# COINCIDENCES FOR MAPS OF SPACES WITH FINITE GROUP ACTIONS

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Abstract. Let $G$ be a finite group acting freely in a Hausdorff, paracompact, connected and locally pathwise connected topological space $X$ such that $H_i(X, Z) = 0$ for $0 < i < m$ and $H^{m+1}(G, Z) \neq 0$. Let $f : X \to Y$ be a map of $X$ to a finite $k$-dimensional CW-complex $Y$. We show that if $m \geq |G|k$, then $f$ has a $(H, G)$-coincidence point for some nontrivial subgroup $H$ of $G$.

## 1. Introduction.

The classic Borsuk-Ulam theorem says that every map of $S^n$ into the euclidean $k$-dimensional space $R^k$ has an antipodal coincidence, that is, it maps some pair of

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antipodal points into a single point if $n \geq k$; moreover, the inequality $n \geq k$ is the best possible, since for $n < k$ there is an inclusion $S^n \to R^k$. This result can be generalized in many ways: $S^n$ and $R^k$ can be replaced by more general spaces $X$ and $Y$, and the antipodal action of $Z_2$ on $S^n$ can be replaced by actions of other groups. We will consider the following setting: suppose that $X, Y$ are topological spaces, $G$ is a group acting freely on $X$ and $f : X \to Y$ is a map. If $H$ is a subgroup of $G$, then $H$ acts on the right on each orbit $Gx$ of $G$ as follows: if $y \in Gx$ and $y = gx, g \in G$, then $hy = ghx$. Following [7], [9] and [11], the concept of $G$-coincidence can be generalized as follows: a point $x \in X$ is said to be a $(H, G)$-coincidence point of $f$ if $f$ sends every orbit of the action of $H$ on the $G$-orbit of $x$ to a single point. Of course, if $H$ is the trivial subgroup, then every point of $X$ is a $(H, G)$-coincidence. If $H = G$, this is the usual definition of coincidence. If $G = Z_r$, $p$ divides $r$, and $H = Z_p$, then a $(H, G)$-coincidence is a $(p, r)$-coincidence in the sense of [9]. If $G = Z_p$ with $p$ prime, then a nontrivial $(H, G)$-coincidence point is a $G$-coincidence point.

Generally speaking, we want to find conditions on $X, Y$ and $G$ which assure that every map $f : X \to Y$ has a $(H, G)$-coincidence point for some nontrivial subgroup $H \subset G$. Recently we have proved the following result ([7]).

**Proposition.** *Suppose that $G$ is a finite group acting freely on a CW-complex $X$ that is a homotopy $m$-dimensional sphere and suppose that $f : X \to Y$ is a map, where $Y$ is a finite $k$-dimensional CW-complex. Then, if $m \geq |G|k$, there is a nontrivial subgroup $H \subset G$ and a $(H, G)$-coincidence point for $f$.*

For $X = S^m$, this question had been solved in [10] and [11] when $G = Z_2$ (compare also [12]), and more generally in [9] and [11] for $G = Z_q$ (in this case $m$ is odd).

The aim of this paper is to show that the above result can be extended to a larger class of spaces $X$. Specifically, we will prove the following theorem.

**Theorem 1.** *Let $X$ be a Hausdorff, paracompact, connected and locally pathwise connected topological space, and let $G$ be a finite group acting freely on $X$. Suppose that $H^{m+1}(G, Z) \neq 0$ for some natural number $m \geq 1$ and $H_i(X, Z) = 0$ for $0 < i < m$. If $Y$ is a finite $k$-dimensional CW-complex with $m \geq |G|k$ and $f : X \to Y$ is a map, then there is a nontrivial subgroup $H \subset G$ and a $(H, G)$-coincidence point for $f$.*

If $G$ is a finite group which acts freely on a CW-complex of the homotopy type of a $m$-dimensional sphere, then by [3, Ch. XII, §11] $H^{m+1}(G, Z) \cong Z_{|G|}$, the cyclic group of order $|G|$; thus the above theorem is in fact an extension of our previous result. In addition, we will show that the inequality $m \geq |G|k$ is the best condition for the existence of $G$-coincidences if $G = Z_p$ with $p$ prime.

