THE BORSUK-ULAM THEOREM FOR GENERAL SPACES

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Abstract. Let X, Y be topological spaces and $T : X \to X$ a free involution. In this context, a question that naturally arises is whether or not all continuous map $f : X \to Y$ has a T-coincidence point, that is, a point $x \in X$ with $f(x) = f(T(x))$. If additionally Y is equiped with a free involution $S : Y \to Y$, another question is concerning the existence of equivariant maps $f : (X, T) \to (Y, S)$. In this paper we obtain results of this nature under cohomological (homological) conditions on the spaces X and Y.

1. Introduction

Let X be a topological space. An *involution* on X is a continuous map $T : X \to X$ which is its own inverse. A classic example is the antipodal map $A : S^n \to S^n$, $A(x) = -x$, where S^n denotes the n-sphere.

Suppose that X and Y are topological spaces with involutions $T : X \to X$ and $S : Y \to Y$. A map f from X to Y is *equivariant* if $Sf = fT$.

One formulation of the Borsuk-Ulam theorem is that there is no map from S^m to S^n equivariant with respect to the antipodal map when $m > n$ (see, for example, [6; 7.2]). Our objective in this paper is to generalize this result replacing spheres by a wide class of topological spaces, and giving the results in terms of cohomological (homological) properties of these spaces. Let us think first in terms of replacing the domain (S^m, A) by a space X equiped with an involution $T : X \to X$ which is free, that is, $T(x) \neq x$ for any $x \in X$. The fact that for $m \leq n$ the inclusion $(S^m, A) \to (S^n, A)$ is equivariant suggests that a reasonable condition to be imposed on X is that its cohomology (or

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homology) is zero in dimensions less than or equal to n. In this direction, J. W. Walker proved in [4] that if X is a Hausdorff and pathwise connected space so that the singular Z_2-homology $H_r(X, Z_2)$ is zero for $1 \leq r \leq n$, then there is no equivariant map $f : (X, T) \to (S^n, A)$. However, in the Walker's proof, the target space (S^n, A) plays a fundamental role, since this proof is based on the construction of special homology classes of X, which in turn depend on the geometric A-equivariant j-dimensional hemispheres of S^n, $0 \leq j \leq n$; loosely speaking, the Walker's theorem shows that the existence of an equivariant map $f : (X, T) \to (S^n, A)$ forces the existence of some nonzero homology class β of $H_j(X, Z_2)$ for some $1 \leq j \leq n$, and such a β is inherited from the geometry of (S^n, A) via f. In the direction of replacing also the target space by more general spaces, our first result will be the following

Theorem 1. *Let X, Y be Hausdorff, pathwise connected and paracompact spaces, equiped with free involutions $T : X \to X$ and $S : Y \to Y$. Suppose that for some natural number $n \geq 1$, $\check{H}^r(X) = 0$ for $1 \leq r \leq n$ and $\check{H}^{n+1}(Y/S) = 0$, where $\check{H}$ denotes Cech cohomology mod 2 and Y/S is the orbit space of Y by S. Then there is no equivariant map $f : (X, T) \to (Y, S)$.*

For example, Y may be any n-dimensional manifold with free involution, which evidently includes (S^n, A). We would like to thank the referee for the statement and proof of the above result. Also the referee pointed us that the result still holds (with the same proof) for X, Y non paracompact spaces, when the principal Z_2-bundles $X \to X/T$ and $Y \to Y/S$ can be trivialized over partitions of unity of X/T and Y/S; additionally, the referee remarked that his method applies to the case where X is the odd dimensional sphere and (more generally) $T : X \to X$ is the standard free periodic homeomorphism of period p, where p is an odd prime (see Section 2, Remark 2).

Even if the two above assumptions (paracompactness and trivializing partition of unity) are simultaneously removed, we can obtain a homological version of the result if in addition we impose that Y is locally pathwise connected (in fact, we do not need the Hausdorff propertie on X in this case). This will follow from the observation that the special homology classes of X considered by Walker in the proof of the above mentioned theorem of [4] have an additional feature which depends only on the fact that $H_r(X, Z_2) = 0$ for $1 \leq r \leq n$, and not on the geometry of S^n and the existence of equivariant maps from X to S^n; in fact, these classes may be considered as belonging to the equivariant *T-homology groups* $H_r(X, T)$, and under this viewpoint we will prove that they have nonzero *Z_2-index of Yang* (Lemma 2 of Section 3). Summarizing, we will prove the following

