A LOCAL HOPF LEMMA AND UNIQUE CONTINUATION FOR THE HELMHOLTZ EQUATION

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Abstract. We prove local unique continuation across the boundary and boundary uniqueness results for solutions of the Helmholtz equation in a neighborhood of the origin in a half space $\mathbb{R}^n_+$.

0. Introduction

Let u be a real valued harmonic function defined on the upper half ball $B^+ \subset\subset \mathbb{R}^n$, continuous up to $T \doteq \{(x_1, \dots, x_{n-1}, 0)\} \cap B$. The Schwarz reflection principle states that if $u(x', 0) = 0$, $|x'| < 1$, then u can be continued across T as a harmonic function and since this is the only possible harmonic extension this phenomenon is usually called unique continuation (across the boundary). A related uniqueness fact concerning a single point in T is given by Hopf's classical boundary point lemma: if u assumes a maximum at $p \in T$ and the normal derivative $\partial_{x_n} u(p)$ exists and is zero, then u must be constant, $u \equiv u(p)$.

Unique continuation for holomorphic functions of one variable with nonnegative real part on a piece of the boundary and local forms of Hopf's lemma were proved in [7], [10] and [11]. The results were used to prove a more general Schwarz reflection principle for holomorphic functions mapping the real line into a totally real manifold or a real analytic set. They were also used to prove unique continuation for CR mappings. Earlier results along this line were obtained in the works [12], [4], [8], [1], and [2]. The paper [2] contains a general local Hopf lemma for holomorphic functions of one variable with applications to unique continuation for CR mappings, see also [9] for an extension of the latter results.

This paper focuses on local unique continuation across the boundary and on local Hopf's lemma for solutions of the Helmholtz equation

$$L_c u = \Delta u + cu = 0, \ c \in \mathbb{R},$$

in an open set of the half space $\mathbb{R}^n_+$ generalizing the results proved in [5] and [6] for harmonic functions, i.e., the case $c = 0$. The hypotheses involve a local boundary sign condition on the product of the solution times a monomial. These uniqueness phenomena, obtained for the Laplace operator in [6] and for the more general operators L_c in this work, extend the classical result of Hopf about the

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nonvanishing of the normal derivative at a boundary point where a nonconstant solution attains an extremum in two directions: first, the assumption is local in nature and only imposes conditions at the boundary; secondly, unique continuation may hold even at a point where a solution changes sign at the boundary.

The Helmholtz equation is very important in physics. Solving many physically significant partial differential equations such as the wave equation, the heat equation, the Klein-Gordon equation, Maxwell's equations and Schrodinger's equation often requires solving Helmholtz's equation. The equation arises in the study of electromagnetic radiation, optics, seismology, and acoustics. It also describes mass transfer processes with volume chemical reactions of the first order. From a mathematical point of view, it represents an eigenvalue problem for the Laplace operator.

In the appendix, we present another proof of one of our results which gives rise to a generalized notion of the Pascal triangle. A future paper will be devoted to extensions to more general second order elliptic operators.

1. Statements and proof of a lemma

We wish to study the boundary unique continuation problem for the solutions of $L_c u = \Delta u + cu$ where c is a real constant. We will say that a continuous function w defined on the half ball $B_r^+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x| < r,\ x_n > 0\}$ is flat at 0 if for every positive integer N, there is a constant C_N such that $|w(x)| \leq C_N |x|^N$.

Theorem 1. *Let u be a solution of $L_c u = 0$ in the half ball*

$$B_r^+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x| < r,\ x_n > 0\},$$

continuous on the closure. Assume that

(1) *for some monomial s^α, $s = (s_1, \ldots, s_{n-1})$, $s^\alpha u(s, 0) \geq 0$ for $|s| \leq r$;*
(2) *for every multi-index β in n variables with $|\beta| \leq d$, where $d = |\alpha|$ is the degree of s^α, the function $x_n \mapsto (\partial_x^\beta u)(0, x_n)$ is flat at $x_n = 0$.*

Then $u(x', 0) \equiv 0$ and hence u extends as a solution to a neighborhood of 0 in $\mathbb{R}^n$.

For $c = 0$, Theorem 1 was proved in [6] with the monomial s^α replaced by a homogeneous polynomial $p(s) = p(s_1, \ldots, s_n)$ under the additional assumption that for every positive integer N, the function $|s|^{-N} u(s, 0)$ is integrable on the ball $\{s \in \mathbb{R}^{n-1} : |s| \leq r\}$.

Corollary 1. *Let u be a solution of $L_c u = 0$ in the half ball B_r^+, continuous on the closure. Assume that*

(1) *for some monomial s^α, $s = (s_1, \ldots, s_{n-1})$, $s^\alpha u(s, 0) \geq 0$ for $|s| \leq r$;*
(2) *the function $u(x)$ is flat at 0.*

Then $u(x) \equiv 0$.

When the dimension $n = 2$, we have:

Corollary 2. *Assume $n = 2$. Let $u(x, y)$ be a solution of $L_c u = 0$ in B_r^+, continuous on the closure and flat at 0. Assume that one of the following conditions holds:*

(1) *$u(x, 0)$ does not change sign;*
(2) *$xu(x, 0)$ does not change sign and the function $y \mapsto \frac{\partial u}{\partial x}(0, y)$ is flat at $y = 0$.*

Then $u \equiv 0$.

Corollary 3. *Let $u(x) = u(x', x_n)$ be a solution of $L_c u = 0$ in B_r^+, continuous on the closure. Assume that $u(0, x_n)$ vanishes to infinite order at zero and $\liminf_{x_n \to 0^+} \frac{u(x)}{x_n} \geq 0$. Then $u \equiv 0$ on B_r^+.*

Corollary 3 was proved for $c = 0$ in [3].

The following lemma will be used in the proof of Theorem 1.

Lemma 1. *Let u be a solution of $L_c u = 0$ in the half ball*

$$B_r^+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x| < r,\ x_n > 0\},$$

continuous on the closure. Assume that

1. *for some monomial s^α, $s = (s_1, \ldots, s_{n-1})$, $s^\alpha u(s, 0) \geq 0$ for $|s| \leq r$;*
2. *for every multi-index β in n variables with $|\beta| \leq d$, where $d = |\alpha|$ is the degree of s^α, the function $x_n \mapsto (\partial_x^\beta u)(0, x_n)$ is flat at $x_n = 0$.*

Then, for every positive integer N, the function $|s|^{-N} s^\alpha u(s, 0)$ is integrable on the ball $\{s \in \mathbb{R}^{n-1} : |s| \leq r\}$. Moreover, the function $x_n \mapsto \partial_x^\alpha u(0, x_n) \in C^\infty([0, r))$.