Recently, in [18], A. Yu. Volovikov studied this question above for $G = Z_p^k = Z_p \oplus ... \oplus Z_p$ ($k$ summands), where $p$ is a prime number, and $X$ is a connected

CW-complex with a $G$-action such that $H^i(X, Z_p) = 0$ for $0 < i < m$; he used the concept of index $i(X)$ of [17]. However, our result does not imply the Volovikov's result when $k > 1$, since in his study $H = G$.

## 2. Proof of the theorem.

**Lemma 1.** *Let $G$ be a finite group such that $H^{m+1}(G, Z) \neq 0$ for some $m \geq 1$, and let $p$ be a prime divisor of the order of $H^{m+1}(G, Z)$. Then the cohomology $H^m(G, Z_p)$ contains $Z_p$ as a summand.*

*Proof.* By the universal coefficient formula,

$$H^{m+1}(G, Z) = \text{free part of}(H_{m+1}(G, Z)) \oplus \text{torsion}(H_m(G, Z)).$$

Since $G$ is a finite group, the $Z$-homology of $G$ has only torsion in dimensions greater than zero, hence $H_m(G, Z) \cong H^{m+1}(G, Z)$. By the universal coefficient theorem,

$$H^m(G, Z_p) \cong Hom(H_m(G, Z), Z_p) \oplus Ext(H_{m-1}(G, Z), Z_p)$$

and $Hom(H_m(G, Z), Z_p)$ has a summand isomorphic to $Z_p$.

□

Let $B(G)$ be the classifying space for $G$. Since $X$ is Hausdorff and paracompact, we can take a classifying map $c : X/G \to B(G)$ for the principal $G$-bundle $\pi : X \to X/G$.

**Lemma 2.** *Let $p$ be a prime divisor of the order of $H^{m+1}(G, Z)$. Then the homomorphism*

$$c^* : H^m(B(G), Z_p) \to H^m(X/G, Z_p)$$

*is injective (in particular, it is nontrivial by Lemma 1).*

*Proof.* Consider the universal $G$-bundle $E(G) \to B(G)$. By [4, Ch. III, §4 page 208] we have a fibration $X \to X_G \xrightarrow{\rho} B(G)$, with base space $B(G)$, fibre map $\rho$ and fiber $X$, and where $X_G = E(G) \times_G X$ (see also [1, Ch. IV, §6, Lemma 6.2 page 146]). Since $G$ is a finite group and acts freely on $X$, then $X_G$ is homotopy equivalent to $X/G$. If $h : X/G \to X_G$ is a homotopy equivalence, $\rho h : X/G \to B(G)$ also classifies the principal $G$-bundle $\pi : X \to X/G$, and so it is homotopic to $c$. Hence it suffices to show that $\rho^* : H^m(B(G), Z_p) \to H^m(X_G, Z_p)$ is injective. Consider the cohomology spectral sequence $\{E_r^{p,q}, \partial^r\}$ associated to the fibration $\rho : X_G \to B(G)$ (see for example Theorem 6 of [14, Ch 9, page 495]). Since $H_q(X, Z) = 0$ for $0 < q < m$, the $E_2^{p,q}$ term of this spectral sequence vanish for all $0 < q < m$. Therefore $E_2^{m,0} = E_3^{m,0} = ... = E_\infty^{m,0}$ and follows that $E_\infty^{m,0} = H^m(B(G), Z_p)$.

By a routine argument of spectral sequence we obtain the following short exact sequence:

$$0 \to E_\infty^{m,0} = H^m(B(G), Z_p) \xrightarrow{\rho^*} H^m(X_G, Z_p) \to E_\infty^{0,m} \to 0.$$

Therefore $\rho^*$ is injective and the result follows.

□

**Lemma 3.** *Let $X$ be a Hausdorff, connected and locally pathwise connected space, and let $G$ be a finite group acting freely on $X$. Then for any $i \geq 0$ and a commutative ring $R$ with a unit, there is a transfer homomorphism $\tau_X : H^i(X, R) \to H^i(X/G, R)$ with the following properties:*

*i) if $X$ is a $k$-dimensional $G$-CW-complex (not necessarily finite), then $\tau_X : H^k(X, R) \to H^k(X/G, R)$ is surjective;*

*ii) if $Y$ is another space that satisfies the hypothesis of the lemma, $h : X \to Y$ is an equivariant map and $\overline{h} : X/G \to Y/G$ is the map induced by $h$, then $\tau_X h^* = \overline{h}^* \tau_Y$.*

