BOUNDS ON COINCIDENCE INDICES ON NON-ORIENTABLE SURFACES

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ABSTRACT. In this work we present several results about bounds for coincidence indices of Nielsen coincidence classes for maps between non-orientable surfaces. We consider the definition of coincidence index in the non-orientable case and we prove that when the domain is not the Klein bottle, is different from the counter-domain and both maps are not homotopic to a covering map, then the coincidence class index is not bounded. We also prove that when the domain is the Klein bottle these indices are bounded.

1. Introduction

The questions about bounds for indices first appeared in the fixed point context. The first results appeared in studies of surface homeomorphisms ([17], [16] and [11]). In [10], [12] and [13] some results about bounds for Nielsen fixed point class indices for self-maps of surfaces are given.

In the coincidence context, bounds for coincidence class indices for pairs of maps in the torus may be found in [1] and [2] and, more recently, in [6] we find many results about bounds for coincidence class indices for orientable surfaces. In particular they studied maps $f_1, f_2 : S_h \mapsto S_g$ with $h \geq g$ and where S_i is the compact orientable surface of genus i.

Using the notation of [6], we define:

$$B(g, h, d_1, d_2) = sup\{|ind(C)| \ \mid \ |deg(f_1)| = d_1, \ |deg(f_2)| = d_2\},$$

where the supremum is taken over all coincidence classes C of all pairs of maps $(f_1, f_2) : S_h \mapsto S_g$ with the given degrees.

We recall the following results:

Date: August 18, 2005.

2000 *Mathematics Subject Classification.* Primary 55M20; Secondary 57N05.

Key words and phrases. Nielsen theory, coincidence theory, coincidence index, surfaces.

This work is part of the doctoral thesis of the author, made at University of São Paulo under the supervision of Professor Daciberg Lima Gonçalves.

This work was supported by FAPESP.

Theorem 1.1. *$B(1, h, d_1, d_2) = \infty$ for $h > 1$.*[6, Proposition 9, page 83]

Theorem 1.2. *When $2 \leq g \leq h$ and $0 \leq d_1, d_2 < \frac{h-1}{g-1}$, hence neither of f_1, f_2 is homotopic to a covering map, then $B(g, h, d_1, d_2) = \infty$.* [6, Theorem 10, page 83]

In general, the case $d_2 = \frac{h-1}{g-1}$ remains open, despite the fact that, under certain conditions on the map f_2 there is a bound for these indices [6, page 75 and Proposition 11, page 85].

To obtain similar results for maps between non-orientable surfaces we need to define an index for Nielsen coincidence classes in this context.

In [3] we find a definition of a semi-index (a non-negative integer) for smooth manifolds, and in [9] a similar construction is described for topological manifolds. These definitions are very "geometric".

In [4] an index for maps from a complex into a manifold was obtained (without hypotheses on orientation) in an algebraic way, using obstruction theory.

The equivalence between the two definitions is shown in [5] under the condition that one of the maps is orientation true. Then a Lefschetz number is defined with the usual properties.

In this paper we will prove some results about bounds for coincidence class index on non-orientable surfaces. In all cases that we study f_2 will be orientation true. Then the Nielsen coincidence class index ($ind(f_1, f_2, C)$) and the Lefschetz number ($L(f_1, f_2)$) are defined for a pair of maps.

We will use the notation $deg(f)$ for the *Absolute degree* of a map, as presented in [7, definition 1.3, pages 314 and 315] and, by abuse of notation, we will call this the *degree of f*. Then for a map f between closed manifolds $deg(f)$ is a non-negative integer.

2. Maps from the Klein Bottle

We will denote the Klein bottle by K and the non-orientable surface constructed by a connected sum of n torus with a Klein bottle as K_n ($K_0 = K$).

In the same way as in the orientable case ([6]), we define:

$$B_K(g, h, d_1, d_2) = sup\{|ind(f_1, f_2, C)| \ \mid \ deg(f_1) = d_1,\ deg(f_2) = d_2\},$$

where the supremum is taken over all coincidences classes C of all pairs of maps $(f_1, f_2) : K_h \mapsto K_g$ with given degrees and with f_2 orientation true.

2.1. **Maps between two Klein bottles.** Fixing: $\pi_1(K) =< \alpha, \beta \mid \alpha \cdot \beta \cdot \alpha \cdot \beta^{-1} >$ and $p : S_1 \mapsto K$ the covering map of torus over the Klein bottle, such that $\pi_1(S_1) =< a, b \mid a \cdot b \cdot a^{-1} \cdot b^{-1} >$, then $p(a) = \alpha$ and $p(b) = \beta^2$.

Theorem 2.1. $B_K(0, 0, d_1, d_2) \leq 1, \forall \;\; d_1, d_2 \geq 0.$

Proof: We take $f_1, f_2 : K \mapsto K$ with f_2 orientation true, then: $f_{2\#}(\alpha) = \alpha^r$ and $f_{2\#}(\beta) = \alpha^s \beta^{2q+1}$.

Case 1 f_1 is not orientation true.

We have $f_{1\#}(\alpha) = 1$, $f_1(\beta) = \alpha^t \beta^{2p}$ and f_1, f_2 lift to maps $\widetilde{f_1}, \widetilde{f_2} : S_1 \mapsto S_1$ such that: $\widetilde{f_1}_{\#}(a) = 1$, $\widetilde{f_1}_{\#}(b) = a^{2t} b^{2p}$, $\widetilde{f_2}_{\#}(a) = a^r$, $\widetilde{f_2}_{\#}(b) = b^{2q+1}$.

We note that the identity map of the Klein bottle lift to the identity map or to a map $\theta : S_1 \mapsto S_1$ such that $\theta_{\#}(a) = a^{-1}$ and $\theta_{\#}(b) = b$.

