\Lambda = \sum_{j=1}^{p-1} jT_\#^j - \sum_{i=2}^{p}(i-1)T_\#^i = T_\# - (p-1)T_\#^p + \sum_{j=2}^{p-1}(j-(j-1))T_\#^j =$$

$$Id + T_\# + \sum_{j=2}^{p-1} T_\#^j = \theta.$$

Suppose then $c \in Z_r(X,T)$ with $c = \theta(d)$. We have $\partial(\Psi(\partial(d))) = \Psi(\partial(\partial(d))) = 0$ and

$$T_\#(\Psi(\partial(d))) = (T_\# + Id - Id)(\Psi(\partial(d))) = (Id - \Lambda)(\Psi(\partial(d))) =$$

$$\Psi(\partial(d)) - \theta(\partial(d)) = \Psi(\partial(d)) - \partial(c) = \Psi(\partial(d)).$$

Hence $\Psi(\partial(d))$ is a $(T, r-1)$-cycle and thus $J(\Psi(\partial(d)))$ makes sense by the inductive hypothesis.

Proposition 2. *The element $J(\Psi(\partial(d))) \in Z_p$ is independent of the choice of d.*

PROOF. Let $c = a_1c_1 + a_2c_2 + ... + a_sc_s \in Z_r(X,T)$ with orbits $\beta_1, ..., \beta_l$, and let $d, d' \in S_r(X, Z_p)$ such that $\theta(d) = \theta(d') = c$. Then we can write $d = a_{i_1}c_{i_1} + a_{i_2}c_{i_2} + ... + a_{i_l}c_{i_l}$, $d' = a_{i_1}T_\#^{v_1}(c_{i_1}) + a_{i_2}T_\#^{v_2}(c_{i_2}) + ... + a_{i_l}T_\#^{v_l}(c_{i_l})$, where each $c_{i_t} \in \beta_t$. Set $v =$maximum$\{v_1, v_2, ..., v_l\}$. For each i, $1 \le i \le v+1$, there exists a sequence $d = d_1, d_2, ..., d_v, d_{v+1} = d'$ of r-chains with $\theta(d_i) = c$. For each i, $2 \le i \le v+1$, there exist r-chains A_i, B_i with $d_{i-1} = A_i + B_i$ and $d_i = A_i + T_\#(B_i)$ (A_i can be zero). Therefore it suffices to show that if $c \in Z_r(X,T)$ satisfies $c = \theta(A+B) = \theta(A + T_\#(B))$, then $J(\Psi(\partial(A+B))) = J(\Psi(\partial(A + T_\#(B))))$. Since by the inductive hypothesis $J : Z_{r-1}(X,T) \to Z_p$ is a Z_p-homomorphism, we get

$$J(\Psi(\partial(A+B))) - J(\Psi(\partial(A+T_\#(B)))) = J(\Psi(\partial(A+B)) - \Psi(\partial(A+T_\#(B)))) =$$

$$J(\Psi(\partial((Id - T_\#)B))) = J(\Psi(\Lambda(\partial(B)))) = J(\theta(\partial(B))).$$

Since $\theta(\partial(B)) = \partial(\theta(B)) \in B_{r-1}(X,T)$ and again by the inductive hypothesis J maps $B_{r-1}(X,T)$ into zero, the result follows. $\square$

Proposition 2) says that the rule $J(c) = J(\Psi(\partial(d)))$, where $c = \theta(d)$, provides a well-defined map $J : Z_r(X,T) \to Z_p$; this map is easily seen to be a Z_p-homomorphism. Further, J maps $B_r(X,T)$ into zero: if $c \in Z_r(X,T)$ satisfies $c = \partial(a)$ with $a = \theta(b)$, then $c = \theta(\partial(b))$, and so $J(c) = J(\Psi(\partial(\partial(b)))) = 0$. In this way, $J([c]) = J([\Psi(\partial(d))])$ is a well-defined homomorphism

$$J : H_r(X,T) \to Z_p,$$

where [ ] denotes homology class. This graded homomorphism can be considered a Z_p-index homomorphism because of the following proposition.