*Proof.* First we define $\tau_X$ at the cochain level. Let $\mu \in S^i(X, R)$ a singular $i$-cochain; that is, $\mu$ is a $R$-homomorphism from the singular $i$-chain $R$-module $S_i(X, R)$ into $R$. Take the standard $i$-simplex $\sigma_i$ and fix a base point $z \in \sigma_i$. Let $\phi : \sigma_i \to X/G$ be a singular $i$-simplex, and

$$\pi^{-1}(\phi(z)) = \{a_1, a_2, ..., a_r\},$$

where $r$ is the order of $G$ and $\pi : X \to X/G$ is the quotient map. Since $\pi : X \to X/G$ is a $r$-fold covering with $X$ locally pathwise connected, and $\sigma_i$ is simply connected, there are unique liftings

$$\phi_1, \phi_2, ..., \phi_r : \sigma_i \to X$$

of $\phi$ such that $\phi_i(z) = a_i$, $i = 1, 2, ..., r$. Define

$$\overline{\mu}(\phi) = \mu(\phi_1) + \mu(\phi_2) + ... + \mu(\phi_r).$$

It is straighforward to verify that the assignment $\mu \mapsto \overline{\mu}$ is $R$-linear and maps cocycles into cocycles and coboundaries into coboundaries, hence it defines an $R$-homomorphism $\tau_X : H^i(X, R) \to H^i(X/G, R)$ (which depends on the base point of $\sigma_i$). The fact that these transfer homomorphisms satisfy ii) follows from the commutativity relation $\overline{h}\pi_X = \pi_Y h$, where $\pi_X, \pi_Y$ are the quotient maps corresponding to $X$ and $Y$.

Suppose now that $X$ is a $k$-dimensional $G$-CW-complex. Consider $S^j(\quad, R) = Hom_R(S_j(\quad, R), R)$, where $S_j(\quad, R)$ is the free $R$-module generated by the open

$j$-dimensional cells. Since $X$ is a $G$-CW-complex, $X/G$ has the induced CW-structure; in particular, if $\{e_\alpha\}$ are the $k$-cells of $X/G$, then for each $\alpha$ there are exactly $r$ $k$-cells $e^1_\alpha, e^2_\alpha, ..., e^r_\alpha$ of $X$ mapped by $\pi$ into $e_\alpha$. In this case the transfer homomorphism $\tau_X : H^k(X, R) \to H^k(X/G, R)$ is the homomorphism induced by the chain map $\tau_X : S^k(X, R) \to S^k(X/G, R)$ given by $\tau_X(\mu)(e_\alpha) = \mu(e^1_\alpha) + \mu(e^2_\alpha) + ... + \mu(e^r_\alpha)$. Take a $k$-cocycle $\eta \in S^k(X/G, R)$ and define a $k$-cochain $\eta' \in S^k(X, R)$ by $\eta'(e^j_\alpha) = \eta(e_\alpha)$ if $j = 1$, and $\eta'(e^j_\alpha) = 0$ if $j \neq 1$, for every $\alpha$. Clearly, $\tau_X(\eta') = \eta$; moreover, since $X$ has dimension $k$, $S^{k+1}(X, R) = 0$, and so every $k$-cochain of $X$ is a $k$-cocycle. Thus $\eta'$ represents a cohomology class of $H^k(X, R)$ that is mapped by $\tau_X$ into $[\eta]$.

□

With the above lemmas, the remaining arguments are similar to those of [7].

*Proof of the Theorem.* Let $G = \{g_1, ..., g_r\}$ be a fixed enumeration of elements of $G$, and $Y^r = Y \times ... \times Y$. Consider the map $G \times Y^r \to Y^r$ given by