If $x_0 \in Coin(f_1, f_2)$ and $\widetilde{x_0}, \widetilde{x_0}' \in S_1$ such that $p(\widetilde{x_0}) = p(\widetilde{x_0}') = x_0$, and if $\widetilde{x_0} \in Coin(\widetilde{f_1}, \widetilde{f_2})$ and γ is a curve on S_1 from $\widetilde{x_0}$ to $\widetilde{x_0}'$, we have $p(\gamma) \in \frac{\pi_1(K)}{p_{\#}(\pi_1(S_1))}$ and then $\widetilde{x_0}' \in Coin(\widetilde{f_1}, \widetilde{f_2})$ if, and only if, $\widetilde{f_1}(\gamma) \circ \widetilde{f_2}(\gamma)^{-1} \in \pi_1(S_1)$ i. e. $f_{1\#}(p(\gamma)) \circ f_{2\#}(p(\gamma))^{-1} \in p_{\#}(\pi_1(S_1))$.

But $\widetilde{x_0}$ and $\widetilde{x_0}'$ belong to the same fiber of p, then $p(\gamma) = \alpha^k \beta^{2l+1}$ and we have:

$$\begin{aligned} f_{1\#}(p(\gamma)) \circ f_{2\#}(p(\gamma))^{-1} &= \alpha^{t(2l+1)} \beta^{(2l+1)(2p)} \beta^{-(2l+1)(2q+1)} \alpha^{-kr-s} \\ &= \alpha^{k'} \beta^{2q'+1} \notin p_{\#}(\pi_1(S_1)). \end{aligned}$$

Therefore $\widetilde{x_0}' \notin Coin(\widetilde{f_1}, \widetilde{f_2})$, in fact $\widetilde{x_0}' \in Coin(\theta \circ \widetilde{f_1}, \widetilde{f_2})$ because we know $\widetilde{f_1}(\widetilde{x_0}')$ and $\widetilde{f_2}(\widetilde{x_0}')$ belong to $p^{-1}(f_1(x_0))$ that have only two elements. So the coincidence classes of f_1, f_2 are in bijection with the coincidence classes of $\widetilde{f_1}, \widetilde{f_2}$ by p.

Now p is orientation true and if C is a coincidence class of f_1, f_2 then $|ind(f_1, f_2, C)| = |ind(\widetilde{f_1}, \widetilde{f_2}, p^{-1}(C))|$ but $|ind(\widetilde{f_1}, \widetilde{f_2}, p^{-1}(C))| \leq 1$ [1, page 125] and then $|ind(f_1, f_2, C)| \leq 1$.

Case 2 f_1 is also orientation true.

In this case we have $f_{1\#}(\alpha) = \alpha^m$, $f_1(\beta) = \alpha^t \beta^{2p+1}$ and $\widetilde{f_1}, \widetilde{f_2} : S_1 \mapsto S_1$ liftings of f_1, f_2 respectively, such that: $\widetilde{f_1}_{\#}(a) = a^m$, $\widetilde{f_1}_{\#}(b) = b^{2p+1}$, $\widetilde{f_2}_{\#}(a) = a^r$, $\widetilde{f_2}_{\#}(b) = b^{2q+1}$.

As in case 1 we obtain: if $\widetilde{x_0} \in Coin(\widetilde{f_1}, \widetilde{f_2})$ then $\widetilde{x_0}' \in Coin(\widetilde{f_1}, \widetilde{f_2})$; further, if $\widetilde{x_0}$ and $\widetilde{x_0}'$ are not in the same coincidence class of $\widetilde{f_1}, \widetilde{f_2}$ and if C is the class that contains x_0

then $|ind(f_1,f_2,C)| = |ind(\widetilde{f_1},\widetilde{f_2},\tilde{C})|$ where $\tilde{C}$ is the coincidence class of $\widetilde{f_1},\widetilde{f_2}$ such that $\widetilde{x_0}\in\tilde{C}$.

If $\widetilde{x_0}$ and $\widetilde{x_0}'$ belong to the same coincidence class[1], all pairs $\tilde{x_1}$, $\tilde{x_2}$ of points such that $p(\tilde{x_1}) = p(\tilde{x_2}) \in Coin(f_1,f_2)$ are, both in the same class of $\widetilde{f_1},\widetilde{f_2}$ or of $\theta\circ\widetilde{f_1},\widetilde{f_2}$.

p is an orientation true local homeomorphism and if $\tilde{C}$ is the coincidence class of the pair $\widetilde{f_1},\widetilde{f_2}$ (or of $\theta\circ\widetilde{f_1},\widetilde{f_2}$) that contain $\widetilde{x_0}$, then $|ind(\widetilde{f_1},\widetilde{f_2},\tilde{C})| = 2|ind(f_1,f_2,C)|$ (or $|ind(\theta\circ\widetilde{f_1},\widetilde{f_2},\tilde{C})| = 2|ind(f_1,f_2,C)|$).

By [5, Remark 2, page 22], $|ind(f_1,f_2,C)|$ is an integer, furthermore, we know that $|ind(\widetilde{f_1},\widetilde{f_2},\tilde{C})|\leq 1$ and $|ind(\theta\circ\widetilde{f_1},\widetilde{f_2},\tilde{C})|\leq 1$, so we have $ind(f_2,f_1,C)=0$. □

2.2. **Counter-domain with genus greater than 2.** We take the Klein bottle to be the square $[0,1]\times[0,1]$ with the identification show in Figure 1.

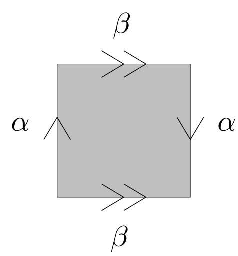


FIGURE 1. Klein bottle.