Theorem 2. *Let $(X, T), (Y, S)$ be spaces with free involutions so that X is pathwise connected and Y is Hausdorff and locally pathwise connected. Suppose that for some natural number $n \geq 1$, $H_r(X, Z_2) = 0$ for $1 \leq r \leq n$ and $H_{n+1}(Y/S, Z_2) = 0$. Then there is no equivariant map $f : (X, T) \to (Y, S)$.*

For a given space Y, let $\triangle = \{(x, y) \in Y \times Y \quad / \quad x = y\}$ be the usual diagonal, and write $Y^* = Y \times Y - \triangle$; note that Y^* admits the free involution $t_Y : Y^* \to Y^*$ given by $t_Y(x, y) = (y, x)$. As a consequence of Theorem 2 one has

Theorem 3. *Let (X, T) be a pathwise connected space with free involution, and let Y be a Hausdorff and locally pathwise connected space. For a natural number $n \geq 1$, suppose that $H_r(X, Z_2) = 0$ for $1 \leq r \leq n$ and that $H_{n+1}(Y^*/t_Y, Z_2) = 0$. Then every continuous map $f : X \to Y$ has a T-coincidence point, that is, a point $x \in X$ such that $f(x) = f(T(x))$.*

For example, suppose Y a finite n-dimensional CW complex. Then Y^*/t_Y admits a structure of a $2n$-dimensional CW complex, hence $H_{2n+1}(Y^*/t_Y, Z_2) = 0$. In this way, every continuous map $f : S^m \to Y$ has an antipodal coincidence point when $m > 2n$. This particular case of Theorem 3 was proved by J. Jaworowski and M. Izydorek in [5].

2. **The Borsuk-Ulam theorem for paracompact spaces**

The aim of this section is to prove Theorem 1. First consider $B(Z_2)$ the classifying space for Z_2, and denote by $\alpha \in H^1(B(Z_2), Z_2) \cong \check{H}^1(B(Z_2))$ the Euler class of the universal principal Z_2-bundle over $B(Z_2)$. Since X is a Hausdorff paracompact space, one can take a classifying map $g : X/T \to B(Z_2)$ for the principal Z_2-bundle $X \to X/T$, and from $g^* : \check{H}^1(B(Z_2)) \to \check{H}^1(X/T)$ one gets the *Euler class* $e = g^*(\alpha) \in \check{H}^1(X/T)$ of $X \to X/T$.

From [3; Section 3.7 (sequence 7.8 of p. 143)] one has the Smith-Gysin exact sequence

$$\check{H}^0(X/T) \xrightarrow{p^*} \check{H}^0(X) \xrightarrow{\tau} \check{H}^0(X/T) \xrightarrow{\cup e} \check{H}^1(X/T) \longrightarrow \dots$$
$$\longrightarrow \check{H}^r(X/T) \xrightarrow{p^*} \check{H}^r(X) \xrightarrow{\tau} \check{H}^r(X/T) \xrightarrow{\cup e}$$
$$\check{H}^{r+1}(X/T) \longrightarrow \dots \quad ,$$

where $p : X \to X/T$ is the quotient map and $\tau : \check{H}^r(X) \to \check{H}^r(X/T)$ is the transfer homomorphism.

Since X is pathwise connected, $p^* : \check{H}^0(X/T) \to \check{H}^0(X)$ is an isomorphism, hence $\cup e : \check{H}^0(X/T) \to \check{H}^1(X/T)$ is injective and thus $e = 1 \cup e$ is a nonzero class. The fact that $\check{H}^r(X) = 0$ for $1 \le r \le n$ implies that $\cup e : \check{H}^r(X/T) \to \check{H}^{r+1}(X/T)$ is an isomorphism for $1 \le r \le n-1$ and injective for $r = n$, hence $e^{n+1} \in \check{H}^{n+1}(X/T)$ is nonzero.

Now suppose $f : (X,T) \to (Y,S)$ an equivariant map, and let $h : Y/S \to B(Z_2)$ be a classifying map for $Y \to Y/S$. Then $h\overline{f}$ can be taken as the previous classifying map $g : X/T \to B(Z_2)$, where $\overline{f} : X/T \to Y/S$ is the map induced by f. It follows that $\overline{f}^*(e'^{n+1}) = e^{n+1}$, where $e' \in \check{H}^1(Y/S)$ is the Euler class of $Y \to Y/S$, and thus $e'^{n+1} \neq 0$, which contradicts the fact that $\check{H}^{n+1}(Y/S) = 0$.

