INVOLUTIONS FIXING $\mathbf{F^n \cup F^2}$

PERGHER, PEDRO L. Q. ; FIGUEIRA, FÁBIO G.

Abstract. Let M^m be a closed and smooth manifold with an involution having fixed point set of the form $F^n \cup F^2$, where F^n and F^2 are submanifolds with dimensions n and 2, respectively, and where $2 < n < m$ and $F^n \cup F^2$ does not bound. The main result of this paper is to establish the upper bound for m, for each n. The existence of these bounds is guaranteed by the famous 5/2-theorem of J. Boardman, which establishes that, under the above hypotheses, $m \leq 5/2n$.

1. Introduction

Suppose M^m is a smooth and closed m-dimensional manifold and $T : M^m \mapsto M^m$ is a smooth involution defined on M^m. The fixed point set of T, F, is a disjoint union of closed submanifolds of M^m, $F = \bigcup_{j=0}^{n} F^j$, where F^j denotes the union of those components of F having dimension j. It is well known, from equivariant bordism theory, that if (M^m, T) is nonbounding then F cannot be too low dimensional. This fact was evidenced from an old result of P. Conner and E. E. Floyd (Theorem 27.1 of [4]), which stated: for each natural number n, there exists a number $\varphi(n)$ with the property that, if (M^m, T) is an involution fixing $F = \bigcup_{j=0}^{n} F^j$ and if $m > \varphi(n)$, then (M^m, T) bounds equivariantly. Later this was explicitly confirmed by the famous 5/2-Theorem of J. Boardman of [3]: if (M^m, T) fixes $F = \bigcup_{j=0}^{n} F^j$ and M^m is nonbounding, then $m \leq \frac{5}{2}n$. A strengthened version of this fact was obtained by R.E. Stong and C. Kosniowski

1991 *Mathematics Subject Classification.* (2.000 Revision) Primary 57R85; Secondary 57R75.

Key words and phrases. involution, fixed data, Stiefel-Whitney class, characteristic number, Wu class, stable cobordism class, splitting principle.

The first author was partially supported by CNPq and FAPESP..

in [2]: if (M^m, T) is a nonbounding involution fixing $F = \bigcup_{j=0}^{n} F^j$, then $m \leq \frac{5}{2}n$. In particular, if $F = \bigcup_{j=0}^{n} F^j$ is nonbounding (which means that at least one F^j is nonbounding) and (M^m, T) fixes F, then $m \leq \frac{5}{2}n$; this follows from the fact that the equivariant cobordism class of (M^m, T) is determined by the cobordism class of its fixed data. The generality of this last result allows the possibility that fixed components of all dimensions j, $0 \leq j \leq n$, occur; in this way, it is natural to ask whether there exists a better upper bound for m when we omit some components of F. This is inspired by the following result of Stong and Kosniwoski of [2]: if (M^m, T) is an involution whose fixed set has constant dimension n, and if $m > 2n$, then (M^m, T) bounds equivariantly. In particular, if $F = F^n$ with constant dimension n is nonbounding, and if (M^m, T) fixes F, then $m \leq 2n$. This bound is best possible, as can be seen by taking the involution $(F^n \times F^n, T)$, where F^n is any nonbounding n-dimensional manifold (with the exception of $n = 1$ and $n = 3$) and T switches coordinates. Thus one has a concrete improvement of the Boardman's bound when we omit all j-dimensional components with $j < n$.

The above considerations can be placed in the following general setting: for each natural number n and each subset $X \subset \{0, 1, 2, \ldots, n-1\}$ (we allow X to be empty), we define $m(n; X)$ as being the number

> $m(n; X) =$ maximum $\{m \mid$ there exists an involution (M^m, T) fixing F such that F does not bound, n is the dimension of the non-empty component of F of largest dimension, and if F^j is a non-empty j-dimensional component of F with $j < n$, then $j \in X\}$.

As it was seen above, this number always exists (but it is not defined if we allow F to be a boundary, since in this case one has involutions fixing F with any codimension); further, if $j \in X$, the number of j-dimensional components of F has no influence in the value of $m(n; X)$, since any involution is equivariantly cobordant to an involution with the property that the j-dimensional part of the fixed set is connected.

Under this setting, the Boardman's bound is stated as "for every n and every $X \subset \{0, 1, 2, \ldots, n-1\}$, $m(n; X) \leq \frac{5}{2}n$", and the Stong-Kosniowski's bound is stated as "for $n \neq 1$ and 3, and $X = \emptyset$, $m(n; X) = 2n$".

Once the case $X = \emptyset$ is established, the next natural step is to consider X containing a single element, which means to consider fixed sets of the form $F = F^n \cup F^j$, $j < n$. For $j = 0$, $F = F^n \cup F^0$ reduces to $F = F^n \cup \{point\}$. Concerning this case, recently Stong and Pergher proved the following result [5]: for each natural number n, write $n = 2^p q$, where $p \geq 0$ and q is odd, and set

$$m(n) = \begin{cases} (2^{p+1} - 1)q + p + 1 = 2n + p - q + 1, & \text{if } p \leq q + 1 \\ (2^{p+1} - 2^{p-q})q + 2^{p-q}(q+1) = 2n + 2^{p-q}, & \text{if } p \geq q. \end{cases}$$

Then, if (M^m, T) is an involution whose fixed set has the form $F = F^n \cup \{point\}$, $m \leq m(n)$; further, there are involutions with $m = m(n)$ fixing a point and some F^n.