Definition 2.2. *Let $w\in\pi_1(K_g)$, $s\in\mathbb{Z}$. Define $f_{w,s}: K\mapsto K_g$ by:*

(1) *$f_{w,s}([0,\frac{1}{s}]\times 0) = \overline{w}$ (with $<\overline{w}>= w$) a local homeomorphism such that: when we "walk" from $(0,0)$ to $(\frac{1}{s},0)$, we cover $\overline{w}$ in the positive way, if $s>0$, or in the negative sense if $s<0$.*
(2) *For $\frac{1}{s} < q \leq 1$ then $f_{w,s}(q,0) = f_{w,s}(q',0)$ with $0\leq q' < \frac{1}{s}$ and $q = q' + \frac{k}{s}$ with $k\in\mathbb{N}$.*
(3) *For $y>0$ then $f_{w,s}(x,y) = f_{w,s}(x,0)$.*

[1]Then there exists a curve $\gamma:[0,1]\to S_1$ such that $\gamma(0)=\widetilde{x_0}$, $\gamma(1)=\widetilde{x_0}'$ and $\widetilde{f_1}(\gamma)\circ\widetilde{f_2}(\gamma)^{-1}$ is homotopic to the trivial curve, but, in this case $f_{2\#}(p(\gamma))\circ f_{1\#}(p(\gamma))^{-1}$ is also homotopic to the trivial curve, as $p(\gamma)=\alpha^k\beta^{2l+1}$ we have:

$$\begin{aligned} f_{1\#}(p(\gamma))\circ f_{2\#}(p(\gamma))^{-1} &= \alpha^{km+t}\beta^{(2l+1)(2p+1)}\beta^{-(2l+1)(2p+1)}\alpha^{-kr} \\ &= \alpha^{k(m-r)+t-s}\beta^{(2l+1)(2p-2q)}. \end{aligned}$$

We conclude that $p=q$ and $L(\widetilde{f_1},\widetilde{f_2})=0$. This is enough to show that all coincidence classes of the pair $\widetilde{f_1},\widetilde{f_2}$ have index zero.

We note that the inverse image of any point of K_g by $f_{w,s}$ is the union of a finite number of segments or it is empty.

Theorem 2.3. $B_K(g,0,0,0) = 0$ *for all* $g \geq 1$.

Proof: Let $f : K \mapsto K_g$ be a map. We know that $\pi_1(K_g)$ may be presented by $2g+2$ generators and only one relation:

$$\pi_1(K_g) =< a_1, b_1, a_2, b_2, \ldots, a_g, b_g, \alpha, \beta \mid \prod_{i=1}^{g}[a_i, b_i] \cdot \alpha \cdot \beta \cdot \alpha \cdot \beta^{-1} >$$

We note that $f_{\#}(\pi_1(K_g))$ is generated by two elements, $w' = f_{\#}(\alpha)$ and $w'' = f_{\#}(\beta)$. They satisfy the relation:

$$w' \cdot w'' \cdot w' \cdot w''^{-1} = 1.$$

Now, using *Freiheitssatz* [15, Theorem 4.10, page 252] we have that $f_{\#}(\pi_1(K))$ is free with only one generator, so there exists $w \in \pi_1(K_g)$ such that $w' = w^r$ and $w'' = w^s$, $r, s \in \mathbb{Z}$ and we have, using the relation above:

$$w^r \cdot w^s \cdot w^r \cdot w^{-s} = 1$$

Then $r = 0$ and $f_{\#}(\alpha) = 1$, the trivial element. Since K is an Eilenberg-MacLane space of type $K(\pi, 1)$, then f is homotopic to a map $f_{w,s}$.

Let $f_1, f_2 : K \mapsto K_g$ be a pair of maps, then $f_1 \sim f_{w_1,s}$ and $f_2 \sim f_{w_2,r}$. If we choose curves $\overline{w_1}, \overline{w_2}$ on K_g such that: $< \overline{w_1} >= w_1$, $< \overline{w_2} >= w_2$ and $\overline{w_1} \cap \overline{w_2} = \{p_1, p_2, \cdots, p_n\}$ is finite, we may suppose that

$$f_{w_1,s}(K) \subset \overline{w_1} \ \text{ and } \ f_{w_2,r}(K) \subset \overline{w_2}$$

and, under these conditions:

$$Coin(f_{w_1,s}, f_{w_2,r}) = \bigcup_{i=1}^{n}(f_{w_1,s}^{-1}(p_i) \cap f_{w_2,r}^{-1}(p_i)).$$

In fact, if $Coin(f_{w_1,s}, f_{w_2,r}) \neq \emptyset$ then, using a small translation of $f_{w_2,r}$ in the direction of β, we obtain $Coin(f_{w_1,s}, f_{w_2,r}) = \emptyset$. This shows that if C is a coincidence class of the pair $(f_{w_1,s}, f_{w_2,r})$ then $ind(f_{w_1,s}, f_{w_2,r}, C) = 0$; we know that $f_1 \sim f_{w_1,s}$ and $f_2 \sim f_{w_2,r}$, so the same occurs for the pair f_1, f_2. □

3. Maps to the Klein bottle

Theorem 3.1. $B_K(0,h,d_1,d_2) = \infty$ *with* $h > 1$, $\forall \ d_1, d_2 \geq 0$.

Proof: We recall that

$$\pi_1(K_h) =< a_1, b_1, a_2, b_2 \ldots, a_h, b_h, \alpha, \beta \mid \prod_{i=1}^{h}[a_i, b_i] \cdot \alpha \cdot \beta \cdot \alpha \cdot \beta^{-1} > .$$

Now, we define homomorphisms[2] $\bar{f}_i : \pi_1(K_h) \mapsto \pi_1(K)$ such that:

$$\begin{array}{ccc} & \bar{f}_1 & \\ a_1 & \mapsto & 1 \\ b_1 & \mapsto & \alpha \\ a_2 & \mapsto & 1 \\ b_2 & \mapsto & \beta^2 \\ a_3 & \mapsto & 1 \\ b_3 & \mapsto & 1 \\ \vdots & \vdots & \vdots \\ a_h & \mapsto & 1 \\ b_h & \mapsto & 1 \\ \alpha & \mapsto & \alpha^{d_1} \\ \beta & \mapsto & \beta^{-1} \end{array} \qquad \begin{array}{ccc} & \bar{f}_2 & \\ a_1 & \mapsto & 1 \\ b_1 & \mapsto & 1 \\ a_2 & \mapsto & \alpha^n \\ b_2 & \mapsto & 1 \\ a_3 & \mapsto & 1 \\ b_3 & \mapsto & 1 \\ \vdots & \vdots & \vdots \\ a_h & \mapsto & 1 \\ b_h & \mapsto & 1 \\ \alpha & \mapsto & \alpha^{d_2} \\ \beta & \mapsto & \beta \end{array}$$

Since K_h is an Eilenberg-MacLane space of type $K(\pi, 1)$ then there exists $f_i : K_h \mapsto K$ such that $f_{i\#} = \bar{f}_i$.