Remark 1. As observed in the introduction, the same argument works when X and Y are not paracompact but $X \to X/T$ and $Y \to Y/S$ can be trivialized over partitions of unity of X/T and Y/S.

Remark 2. The above method also works when X is the $(n+1)$-dimensional sphere S^{n+1}, with $n+1$ odd, equiped with the free action of the cyclic group Z_p generated by the standard periodic homeomorphism $T : S^{n+1} \to S^{n+1}$ of period p, where p is an odd prime. In fact, suppose in this case Y a Hausdorff and paracompact space, equiped with a free action of Z_p generated by $S : Y \to Y$, and with $\check{H}^{n+1}(Y/S) = 0$, where $\check{H}$ denotes Cech cohomology mod p. We recall that the orbit space S^{n+1}/T is the generalized lens space L_p^{n+1}. A model for $B(Z_p)$, the classifying space for Z_p, is the infinite lens space $L_p^\infty = S^\infty/T$. Since p is an odd prime, $\check{H}^i(L_p^\infty) \cong H^i(L_p^\infty, Z_p) \cong Z_p$ for any $i \geq 0$, and given any nonzero element $\alpha \in \check{H}^1(L_p^\infty)$, one has that $x_i = \alpha(\beta(\alpha))^i$ is a nonzero element of $\check{H}^{2i+1}(L_p^\infty)$, where $\beta : \check{H}^1(L_p^\infty) \to \check{H}^2(L_p^\infty)$ is the Bockstein homomorphism. Moreover, if $g : L_p^{n+1} \to L_p^\infty$ is any classifying map for the principal Z_p-bundle $S^{n+1} \to L_p^{n+1}$, then $g^*(x_i)$ is a nonzero element of $\check{H}^{2i+1}(L_p^{n+1}) \cong Z_p$ for any $0 \leq i \leq n/2$.

Suppose then $f : (S^{n+1}, T) \to (Y, S)$ an equivariant map. Following the same lines of the above proof, let $h : Y/S \to B(Z_p)$ be a classifying map for $Y \to Y/S$. Then $h\overline{f}$ is a classifying map for $S^{n+1} \to L_p^{n+1}$, and $(h\overline{f})^*(x_{n/2})$ is a nonzero element of $\check{H}^{n+1}(L_p^{n+1})$. This means that $h^*(x_{n/2}) \in \check{H}^{n+1}(Y/S)$ is a nonzero element, which is impossible.

Remark 3. In [1], C. T. Yang defined the Z_2-index of a free involution (X, T) as being the integer n such that $\nu(H_r(X, T)) = Z_2$ for $0 \leq r \leq n$ and $= 0$ for $r > n$ (for the definitions of $H_r(X, T)$ and $\nu : H_r(X, T) \to Z_2$ see Section 3), and proved the following result (corollary 4.2, page 270):

"Let (X, T) be a free involution of Z_2-index n. Then a continuous map from X into the euclidean k-space R^k maps some pair $\{x, T(x)\}$ into a single point if $n \geq k$."

This generalizes the classic Borsuk-Ulam theorem which says that every continuous map of S^n into R^k maps some pair of antipodal points into a single point if $n \geq k$. In Section 3 we will prove that if (X, T) is a free involution with

X pathwise connected and with $H_r(X, Z_2) = 0$ for $1 \leq r \leq n-1$, then the Z_2-index of (X, T) is $\geq n$ (Lemma 2). Together with the above Yang's result, this will immediately give the following generalization of the Borsuk-Ulam theorem:

Theorem A. *Let (X, T) be a free involution with X pathwise connected, and let $f : X \to R^k$ be a continuous map. Suppose that $H_r(X, Z_2) = 0$ for $1 \leq r \leq n-1$. Then if $n \geq k$ there is a point $x \in X$ such that $f(x) = f(T(x))$.*

The following cohomological version of Theorem A can be proved directly from Theorem 1.

Theorem A'. *Let (X, T) be a free involution with X Hausdorff, pathwise connected and paracompact, and let $f : X \to R^k$ be a continuous map. Suppose that $\check{H}^r(X) = 0$ for $1 \leq r \leq n-1$. Then if $n \geq k$ there is a point $x \in X$ such that $f(x) = f(T(x))$.*

In fact, suppose $f(x) \neq f(T(x))$ for any $x \in X$. Then one has the equivariant map $F : (X, T) \to (S^{k-1}, A)$ given by

$$F(x) = \frac{f(x) - f(T(x))}{|f(x) - f(T(x))|} \quad .$$

Since $n \geq k$, $\check{H}^n(S^{k-1}/A) = 0$, which contradicts Theorem 1 (we would like to thank Professor Carlos Biasi of the ICMC-USP-São Carlos for the statement of Theorem A and its cohomological version, and for this direct proof; evidently, this direct proof also works for Theorem A, thus avoiding the above Yang's result).