Proof. Since the other cases are similar, we consider L_{-1} and assume that the dimension $n = 2p \geq 4$. The case $n = 2$ will be treated later. We proceed by induction and assume that the lemma has been proved for monomials of degree less than d. If one of the coordinates of $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ is greater than one, then for some multi-index γ with $|\gamma| < d$, $s^\gamma u(s, 0) \geq 0$ and therefore the induction assumption would imply the assertion of the Lemma. We may therefore assume that each $\alpha_i \leq 1$. Since the proof below will show that the other cases can be handled in the same way (including the case $\alpha = 0$), we assume henceforth that each $\alpha_i = 1$. The proof of Theorem 1 in Section 2 will show that the Poisson kernel for L_{-1} in the upper half space has the form

$$P = y_n \left(\sum_{m=0}^{p-1} A_m r^{2m-n} + f_1(r) \log r + g_2(r) \right), \quad r = \sqrt{|x' - y'|^2 + y_n^2},$$

where A_m are constants, $A_0 \neq 0$, f_1 and g_2 real analytic functions of r^2. Let $0 \leq \psi(x') \leq 1$ be a smooth function on $\mathbb{R}^{n-1}$, $\psi(x') \equiv 1$ for $|x'| \leq \frac{r}{2}$ and supported in $|x'| < \frac{3r}{2}$. Let $f(x') = \psi(x') u(x', 0)$ and define

$$v(y) = \int_{\mathbb{R}^{n-1}} P(x', 0, y) f(x')\, dx'$$

where P is the Poisson kernel for L_{-1} in the upper half space. We have

$$\frac{v(y)}{y_n} = \int_{\mathbb{R}^{n-1}} f(x') A(x', y)\, dx',\ y \in B_r^+ \tag{1.1}$$

where

$$A(x', y) = \sum_{m=0}^{p-1} A_m r^{2m-n} + f_1(r) \log r + g_2(r). \tag{1.2}$$

We consider $D(A(x',y))$ at $y'=0$ where $D=\partial_{y'}^{\alpha}=\partial_{y_{n-1}}\cdots\partial_{y_1}$. For any integer $j\geq 1$, we have

$$D(r^{-j})|_{y'=0}=\frac{j(j+2)\cdots(j+2(n-2))}{r^{j+2(n-1)}}x_1\cdots x_{n-1}. \tag{1.3}$$

Observe next that since $f_1(r)=f(r^2)$ for some analytic function f, the function $D(f_1(r)\log(r))$ is a finite sum of terms (with constant coefficients) of the type $h_0(r)(\log r)(x_1-y_1)\cdots(x_{n-1}-y_{n-1})$ and $\frac{h_j(r)}{r^{2j}}(x_1-y_1)\cdots(x_{n-1}-y_{n-1})$ for some real analytic functions h_j of r^2, $1\leq j\leq n-1$. It follows that for some constants R_j,

$$D(f_1(r)\log(r))\big|_{y'=0}=\left(R_0h_0(r)\log r+\sum_{j=1}^{n-1}R_j\frac{h_j(r)}{r^{2j}}\right)x_1\cdots x_{n-1}. \tag{1.4}$$

We also have

$$D(g_2(r))|_{y'=0}=h(r^2)x_1\cdots x_{n-1}$$

for some real analytic function h. We can thus write

$$D\left(\frac{v(y)}{y_n}\right)|_{y'=0}=\int_{\mathbb{R}^{n-1}}B(x',y_n)x_1\cdots x_{n-1}f(x')\,dx' \tag{1.5}$$

where for some constants B_m

$$B(x',y_n)=\sum_{m=0}^{p-1}\frac{A_mB_m}{(|x'|^2+y_n^2)^{\frac{3n}{2}-m-1}}+R_0h_0(r)\log r+\sum_{j=1}^{n-1}R_j\frac{h_j(r)}{r^{2j}}+h(r^2). \tag{1.6}$$

For any N, we will use the Taylor approximation:

$$\begin{aligned}\frac{1}{(|x'|^2+y_n^2)^{\frac{3n}{2}-m-1}}&=\sum_{j=0}^{N}(-1)^jM_j^m\frac{y_n^{2j}}{|x'|^{2j+3n-2m-2}}\\ &+(-1)^{N+1}(N+1)M_{N+1}^m\left(\int_0^1\frac{(1-t)^N}{(ty_n^2++|x'|^2)^{\frac{3n}{2}-m+N}}\,dt\right)y_n^{2N+2}\end{aligned} \tag{1.7}$$

where

$$M_j^m=\frac{(\frac{3n-m-2}{2})(\frac{3n-m-2}{2}+1)\cdots(\frac{3n-m-2}{2}+j-1)}{j!},\ M_0^m=1.$$

It follows that, in a formal sense, for every integer N,

$$\begin{aligned}D\left(\frac{v(y)}{y_n}\right)|_{y'=0}&=\sum_{m=0}^{p-1}\sum_{j=0}^{N}(-1)^jM_j^mA_mB_m\left(\int_{\mathbb{R}^{n-1}}\frac{x_1x_2\ldots x_{n-1}f(x')}{|x'|^{2j+3n-2m-2}}\,dx'\right)y_n^{2j}\\ &+(-1)^{N+1}(N+1)\times\\ &\sum_{m=0}^{p-1}M_{N+1}^mA_mB_m\left(\int_{\mathbb{R}^{n-1}}\int_0^1\frac{(1-t)^Nx_1\cdots x_{n-1}f(x')}{(ty_n^2++|x'|^2)^{\frac{3n}{2}-m+N}}\,dt\,dx'\right)y_n^{2N+2}\\ &+R_0\int_{\mathbb{R}^{n-1}}h_0(|x'|^2+y_n^2)\log(|x'|^2+y_n^2)x_1\cdots x_{n-1}f(x')\,dx'\\ &+\sum_{j=1}^{n-1}R_j\int_{\mathbb{R}^{n-1}}\frac{h_j(r)}{r^{2j}}x_1\cdots x_{n-1}f(x')\,dx'+\int_{\mathbb{R}^{n-1}}h(r^2)x_1\cdots x_{n-1}f(x')\,dx'.\end{aligned} \tag{1.8}$$