Both f_1 and f_2 are orientation true and they lift to: $\widetilde{f}_i : S_{2h+1} \mapsto S_1$ between the double orientable coverings.

We may check that $\widetilde{f}_i$ has degree d_i and then,using [7, Definition 1.3, pages 314 and 315] $deg(f_i) = d_i$.

Using the notation of [8] we have: $f_2^*(\tilde{\alpha}) = n\tilde{a_2} + d_2\tilde{\alpha}$, $f_2^*(\beta) = \beta$ and $f_1^*(\beta) = 2b_2 - \beta$. By [8, Theorem 2.5, page 163], we have:

$$L(f_1, f_2) = 2 \cdot (d_1 - n).$$

We know if $w \in \pi_1(K)$ then $w = \alpha^p \cdot \beta^q$ and we have:

$$w = \alpha^p \cdot \beta^q \sim f_1(a_2^{-p}) \cdot \alpha^p \cdot \beta^q \cdot (f_2(a_2^{-p}))^{-1} = \alpha^{-p} \cdot \alpha^p \cdot \beta^q = \beta^q$$

As $q = 2q'$ or $q = 2q' + 1$ and

$$\beta^q \sim f_1(\beta^{-q'}) \cdot \beta^q \cdot (f_2(\beta^{-q'}))^{-1} = \beta^{-q'} \cdot \beta^q \cdot \beta^{-q'}.$$

Hence if $q = 2q'$ then $\beta^q \sim 1$ and if $q = 2q' + 1$ then we have $\beta^q \sim \beta$.

There exist only two Reidemeister classes and the sum of their indices is $|2 \cdot (d_1 - n)|$ (see [5, Theorem 4.5, page 13; Definition 5.1, page 19 and Theorem 5.5, page 21]), which means that $B_K(0, h, d_1, d_2) = \infty$. □

[2] $\bar{f}_i(\prod_{i=1}^{h} [a_i, b_i] \cdot \alpha \cdot \beta \cdot \alpha \cdot \beta^{-1}) = 1$

4. The general case

4.1. **Maps of degree zero.**

Theorem 4.1. *$B_K(g, h, 0, 0) = \infty$ for $h, g \geq 1$.*

Proof:

Case 1 ($g = h = 1$) Using the same notation as above we define $f_1, f_2 : K_1 \mapsto K_1$ by[3]:

$$
\begin{array}{ccc}
 & f_{1\#} & \\
a_1 & \mapsto & 1 \\
b_1 & \mapsto & \beta^n \\
\alpha & \mapsto & 1 \\
\beta & \mapsto & 1
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & f_{2\#} & \\
a_1 & \mapsto & \alpha \\
b_1 & \mapsto & 1 \\
\alpha & \mapsto & 1 \\
\beta & \mapsto & \beta
\end{array}
$$

We observe if w is a Reidemeister class associated to a Nielsen class, then w is a word in α and β. Now we have:

$$w = w' \cdot \alpha^k \sim f_1(a_1^k) \cdot w' \cdot \alpha^k \cdot (f_2(a_1^k))^{-1} \sim w' \cdot \alpha^k \cdot \alpha^{-k} \sim w'.$$

$$w = w' \cdot \beta^k \sim f_1(\beta^k) \cdot w' \cdot \beta^k \cdot (f_2(\beta^k))^{-1} \sim w' \cdot \beta^k \cdot \beta^{-k} \sim w'.$$

Then $w \sim 1$ and this shows that the pair (f_1, f_2) has only one Nielsen class C.

With the notation of [8] we have:

$$f_2^{*1}(\ ,\mathbb{Q}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad f_2^{*1}(\ ,\tilde{\mathbb{Q}}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{e} \quad f_1^{*1}(\ ,\mathbb{Q}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and by [8, Theorem 2.8, page 165] we obtain $L(f_1, f_2) = n$ and then $|ind(f_1, f_2, C)| = n$.

[3] $f_{i\#}([a_1, b_1] \cdot \alpha \cdot \beta \cdot \alpha \cdot \beta^{-1}) = 1$

Case 2 ($g = 1$ and $h = 2$) As in case 1 we define $f_1, f_2 : K_2 \mapsto K_1$ by:

$$\begin{array}{ccc} & f_{1\#} & \\ a_1 & \mapsto & 1 \\ b_1 & \mapsto & b_1^{-n} \\ a_2 & \mapsto & 1 \\ b_2 & \mapsto & 1 \\ \alpha & \mapsto & 1 \\ \beta & \mapsto & 1 \end{array} \qquad \begin{array}{ccc} & f_{2\#} & \\ a_1 & \mapsto & a_1 \\ b_1 & \mapsto & 1 \\ a_2 & \mapsto & b_1 \\ b_2 & \mapsto & 1 \\ \alpha & \mapsto & 1 \\ \beta & \mapsto & \beta \end{array}$$

In this case, if w is a Reidemeister class associated to a Nielsen class, then w is a word in a_1, b_1 and β. Now we have:

$$w = w' \cdot a_1^k \sim f_1(a_1^k) \cdot w' \cdot a_1^k \cdot (f_2(a_1^k))^{-1} \sim w' \cdot a_1^k \cdot a_1^{-k} \sim w'.$$

$$w = w' \cdot b_1^k \sim f_1(b_2^k) \cdot w' \cdot a_2^k \cdot (f_2(b_2^k))^{-1} \sim w' \cdot b_1^k \cdot b_1^{-k} \sim w'.$$

$$w = w' \cdot \beta^k \sim f_1(\beta^k) \cdot w' \cdot \beta^k \cdot (f_2(\beta^k))^{-1} \sim w' \cdot \beta^k \cdot \beta^{-k} \sim w'.$$

Again $w \sim 1$ and there exists a unique Nielsen class with index $L(f_1, f_2)$, by [8, Theorem 2.8, page 165] $L(f_1, f_2) = n$.