Remark 4. In [2], F. Cohen and J. E. Connett obtained a Borsuk-Ulam result for maps $f : X \to R^n$ $(n \geq 2)$, where X is a Hausdorff space which supports a free Z_p-action $(p \geq 2)$, that restricted to the particular case $p = 2$ has the following statement: if (X, T) is a free involution with X $(n-1)$-connected and $f : X \to R^n$ is a continuous map, then there is $x \in X$ such that $f(x) = f(T(x))$. It is interesting to note that Theorem A and its cohomological version are stronger than this fact, since a $(n-1)$-connected space has $H_r(X, Z_2)$ $(\check{H}^r(X))$ equal to zero for $1 \leq r \leq n-1$.

3. **A homological version of the Borsuk-Ulam theorem for non paracompact spaces**

In this section we prove Theorems 2 and 3. We need first to recall the concepts, mentioned in Section 2 and developed by C. T. Yang in [1], of *T-homology* and *Z_2-index* associated to pairs (X, T), where X is a topological space and $T : X \to X$ is a free involution; such a pair is called a *T-pair* .

Let $S_p(X, Z_2)$ be the singular chain group of X with coefficients in Z_2, and consider the induced chain map $T_\# : S_p(X, Z_2) \to S_p(X, Z_2)$. A p-chain $c \in S_p(X, Z_2)$ is called a *(T, p)-chain* if $T_\#(c) = c$. All the (T, p)-chains form a subgroup $S_p(X, T) \subset S_p(X, Z_2)$, and the boundary operator $\partial : S_p(X, Z_2) \to S_{p-1}(X, Z_2)$ maps $S_p(X, T)$ into $S_{p-1}(X, T)$; hence one has the homology groups

$$H_p(X, T) = \frac{Z_p(X, T)}{B_p(X, T)},$$

where $Z_p(X, T) = \{c \in S_p(X, T)/\partial(c) = 0\}$ and $B_p(X, T) = \partial(S_{p+1}(X, T))$. $H_p(X, T)$ is called the p^{th} *T-homology group* of the T-pair (X, T). An equivariant map $f : (X, T) \to (Y, S)$ of T-pairs induces a homomorphism $f_* : H_p(X, T) \to H_p(Y, S)$.

The *Z_2-index homomorphism* $\nu : H_p(X, T) \to Z_2 = \{0, 1\}$ can be defined with the support of the following fact: a p-chain $c \in S_p(X, Z_2)$ is a (T, p)-chain if and only if $c = d + T_\#(d)$ for some p-chain $d \in S_p(X, Z_2)$. Then ν is defined by recurrence at the (T, p)-cycles level as follows: let $z = c + T_\#(c) \in Z_p(X, T)$. If $p = 0$, write $c = x_1 + x_2 + ... + x_r$, where each x_i is a point of X. Then $\nu(z) = 0$ if r is even and $\nu(z) = 1$ if r is odd. If $p > 0$, $\nu(z) = \nu(\partial(c))$.

A fundamental fact about the Z_2-index is that if $f : (X, T) \to (Y, S)$ is an equivariant map of T-pairs, then $\nu(f_*(\xi)) = \nu(\xi)$ for every $\xi \in H_p(X, T)$.

The proofs of Theorems 2 and 3 will follow easily from the following lemmas.

Lemma 1. *If (Y, S) is a T-pair with Y Hausdorff and locally pathwise connected, then $H_p(Y, S)$ is isomorphic to $H_p(Y/S, Z_2)$.*

Lemma 2. *Let (X,T) be a T-pair with X pathwise connected. For a natural number $n \geq 1$, suppose $H_r(X,Z_2)=0$ for $1 \leq r \leq n$. Then there exist classes with nonzero Z_2-index in $H_{n+1}(X,T)$.*

In fact, for Theorem 2, suppose by contradiction that there exists an equivariant map $f:(X,T) \to (Y,S)$. One then has the induced homomorphism $f_*: H_{n+1}(X,T) \to H_{n+1}(Y,S)$. By Lemma 2, there exists $\xi \in H_{n+1}(X,T)$ with $\nu(\xi)=1$, hence $\nu(f_*(\xi))=1$. But this is impossible, since by Lemma 1 $H_{n+1}(Y,S) \cong H_{n+1}(Y/S,Z_2)=0$.