Let $w(x', y_n) = v(x', y_n) - u(x', y_n), \quad y_n \geq 0$. Then w extends as a solution on a neighborhood of the origin by defining $w(x', y_n) = -w(x', -y_n)$ for $y_n < 0$. Since $w(x', 0) \equiv 0$ and $y_n \mapsto w(x', x_n)$ is odd for $|x'| < \frac{r}{2}$, $\partial_{y_n}^{2j} w(x', 0) \equiv 0$ and hence we have

$$D\left(\frac{w(x', y_n)}{y_n}\right)\Big|_{x'=0} = \sum_{k=0}^{\infty} c_k y_n^{2k}.$$

By hypothesis, $D\left(\frac{u(x', y_n)}{y_n}\right)\Big|_{x'=0}$ is flat at $y_n = 0$. Therefore, for every N,

$$D\left(\frac{v(x', y_n)}{y_n}\right)\Big|_{x'=0} = \sum_{k=0}^{N} c_k y_n^{2k} + O(y_n^{2N+2}). \tag{1.9}$$

This implies that

$$\lim_{y_n \to 0+} D\left(\frac{v(x', y_n)}{y_n}\right)\Big|_{x'=0} = c_0. \tag{1.10}$$

From (1.5) and (1.10), using the assumption that $x_1 \cdots x_{n-1} f(x') \geq 0$ and the monotone convergence theorem, we get

$$\begin{aligned} c_0 = \sum_{m=0}^{p-1} M_0^m A_m B_m \int_{\mathbb{R}^{n-1}} \frac{x_1 x_2 \ldots x_{n-1} f(x')}{|x'|^{3n-2m-2}}\, dx' \\ + R_0 \int_{\mathbb{R}^{n-1}} h_0(|x'|^2) \log(|x'|^2) x_1 \cdots x_{n-1} f(x')\, dx' \\ + \sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} \frac{h_j(r)}{r^{2j}} x_1 \cdots x_{n-1} f(x')\, dx' \\ + \int_{\mathbb{R}^{n-1}} h(r^2) x_1 \cdots x_{n-1} f(x')\, dx'. \end{aligned} \tag{1.11}$$

Next note that

$$D\left(\frac{v(x', y_n)}{y_n}\right)|_{x'=0} - c_0 = c_1 y_n^2 + O(y_n^4)$$

and so

$$\lim_{y_n \to 0+} \frac{1}{y_n^2}\left[D\left(\frac{v(x', y_n)}{y_n}\right)|_{x'=0} - c_0\right] = c_1. \tag{1.12}$$

From (1.8) and (1.12), we conclude that

$$c_1 = -\sum_{m=0}^{p-1} M_1^m A_m B_m \int_{\mathbb{R}^{n-1}} \frac{x_1 x_2 \ldots x_{n-1} f(x')}{|x'|^{3n-2m}}\, dx' \tag{1.13}$$

and in particular, $\frac{x_1 \cdots x_{n-1} f(x')}{|x'|^{3n}} \in L^1(\mathbb{R}^{n-1})$. Assume now that for some $N \geq 2$, $\frac{x_1 \cdots x_{n-1} f(x')}{|x'|^{2N+3n-2}} \in L^1(\mathbb{R}^{n-1})$ and

$$c_j = \sum_{m=0}^{p-1} (-1)^j M_j^m A_m B_m \int_{\mathbb{R}^{n-1}} \frac{x_1 x_2 \ldots x_{n-1} f(x')}{|x'|^{2j+3n-2m-2}}\, dx', \quad \text{for } j \leq N. \tag{1.14}$$

Then

$$D\left(\frac{v(x',y_n)}{y_n}\right)\Big|_{x'=0} - \sum_{j=0}^{N} c_j y_n^{2j} = (-1)^{N+1}(N+1)\times$$

$$\sum_{m=0}^{p-1} M_{N+1}^m A_m B_m \left(\int_{\mathbb{R}^{n-1}} \int_0^1 \frac{(1-t)^N x_1\cdots x_{n-1} f(x')}{(ty_n^2 + +|x'|^2)^{\frac{3n}{2}-m+N}}\, dt\, dx'\right) y_n^{2N+2}$$

$$= c_{N+1} y_n^{2N+2} + O(y_n^{2N+4}).$$

It follows that

$$\lim_{y_n\to 0+} \frac{D\left(\frac{v(x',y_n)}{y_n}\right)|_{x'=0} - \sum_{j=0}^{N} c_j y_n^{2j}}{y_n^{2N+2}} = c_{N+1}.$$

Thus (1.14) holds for all j, and $\frac{x_1\cdots x_{n-1} f(x')}{|x'|^N} \in L^1(\mathbb{R}^{n-1})$ for all N.
From (1.5), we have

$$\partial^\alpha v(y)\big|_{y'=0} = \int_{\mathbb{R}^{n-1}} B(x',y_n) y_n x_1\cdots x_{n-1} f(x')\, dx'$$

and so the integrability of $\frac{x_1\cdots x_{n-1} f(x')}{|x'|^N}$ for every N shows that $D\left(\frac{v(y)}{y_n}\right)\Big|_{y'=0}$ is smooth up to $y_n = 0$ and hence the same holds for $D\left(\frac{u(y)}{y_n}\right)\Big|_{y'=0}$.
When the dimension $n = 2$, we can use the Poisson kernel which has the form

$$P(x',0,y) = \left(-\frac{2}{r^2} + f_1(r)\log r + g_2(r)\right)$$

and argue as in the case $n \geq 4$. □

2. Proof of Theorem 1

Proof of Theorem 1. The case $c = 0$ was done in the work [6] . When $c \neq 0$, it suffices to prove the theorem for L_{-1} and L_1. We will consider L_{-1} first. We assume first that the degree of the monomial s^α is 0. Thus $u(x',0) \geq 0$ and $u(0,x_n)$ is flat at 0.