Case 3 ($g \geq 1$ and $h > 2$) This case is a simple generalization of case 2.

In all cases $|f_{i\#}(\pi_1(K_h)) : \pi_1(K_g)| = \infty$ and then $deg(f_i) = 0$. □

4.2. **Maps of non-zero degree.**

Theorem 4.2. *$B_K(g, h, d_1, d_2) = \infty$ for $h \geq g \geq 1$, $1 \leq d_1 \leq \frac{h}{g} - 1$ and $1 \leq d_2 < \frac{h}{g} - 1$.*

Proof: Denoting: $\pi_1(K_n) =< \gamma_1, \gamma_2, \cdots, \gamma_{2n+2} \mid \prod_{i=1}^{2n+2} \gamma_i^2 >$ and $\theta = \gamma_{2g+1}\gamma_{2g+2}$, we define $f_{1\#}, f_{2\#} : \pi_1(K_h) \mapsto \pi_1(K_g)$:

If $d_1 \neq 1$:

$$f_{1\#}(\gamma_i) = \begin{cases} 1 & \text{if} \quad i = 1 \ \text{ or } \ 2d_1g + 4 \leq i \leq 2h + 1 \\ \gamma_{i-1} & \text{if} \quad 2 \leq i \leq 2g + 2 \\ \gamma_{2g+1}\theta^{d_1} & \text{if} \quad i = 2g + 3 \\ \theta^{-e(i)}\gamma_{s(i)}\theta^{e(i)} & \text{if} \quad 2g + 4 \leq i \leq 2d_1g + 2 \\ \theta^{-1}\gamma_{2g}^{1-k}\theta & \text{if} \quad i = 2d_1g + 3 \\ \theta^{-1}\gamma_{2g}^{k}\theta & \text{if} \quad i = 2h + 2. \end{cases}$$

where[4] $e(i) = d_1 - [\frac{i-4}{2g}]$ and $s(i) = (i \mod 2g) - 3$.

If $d_1 = 1$:

$$f_{1\#}(\gamma_i) = \begin{cases} 1 & \text{if } i = 1 \text{ or } 2g+4 \leq i \leq 2h+1 \\ \gamma_{i+1} & \text{if } 2 \leq i \leq 2g+1 \\ \gamma_1 & \text{if } i = 2g+2 \\ \gamma_2^{1-k} & \text{if } i = 2g+3 \\ \gamma_2^k & \text{if } i = 2h+3 \end{cases}$$

If $d_2 \neq 1$:

$$f_{2\#}(\gamma_i) = \begin{cases} \gamma_i & \text{if } 1 \leq i \leq 2g+2 \\ \theta^{-p(i)}\gamma_{q(i)}\theta^{p(i)} & \text{if } 2g+3 \leq i \leq 2d_2g+2 \\ \gamma_1^{(-1)^{i+1}} & \text{if } 2d_2g+3 \leq i \leq 2h \\ \theta^{d_2-1}\gamma_{2g+2}^{-1} & \text{if } i = 2h-1 \\ \gamma_{2g+2} & \text{if } i = 2h+2 \end{cases}$$

where $p(i) = [\frac{i-(2g+3)}{2g}]$ and $q(i) = (i \mod 2g) - 2$.

If $d_2 = 1$:

$$f_{2\#}(\gamma_i) = \begin{cases} \gamma_i & \text{if } 1 \leq i \leq 2g+2 \\ \gamma_1^{(-1)^{i+1}} & \text{if } 2g+3 \leq i \leq 2h+2 \end{cases}$$

We have[5] $f_{i\#}(\prod_{j=1}^{2h+2} \gamma_i^2) = 1$ for $i = 1, 2$. This is enough to show that $f_{i\#}$ are homomorphisms and can be extended to maps $f_i : K_h \mapsto K_g$. By construction $deg(f_1) = d_1$, $deg(f_2) = d_2$, and f_2 is orientation free.

Observe that if $w \in \pi_1(K_h)$ and w can be written as a word in $< \gamma_1, \gamma_2, \cdots, \gamma_t >$ with $t \leq 2g+2$ then $f_{2\#}(w) = w$, and $f_{1\#}(w)$ can also be written as a word in $< \gamma_1, \gamma_2, \cdots, \gamma_{t-1} >$.

Now if $w \in \pi_1(K_g)$ then w is a word in $< \gamma_1, \gamma_2, \cdots, \gamma_{2g+2} >$, we take $w' \in \pi_1(K_h)$ to be the same word, and we have:

---

[4] $[\frac{a}{b}]$ denotes the greatest integer less or equal the rational number $\frac{a}{b}$.