Together with the case $X = \emptyset$, this result says that

$$m(n; \{0\}) = \text{ maximum } \{m(n), 2n\} \text{ if } n \neq 3, \text{ and } m(3; \{0\}) = 4.$$

The objective of this paper is to calculate $m(n; \{2\})$. Specifically, we shall prove that $m(n; \{2\}) = \text{maximum } \{m(n-2) + 4, 2n\}$ when $n \geq 3$.

Concerning $m(n; \{1\})$, in her doctoral thesis [6] (and in [7]), S. Kelton studied bounds for involutions (M^m, T) whose fixed set has the form $F = F^n \cup \mathbb{R}\mathrm{P}^j$, where $\mathbb{R}\mathrm{P}^j$ is the j-dimensional real projective space. Among the results, one finds: suppose (M^m, T) is an involution whose fixed set has the form $F = F^n \cup \mathbb{R}\mathrm{P}^1$ and the normal bundle of $\mathbb{R}\mathrm{P}^1$ in M^m is nonbounding. Then, if n is odd, $m \leq m(n-1) + 1$, and if n is even, $m \leq m(n-1) + 2$; further, these bounds are best possible. Since $F^n \cup F^1$ reduces to $F^n \cup \mathbb{R}\mathrm{P}^1$, these results give (for $n > 1$):

$$m(n; \{1\}) = \begin{cases} \text{maximum } \{m(n-1) + 1, 2n\}, & \text{if } n \text{ is odd;} \\ \text{maximum } \{m(n-1) + 2, 2n\}, & \text{if } n \text{ is even.} \end{cases}$$

We remark that, in the cases $F = F^n \cup F^0$ and $F = F^n \cup F^1$, one has an unique nonbounding stable cobordism class of bundles over F^j, $j = 0$ or 1 (the trivial bundle when $j = 0$, and the stable cobordism class of the canonical line bundle over $\mathbb{R}\mathrm{P}^1$ when $j = 1$). As we will see, the technical difficulty in the calculation of $m(n;\{2\})$ lies in the fact that one has a lot of possible stable cobordism classes of bundles over F^2.

2. Computation of $m(n;\{2\})$

In this section we will show that $m(n;\{2\}) = \text{maximum}\{m(n-2)+4, 2n\}$, where $n \geq 3$. By the definition of $m(n;X)$, one needs to consider involutions (M^m, T) for which the fixed set F does not bound and has the form $F = F^n$ or $F = F^n \cup F^2$, and one knows that F^n and F^2 can be assumed to be connected. The first thing to do is to exhibit, for each $n \geq 3$, involutions (M^m, T) with $m = 2n$ and $m = m(n-2)+4$, and with F having the form described above. As already remarked, taking any n-dimensional nonbounding manifold F^n, the twist involution on $F^n \times F^n$ provides an example with $m = 2n$. On the other hand, and as remarked in the previous section, in [5] Stong and Pergher constructed, for each $n \geq 1$, a special involution $(M^{m(n)}, T_n)$ for which the fixed set has the form $F^n \cup \{point\}$. Given $n \geq 3$, consider the involution $(M^{m(n-2)} \times \mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2, T)$, where $T(x,y,z) = (T_{n-2}(x), z, y)$. The fixed set of T has the form

$$(F^{n-2} \cup \{point\}) \times \mathbb{R}\mathrm{P}^2 = F^{n-2} \times \mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^2,$$

and since $\mathbb{R}\mathrm{P}^2$ does not bound, this provides an example with $m = m(n-2)+4$. Since $m(3-2)+4 = 6 = 2 \cdot 3$, this approach causes no problem when $n = 3$.

With these examples on hand and taking into account the Stong-Kosniowski's bound for connected fixed sets, all that remains is to show the following fact: if (M^m, T) is an involution whose fixed set F does not bound and has the form $F = F^n \cup F^2$, then either $m \leq 2n$ or $m \leq m(n-2)+4$. Let $\eta \mapsto F^n$, $\mu \mapsto F^2$ denote the normal bundles of F^n and F^2 in M^m. If $\mu \mapsto F^2$ bounds, it can be equivariantly removed to give an involution (N^m, T'), equivariantly cobordant to (M^m, T), and with fixed data $\eta \mapsto F^n$. Since F^2 bounds, F^n does not bound and so $m \leq 2n$. Thus the computation of $m(n; \{2\})$ is reduced to the following

Theorem 2.1. *Suppose that (M^m, T) is an involution having fixed set F which does not bound and has the form $F = F^n \cup F^2$. If the normal bundle over the component F^2 does not bound, then $m \leq m(n-2)+4$.*

Remark. As we will see, the hypothesis "F does not bound" is really not necessary to the proof.

As above, denote by $(\eta \mapsto F^n) \cup (\mu \mapsto F^2)$ the fixed data of (M^m, T). If $\mu \mapsto F^2$ is cobordant to $\mu' \mapsto F^{2'}$, then there exists an involution (N^m, T'), cobordant to (M^m, T) and with fixed data $\eta \cup \mu'$. Thus, since we will be working with characteristic numbers, our first task will be to describe a complete list of explicit representatives for the possible nonbounding cobordism classes of bundles over 2-dimensional closed manifolds. We need some notations: if ξ is a vector bundle and n is a natural number, $n\xi$ will denote the Whitney sum of n copies of ξ. We will use ε^r to denote the trivial r-dimensional vector bundle over any base space. For any vector bundle over a closed 2-dimensional manifold, $\mu \mapsto F^2$, one lets $\mathbb{W}(F^2) = 1 + w_1 + w_2$ be the Stiefel-Whitney class of F^2 and $\mathbb{W}(\mu) = 1 + v_1 + v_2$ be the Stiefel-Whitney class of μ.