Proposition 3. *Let $(X,T),(Y,S)$ be Z_p-spaces and $f : (X,T) \to (Y,S)$ an equivariant map. Then, for any (T,r)-cycle $c \in Z_r(X,T)$, one has $J(f_*([c])) = J([c])$.*

PROOF. In fact, the result is true at the (T,r)-cycle level, and can be proved by induction on r. First note that, since f is equivariant, $f_\# \theta_T = \theta_S f_\#$ and $f_\# \Psi_T = \Psi_S f_\#$.

For $r = 0$, let $c = \theta(d) \in Z_0(X,T)$, where $d = a_1 d_1 + a_2 d_2 + ... + a_s d_s$. Then

$$J(f_\#(c)) = J(f_\#(\theta(d))) = J(\theta(f_\#(d))) = J(\theta(\sum_{i=1}^{s} a_i f(d_i))) = \sum_{i=1}^{s} a_i = J(c).$$

Suppose that the statement is true for $r-1$, and take $c = \theta(d) \in Z_r(X,T)$. Then

$$J(f_\#(c)) = J(f_\#(\theta(d))) = J(\theta(f_\#(d))) = J(\Psi(\partial(f_\#(d)))) = J(f_\#(\Psi(\partial(d)))),$$

and by the induction hypothesis, and the fact that $\Psi(\partial(d))$ is a $(T,r-1)$-cycle,

$$J(f_\#(\Psi(\partial(d)))) = J(\Psi(\partial(d))) = J(c). \qquad \square$$

Remark. For $p = 2$, note that if $c = \theta(d) \in Z_0(X,T)$ with $d = d_1 + d_2 + ... + d_s$, where each d_i is a point of X, then $J(c) = 0$ if s is even and $J(c) = 1$ if s is odd. If $r > 0$ and $c = \theta(d) \in Z_r(X,T)$, then $J(c) = J(\Psi(\partial(d))) = J(T_\#(\partial(d)))$, and since $c = d + T_\#(d) = T_\#(d) + T_\#(T_\#(d))$, one has $J(c) = J(T_\#(\partial(T_\#(d)))) = J(\partial(d))$. This coincides with the definition of the Z_2-index $\nu : H_r(X,T) \to Z_2$ given by Yang in **[CY]** . Now suppose that $p > 2$ and $c = \theta(d) \in Z_r(X,T)$. Although $\partial(d)$ is an $(r-1)$-cycle, it might not belong to $Z_{r-1}(X,T)$, and in this case $J(\partial(d))$ is not defined. This justifies, as mentioned in the introduction, why an extension for $p > 2$ of the Yang's index is not direct.

Now we prove the following proposition.

Proposition 4. *Let (X, T) be a Z_p-space, with X pathwise connected. For a natural number $n \geq 1$, suppose $H_r(X, Z_p) = 0$ for all r, $1 \leq r \leq n$. If $p = 2q$ with q odd, then $J(H_r(X, T)) \neq 0$ for all r, $0 \leq r \leq n+1$.*

PROOF. We first need to recall the construction of certain special singular j-chains $c_j \in S_j(X, Z_p)$, $0 \leq j \leq n+1$, considered by T. Kobayashi in **[TK]** . Note that the chain maps θ and Λ satisfy $\theta\theta = 0$ and $\theta\Lambda = \Lambda\theta = 0$. Pick a point in X and call c_0 the 0-chain corresponding to this point. Since X is pathwise connected, there is a path joining $T(c_0)$ to c_0, and we call c_1 the 1-chain corresponding to this path. Note that $\partial(c_1) = c_0 - T_\#(c_0) = \Lambda(c_0)$ and $\partial(\theta(c_1)) = \theta(\Lambda(c_0)) = 0$, that is, $\theta(c_1)$ is a 1-cycle. Since $H_1(X, Z_p) = 0$, there exists a 2-chain c_2 so that $\partial(c_2) = \theta(c_1)$. We can proceed inductively: suppose that, for some j, $2 \leq j \leq n$, one has constructed $c_0, c_1, c_2, ..., c_j$ so that $\partial(c_j) = \theta(c_{j-1})$ if j is even and $\partial(c_j) = \Lambda(c_{j-1})$ if j is odd. Then if j is even one has $\partial(\Lambda(c_j)) = \Lambda(\theta(c_{j-1})) = 0$, and since $H_j(X, Z_p) = 0$ there exists a $(j+1)$-chain c_{j+1} so that $\partial(c_{j+1}) = \Lambda(c_j)$. Similarly, if j is odd, one has $\partial(\theta(c_j)) = \theta(\Lambda(c_{j-1})) = 0$, hence there exists c_{j+1} so that $\partial(c_{j+1}) = \theta(c_j)$. In this way, one obtains j-chains c_j, $0 \leq j \leq n+1$, satisfying $\partial(c_j) = \theta(c_{j-1})$ for j even and $\partial(c_j) = \Lambda(c_{j-1})$ for j odd.