[5] We note that:

$$\begin{aligned} f_{1\#}(\gamma_{2g+2}^2\gamma_{2g+3}^2) &= \gamma_{2g+1}^2\gamma_{2g+1}^{-1}(\gamma_{2g+1}\gamma_{2g+2})^{d_1}\gamma_{2g+1}^{-1}(\gamma_{2g+1}\gamma_{2g+2})^{d_1} \\ &= \prod_{i=0}^{d_1-1}(\gamma_{2g+1}\gamma_{2g+2})^{-i}\gamma_{2g+1}^2\gamma_{2g+2}^2(\gamma_{2g+1}\gamma_{2g+2})^{i}. \end{aligned}$$

$$\begin{aligned} f_{2\#}(\gamma_{2h+1}^2\gamma_{2h+2}^2) &= (\gamma_{2g+1}\gamma_{2g+2})^{d_2-1}\gamma_{2g+2}^{-1}(\gamma_{2g+1}\gamma_{2g+2})^{d_2-1}\gamma_{2g+2}^{-1}\gamma_{2g+2}^2 \\ &= \prod_{i=d_2-2}^{0}(\gamma_{2g+1}\gamma_{2g+2})^{i}\gamma_{2g+1}^2\gamma_{2g+2}^2(\gamma_{2g+1}\gamma_{2g+2})^{-i}. \end{aligned}$$

$$w \sim f_{1\#}w'^{-1} \cdot w \cdot f_{2\#}w'^{-1} \sim f_{1\#}w'^{-1}.$$

This word can be written in $< \gamma_1, \gamma_2, \cdots, \gamma_{2g+1} >$, by repetition of the same argument, at most $2g+1$ times, we obtain $w \sim 1$. This shows that there exists only one Reidemeister class for the pair (f_1, f_2). Once more, we will use the fact that the index of this class is $L(f_1, f_2)$.Furthermore, we will show that changing k, that we used in the definition of $f_{1\#}$, we can obtain $L(f_1, f_2)$ as big as we want.

To calculate $L(f_1, f_2)$ we need to know how these homomorphisms act on sets of generators of $H_1(K_h, \mathbb{Z})$ and $H_1(K_g, \mathbb{Z})$ with the properties described in [8, proof of Proposition 2.2, page 160].

We take a system of generators of $H_1(K_n, \mathbb{Z})$ given by:

$$\{\gamma_1\gamma_2, \gamma_2^{-1}\gamma_{2n+2}, \gamma_3\gamma_4, \gamma_4^{-1}\gamma_{2g+2}, \cdots, \gamma_{2n}^{-1}\gamma_{2n+2}, \gamma_1^{-1}\gamma_2^{-1}\cdots\gamma_{2n}^{-1}\gamma_{2n+1}\gamma_{2n+2}, \gamma_{2n+2}^{-1}\}.$$

Now we can describe $f_1 : H_1(K_h) \mapsto H_1(K_g)$ as the transposed matrix of the product $T^{-1} \cdot M_1 \cdot T$ where M_1 in the case $d_1 \neq 1$ is the $(2g+2) \times (2h+2)$ matrix:

$$\begin{pmatrix}
0 & & & 0 & 0 & & & & & & & & & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & & & 0 & 0 & & & & & & & & & 0 & 1 & & 0 & \cdots & 0 & 0 \\
\vdots & & I_{2g}^{\{1\}} & \vdots & \vdots & & I_{2g}^{\{2\}} & & \cdots & \cdots & & I_{2g}^{\{d_1-1\}} & & & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\cdot & & & 0 & 0 & & & & & & & & & 0 & \cdots & 1-k & 0 & \cdots & 0 & k \\
0 & 0 & \cdots & 1 & d_1-1 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & d_1 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}$$

and T is the $2h$ or $2g$-square matrix:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1
\end{pmatrix}.$$

In this way we obtain, for $d_1 \neq 1$ (with notation of [8]):

$$
f_{1\#}(\ ,\mathbb{Q}) = \left(\begin{array}{ccccccccc}
 & & & & & & & 0 & 1 \\
 & & & & & & & 0 & -1-k \\
\multicolumn{7}{c}{M_{2g-2}^{\{1\}}} & \vdots & \vdots \\
 & & & & & & & 0 & 0 \\
 & & & & & & & 0 & -1-k \\
1 & 0 & 1 & 0 & \cdots & 1 & -1 & 1 & 0 \\
-1 & 0 & -1 & 0 & \cdots & -1 & -k & -1 & -1-k \\
d_1 & 1 & d_1-1 & 0 & \cdots & d_1-1 & 0 & d_1-1 & 0 \\
-1 & -1 & 0 & 0 & \cdots & 0 & -k & 0 & -1-k \\
0 & -1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & \cdots & 0 & -k & 0 & -1-k \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & -1 & -1-k & 0 & -1-k \\
 & & & & & & & 0 & 0 \\
 & & & & & & & 0 & -1-k \\
\multicolumn{7}{c}{M_{2g-2}^{\{3\}}} & \vdots & \vdots \\
 & & & & & & & 0 & 0 \\
 & & & & & & & 0 & -1-k \\
 & & & & & & & 0 & 0 \\
 & & & & & & & 0 & -1-k \\
\multicolumn{7}{c}{M_{2g-2}^{\{4\}}} & \vdots & \vdots \\
 & & & & & & & 0 & 0 \\
 & & & & & & & 0 & -1-k \\
\multicolumn{7}{c}{\vdots} & \vdots & \vdots \\
 & & & & & & & 0 & 0 \\
 & & & & & & & 0 & -1-k \\
\multicolumn{7}{c}{M_{2g-2}^{\{d_1\}}} & \vdots & \vdots \\
 & & & & & & & 0 & 0 \\
 & & & & & & & 0 & -1-k \\
0 & 0 & 0 & 0 & \cdots & 0 & -1+k & 0 & -1+k \\
0 & 0 & 0 & 0 & \cdots & 0 & -k & 0 & -k \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -k & 0 & -k \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -k & 0 & -k \\
-2.d_1 & 0 & -2.d_1 & 0 & \cdots & -2.d_1 & -2k & -d_1 & -2k \\
0 & 0 & 0 & 0 & \cdots & 0 & k & 0 & k
\end{array}\right),
$$

where:

$$M_{2g-2} = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 0 & 0 & \cdots & 0 & -k \\ 0 & -1 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & -1 & \cdots & 0 & -k \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & -1-k \end{pmatrix}.$$

and for $d_1 = 1$ we obtain:

$$f_{1\#}(\ ,\mathbb{Q}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & -k & -1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & -1-k \\ 0 & 0 & 0 & -1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 & -1 & -1 & \cdots & 0 & 0 & 0 & -1-k \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ 0 & -k & 0 & 0 & 0 & 0 & \cdots & -1 & -1 & 0 & -1-k \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & -1 & 1 & 0 \\ -1 & -k & -1 & 0 & -1 & 0 & \cdots & -1 & 0 & 0 & -1-k \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -1-k & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & -1-k \\ 0 & -1+k & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1+k \\ 0 & -k & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1-k \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1-k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1-k \\ -2 & -2k & -2 & 0 & -2 & 0 & \cdots & -2 & 0 & -1 & -2k \\ 0 & k & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & k \end{pmatrix},$$