Lemma 2.2. *For vector bundles as above, one has $w_1^2 = w_2$ and $v_1^2 = w_1 v_1$.*

Proof. F^2 is either a boundary or cobordant to $\mathbb{R}\mathrm{P}^2$. Since $\mathbb{R}\mathrm{P}^2$ and any manifold which bounds satisfy $w_1^2 = w_2$, this is also true for F^2. Now let $U = 1 + u$ be the Wu class of F^2; one knows that $u = w_1$. Then $Sq^1(v_1) = uv_1 = w_1 v_1$, where Sq is the Steenrod operation; but also $Sq^1(v_1) = v_1^2$, and the result follows. □

The cobordism class of $\mu \mapsto F^2$ is determined by its characteristic numbers. By the above lemma, these numbers are reduced to the ones obtained from w_1^2 $(= w_2)$, v_2 and v_1^2 $(= w_1 v_1)$. This gives at most seven possibilities for nonbounding classes. Next we describe examples realizing each one of these possibilities. Denote by $\xi \mapsto \mathbb{R}\mathrm{P}^2$ the canonical line bundle. Then one has the bundles:

1) the 0-dimensional bundle $0 \mapsto \mathbb{R}\mathrm{P}^2$, with $w_1^2 \neq 0,\ v_2 = 0$ and $v_1^2 = 0$;

2) $\xi \mapsto \mathbb{R}\mathrm{P}^2$, with $w_1^2 \neq 0,\ v_2 = 0$ and $v_1^2 \neq 0$;

3) $2\xi \mapsto \mathbb{R}\mathrm{P}^2$, with $w_1^2 \neq 0,\ v_2 \neq 0$ and $v_1^2 = 0$;

4) $3\xi \mapsto \mathbb{R}\mathrm{P}^2$, with $w_1^2 \neq 0,\ v_2 \neq 0$ and $v_1^2 \neq 0$.

Now consider $\xi \oplus \varepsilon^1 \mapsto \mathbb{R}\mathrm{P}^1$, where again ξ denotes the canonical line bundle. Consider $\mathbb{R}\mathrm{P}(\xi \oplus \varepsilon^1) \mapsto \mathbb{R}\mathrm{P}^1$ the real projective space bundle associated to $\xi \oplus \varepsilon^1$, and denote by $\lambda \mapsto \mathbb{R}\mathrm{P}(\xi \oplus \varepsilon^1)$ the line bundle of the double cover $S(\xi \oplus \varepsilon^1) \mapsto \mathbb{R}\mathrm{P}(\xi \oplus \varepsilon^1)$, $S(\xi \oplus \varepsilon^1)$ the sphere bundle of $\xi \oplus \varepsilon^1$. Note that $K^2 = \mathbb{R}\mathrm{P}(\xi \oplus \varepsilon^1)$ is a closed 2-dimensional manifold, and one has the following examples with its respective characteristic numbers, obtained from standard computations in the cohomology of K^2:

5) $\lambda \mapsto K^2$, with $w_1^2 = 0,\ v_2 = 0$ and $v_1^2 \neq 0$;

6) $2\lambda \mapsto K^2$, with $w_1^2 = 0,\ v_2 \neq 0$ and $v_1^2 = 0$;

7) $3\lambda \mapsto K^2$, with $w_1^2 = 0,\ v_2 \neq 0$ and $v_1^2 \neq 0$.

We denote by $\beta_i,\ 1 \leq i \leq 7$, the stable cobordism classes corresponding to these examples. The following lemma will be crucial to our purposes:

Lemma2.3. *If $m > m(n-2)+4$, then $w_1^2 = v_1^2$ and $v_2 = 0$.*

Note that the unique β_i satisfying $w_1^2 = v_1^2$ and $v_2 = 0$ is β_2. Thus this lemma will reduce our task to the following

Theorem 2.4. *Let (M^m, T) be an involution having fixed set F of the form $F = F^n \cup F^2$. If the normal bundle $\mu \mapsto F^2$ represents β_2, then $m \leq m(n-2)+4$.*

The following basic fact from [4] will be needed for the proof of Lemma 2.3: the projective space bundles $\mathbb{R}\mathrm{P}(\eta)$ and $\mathbb{R}\mathrm{P}(\mu)$, with its standard line bundles $\lambda \mapsto \mathbb{R}\mathrm{P}(\eta)$ and $\nu \mapsto \mathbb{R}\mathrm{P}(\mu)$, are cobordant as elements of the bordism group $\mathcal{N}_{m-1}(BO(1))$. Then any class of dimension $m-1$, given by a product of the classes $w_i(\mathbb{R}\mathrm{P}(\eta))$ and $w_1(\lambda)$, evaluated on the fundamental homology class

$[\mathbb{R}\mathrm{P}(\eta)]$, gives the same characteristic number as the one obtained by the corresponding product of the classes $w_i(\mathbb{R}\mathrm{P}(\mu))$ and $w_1(\nu)$, evaluated on $[\mathbb{R}\mathrm{P}(\mu)]$. We will apply this using some very special classes. Set $k = m - n$, and write

$$\begin{aligned} \mathbb{W}(F^n) &= 1 + \theta_1 + \cdots + \theta_n, \\ \mathbb{W}(\eta) &= 1 + u_1 + \cdots + u_k \quad \text{and} \\ \mathbb{W}(\lambda) &= 1 + c \end{aligned}$$

for the Stiefel-Whitney classes of F^n, η and λ, respectively. From [1] one knows that

$$\mathbb{W}(\mathbb{R}\mathrm{P}(\eta)) = (1 + \theta_1 + \cdots + \theta_n)\{(1+c)^k + (1+c)^{k-1}u_1 + \cdots + (1+c)u_{k-1} + u_k\},$$

where here we are suppressing bundle maps. For any integer r, one lets

$$W[r] = \frac{\mathbb{W}(\mathbb{R}\mathrm{P}(\eta))}{(1+c)^{k-r}}$$

Note that each class $W[r]_j$ is a polynomial in the classes $w_i(\mathbb{R}\mathrm{P}(\eta))$ and c. Further, these classes satisfy the following special properties (see [5], Section 2):

$$\begin{aligned} W[r]_{2r} &= \theta_r c^r + \text{ terms with smaller } c \text{ powers}, \\ W[r]_{2r+1} &= (\theta_{r+1} + u_{r+1})c^r + \text{ terms with smaller } c \text{ powers}. \end{aligned}$$