Since $\theta\theta = 0$ and $\theta\Lambda = 0$, it is easy to see that each $\theta(c_j)$ is a j-cycle, hence $J(\theta(c_j))$ makes sense. Our next step is to show that $J(\theta(c_j)) \neq 0$ for all $0 \leq j \leq n+1$. First note that $T^i_\#\theta = \theta$ for each $1 \leq i \leq p-1$, and thus

$$\Psi\theta = \sum_{i=1}^{p-1} iT^i_\#\theta = \sum_{i=1}^{p-1} i\theta = \frac{p(p-1)}{2}\theta.$$

Now since c_0 consists of a single point, $J(\theta(c_0)) = 1$ by definition. It follows that

$$J(\theta(c_1)) = J(\Psi(\partial(c_1))) = J(\Psi(\Lambda(c_0))) = J(\theta(c_0)) = 1.$$

Now

$$J(\theta(c_2)) = J(\Psi(\partial(c_2))) = J(\Psi(\theta(c_1))) = \frac{p(p-1)}{2}J(\theta(c_1)) = \frac{p(p-1)}{2}.$$

Since $p-2$ is even and

$$\frac{p(p-1)}{2} = (p-2)\frac{p}{2} + \frac{p}{2},$$

one has $\frac{p(p-1)}{2} \equiv \frac{p}{2} \bmod p$. Thus

$$J(\theta(c_2)) = \frac{p}{2} \neq 0.$$

Suppose inductively that for some j, $2 \le j \le n$, $J(\theta(c_j)) = \frac{p}{2} = q$. Then if j is even one has

$$J(\theta(c_{j+1})) = J(\Psi(\partial(c_{j+1}))) = J(\Psi(\Lambda(c_j))) = J(\theta(c_j)) = q.$$

On the other hand, if j is odd one has

$$J(\theta(c_{j+1})) = J(\Psi(\partial(c_{j+1}))) = J(\Psi(\theta(c_j))) = \frac{p(p-1)}{2} J(\theta(c_j)) = q^2.$$

Since q is odd, $q^2 \equiv q \bmod 2q$, hence $J(\theta(c_{j+1})) = q$ and the result is proved. □

Remark. If $p = 2q$ and q is even, the above argument shows that $J(H_j(X,T)) \neq 0$ for $0 \le j \le 3$, but since $q^2 \equiv 0 \bmod p$ in this case, it does not show that $J(H_j(X,T)) \neq 0$ for $j \ge 4$. If p is odd, $\frac{p(p-1)}{2} \equiv 0 \bmod p$, hence the argument shows that $J(H_1(X,T)) \neq 0$ but does not show that $J(H_j(X,T)) \neq 0$ when $j \ge 2$. For these remaining cases, the question remains open as to the existence of classes $\beta \in H_r(X,T)$ such that $J(\beta) \neq 0$.

The following technical result will be important for the remainder of the work.