Suppose the dimension $n = 2p \geq 4$ is an even integer. The case $n = 2$ will be treated later. Assume that $\alpha = 0$. We seek a fundamental solution $K(x,y)$ of the form

$$K(x,y) = r^{2-n}\left(\sum_{m=0}^{p-2} C_m r^{2m}\right) + \sum_{m=1}^{\infty} (A_m \log r + B_m)\, r^{2m-2}$$

where $r = |x-y|$ and the coefficients are to be determined. Let $A_1 = 1$ and for $m \geq 2$, let

$$A_m = \frac{1}{2^{m-1}(m-1)!n(n+2)\cdots(n+2(m-2))}.$$

Let $B_1 = 0$ and for $m \geq 2$, define B_m by the recursive relation

$$B_m = \frac{B_{m-1} - (2(2m-2)+n-2)\, A_m}{(2m-2)(2m+n-4)}.$$

For $m \geq 1$, define

$$u_m(r) = A_m r^{2m-2} \log r + B_m r^{2m-2}.$$

Using

$$\Delta(r^m) = m(m+n-2)r^{m-2},$$

$$\Delta(r^m \log r) = m(m+n-2)r^{m-2} \log r + (2m+n-2)r^{m-2} \tag{2.1}$$

we get $\Delta u_1 = \Delta(\log r) = (n-2)r^{-2}$ and for $m \geq 2$, $\Delta u_m = u_{m-1}$. Let $C_{p-2} = n-2$ and since $n \geq 4$, we can choose $C_1, \dots, C_{p-3}$ that satisfy

$$\Delta\left(\frac{C_m}{r^{n-2-2m}}\right) = \frac{C_{m-1}}{r^{n-2-2(m-1)}},\ 1 \leq m \leq p-2. \tag{2.2}$$

For any m, $|A_m| \leq \frac{1}{m!}$ and by induction, it can be seen that $|B_m| \leq \frac{2}{m!}$. It follows that the series $\sum_{m=1}^{\infty} A_m r^{2m-2} \log r$ and $\sum_{m=1}^{\infty} B_m r^{2m-2}$ converge absolutely for any $r > 0$. For $x \neq y$, let $K(x,y)$ be as above with these coefficients. Observe that

$$\Delta K = C_0 \Delta(r^{2-n}) + \sum_{m=0}^{p-2} C_m r^{2m+2-n} + \sum_{m=1}^{\infty} u_m(r) = c\delta_0 + K$$

for some nonzero constant $c \neq 0$. Thus, modulo a constant factor, K is a fundamental solution for L_{-1}. We consider next the Poisson kernel P of L_{-1} for $\mathbb{R}^n_+$:

$$P(x', 0, y) = \frac{\partial}{\partial x_n} \left(K(x,y) - K(x,y^*)\right)(x', 0, y),\ y^* = (y', -y_n).$$

Since

$$\begin{aligned} \frac{\partial}{\partial x_n} K(x,y) = &\sum_{m=0}^{p-2} (2m+2-n) C_m r^{2m-n}(x_n - y_n) \\ &+ \frac{f(r)}{r^2}(x_n - y_n) + \frac{f'(r)\log r}{r}(x_n - y_n) + \frac{g'(r)}{r}(x_n - y_n), \end{aligned}$$

we get

$$P(x', 0, y) = -2\sum_{m=0}^{p-2} (2m+2-n) C_m r^{2m-n} y_n - 2\frac{f(r)}{r^2} y_n - 2\frac{f'(r)\log r}{r} y_n - 2\frac{g'(r)}{r} y_n,$$

where in the latter, $r = \left(|x'-y'|^2 + y_n^2\right)^{\frac{1}{2}}$. Note that

$$f_1(r) = \frac{-2f'(r)}{r} = -2\sum_{m=2}^{\infty} (2m-2) A_m r^{2m-4}$$

and

$$g_1(r) = \frac{-2g'(r)}{r} = -2\sum_{m=2}^{\infty} (2m-2) B_m r^{2m-4}$$

are even functions. We can write

$$P(x',0,y) = -2\sum_{m=0}^{p-2}(2m+2-n)C_m r^{2m-n} y_n + \frac{4(n-2)}{r^2} y_n + f_1(r)\log r y_n + g_2(r) y_n$$

where $g_2(r) = g_1(r) - 2\sum_{m=1}^{\infty} A_{m+1} r^{2m-2}$ is real analytic. For $k \geq 1$ we will estimate $\left(\frac{\partial}{\partial y_n}\right)^{2k+1} P(x',0,y)$ at $y_n = 0$. Observe that when $y_n = 0$,

$$\begin{aligned}\partial_{y_n}^{2k+1} P(x',0,y) = (2k+1)\partial_{y_n}^{2k}\Big(-2\sum_{m=0}^{p-2}(2m+2-n)C_m r^{2m-n} \\ + \frac{4(n-2)}{r^2} + f_1(r)\log r + g_2(r)\Big).\end{aligned} \tag{2.3}$$

Since $g_2(r)$ is real analytic, for r near 0, there is a constant $C > 0$ such that

$$\left|(2k+1)\left(\frac{\partial}{\partial y_n}\right)^{2k} g_2(r)\right| \leq C^{2k+1}(2k+1)!\,. \tag{2.4}$$

Consider next the term $f_1(r)\log r$. The analyticity of f_1 implies that near the origin, for some constant which we will still denote by C,

$$\left|\left(\frac{\partial}{\partial y_n}\right)^{m} f_1(r)\right| \leq C^{m+1} m!\,. \tag{2.5}$$

We have

$$\left(\frac{\partial}{\partial y_n}\right)^{2k} f_1(r)\log r = \sum_{j=0}^{2k}\frac{(2k)!}{j!(2k-j)!}\left(\frac{\partial}{\partial y_n}\right)^{j}\log r\left(\frac{\partial}{\partial y_n}\right)^{2k-j} f_1(r). \tag{2.6}$$

We need to estimate (at $y_n = 0$)

$$\left(\frac{\partial}{\partial y_n}\right)^{j}\log r = \frac{1}{2}\left(\frac{\partial}{\partial y_n}\right)^{j}\log r^2, \;\; r^2 = |x'-y'|^2 + y_n^2.$$

For $a > 0$ and $t \geq 0$, consider the function

$$\log(a+t^2) = F\circ g(t) \text{ where } g(t) = t^2, \text{ and } F(s) = \log(a+s).$$