$$(g, (y_1, ..., y_r)) \to (y_{\sigma_g(1)}, ..., y_{\sigma_g(r)}),$$

where the permutation $\sigma_g$ is defined by $g_i g = g_{\sigma_g(i)}$; this map is a left G-action on $Y^r$. For a subgroup $H \subset G$, let $(Y^r)^H$ be the fixed point set of $H$ and $F = \bigcup_H (Y^r)^H$, where $H$ runs over all nontrivial subgroups of $G$. Let $Y_0^{(r)} = Y^r - F$; it is precisely the part of $Y^r$ where the $G$-action is free. With this $G$-action on $Y^r$, $Y_0^{(r)}$ inherits a $k$-dimensional $G$-CW-structure from the CW-structure of $Y$. In fact, consider the usual product CW-structure on $Y^r$. If $e^j$ is an open $j$-cell of $Y^r$ with $e^j \cap F \neq \emptyset$, then $e^j - F$ is a disjoint union of open $s$-cells $e^s_1$,...,$e^s_{t_s}$, where $0 \leq s \leq j$, so that each $e^s_i$ can be subdivided into open cells in such a way that they are freely interchanged by $G$. The required $G$-structure on $Y_0^{(r)}$ is given then by the union of these cells with the open cells $e^j$ of $Y^r$ such that $e^j \cap F = \emptyset$.

If $X$ is any space with a $G$-action, then a map $f : X \to Y$ induces an equivariant map $\phi : X \to Y^r$,

$$\phi(x) = (f(g_1 x), ..., f(g_r x)).$$

Suppose that $f : X \to Y$ has no $(H, G)$-coincidence points for any nontrivial subgroup $H \subset G$. Then $\phi(X) \subset Y_0^{(r)}$, so $\phi$ factors through $Y_0^{(r)}$, $\phi = X \xrightarrow{\phi_0} Y_0^{(r)} \hookrightarrow Y^{(r)}$; $\phi_0$ is an equivariant map and thus induces a map $\overline{\phi}_0 : X/G \to Y_0^{(r)}/G$.

Let $c_Y : Y_0^{(r)}/G \to B(G)$ be a classifying map for the principal $G$-bundle $\pi_{Y_0^{(r)}} : Y_0^{(r)} \to Y_0^{(r)}/G$. Then $c = c_Y \overline{\phi}_0 : X/G \to B(G)$ is a classifying map for the principal $G$-bundle $\pi_X : X \to X/G$ of the previous section.

Now let $p$ be a prime that divides the order of $H^{m+1}(G, Z)$. First, if $m > rk$, then the topological dimension of $Y_0^{(r)}/G$ is less than $m$, hence $c^*_Y : H^m(B(G), Z_p)$

$\to H^m(Y_0^{(r)}/G, Z_p)$ is zero and $c^* = \overline{\phi}_0^* c_Y^* = 0$, which contradicts Lemma 2. Thus we can assume that $m = rk$. We will again obtain a contradiction to Lemma 2 by showing that $\overline{\phi}_0^* : H^m(Y_0^{(r)}/G, Z_p) \to H^m(X/G, Z_p)$ is zero.

The map $i^* : H^m(Y^r, Z_p) \to H^m(Y_0^{(r)}, Z_p)$ is a part of the cohomology sequence of the pair $(Y^r, Y_0^{(r)})$ and $H^m(Y^r, Y_0^{(r)}) = 0$, so $i^*$ is surjective. Since $H_i(X, Z) = 0$ for $0 < i < m$, the universal coefficient theorem implies that $H^i(X, Z_p) = 0$ for $0 < i < m$. On the other hand, by the Künneth formula, any nonzero class of $H^m(Y^r, Z_p)$ is a cup product of $k$-dimensional classes and $0 < k < m$ (recall that $r \geq 2$), which implies that $\phi^* : H^m(Y^r, Z_p) \to H^m(X, Z_p)$ is zero. Since $\phi_0 i = \phi$, it follows that $\phi_0^* : H^m(Y_0^{(r)}, Z_p) \to H^m(X, Z_p)$ is zero.

Now consider the transfer homomorphisms $\tau_X : H^m(X, Z_p) \to H^m(X/G, Z_p)$ and $\tau_{Y_0^{(r)}} : H^m(Y_0^{(r)}, Z_p) \to H^m(Y_0^{(r)}/G, Z_p)$ given by Lemma 3. By ii) of that lemma we have the following commutative rectangle

$$\begin{array}{ccc} H^m(X, Z_p) & \xrightarrow{\tau_X} & H^m(X/G, Z_p) \\ \uparrow \phi_0^* & & \uparrow \overline{\phi}_0^* \\ H^m(Y_0^{(r)}, Z_p) & \xrightarrow{\tau_{Y_0^{(r)}}} & H^m(Y_0^{(r)}/G, Z_p), \end{array}$$

which implies that

$$\overline{\phi}_0^* \tau_{Y_0^{(r)}} : H^m(Y_0^{(r)}, Z_p) \to H^m(X/G, Z_p)$$

is zero. Since $Y_0^{(r)}$ is a $m$-dimensional $G$-CW-complex, part i) of Lemma 3 implies that $\tau_{Y_0^{(r)}}$ is surjective and so $(\overline{\phi}_0)^* : H^m(Y_0^{(r)}/G, Z_p) \to H^m(X/G, Z_p)$ is the zero homomorphism. □