For Theorem 3, again by contradiction, suppose that $f(x) \neq f(T(x))$ for any $x \in X$. Then the map $F: X \to Y \times Y$ given by $F(x)=(f(x), f(T(x)))$ defines an equivariant map $F:(X,T) \to (Y^*, t_Y)$. Since Y^* is Hausdorff and locally pathwise connected, this contradicts Theorem 2.

In this way, all that remains is to prove Lemmas 1 and 2.

Proof of Lemma 1. This is asserted by C. T. Yang in [1] for the Cech homology mod 2 $\check{H}_p(Y/S)$, but it is easily seen to be valid for the singular Z_2-homology if Y is Hausdorff and locally pathwise connected. In fact, one has a chain map $P: S_p(Y,S) \to S_p(Y/S,Z_2)$ given by $P(d+T_\#(d))=p_\#(d)$, where $p: Y \to Y/S$ is the quotient map. It is straightforward to see that P is one-to-one. Now take σ_p the standard p-simplex and consider $\phi: \sigma_p \to Y/S$ any singular p-simplex. Since $p: Y \to Y/S$ is a two-fold covering with Y Hausdorff and locally pathwise connected, and σ_p is simply connected, one has by the lifting theorem (for example, see [7;page 89]) that there is $\phi': \sigma_p \to Y$ such that $\pi\phi' = \phi$, which shows that P is onto; consequently, $P_*: H_p(Y,S) \to H_p(Y/S,Z_2)$ is an isomorphism. /

Proof of Lemma 2. We need first to recall the construction of certain special singular j-chains $c_j \in S_j(X,Z_2)$, $0 \leq j \leq n+1$, considered by J. W. Walker in [4]. Consider the chain map $\theta = Id_\# + T_\# : S_j(X,Z_2) \to S_j(X,Z_2)$, where "$Id$" denotes the identity map. This chain map satisfies $\theta\theta=0$. The chains c_j are inductively constructed so that $\partial(c_j)=\theta(c_{j-1})$ for $1 \leq j \leq n+1$, and consequently each $\theta(c_j)$ will be a j-cycle. First, we pick a point in X, and call c_0 the 0-chain corresponding to this point. Of course, $\theta(c_0) = c_0 + T_\#(c_0)$ is a 0-cycle. Since X is pathwise connected, there is a path joining c_0 to $T_\#(c_0)$, and we call c_1 the 1-chain corresponding to this path. Then $\partial(c_1) = T_\#(c_0) + c_0 = \theta(c_0)$, and additionally $\partial\theta(c_1) = \theta\partial(c_1) = \theta\theta(c_0) = 0$, that is, $\theta(c_1)$ is a 1-cycle.

Suppose inductively that, for some $1 \leq j \leq n$, one has constructed $c_0, c_1, ..., c_j$ so that $\partial(c_j) = \theta(c_{j-1})$. Then $\partial\theta(c_j) = \theta\partial(c_j) = \theta\theta(c_{j-1}) = 0$, so $\theta(c_j)$ is a j-cycle. Since $H_j(X, Z_2) = 0$, there exists a $j+1$-chain c_{j+1} so that $\partial(c_{j+1}) = \theta(c_j)$; additionally, $\partial\theta(c_{j+1}) = \theta\partial(c_{j+1}) = \theta\theta(c_j) = 0$, so $\theta(c_{j+1})$ is a $j+1$-cycle. This completes the inductive construction of $c_0, c_1, ..., c_{n+1}$.

Now observe that a j-chain $c \in S_j(X, Z_2)$ is a (T, j)-chain if and only if $\theta(c) = 0$. Since $\theta\theta = 0$, each $\theta(c_j)$ is then a (T, j)-cycle and thus $\nu(\theta(c_j))$ makes sense. We assert that $\nu(\theta(c_j)) = 1$ for each $0 \leq j \leq n+1$. In fact, since c_0 consists of a single point, $\nu(\theta(c_0)) = 1$ by definition. Suppose inductively that $\nu(\theta(c_j)) = 1$ for some $0 \leq j \leq n$. Then

$$\nu(\theta(c_{j+1})) = \nu(c_{j+1} + T_\#(c_{j+1})) = \nu(\partial(c_{j+1})) = \nu(\theta(c_j)) = 1.$$

This completes the proof. /

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