To find its higher order derivatives, we use Faà di Bruno's formula (the chain rule for higher order derivatives) that says that

$$\frac{d^m}{dt^m}F(g(t)) = \sum \frac{m!}{m_1!m_2!\cdot m_m!}F^{(m_1+\cdots+m_m)}(g(t))\prod_1^m\left(\frac{g^{(j)}(t)}{j!}\right)^{m_j} \tag{2.7}$$

where the sum is over all m-tuples of nonnegative integers $(m_1,\ldots,m_m)$ satisfying the constraint

$$m_1 + 2\,m_2 + 3\,m_3 + \cdots + m\,m_m = m.$$

This formula is greatly simplified for $g(t) = t^2$ because $g'(t) = 2t$, $g''(t) = 2$ and $g^{(j)} = 0$ for $j > 2$. Writing $m_1 = j$, $m_2 = i$ and $m_j = 0$ for $j > 2$ we get

$$\frac{d^m}{dt^m} F(t^2) = \sum_{j+2i=m} \frac{m!}{j!i!} F^{(j+i)}(g(t)) \frac{(2t)^j}{2}. \tag{2.8}$$

If $m = j + 2i$ is odd then j is odd so the right hand side in (2.8) contains a power of t and we see that

$$\left| \frac{d^m}{dt^m} F(g(t)) \right|_{t=0} = 0, \quad m = 2i+1,\ i = 1, 2, \dots. \tag{2.9}$$

If $m = j + 2i$ is even, only the term with $j = 0$ will not vanish at $t = 0$. The conclusion is that for $m = 2i \geq 2$

$$\left| \frac{d^{2i}}{dt^{2i}} F(g(t)) \right|_{t=0} = \frac{(2i)!}{2i!} F^{(i)}(0). \tag{2.10}$$

Since

$$\frac{d^i}{dt^i} \log(a+s)|_{s=0} = \frac{(-1)^{i+1}(i-1)!}{a^i}, \quad \text{for } i \geq 1$$

it follows that

$$\left| \frac{d^m}{dt^m} F(g(t)) \right|_{t=0} = \left| \frac{d^{2i}}{dt^{2i}} F(g(t)) \right|_{t=0} \leq \frac{(2i)!(i-1)!}{2i!a^i}, \ m = 2i \geq 2. \tag{2.11}$$

We thus have

$$\begin{aligned} \left| \left(\frac{\partial}{\partial y_n} \right)^j \log r \right|_{y_n=0} &= \frac{1}{2} \left| \left(\frac{\partial}{\partial y_n} \right)^j \log r^2 \right|_{y_n=0} \\ &\leq \frac{j!(j'-1)!}{2(j'!)a^{j'}}, \ j = 2j' \\ &\leq \frac{j!}{2a^{j'}} \quad \text{for } j = 2j' \geq 0 \end{aligned} \tag{2.12}$$

and it is zero when j is odd. Setting $r = \sqrt{|x'|^2 + y_n^2}$,

$$\left| \left(\frac{\partial}{\partial y_n} \right)^j \log r \right|_{y_n=0} \leq \frac{j!}{2|x'|^j}. \tag{2.13}$$

Hence from (2.5), (2.6), and (2.13),

(2.14)

$$\begin{aligned} \left| \left(\frac{\partial}{\partial y_n} \right)^{2k} f_1(r) \log r \right|_{y_n=0} &\leq C^{2k+1}(2k)! \sum_{j=0, j=2j'}^{2k} \left(\frac{1}{C|x'|} \right)^j \\ &= C^{2k+1}(2k)! \sum_{i=0}^{k} \left(\frac{1}{C|x'|} \right)^{2i} \\ &\leq C^{2k+1}(2k)!(k+1) \left(\frac{1}{C|x'|} \right)^{2k} \quad (\text{for } |x'| \text{ small}) \\ &\leq C(2k)!(k+1) \left(\frac{1}{|x'|} \right)^{2k}. \end{aligned}$$

Consider next

$$\left(\frac{\partial}{\partial y_n}\right)^{2k} r^{2i-n}|_{y_n=0}, \quad \text{for } 0 \le i \le p-2,\ r = \sqrt{|x'|^2 + y_n^2}.$$

Let $g(t) = t^2$ and for $s \ge 0$, let $F(s) = (a+s)^{i-p}$ for some $a > 0$. Using Faà di Bruno's formula, we get

$$\begin{aligned}\left|\left(\frac{d}{dt}\right)^{2k} F \circ g(t)\right| &= \frac{(2k)!}{2(k!)}|F^{(k)}(0)| \\ &= \frac{(2k)!(p-i)(p-i+1)\cdots(p-i+k-1)}{2(k!)a^{k+p-i}}\end{aligned}$$

and since the latter is maximized at $i = 0$,

$$\begin{aligned}&\left|(2k+1)\partial_{y_n}^{2k}\left(\sum_{m=1}^{p-2}(2m+2-n)C_m r^{2m-n} + \frac{4(n-2)}{r^2}\right)\right|_{y_n=0} \\ &\qquad \le Ea(2k+1)\left(\frac{(2k)!(p+k-1)!}{2(k!)(p-1)!a^{k+p}}\right) \qquad (2.15)\end{aligned}$$

for some constant E independent of a (for $0 < a \le \frac{1}{2}$ say) while

$$\left|(2k+1)\left(\frac{\partial}{\partial y_n}\right)^{2k}\left(2(2-n)r^{-n}\right)\right| = \left(\frac{(2k+1)!(p+k-1)!}{2(k!)(p-1)!a^{k+p}}\right). \qquad (2.16)$$

From (2.4) and (2.14) we have

$$\begin{aligned}&\left|(2k+1)\left(\frac{\partial}{\partial y_n}\right)^{2k}(f_1(r)\log r + g_2(r))\right| \\ &\qquad \le C^{2k+1}(2k+1)! + C(2k+1)!\left(\frac{1}{|x'|}\right)^{2k}. \qquad (2.17)\end{aligned}$$