Proposition 5. *Suppose (X,T) a Z_p-space, where X is Hausdorff, connected and locally pathwise connected. Then $H_r(X,T)$ is isomorphic to $H_r(X/T, Z_p)$.*

PROOF. Consider $\Gamma : S_r(X,T) \to S_r(X/T, Z_p)$ given by $\Gamma(c) = \pi_{\#}(d)$, where $c = \theta(d)$ and $\pi : X \to X/T$ is the quotient map. Since $\pi T^j = \pi$ for any $1 \le j \le p-1$, Γ does not depend on the choice of d. Further, Γ is a chain map. We assert that Γ is one-to-one. To see this, we need first the following general fact: if σ_r is the standard r-simplex and $d_1, d_2 : \sigma_r \to X$ are singular r-simplexes such that $\pi d_1 = \pi d_2$, then there is a k, $0 \le k \le p-1$, so that $d_1 = T^k d_2$. In fact, pick $x_0 \in \sigma_r$. Then there is a k, $0 \le k \le p-1$, such that $d_1(x_0) = T^k d_2(x_0)$. Since X is Hausdorff, $\pi : X \to X/T$ is a p-fold covering with sheets around the points of a fibre being interchanged by the powers of T. In this way, the set $\{x \in \sigma_r | d_1(x) = T^k d_2(x)\}$ is an open and nonempty set of σ_r. Since this set is also closed and σ_r is connected, the fact follows.

Now suppose $c \in S_r(X,T)$ a nonzero (T,r)-chain. Then $c = \theta(d)$ where d is a nonzero chain. It may be assumed that $d = a_1 d_1 + a_2 d_2 + ... + a_s d_s$, where each d_i is a singular r-simplex with $d_i \neq d_j$ for $i \neq j$, and each $a_i \neq 0$. Additionally, it may be assumed that d_i and d_j belong to different orbits. It follows that $\pi d_i \neq \pi d_j$ if $i \neq j$. Hence $\Gamma(c) = \sum_{i=1}^{s} a_i(\pi d_i)$ is a nonzero chain.

Let $\phi : \sigma_r \to X/T$ be a singular r-simplex. Since $\pi : X \to X/T$ is a p-fold covering with X connected and locally pathwise connected, and σ_r is simply connected, one has by the lifting theorem (for example, see **[SH]** , page 89) that there is $\phi' : \sigma_r \to X$ such that $\pi\phi' = \phi$, which shows that Γ is surjective. Consequently, $\Gamma_* : H_r(X,T) \to H_r(X/T, Z_p)$ is an isomorphism. □

A well known fact about the $(n+1)$-dimensional real projective space RP^{n+1} is that its *rth* singular Z_2-homology is nonzero for $1 \leq r \leq n+1$. The result that follows is a generalization of this fact.

Corollary. *Suppose (X,T) a Z_p-space, where X is Hausdorff, connected and locally pathwise connected. For a natural number $n \geq 1$, suppose $H_r(X, Z_p) = 0$ for $1 \leq r \leq n$. If $p = 2q$ with q odd, then $H_r(X/T, Z_p) \neq 0$ for all r, $1 \leq r \leq n+1$.*

Remark. By the remark after Proposition 4, the above corollary still holds for any even p and $1 \leq r \leq 3$ (in this case we need only that $H_r(X, Z_p) = 0$ for $1 \leq r \leq 2$), and for any p and $r = 1$ (in this case no requirement on $H_r(X, Z_p)$ for $r \geq 1$ is needed).

Remark. For $p = 2$ and X a Hausdorff and paracompact space, the above corollary can be proved in a more direct way considering the Cech cohomology mod 2 instead the singular Z_2-homology. In fact, we can use in this case the Smith-Gysin exact sequence (which can be taken as the Z_2-version for real line bundles of the sequence (10.5) of **[GB]** , Section 3.10, page 161; alternatively, see **[JM]** , corollary 12.3, page 145)

$$\check{H}^0(X/T) \xrightarrow{\pi^*} \check{H}^0(X) \xrightarrow{\tau} \check{H}^0(X/T) \xrightarrow{\cup e} \check{H}^1(X/T) \longrightarrow ...$$
$$\longrightarrow \check{H}^r(X/T) \xrightarrow{\pi^*} \check{H}^r(X) \xrightarrow{\tau} \check{H}^r(X/T) \xrightarrow{\cup e}$$
$$\check{H}^{r+1}(X/T) \longrightarrow ... \quad .$$