**An example.** Suppose $G = Z_p$, where $p$ is a prime. We can define a natural free action of $G$ on $S^{p-2}$ as follows. Let $G$ act on $\mathbb{R}^p$, the $p$-dimensional euclidean space, by cyclic permutations of coordinates; the unit sphere $S^{p-1} \subset \mathbb{R}^p$ is invariant under this action. If we consider $\mathbb{R}^p$ as the space $Map(G, \mathbb{R}) = \{\lambda : G \to \mathbb{R}\}$ of maps, then the action is given by $g\lambda(h) = \lambda(g^{-1}h)$, for $g \in G$ and $\lambda \in Map(G, \mathbb{R})$. Let $\triangle$ be the diagonal of $\mathbb{R}^p$, $\triangle = \{(x_1, ..., x_p) \in \mathbb{R}^p \mid x_i = x_j, 1 \leq i, j \leq p\}$. Then the action is free on $\mathbb{R}^p - \triangle$, and in particular it is free on the unit sphere $S^{p-2}$ in the orthogonal complement of the one-dimensional subspace $\triangle$.

Let $\varphi : S^{p-2} \to \mathbb{R}$ be defined by the first projection $\varphi(x_1, \ldots, x_p) = x_1$. Clearly $\varphi$ has no $G$-coincidence points and $\varphi(S^{p-2}) \subset [-1, 1]$.

Now let $T$ be the one-dimensional CW-complex on the 2-plane that is the union of the segment $I_1 = [-1, 1]$ of the $x$-axis with the segment $I_2 = [0, 1]$ of the $y$-axis; thus $\varphi(S^{p-2}) \subset I_1$. Next take the join space $X^{p-1} = S^{p-2} * G$; it is a $(p-1)$-dimensional CW-complex. If $p = 2$, then $X^1$ is homeomorphic to the circle $S^1$.

The points of $X^{p-1}$ can be written as $[\lambda, t, g]$, where $\lambda \in S^{p-2}$, $t \in [0,1]$ and $g \in G$. An element $h \in G$ acts on $[\lambda, t, g]$ as $h[\lambda, t, g] = [h\lambda, t, hg]$, and this action of $G$ on $X^{p-1}$ is free. Define $\psi : X^{p-1} \to T$ by

$$\psi[\lambda, t, g] = \left\{ \begin{array}{l} t\lambda(e) \in I_1,\ g \neq e, \\ (2t-1)\lambda(e) \in I_1,\ g = e,\ 1/2 \leq t \leq 1, \\ 1 - 2t \in I_2,\ g = e,\ 0 \leq t \leq 1/2 \end{array} \right\}$$

where $e$ is the unity element of $G$. A direct calculation shows that $\psi$ has no $G$-coincidence points.

Now consider the join $X = X^{p-1} * ... * X^{p-1}$ of $m$ copies of $X^{p-1}$ . It is a CW-complex of dimension $mp-1$ with $H_i(X, Z) = 0$ for each $0 < i < mp-1$ satisfying all the remaining hypotheses of our theorem. A point of $X$ can be written as $[t_1x_1 + ... + t_mx_m]$, where $x_i \in X$, $t_1 + ... + t_m = 1$ and $t_i \geq 0$, $i = 1, ..., m$, and $G$ acts freely on $X$ through the diagonal action

$$g.[t_1x_1 + ... + t_mx_m] = [t_1(g.x_1) + ... + t_m(g.x_m)].$$

We can define a map $F : X \to T^m$ as

$$F([t_1x_1 + ... + t_mx_m]) = (t_1\psi(x_1), ..., t_m\psi(x_m)).$$

The fact that $\psi$ has no $G$-coincidence points implies that $F$ has no $G$-coincidence points.

This construction shows that the hypothesis $m \geq |G|k$ has the following properties:

a) It is the best condition for the existence of $(H, G)$-coincidences if all finite groups $G$ are considered.

b) It is the best condition for the existence of $G$-coincidences if $G = Z_p$ with $p$ prime.