For $n \geq 3$, write $n - 2 = 2^p q$, where $p \geq 0$ and q is odd, and suppose first that $p < q + 1$. Consider the list of integers $r_1, r_2, \ldots, r_p$, where $r_i = 2^p - 2^{p-i}$, and take the class

$$X = W[2^p - 1]_{2^{p+1}-1}^{q+1-p} \cdot W[r_1]_{2r_1} \cdot W[r_2]_{2r_2} \cdots W[r_p]_{2r_p}$$

(if $p = 0$, this class reduces to $X = W[0]_1^{q+1}$). The dimension of X is

$$(q + 1 - p)(2^{p+1} - 1) + 2\sum_{i=1}^{p}(2^p - 2^{p-i}) = (2^{p+1} - 1)q + p + 1 = m(n-2).$$

From the properties above listed, one has

$$\begin{aligned} X &= ((\theta_{2^p} + u_{2^p})c^{2^p-1} + \text{ terms with smaller } c \text{ powers })^{q+1-p} \cdot \\ &\quad \cdot(\theta_{r_1} c^{r_1} + \text{ terms with smaller } c \text{ powers })\cdot \\ &\quad \cdots \\ &\quad \cdot(\theta_{r_p} c^{r_p} + \text{ terms with smaller } c \text{ powers }) = \\ &= ((\theta_{2^p} + u_{2^p})^{q+1-p} \cdot \theta_{r_1} \cdot \theta_{r_2} \cdots \theta_{r_p})c^{(q+1-p)(2^p-1)+\sum_{i=1}^{p} r_i} + \\ &\quad + \text{terms with smaller } c \text{ powers.} \end{aligned}$$

Note that

$$(q+1-p)\cdot 2^p + \sum_{i=1}^{p} r_i = (q+1-p)2^p + p2^p - 2^p + 1 = 2^p q + 1 = n-1.$$

Thus X has the form

$$X = A_{n-1}\cdot c^{m(n-2)-n+1} + \text{ terms with smaller } c \text{ powers},$$

where A_{n-1} is a class of dimension $n-1$ coming from the cohomology of F^n.

Now suppose $p \geq q+1$, and consider the list $r_1, r_2, \ldots, r_{q+1}$, where again $r_i = 2^p - 2^{p-i}$. In this case, take

$$X = W[r_1]_{2r_1}\cdot W[r_2]_{2r_2}\cdots W[r_{q+1}]_{2r_{q+1}}.$$

The dimension of X is

$$\begin{aligned} 2\sum_{i=1}^{q+1} r_i &= \sum_{i=1}^{q+1}(2^{p+1} - 2^{p-i+1}) \\ &= (q+1)2^{p+1} - 2^{p+1} + 2^{p-q} = q2^{p+1} + 2^{p-q} \\ &= (2^{p+1} - 2^{p-q})q + 2^{p-q}(q+1) = m(n-2) \end{aligned}$$

and

$$X = \theta_{r_1}\cdot\theta_{r_2}\cdots\theta_{r_{q+1}}\cdot c^{r_1+\cdots+r_{q+1}} + \text{ terms with smaller } c \text{ powers}.$$

Note that

$$\begin{aligned} \sum_{i=1}^{q+1} r_i &= \sum_{i=1}^{q+1}(2^p - 2^{p-i}) &&= (q+1)2^p - 2^p + 2^{p-q-1} \\ &= 2^p q + 2^{p-q-1} &&= n-2+2^{p-q-1} \geq n-1. \end{aligned}$$

Thus, for every $n \geq 3$, X is a class of dimension $m(n-2)$ which has the form

$$X = A_l \cdot c^{m(n-2)-l} + \text{ terms with smaller } c \text{ powers},$$

where A_l has dimension $l \geq n-1$ and comes from the cohomology of F^n.

Next we shall introduce some special classes of dimension 4 associated to line bundles $\lambda \mapsto B^s$, where B^s is a closed s-dimensional manifold. Using the splitting principle, write

$$\mathbb{W}(B^s) = (1+x_1)\cdot(1+x_2)\cdots(1+x_s)$$

and $\mathbb{W}(\lambda) = 1 + c$. Consider the symmetric polynomials in the variables $x_1, x_2, \ldots, x_s, c$, of degree 4, given by

$$f_{\omega_1} = \sum_{i<j} x_i(x_i + c)x_j(x_j + c)$$

and

$$f_{\omega_2} = \sum_{i} x_i^2(x_i + c)^2$$

Then f_{ω_1} and f_{ω_2} determine polynomials of dimension 4 in the classes $w_i(B^s)$ and $w_1(\lambda) = c$. Returning to $\lambda \mapsto \mathbb{R}\mathrm{P}(\eta)$, write

$$\begin{array}{ll} \mathbb{W}(F^n) & = (1 + x_1) \cdot (1 + x_2) \cdots (1 + x_n) \quad \text{and} \\ \mathbb{W}(\eta) & = (1 + y_1) \cdot (1 + y_2) \cdots (1 + y_k). \end{array}$$