Here, $\tau : \check{H}^r(X) \to \check{H}^r(X/T)$ is the transfer homomorphism and $e \in \check{H}^1(X/T)$ is the *Euler class* of $X \to X/T$. In other words, $e = g^*(\alpha)$, where $\alpha \in H^1(B(Z_2), Z_2)$ is the Euler class of the universal principal Z_2-bundle over the Z_2-classifying space $B(Z_2)$, and $g : X/T \to B(Z_2)$ is a classifying map for $X \to X/T$ (we observe that $H^1(B(Z_2), Z_2)$ is isomorphic to $\check{H}^1(B(Z_2))$).

3. ON THE EXISTENCE OF $\mathbf{Z_p}$-EQUIVARIANT MAPS

In this section we are concerned with the so called Borsuk-Ulam Problem which deals with the existence of equivariant maps between two given Z_p-spaces (X, T) and (Y, S).

Let $A_k : S^k \to S^k$ be the antipodal map from the k-sphere to itself. The classical Borsuk-Ulam Theorem can be formulated as follows. If $f : (S^m, A_m) \to (S^n, A_n)$ is an equivariant map, then $m \leq n$ (see, for example, **[MA]** , 7.2). It is reasonable to investigate to what extent the geometry of the special pairs (S^m, A_m) and (S^n, A_n) is essential to the result.

In the direction of replacing the domain (S^m, A_m) by a more general Z_p-space, J. W. Walker proved in **[JW]** the following generalization of the classical Borsuk-Ulam Theorem. If (X, T) is any Z_2-space and $f : (X, T) \to (S^n, A_n)$ is an equivariant map, then there exists an r, $1 \leq r \leq n$, such that $H_r(X, Z_2) \neq 0$. Somewhat later, T. Kobayashi established a related result for $p > 2$ (see **[TK]**); specifically, Kobayashi proved the following result. If (X, T) is any Z_p-space and $f : (X, T) \to (S^{2n+1}, T_p)$, where T_p is defined in Section 1, is an equivariant map, then there exists an r, $1 \leq r \leq 2n + 1$, such that $H_r(X, Z_p) \neq 0$.

Based upon Section 2, we have the following result which is a generalization of Walker's theorem and it also includes a special case of Kobayashi's theorem.

Theorem. *Let (X, T) and (Y, S) be Z_p-spaces with $p = 2q$ and q odd. Suppose that*

i) X is pathwise connected and Y is Hausdorff, connected and locally pathwise connected;

ii) for some natural number $n \geq 1$, $H_{n+1}(Y/S, Z_p) = 0$.

Then, if $f : (X, T) \to (Y, S)$ is an equivariant map, there exists an r, $1 \leq r \leq n$, such that $H_r(X, Z_p) \neq 0$.

PROOF. Suppose by contradiction that $H_r(X, Z_p) = 0$ for all r, $1 \leq r \leq n$. By Proposition 4), one then has that $J(H_{n+1}(X, T)) \neq 0$. Consider the induced Z_p-homomorphism $f_* : H_{n+1}(X, T) \to H_{n+1}(Y, S)$. By Proposition 3), it follows that $Jf_*(H_{n+1}(X, T)) \neq 0$, and hence that $H_{n+1}(Y, S) \neq 0$. But this is impossible, since by Proposition 5) $H_{n+1}(Y, S)$ is isomorphic to $H_{n+1}(Y/S, Z_p)$. □

This result can be restated as follows. If (X, T) and (Y, S) are Z_p-spaces as described above, and $H_r(X, Z_p) = 0$ for $1 \leq r \leq n$, then there is no equivariant map $f : (X, T) \to (Y, S)$. For example, Y can be any n-dimensional manifold equipped with a free periodic homeomorphism of period p, where $p = 2q$ with q odd.

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