It is an interesting question whether this condition is the best possible for other finite groups $G$.

Expanding further on the above constructions, the first example of a map $f : S^{2m-1} \to Y$ of a sphere $S^{2m-1}$ into a contractible $m$-dimensional polyhedron $Y$ such that no pair of antipodes is identified was found in 1937 by H. Hopf [5]. Such examples were rediscovered in 1974 by Ščepin [12] and Stesin [15]; moreover, Ščepin's construction coincides with Hopf's construction but differs from Stesin's construction. Later such constructions were studied in [16], [13] and [10]. Also, Ščepin [12] proved that any map $S^{2m} \to Y$ to $m$-dimensional polyhedron $Y$ identifies some pair of antipodes (thus Ščepin proved Hopf's conjecture [5]). Stesin [15] obtained the same result for maps homotopic to zero. Another proof of Ščepin result was found by Jaworowski in [11].

A map of a sphere $S^{2m-1}$ into a contractible $m$-dimensional polyhedron which does not identify antipodes could be easily constructed using the result of Flores of [6] (probably Hopf was not acquainted with the result of Flores, in other case he could easily obtain the needed map; however, Hopf defined a much simpler map). In fact, let $\Delta^{2m}_{m-1}$ be the $(m-1)$-skeleton of the $2m$-dimensional simplex $\Delta^{2m}$. Consider the join $\Delta^{2m}_{m-1} * \Delta^{2m}_{m-1}$ of two copies of $\Delta^{2m}_{m-1}$ and the subspace $S \subset \Delta^{2m}_{m-1} * \Delta^{2m}_{m-1}$ (called the *deleted join* ) given by the union of all $\sigma_1 * \sigma_2$, where $\sigma_1$, $\sigma_2$ are nonempty simplexes in $\Delta^{2m}_{m-1}$ such that $\sigma_1 \cap \sigma_2$ is empty. The *interchanging* involution is free on $S$. Flores proved that $S$ is equivariantly homeomorphic to $S^{2m-1}$ considered with the antipodal involution (he used this nontrivial fact and Borsuk–Ulam theorem to prove that $\Delta^{2m}_{m-1}$ could not be embedded into the $(2m-2)$-dimensional Euclidean space). Consider a map $\Delta^{2m}_{m-1} * \Delta^{2m}_{m-1} \to \Delta^{2m}_{m-1} * pt$ to the cone over $\Delta^{2m}_{m-1}$ (here $pt$ denotes the one-point space), which is defined by collapsing one of $\Delta^{2m}_{m-1}$. Then its restriction on $S$, composed with the equivariant homeomorphism $S^{2m-1} \to S$, is a map of a sphere $S^{2m-1}$ into the contractible $m$-dimensional polyhedron $\Delta^{2m}_{m-1} * pt$, and this map does not identify antipodes.

This construction is close to examples of Vaisala [16], Scepin [13] and Izydorek & Jaworowski [10].

If $X$ and $Y$ are spaces as in the theorem and $f : X \to Y$ a map, let $C_H$ be the set of all $(H, G)$-coincidence points of $f$. It is natural to study the size of $C = \bigcup_H C_H$, where $H$ runs over all nontrivial subgroups of $G$. In [8], we studied the size of $C_H$ in terms of the cohomological dimension of $C_H$, when $H$ is a nontrivial normal cyclic subgroup of $G$ with prime order $p$ and $X$ is the $m$-dimensional sphere. By arguments similar to those used in Section 3 of [8], p. 467-469, we can prove the following result.

**Theorem 2.** *Suppose that $X, Y, G$ and $f$ are as in Theorem 1 and $H$ is a nontrivial normal cyclic subgroup $H$ of a prime order $p$. Then $H^{m-|G|k}(C_H, Z_p) \neq 0$.*

*Sketch Proof.*

The proof of the main theorem of [8], Section 3, p. 467-469, can be adjusted to the more general situation of this paper by replacing the top-dimensional generator of $H^m(S^m/Z_p, Z_p)$ of [8] with the nonzero element $g^*(\alpha) \in H^m(X/Z_p, Z_p)$ given by Lemma 2.

□

**Corollary.** *$dim(C) \geq m - |G|k$, where $dim(C)$ is the covering dimension.*

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