Then

$$\mathbb{W}(\mathbb{R}\mathrm{P}(\eta)) = (1 + x_1) \cdots (1 + x_n)(1 + c + y_1) \cdots (1 + c + y_k).$$

It follows that

$$\begin{aligned} f_{\omega_1}(\lambda \mapsto \mathbb{R}\mathrm{P}(\eta)) & = \sum_{i<j} x_i(x_i + c)x_j(x_j + c) + \\ & + \sum_{t<l} y_t(y_t + c)y_l(y_l + c) + \\ & + \sum_{i,t} x_i(x_i + c)y_t(y_t + c) = \\ & = (\sum_{i<j} x_i x_j + \sum_{t<l} y_t y_l + \sum_{i,t} x_i y_t) \cdot c^2 + \\ & + \text{terms with smaller } c \text{ powers}, \end{aligned}$$

and

$$\begin{aligned} f_{\omega_2}(\lambda \mapsto \mathbb{R}\mathrm{P}(\eta)) & = \sum_{i} x_i^2(x_i + c)^2 + \sum_{t} y_t^2(y_t + c)^2 = \\ & = (\sum_{i} x_i^2 + \sum_{t} y_t^2) \cdot c^2 + \sum_{i} x_i^4 + \sum_{t} y_t^4. \end{aligned}$$

Therefore every term of f_{ω_1} and f_{ω_2} has a factor of dimension at least 2 from the cohomology of F^n. We have seen that each term of our previous class X has a factor of dimension at least $n-1$ from the cohomology of F^n, which means that, for $i = 1, 2$, $f_{\omega_i} \cdot X$ is a class in $H^{m(n-2)+4}(\mathbb{R}\mathrm{P}(\eta), Z_2)$ with each one of its terms having a factor of dimension at least $n+1$ from F^n. Thus $f_{\omega_i} \cdot X = 0$.

Since $m > m(n-2)+4$, one can form the class $f_{\omega_i} \cdot X \cdot c^{m-1-(m(n-2)+4)}$, which yields the zero characteristic number $f_{\omega_i} \cdot X \cdot c^{m-1-(m(n-2)+4)}[\mathbb{R}\mathrm{P}(\eta)]$.

Our next task is to analyse the class associated to $\nu \mapsto \mathbb{R}\mathrm{P}(\mu)$ which corresponds to $f_{\omega_i} \cdot X \cdot c^{m-1-(m(n-2)+4)}$. Setting $\mathbb{W}(\nu) = 1 + d$, this class is

$$f_{\omega_i}(\nu \mapsto \mathbb{R}\mathrm{P}(\mu)) \cdot Y \cdot d^{m-1-(m(n-2)+4)},$$

where Y is obtained from X by replacing each $W[r]_i$ by $W[n+r-2]_i$. The Stiefel-Whitney class of $\mathbb{R}\mathrm{P}(\mu)$ is

$$\mathbb{W}(\mathbb{R}\mathrm{P}(\mu)) = (1+w_1+w_2)\{(1+d)^{n+k-2} + (1+d)^{n+k-3}v_1 + (1+d)^{n+k-4}v_2\}.$$

Writing $\mathbb{W}(F^2) = (1+x_1)(1+x_2)$ and $\mathbb{W}(\mu) = (1+y_1)(1+y_2)$, one has

$$\begin{aligned} \mathbb{W}(\mathbb{R}\mathrm{P}(\mu)) &= (1+d)^{n+k-4}\{(1+w_1+w_2)\{(1+d)^2 + (1+d)v_1 + v_2\}\} \\ &= (1+d)^{n+k-4}(1+x_1)(1+x_2)(1+d+y_1)(1+d+y_2). \end{aligned}$$

Noting that the part $(1+d)^{n+k-4}$ does not contribute to f_{ω_i}, we get

$$\begin{aligned} f_{\omega_1}(\nu \mapsto \mathbb{R}\mathrm{P}(\mu)) &= x_1(x_1+d)x_2(x_2+d) + y_1(y_1+d)y_2(y_2+d) + \\ &+ \sum_{i,j} x_i(x_i+d)y_j(y_j+d) = \\ &= (x_1x_2 + y_1y_2 + \sum_{i,j} x_iy_j)d^2 + \\ &+ \text{terms with smaller } c \text{ powers,} \end{aligned}$$

and

$$\begin{aligned} f_{\omega_2}(\nu \mapsto \mathbb{R}\mathrm{P}(\mu)) &= x_1^2(x_1+d)^2 + x_2^2(x_2+d)^2 + y_1^2(y_1+d)^2 + y_2^2(y_2+d)^2 = \\ &= (x_1+x_2+y_1+y_2)^2d^2 + (x_1+x_2+y_1+y_2)^4. \end{aligned}$$