Observe that for $n \ge 4$, $(p+k-1)! \ge (k+1)!$ and so (2.16) implies (at $y_n = 0$):

$$\left|(2k+1)\left(\frac{\partial}{\partial y_n}\right)^{2k}\left(2(2-n)r^{-n}\right)\right| \ge \frac{(k+1)(2k+1)!}{2(p-1)!|x'|^{2k+n}}. \qquad (2.18)$$

From (2.3), (2.14), (2.15), (2.17), and (2.18), we conclude that given $0 < \delta_1 < 1$, there is $\delta_2 > 0$ such that for $|x'| < \delta_2$,

$$\left|\left(\frac{\partial}{\partial y_n}\right)^{2k+1} P(x', 0, y)\right|_{y_n=0} > (1-\delta_1)\frac{(k+1)(2k+1)!}{2(p-1)!|x'|^{2k+n}}. \qquad (2.19)$$

Assume now $n = 2$. In this case, the fundamental solution is given by

$$K = \sum_{m=1}^{\infty} u_m(r)$$

and the Poisson kernel

$$P(x',0,y) = -\frac{2}{r^2}y_n + f_1(r)\log r y_n + g_2(r)y_n$$

where the u_j, f_1 and g_2 are as before. Faà di Bruno's formula leads to the equation

$$\left|\left(\frac{\partial}{\partial y_n}\right)^{2k+1} r^{-2}\right| = \frac{(2k+1)!(k+1)}{2|x'|^{2k+4}}$$

which together with estimate (2.17) imply (2.19).

Let $w(x') = \psi(x')u(x',0)$ where $\psi(x')$ is a smooth function supported in the ball $|x'| < 2\epsilon$, $\psi(x') \equiv 1$ on $|x'| < \epsilon$, $0 \le \psi(x') \le 1$, and ϵ to be chosen later. Define

$$v(y) = \int_{\mathbb{R}^{n-1}} w(x')P(x',0,y)\,dx',\ \ y \in B_r^+. \tag{2.20}$$

Then $L_{-1}v = \Delta v - v = 0$ and $v(y',0) = \lim_{y_n\to 0^+} v(y) = w(y)$. Thus $h = u - v$ is a solution of L_{-1} which is zero on the flat hyperplane near the origin. The function $h = u - v$ extends as an analytic function on a neighborood of the origin. By Lemma 1, the function $v(0, y_n)$ is smooth up to $y_n = 0$. Since $u(0, y_n)$ is flat at $y_n = 0$, we can find $F > 0$ such that

$$\left|\left(\frac{\partial}{\partial y_n}\right)^{2k+1} v(0)\right| = \left|\left(\frac{\partial}{\partial y_n}\right)^{2k+1} (u-v)(0)\right| \le F^{2k+2}(2k+1)!\,. \tag{2.21}$$

We also have, letting $\delta_1 = \epsilon$,

$$\begin{aligned}
\left|\left(\frac{\partial}{\partial y_n}\right)^{2k+1} v(0)\right| &= \int_{\mathbb{R}^{n-1}} w(x')\left|\left(\frac{\partial}{\partial y_n}\right)^{2k+1} P(x',0,0)\right| dx' \\
&\ge (1-\delta_1)\frac{(k+1)(2k+1)!}{2(p-1)!}\int_{\mathbb{R}^{n-1}} \frac{w(x')}{|x'|^{2k+n}}\,dx' \\
&\ge (1-\delta_1)\frac{(k+1)(2k+1)!}{2(p-1)!}\int_{|x'|<\epsilon} \frac{u(x',0)}{|x'|^{2k+n}}\,dx' \\
&\ge (1-\delta_1)\frac{(k+1)(2k+1)!}{2(p-1)!\epsilon^{2k+n}}\int_{|x'|<\epsilon} u(x',0)\,dx'.
\end{aligned} \tag{2.22}$$

By choosing ϵ small enough (depending only on F), since (2.21) and (2.22) hold for any k, by taking the $k-$ th root and letting $k \to \infty$, we conclude that $u(x',0) \equiv 0$ in some neighborhood of the origin in $\mathbb{R}^{n-1}$.

Suppose now the degree $d = |\alpha|$ of the monomial s^α, $s = (s_1, \dots, s_{n-1})$ is positive. We proceed by induction and assume that Theorem 1 has been proved for monomials of degree less than d. If one of the coordinates of $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ is greater than one, then for some multi-index γ with $|\gamma| < d$, $s^\gamma u(s,0) \ge 0$ and therefore the induction assumption would imply that $u(x',0) \equiv 0$. We may therefore assume that each $\alpha_i \le 1$. Since the proof below will show that the other cases can be handled in the same way, we assume henceforth that each $\alpha_i = 1$. Recall the function

$$v(y) = \int_{\mathbb{R}^{n-1}} w(x')P(x',0,y)\,dx',\ \ y \in B_r^+$$

defined in (2.20) and write

$$\frac{v(y)}{y_n} = \int_{\mathbb{R}^{n-1}} w(x')A(x',y)\,dx',\ y \in B_r^+$$

where

$$A(x',y) = -2\sum_{m=0}^{p-2}(2m+2-n)C_m r^{2m-n} + \frac{4(n-2)}{r^2} + f_1(r)\log r + g_2(r).$$

We consider $D(A(x',y))$ at $y' = 0$ where $D = \partial_{y'}^{\alpha} = \partial_{y_{n-1}} \cdots \partial_{y_1}$. Recall from Section 1 that for any integer $j \geq 1$,

$$D(r^{-j})|_{y'=0} = \frac{j(j+2)\cdots(j+2(n-2))}{r^{j+2(n-1)}} x_1 \cdots x_n \tag{2.23}$$

and for some constants R_j,

$$D(f_1(r)\log(r))|_{y'=0} = \left(R_0 h_0(r)\log r + \sum_{j=1}^{n-1} R_j \frac{h_j(r)}{r^{2j}}\right) x_1 \cdots x_{n-1}. \tag{2.24}$$

From (2.15), (2.18), (2.23), (2.24), and the analyticity of $g_2(r)$, we conclude that for some $\delta > 0$ and any $k \geq 1$,

$$\frac{1}{2k!}\partial_{y_n}^{2k}\left\{D\left(\frac{v(y)}{y_n}\right)\right\}|_{y=0} \geq \delta \int_{\mathbb{R}^{n-1}} \frac{x_1 \cdots x_{n-1} u(x',0)}{|x'|^{3n-2+2k}}\,dx'. \tag{2.25}$$