In this way $f_2 : H_1(K_h) \mapsto H_1(K_g)$ can be described as the transpose of the product $T^{-1} \cdot M_2 \cdot T$, where M_2 is the matrix, for $d_2 \neq 1$, $(2g+2) \times (2h+2)$:

$$\left(\begin{array}{ccccccccccccccccccccccc} & & & & & & & & & & & & & & & & 1 & -1 & \cdots & 1 & -1 & 0 & 0 \\ & & I_{2g}^{\{1\}} & & & I_{2g}^{\{2\}} & & & I_{2g}^{\{3\}} & \cdots & \cdots & & & I_{2g}^{\{d_2-1\}} & & & 0 & 0 & \cdots & 0 & 0 & \vdots & \vdots \\ I_{2g+2} & & & & & & & & & & & & & & & & 0 & 0 & & 0 & 0 & 0 & 0 \\ & & & & & & & & & & & & & & & & \vdots & \vdots & & \vdots & \vdots & 0 & 0 \\ & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & d_2-1 & 0 \\ & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & d_2-2 & 1 \end{array}\right)$$

and for $d_2 = 1$, M_2 is the $(2g+2)\times(2h+2)$ matrix:

$$\begin{pmatrix} & 0 & \cdots & 0 \\ & 0 & \cdots & 0 \\ I_{2g+2} & \vdots & & \vdots \\ & 0 & \cdots & 0 \\ & 0 & \cdots & 0 \end{pmatrix}$$

Then we have, for $d_2 \neq 1$:

$$f_2^{*1}(\,,\mathbb{Q}) = \begin{pmatrix} & & & & & 0 \\ & & & & & 0 \\ & & I_{2g-2}^{\{1\}} & & & \vdots \\ & & & & & 0 \\ & & & & & 0 \\ 1 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & I_{2g-2}^{\{2\}} & & & \vdots \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & I_{2g-2}^{\{3\}} & & & \vdots \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & \vdots & & & \vdots \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & I_{2g-2}^{\{d_2\}} & & & \vdots \\ & & & & & 0 \\ & & & & & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \quad f_2^{*1}(\,,\tilde{\mathbb{Q}}) = \begin{pmatrix} & & & & & 0 \\ & & & & & 0 \\ & & I_{2g-2}^{\{1\}} & & & \vdots \\ & & & & & 0 \\ & & & & & 0 \\ 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & I_{2g-2}^{\{2\}} & & & \vdots \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & I_{2g-2}^{\{3\}} & & & \vdots \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & \vdots & & & \vdots \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & I_{2g-2}^{\{d_2\}} & & & \vdots \\ & & & & & 0 \\ & & & & & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 & 0 \\ -2 & 0 & \cdots & -2 & 0 & d_2-2 \end{pmatrix}.$$

And for $d_2 = 1$:

$$f_2^{*1}(\ ,\mathbb{Q}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & & 0 \\ -1 & -1 & & & \cdot \\ \vdots & \vdots & I_{2g-4} & & \vdots \\ 0 & 0 & & & \cdot \\ -1 & -1 & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 1 \end{pmatrix}, \quad f_2^{*1}(\ ,\tilde{\mathbb{Q}}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & & & 0 \\ -1 & -1 & & & \cdot \\ \vdots & \vdots & I_{2g-4} & & \vdots \\ 0 & 0 & & & \cdot \\ -1 & -1 & & & 0 \\ 1 & 0 & \cdots & 1 & 0 \\ -1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -2 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

By calculation, using [8, Theorem 2.8, page 165], we obtain:

$$L(f_1, f_2) = \begin{cases} -3 + 4k & \text{if} \quad d_2 = 1, \ \ d_1 = 1, \ \ g \geq 2; \\ -2 + 4k & \text{if} \quad d_2 = 1, \ \ d_1 > 1, \ \ g \geq 2; \\ d_2 - 1 + 4k & \text{if} \quad d_2 > 1, \ \ d_1 = 1, \ \ g \geq 2; \\ -2(d_2 - d_1) - 1 + 4k & \text{if} \quad d_1 \geq d_2 - 1 > 1, \ \ g = 2, \ \ d_1 \neq h - 2; \\ d_2 - d_1 + 4k & \text{if} \quad d_2 \geq d_1 > 1, \ \ g \geq 3; \\ 2(d_2 - d_1) + 4k & \text{if} \quad (d_1 \geq d_2 > 1, \ \ g \geq 3) \ \text{ or } \ (g = 2, \ \ d_1 = h - 2). \end{cases}$$

□

4.3. **Other surfaces.** The same results are true under similar conditions for maps between other non-orientable surfaces. We can construct examples for this using the same technics. Some of then may be found in [18] but we do not have a simple form including these cases.

5. Some boundedness results

5.1. **Covering maps.** The result that we will show in this subsection is an adaptation, including the non-orientable case, of [6, Proposition 11, page 86] where one of the maps is a covering map. Under the same assumptions of [6] we will obtain a boundary for the index of a coincidence class of surfaces, without the orientable condition. We need to remember that, in our case, one map will be orientation true.