If $\tau \mapsto \mathbb{R}\mathrm{P}(\mu)$ is the tangent bundle over F^2, the factored form of $\mathbb{W}(\tau \oplus \mu)$ is

$$\mathbb{W}(\tau \oplus \mu) = (1+x_1)(1+x_2)(1+y_1)(1+y_2).$$

Setting $\mathbb{W}(\tau \oplus \mu) = 1 + V_1 + V_2$ and noting that if a term (with dimension 4) has power of d less than 2, it necessarily has a factor of dimension greater than 2 from the cohomology of F^2, one then has $f_{\omega_1}(\nu) = V_2d^2$ and $f_{\omega_2}(\nu) = V_1^2d^2$. Denoting by $\mathscr{I}$ the ideal of $H^*(\mathbb{R}\mathrm{P}(\mu), Z_2)$ generated by the classes coming from F^2 and with positive dimension, one has that $f_{\omega_i} \cdot A = 0$ for each $A \in \mathscr{I}$. Thus, in the computation of Y, one needs to consider only that

$$\mathbb{W}(\mathbb{R}\mathrm{P}(\mu)) \equiv (1+d)^{n+k-2} \bmod \mathscr{I}$$

and for each integer l

$$W[l] \equiv (1+d)^l \bmod \mathscr{I}.$$

For $r_i = 2^p - 2^{p-i},\ i = 1, 2, \ldots, p$, set $l_i = n + r_i - 2 = 2^p q + 2 + 2^p - 2^{p-i} - 2 = 2^p q + 2^p - 2^{p-i}$. Then

$$W[l_i]_{2r_i} \equiv \binom{2^p q + 2^p - 2^{p-i}}{2^{p+1} - 2^{p-i+1}} d^{2r_i} \bmod \mathscr{I}.$$

Also, if $r = 2^p - 1,\ l = n + r - 2 = 2^p q + 2^p - 1$ and

$$W[l]_{2r+1} \equiv \binom{2^p q + 2^p - 1}{2^{p+1} - 1} d^{2r+1} \bmod \mathscr{I}.$$

The lesser term of the 2-adic expansion of $2^p q + 2^p$ is 2^{p+1}. Using the fact that a binomial coefficient $\binom{a}{b}$ is nonzero modulo 2 if and only if the 2-adic expansion of b is a subset of the 2-adic expansion of a, we conclude that the above binomial coefficients are nonzero modulo 2. It follows that all classes $W[r]_i$ occurring in Y satisfy $W[r]_i \equiv d^i \bmod \mathscr{I}$, which implies that $Y \equiv d^{m(n-2)} \bmod \mathscr{I}$. Since $H^*(\mathbb{R}\mathrm{P}(\mu), Z_2)$ is the free $H^*(F^2, Z_2)$-module on $1, d, d^2, \ldots, d^{n+k-3}$, we get

$$f_{\omega_1}(\nu) \cdot Y \cdot d^{m-1-(m(n-2)+4)}[\mathbb{R}\mathrm{P}(\nu)] = d^{m-3} V_2[\mathbb{R}\mathrm{P}(\nu)] = V_2[F^2]$$

and

$$f_{\omega_2}(\nu) \cdot Y \cdot d^{m-1-(m(n-2)+4)}[\mathbb{R}\mathrm{P}(\nu)] = V_1^2[F^2].$$

Putting together with the previous calculations on F^n, we conclude that $V_2 = 0$ and $V_1^2 = 0$. Since $V_1 = v_1 + w_1$, we get $v_1^2 = w_1^2$, and since

$$V_2 = v_1 w_1 + v_2 + w_2 = Sq^1(v_1) + v_2 + w_2 = v_1^2 + v_2 + w_2 = w_1^2 + v_2 + w_1^2 = v_2$$

we get $v_2 = 0$. Thus Lemma 2.3 is proved. □

Now we prove Theorem 2.4. One is considering an involution (M^m, T) with fixed set F of the form $F = F^n \cup F^2$, where the normal bundle $\mu \mapsto F^2$ represents β_2, and wants to show that $m \leq m(n-2) + 4$. We maintain the previous notations for the characteristic classes referring to the component F^n, and we can suppose with no loss that $\mu \mapsto F^2 = \xi \oplus \varepsilon^{m-3} \mapsto \mathbb{R}\mathrm{P}^2$. We repeat the notations $\nu \mapsto \mathbb{R}\mathrm{P}(\mu)$ and $\mathbb{W}(\nu) = 1 + d$ for the standard line bundle over $\mathbb{R}\mathrm{P}(\mu)$ and its characteristic class. Let $\alpha \in H^1(F^2, Z_2)$ be the generator. Since $H^*(\mathbb{R}\mathrm{P}(\mu), Z_2)$ is the free $H^*(F^2, Z_2)$-module on $1, d, d^2, \ldots, d^{m-3}$ subject to the relation $d^{m-2} + d^{m-3}\alpha = 0$, one has that $d^{m-1} = d^{m-2}\alpha = d^{m-3}\alpha^2$ is the generator (top) of $H^{m-1}(\mathbb{R}\mathrm{P}(\mu), Z_2)$. Our strategy will consist in showing that, if $m > m(n-2) + 4$, then it is possible to find polynomials in the characteristic classes so that the corresponding characteristic numbers are zero on F^n and nonzero on F^2. First consider n odd. In this case, we will obtain a stronger result, noting that $m(n-2) + 4 = n + 3$.

Lemma 2.5 *If (M^m, T) is an involution fixing $F = F^n \cup F^2$, where n is odd and $\mu \mapsto F^2 = \xi \oplus \varepsilon^{m-3} \mapsto \mathbb{R}\mathrm{P}^2$, then $m \leq n + 1$ (hence $m = n + 1$).*

Proof. On F^n one has

$$W[0] = (1 + \theta_1 + \theta_2 + \cdots + \theta_n)\{1 + \frac{u_1}{1+c} + \cdots + \frac{u_k}{(1+c)^k}\}.$$