Write

$$\frac{v(y)}{y_n} = \frac{v(y)-u(y)}{y_n} + \frac{u(y)}{y_n}.$$

Since $v(y',0) - u(y',0) \equiv 0$ and $v - u$ extends as a real analytic function to a neighborhood of the origin in $\mathbb{R}^n$, the function $\frac{v(y)-u(y)}{y_n}$ is also real analytic on a neighborhood of 0 in $\mathbb{R}^n$. By Lemma 1, the function $D\left(\frac{v(y)}{y_n}\right)$ is smooth on $[0,r)$ and since $u(0,y_n)$ is flat at zero, it follows that all the derivatives of $\frac{u(0,y_n)}{y_n}$ vanish at 0. Hence, for some constant C,

$$\left|\frac{1}{2k!}\partial_{y_n}^{2k}\left\{D\left(\frac{v(y)}{y_n}\right)\right\}\right|_{y=0} \leq C^{k+1}. \tag{2.26}$$

As in the case where the degree of the monomial was zero, the estimates (2.25) and (2.26) imply that $u(x',0) \equiv 0$ for x' near zero in $\mathbb{R}^{n-1}$.

Suppose now the dimension n is odd. We assume first that the degree of α is 0, that is, $u(x',0) \geq 0$. For each $m \geq 1$, define

$$E_m(r) = a_m r^{2m-n} \quad \text{where}$$

$$a_m = \frac{1}{2^{m-1}(m-1)!(2m-n)(2m-2-n)\cdots(4-n)},\ r = |x-y|.$$

Then $\Delta E_1 = c\delta_0$ for some $c \neq 0$, and for $m \geq 1$, $\Delta E_{m+1} = E_m$. Let

$$K(x,y) = \sum_{m=1}^{\infty} E_m(r).$$

Clearly this series converges absolutely for any $r \neq 0$ and

$$L_{-1}K = \Delta K - K = c\delta_0.$$

Let

$$G(x,y) = K(x,y) - K(x,y^*) = F(|x-y|) - F(|x-y^*|).$$

Then except for a constant factor, the Poisson kernel P of L_{-1} for $\mathbb{R}^n_+$ is given by

$$\begin{aligned} P(x',0,y) &= \frac{\partial G}{\partial x_n}(x',0,y) \\ &= \frac{-2F'(r)}{r} y_n \\ &= \left(\frac{c_n}{r^n} + \cdots \frac{c_1}{r} + rh(r)\right) y_n \end{aligned}$$

where $h(r)$ is even in r, analytic everywhere and $c_n \neq 0$. We will first estimate

$$\left(\frac{\partial}{\partial y_n}\right)^{2k+1} (rh(r)y_n)|_{y_n=0} = (2k+1)\left(\frac{\partial}{\partial y_n}\right)^{2k} (rh(r))|_{y_n=0}. \tag{2.27}$$

We have

$$\left(\frac{\partial}{\partial y_n}\right)^{2k} (rh(r)) = \sum_{j=0}^{2k} \frac{(2k)!}{j!(2k-j)!} \left(\frac{\partial}{\partial y_n}\right)^{j} r \left(\frac{\partial}{\partial y_n}\right)^{2k-j} h(r). \tag{2.28}$$

For r in a bounded neighborhood of the origin, there is $C > 0$ that satisfies

$$\left|\left(\frac{\partial}{\partial y_n}\right)^{m} h(r)\right| \leq C^{m+1} m!, \;\; m = 1, 2, \ldots. \tag{2.29}$$

Write

$$r = \sqrt{a+t^2} = F \circ g(t), \;\; g(t) = t^2, \;\; F(s) = \sqrt{a+s}.$$

By Faà di Bruno's formula,

$$\left(\frac{\partial}{\partial y_n}\right)^{j} r|_{y_n=0} = 0 \quad \text{when } j \text{ is odd}$$

and when $j = 2j'$ is even,

$$\left|\left(\frac{\partial}{\partial y_n}\right)^{j} r|_{y_n=0}\right| = \frac{j!}{j'!}|F^{(j')}(0)| \leq \frac{j!}{a^{j-1}}. \tag{2.30}$$

From (2.28), (2.29), and (2.30), we see that

$$\left|\left(\frac{\partial}{\partial y_n}\right)^{2k}(rh(r))\right|_{y_n=0} | \leq C^{2k+1}(2k)!\sum_{i=0}^{k}\frac{1}{a^{2i-1}} \tag{2.31}$$
$$\leq D^{2k+1}\frac{(2k)!}{a^{2k}}$$

for some constant D which is independent of a for small a. Thus

$$\left|\left(\frac{\partial}{\partial y_n}\right)^{2k+1}(rh(r)y_n)|_{y_n=0}\right| \leq D^{2k+1}\frac{(2k+1)!}{a^{2k}}. \tag{2.32}$$

We recall that the principal singular term of $\left(\frac{\partial}{\partial y_n}\right)^{2k+1} P(x',0,y)|_{y_n=0}$, namely, $(2k+1)\left(\frac{\partial}{\partial y_n}\right)^{2k} r^{-n}$ satisfied estimate (2.18):

$$\left|(2k+1)\left(\frac{\partial}{\partial y_n}\right)^{2k}\left(2(2-n)r^{-n}\right)\right| \geq \frac{(k+1)(2k+1)!}{2(p-1)!|x'|^{2k+n}}$$

and it is clear from what was shown in the even dimensional case that the lower order singular terms r^{-j} $(1 \leq j \leq n-1)$ have higher order derivatives that are dominated by those of the principal term. We can therefore argue as in the even dimensional case to conclude that $u(x',0) \equiv 0$ on a neighborhood of the origin in $\mathbb{R}^{n-1}$. When the degree of α is positive, an induction argument as in the even dimensional case leads us to the same conclusion.