Theorem 5.1. *Let S_A and S_B two compact surfaces, $f : S_A \mapsto S_B$ a map and $p : S_A \mapsto S_B$ a covering map. If there exists a subgroup $K \subset \pi_1(S_A)$ such that:*

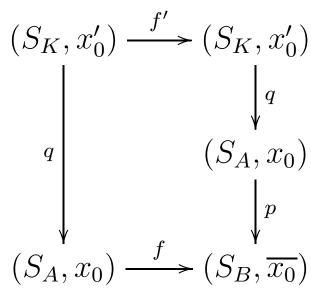
- $[\pi_1(S_A) : K] = n < \infty$;
- $f_\pi(K) \subset p_\pi(K)$

then for all coincidence Nielsen classes C of the pair (f, p), $|ind(f, p, C)| \leq |2n\chi(S_A) - 1|$.

Proof: Let[6] $x_0 \in Coin(f,p) \subset S_A$, and let be C the coincidence class such that $x_0 \in C$.

Since $[\pi_1(S_A) : K] = n < \infty$, there exist a compact surface S_K and a covering map $q : (S_K, x_0') \mapsto (S_A, x_0)$ such that $\chi(S_K) = n\chi(S_A)$ and $q_{\#}(\pi_1(S_K, x_0')) = K$.

If $\overline{x_0} = f(x_0) = p(x_0)$ then there exists a map $f' : S_K \mapsto S_K$ which makes the following diagram commute:



We will denote by C' the coincidence class of the pair (f', id) such that $x_0' \in C'$.

Observe that $Coin(f', id)$ is included naturally in $Coin(p\circ q\circ f', p\circ q)$ and since $p \circ q$ is a covering map, by [6, Lemma 1(iv), pages 75 and 76,], C' is a coincidence class of pair $(p \circ q \circ f', p \circ q)$. Then, using the commutativity of the above diagram, we have that C' is a coincidence class of pair $(f \circ q, p \circ q)$ and $|ind(f', id, C)| = |ind(f \circ q, p \circ q, C')|$.

Since q is a covering map by [6, Lemma 2(iii), page 76], C' covers C, n' times with $1 \leq n' \leq n$. Using [3, Lemma 2.1, corolary 2.2 and Lemma 2.3, pages 77 and 78] and [5, Lemma 5.3, page 20] we obtain that $|ind(f \circ q, p \circ q, C')| = n'|ind(f, p, C)|$ also in the non-orientable case[7]. So we have $|ind(f, p, C)| \leq |ind(f\circ q, p\circ q, C')| = |ind(f', id, C')|$.

Since C' is a coincidence class of the pair (f', id), C' is a fixed point class of $f' : S_K \mapsto S_K$ and using [10, Theorem 2, page 477] we obtain $|ind(f', id, C')| \leq |2\chi(S_K) - 1|$ and then $|ind(f, p, C)| \leq |2n\chi(S_A) - 1|$. □

5.2. **Self maps.** We will denote by P_n the non-orientable surface constructed by connected sum of n tori with a Projective Plane.

By analogy with $B_K(g, h, d_1, d_2)$ we define:

$$B_P(g, h, d_1, d_2) = sup\{|ind(f_1, f_2, C)| \ \ | \ \ deg(f_1) = d_1, \ deg(f_2) = d_2\},$$

[6] If $Coin(f, p) = \emptyset$ we have nothing to do.

[7] We like to remember the equivalence between the semi-index defined in [3] and the absolute value of the index used here (see [5]).

where the supremum is taken over all coincidence classes C of all pairs of maps $(f_1, f_2) : P_h \mapsto P_g$ with given degrees and with f_2 orientation true.

Theorem 5.2. *$B_K(g, g, 0, 1) = 2g-1$ and $B_P(g, g, 0, 1) = 2g$ for $g \geq 1$.*

Proof: Let M_n be the non-orientable surface that admits the surface S_n (with *genus* n) as a double orientable covering. Given a pair $f_1, f_2 : M_n \mapsto M_n$, such that f_2 has degree 1, we know that f_2 is homotopic to a homeomorphism f_2' and the inclusion $Coin(f_1, f_2) \mapsto Fix((f_2')^{-1} \circ f_1)$ preserves the Nielsen classes indices. Further $|deg(f_1)| = |deg((f_2')^{-1} \circ f_1)|$. We can take the pair (f_1, f_2) to be $((f_2')^{-1} \circ f_1, id)$. Now we can use the definition and the properties of the Nielsen fixed point class index. For simplicity, we will denote $(f_2')^{-1} \circ f_1$ by f.

- First we will show that given $C \subset Fix(f)$, a Nielsen class, then $|ind(C)| \leq n$. For this we observe that if $deg(f) = 0$ then f can be factored through the 1-skeleton of M_n. By [14, Proposition 6.6, page 53] we have that $f_\#(\pi_1(M_n))$ is a free subgroup with rank[8] at most $\frac{n+1}{2}$, then we can describe f as a map between two wedges of circles, $\bigvee S^1$, of, at most $\frac{n+1}{2}$ copies of S^1. Now using the results of [10] and the commutativity property of the fixed point class we see that $|ind(C)| \leq n$.
- Now we will construct an example under the conditions of theorem and with $|ind(C)| = n$. Given n odd, we consider $f : \bigvee_i^{\frac{n+1}{2}} S^1 \mapsto \bigvee_i^{\frac{n+1}{2}} S^1$, the map that sends each loop of S^1 over itself with degree 2, and having only one fixed point in the intersection of all the copies of S^1. We can take $\bigvee_i^{\frac{n+1}{2}} S^1 \subset M_n$ and we have that f can be extended to a map $\widetilde{f} : M_n \mapsto M_n$ with degree zero and having only one fixed point class C. By the Lefschetz theorem, we obtain $|ind(C)| = n$.

Now we observe:

$$M_n = \begin{cases} K_{\frac{n-1}{2}} & \text{if } n \text{ odd} \\ P_{\frac{n}{2}} & \text{if } n \text{ even} \end{cases}$$

finishing the proof. □

[8] $\pi_1(M_n)$ has $n+1$ generators $\{\gamma_1, \gamma_2, \cdots, \gamma_{n+1}\}$ and they satisfy $\gamma_1^2\gamma_2^2 \cdots \gamma_{n+1}^2 = 1$

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