If $m > n + 1$, one can form the class $W[0]_1^{n+1} c^{m-1-(n+1)}$ of dimension $m - 1$. Since $W[0]_1^{n+1} = (\theta_1 + u_1)^{n+1}$ comes from F^n, this gives a zero characteristic number. The class over F^2 corresponding to $W[0]$ is $W[n-2]$. Now

$$\mathbb{W}(\mathbb{R}\mathrm{P}(\mu)) = (1 + \alpha + \alpha^2)\{(1+d)^{m-2} + (1+d)^{m-3}\alpha\}$$

and

$$W[n-2] = (1 + \alpha + \alpha^2)\{(1+d)^{n-2} + (1+d)^{n-3}\alpha\}.$$

Since n is odd,

$$W[n-2]_1 = \binom{n-2}{1} d + \alpha + \alpha = d,$$

which gives the nonzero characteristic number

$$W[n-2]_1^{n+1} d^{m-1-(n+1)}[\mathbb{R}\mathrm{P}(\mu)] = d^{m-1}[\mathbb{R}\mathrm{P}(\mu)].$$

□

Now we consider n even, which means in particular that $n \geq 4$. Write $n - 2 = 2^p q$, where $p, q \geq 1$. Over F^n one takes the same class X considered before; that is , $X \in H^{m(n-2)}(\mathbb{R}\mathrm{P}(\eta), Z_2)$ and each term of X has a factor of dimension at least $n-1$ from the cohomology of F^n. Note that, on F^n, $W[0]_2 = \theta_2 + \theta_1 u_1 + u_1 c + u_2$. Hence every term of $W[0]_2^2 = \theta_2^2 + \theta_1^2 u_1^2 + u_1^2 c^2 + u_2^2$ has a factor of dimension at least 2 from F^n. If $m > m(n-2)+4$, one then has the zero characteristic number

$$X \cdot W[0]_2^2 \cdot c^{m-1-(m(n-2)+4)}[\mathbb{R}\mathrm{P}(\eta)].$$

Our next and final task will be to show that, over F^2, the corresponding characteristic number

$$Y \cdot W[n-2]_2^2 \cdot d^{m-1-(m(n-2)+4)}[\mathbb{R}\mathrm{P}(\mu)]$$

is nonzero. First note that a general element of $H^t(\mathbb{R}\mathrm{P}(\mu), Z_2)$ is of the form $a_0 d^t + a_1 \alpha d^{t-1} + a_2 \alpha^2 d^{t-2}$, where $a_i = 0$ or 1. In particular, for the top-generator of $H^{m-1}(\mathbb{R}\mathrm{P}(\mu), Z_2)$, the number of 1's in $\{a_0, a_1, a_2\}$ is 1 or 3. From

$$\mathbb{W}(\mathbb{R}\mathrm{P}(\mu)) = (1+\alpha+\alpha^2)\{(1+d)^{m-2} + (1+d)^{m-3}\alpha\}$$

we get

$$W[l] = (1+\alpha+\alpha^2)\{(1+d)^l + (1+d)^{l-1}\alpha\}$$

and

$$W[l]_t = \binom{l}{t} d^t + \left\{ \binom{l-1}{t-1} + \binom{l}{t-1} \right\} \alpha d^{t-1} + \left\{ \binom{l-1}{t-2} + \binom{l}{t-2} \right\} \alpha^2 d^{t-2}.$$

To compute Y, now write $r_i = 2^p - 2^i,\ i = 0, 1, \ldots, p-1$, and set as before $l_i = n + r_i - 2 = 2^p q + 2^p - 2^i$. Then

$$\begin{aligned} W[l_i]_{2r_i} &= \binom{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1}} d^{2r_i} + \\ &+ \left\{ \binom{2^p q + 2^p - 2^i - 1}{2^{p+1} - 2^{i+1} - 1} + \binom{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1} - 1} \right\} \alpha d^{2r_i - 1} + \\ &+ \left\{ \binom{2^p q + 2^p - 2^i - 1}{2^{p+1} - 2^{i+1} - 2} + \binom{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1} - 2} \right\} \alpha^2 d^{2r_i - 2}. \end{aligned}$$

By inspection of 2-adic expansions, one gets the following values for the above binomial coefficients:

i) $\binom{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1}} \equiv 1 \bmod 2,$

ii) $\binom{2^p q + 2^p - 2^i - 1}{2^{p+1} - 2^{i+1} - 1} \equiv 0 \bmod 2,$

iii) $\binom{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1} - 1} \equiv \begin{cases} 1 \bmod 2, \text{ if } i = 0, \\ 0 \bmod 2, \text{ if } i \geq 1, \end{cases}$

iv) $\binom{2^p q + 2^p - 2^i - 1}{2^{p+1} - 2^{i+1} - 2} \equiv \begin{cases} 1 \bmod 2, \text{ if } i = 0, \\ 0 \bmod 2, \text{ if } i \geq 1, \end{cases}$

and

v) $\binom{2^p q + 2^p - 2^i}{2^{p+1} - 2^{i+1} - 2} \equiv \begin{cases} 1 \bmod 2, \text{ if } i = 0 \text{ or } 1, \\ 0 \bmod 2, \text{ if } i \geq 2. \end{cases}$

It follows that

$$W[l_i]_{2r_i} \equiv \begin{cases} d^{2r_i} + \alpha d^{2r_i - 1}, \text{ if } i = 0, \\ d^{2r_i} + \alpha^2 d^{2r_i - 2}, \text{ if } i = 1, \\ d^{2r_i}, \text{ if } i \geq 2. \end{cases}$$

For $r = 2^p - 1,\ l = n + r - 2 = 2^p q + 2^p - 1$ and

$$\begin{aligned} W[l]_{2r+1} &= \binom{2^p q + 2^p - 1}{2^{p+1} - 1} d^{2r+1} + \\ &+ \left\{ \binom{2^p q + 2^p - 2}{2^{p+1} - 2} + \binom{2^p q + 2^p - 1}{2^{p+1} - 2} \right\} \alpha d^{2r} + \\ &+ \left\{ \binom{2^p q + 2^p - 2}{2^{p+1} - 3} + \binom{2^p q + 2^p - 1}{2^{p+1} - 3} \right\} \alpha^2 d^{2r-1}. \end{aligned}$$

In the above expression, the unique binomial coefficient which is zero is $\binom{2^p q + 2^p - 2}{2^{p+1} - 3}$. Thus $W[l]_{2r+1} = d^{2r+1} + \alpha^2 d^{2r-1}$. With these $l_{i's}$ and l, and for $p \leq q + 1$, one then has that

$$\begin{aligned} Y &= (W[l])^{q+1-p} \cdot \prod_{i=0}^{p-1} W[l_i]_{2r_i} = \\ &= (d^{2r+1} + \alpha^2 d^{2r-1})^{q+1-p} \cdot (d^{2r_0} + \alpha d^{2r_0 - 1}) \cdot (d^{2r_1} + \alpha^2 d^{2r_1 - 2}) \cdot d^{2(r_2 + \cdots + r_{p-1})}. \end{aligned}$$

Because of the rule

$$(d^t+\alpha^2 d^{t-1})^s = \begin{cases} d^{ts}, \text{ if } s \text{ is even,} \\ d^{ts}+\alpha^2 d^{ts-2}, \text{ if } s \text{ is odd,} \end{cases}$$

and the fact that $q+1-p \equiv p \bmod 2$, we get that

$$Y = \begin{cases} d^{m(n-2)}+\alpha d^{m(n-2)-1}+\alpha^2 d^{m(n-2)-2}, \text{ if } p \text{ is even, and} \\ d^{m(n-2)}+\alpha d^{m(n-2)-1}, \text{ if } p \text{ is odd.} \end{cases}$$

For $p > q+1$, one has

$$Y = \prod_{i=p-(q+1)}^{p-1} W[l_i]_{2r_i} = \begin{cases} d^{m(n-2)}+\alpha^2 d^{m(n-2)-2}, \text{ if } p-(q+1)=1, \text{ and} \\ d^{m(n-2)}, \text{ if } p-(q+1)>1. \end{cases}$$

With the values of Y on hand, the final step is the calculation of $W[n-2]_2^2$ on F^2. One has

$$W[n-2] = (1+\alpha+\alpha^2)\{(1+d)^{n-2}+(1+d)^{n-3}\alpha\}$$

and

$$\begin{aligned} W[n-2]_2^2 &= \left(\binom{n-2}{2}d^2+\left(\binom{n-2}{1}+\binom{n-3}{1}\right)\alpha d\right)^2 = \\ &= \binom{2^p q}{2}d^4+\alpha^2 d^2 = \begin{cases} \alpha^2 d^2, \text{ if } p>1, \\ d^4+\alpha^2 d^2, \text{ if } p=1. \end{cases} \end{aligned}$$

Since Y has the form $d^t, d^t+\alpha d^{t-1}, d^t+\alpha^2 d^{t-2}$ or $d^t+\alpha d^{t-1}+\alpha^2 d^{t-2}$, for $p>1$ one has $Y \cdot W[n-2]_2^2 = \alpha^2 d^{m(n-2)+2}$. If $p=1$, $Y = d^{m(n-2)}+\alpha d^{m(n-2)-1}$ and

$$\begin{aligned} Y \cdot W[n-2]_2^2 &= (d^{m(n-2)}+\alpha d^{m(n-2)-1}) \cdot (d^4+\alpha^2 d^2) = \\ &= d^{m(n-2)+4}+\alpha d^{m(n-2)+3}+\alpha^2 d^{m(n-2)+2}. \end{aligned}$$

In any case, $Y \cdot W[n-2]_2^2 \cdot d^{m-1-(m(n-2)+4)}[\mathbb{R}\mathrm{P}(\mu)]$ is a nonzero characteristic number, and our task is ended.

ACKNOWLEDGEMENTS. We would like to express our sincere gratitude and indebtedness to Professor Robert E. Stong of the University of Virginia for the valuable help given by a lot of crucial suggestions.

References

[1] A. Borel and F. Hirzebruch, *On characteristic classes and homogeneous spaces*, I, Amer. J. Math. 80, (1958), 458-538.

[2] C. Kosniowski and R. E. Stong, *Involutions and characteristic numbers*, Topology 17, (1978), 309-330.

[3] J. M. Boardman, *On manifolds with involution*, Bulletin Amer. Math. Soc. 73, (1967), 136-138.

[4] P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Springer-Verlag, Berlin, (1964).

[5] P. L. Q. Pergher and R. E. Stong, *Involutions fixing* $\{point\} \cup F^n$, Transformation Groups 6, (2001), 78-85.

[6] Suzanne M. Kelton, *Involutions fixing* $\mathbb{R}P^j \cup F^n$, Doctoral Thesis, University of Virginia, (2001).

[7] Suzanne M. Kelton, *Involutions fixing* $\mathbb{R}P^j \cup F^n$, Topology Appl. 142, (2004), 197-203.

Pedro L. Q. Pergher
Departamento de Matemática
Universidade Federal de São Carlos
Caixa Postal 676
São Carlos, SP 13565-905, Brazil
E-mail: *pergher@dm.ufscar.br*

Fábio G. Figueira
Departamento de Matemática
Universidade Federal de São Carlos
Caixa Postal 676
São Carlos, SP 13565-905, Brazil
E-mail: *fabio@dm.ufscar.br*