Consider next the case when $c = 1$, $L_1 u = \Delta u + u = 0$. We recall that both in even and odd dimensions, we constructed distributions F_k, $k = 1, 2, \ldots$ with the properties:

$$\Delta F_1 = \delta_0, \ \ \Delta F_{k+1} = F_k \quad \text{for } k \geq 1.$$

Let $K = \sum_{j=1}^{\infty}(-1)^{j+1}F_j(r)$. The series converges absolutely for $r \neq 0$ and K is a distribution that satisfies the equation $L_1 K = \Delta K + K = \delta_0$. Therefore, the analysis for L_{-1} shows that the higher order derivatives of the Poisson kernel for L_1 in the normal direction satisfy similar estimates leading to the conclusion that $u(x',0) \equiv 0$ near the origin in $\mathbb{R}^{n-1}$. □

Proof of Corollary 1. We use the well known gradient estimate for solutions of Poisson's equation, $\Delta w = f$ in a domain $\Omega \subset \mathbb{R}^n$ where $w \in C^2(\Omega)$. For some constant $C = C(n)$,

$$\sup_{\Omega} d_x|Dw(x)| \leq C\left(\sup_{\Omega}|w| + \sup_{\Omega} d_x^2|f(x)|\right), \ \forall x \in \Omega,$$

where $d_x = \operatorname{dist}(x, \partial\Omega)$. Since $w = D_{x_i}u$ is a solution of $L_c w = 0$ whenever u is, iterating this estimate we observe that if in addition u is flat at the origin, so is any derivative $\partial^\beta u$. Hence we may apply Theorem 1 to extend $u(x)$ across $x_n = 0$ as a real analytic function which must vanish identically. □

Proof of Corollary 3. The assumption implies that $u(x',0) \geq 0$ and so, by an application of Theorem 1 with $|\alpha| = d = 0$, $u(x',0) \equiv 0$ near the origin and u extends as a real analytic function past the origin. Let $h(x) = u_{x_n}(x)$. Observe that $L_c(h) = 0$, and h satisfies the hypotheses of Theorem 1. It follows that $h(x',0) \equiv 0$ near the origin. Thus $u(x',0) \equiv u_{x_n}(x',0) \equiv 0$ near the origin in $\mathbb{R}^{n-1}$. Using the equation that u satisfies we conclude that all derivatives of u vanish at the origin and hence $u \equiv 0$. □

3. Appendix

Here we will present another proof of Theorem 1 in which the coefficients in Pascal's Triangle arise. For simplicity, we assume that the degree α of the monomial in Theorem 1 is 0 and that $n = 3$.

Let $Lh = \Delta h - h$ and P denote the Poisson kernel of L in $\mathbb{R}^3_+$. For $x' \in \mathbb{R}^2$, $y \in \mathbb{R}^3$, modulo a constant factor,

$$P(x',0,y) = \frac{y_3}{r^3} - \frac{y_3}{2r} + y_3 f(r)$$

where $f(t)$ is a real analytic function on $\mathbb{R}$. We will use the notation

$$Q = \frac{\partial P}{\partial y_3} = DP.$$

Observe that $LQ(r) = \Delta Q(r) - Q(r) = 0$ for $r \neq 0$ and $\Delta = \Delta_y$. Thus

$$D^2 Q = Q - AQ, \; A = \Delta_{y'}\,. \tag{3.1}$$

Applying D^2 to (3.1) leads to

$$D^4 Q = Q - 2AQ + A^2 Q. \tag{3.2}$$

For k a positive integer assume that

$$D^{2k} Q = \sum_{i=0}^{k} (-1)^i m_i A^i Q$$

where m_i is the i^{th} column in the k^{th} row of Pascal's Triangle. In particular, $m_0 = m_k = 1$. This clearly holds for $k \leq 2$. Then using (3.1),

$$\begin{aligned}
D^{2k+2} Q &= \sum_{i=0}^{k} (-1)^i m_i A^i (D^2 Q) \\
&= \sum_{i=0}^{k} (-1)^i m_i A^i (Q - AQ) \\
&= \sum_{i=0}^{k} (-1)^i m_i A^i (Q) + \sum_{j=1}^{k+1} (-1)^i m_{j-1} A^j (Q) \\
&= Q + \sum_{i=1}^{k} (m_i + m_{i-1}) (-1)^i A^i Q + (-1)^{k+1} A^{k+1} Q \\
&= \sum_{i=0}^{k+1} (-1)^i n_i A^i Q
\end{aligned}$$

where n_i is the i^{th} column in the $(k+1)^{\text{th}}$ row of Pascal's Triangle (note that more generally, the equation $L_c u = \Delta u + cu = 0$ would lead to a generalization of Pascal's Triangle).

For m a positive integer, we will estimate $A^m Q$ when $y_3 = 0$. Since $A = \Delta_{y'}$, we can use the formula

$$Q = Q(x', y', 0) = \frac{1}{r^3} - \frac{1}{2r} + f(r), \ r = |x' - y'|.$$

Recall that if $h = h(r)$ is a radial function, in dimension 2,

$$Ah(r) = h^{''}(r) + \frac{1}{r} h'(r).$$

It follows that for any p, $A(r^p) = p^2 r^{p-2}$ and therefore, for $j \geq 1$,

$$A^j(r^p) = p^2(p-2)^2 \cdots (p - 2(j-1))^2 r^{p-2j}. \tag{3.3}$$

Write $f(r) = \sum_{m=1}^{\infty} a_m r^m$. Then using (3.3),

$$A^j f(r) = \sum_{m=1}^{\infty} a_m m^2 (m-2)^2 \cdots (m - 2(j-1))^2 r^{m-2j}.$$

To estimate this term, we only need to consider m odd. Observe that if $2(j-1) \leq m = 2l + 1$, then

$$m^2(m-2)^2 \cdots (m - 2(j-1))^2 \leq m^{2j} \leq (2j)! e^m \leq 2^{2j} (j!)^2 \tag{3.4}$$

while if $m = 2l + 1 < 2(j-1)$, then

$$\begin{aligned} m^2(m-2)^2 \cdots (m - 2(j-1))^2 &\leq m^{2j} \leq m^{2l}((j-l)!)^2 \\ &\leq e^m (2l)!((j-l)!)^2 \\ &\leq e^m 2^{2l} (l!(j-l)!)^2 \\ &\leq (2e)^m (j!)^2. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we see that

$$|A^j f(r)| \leq C_1 2^{2j} (j!)^2 r^{-2j+1} \tag{3.6}$$

with C_1 a constant independent of j.
We have

$$A^j(r^{-3}) = 3^2 \cdot 5^2 \cdots (3 + 2(j-1))^2 r^{-2j-3} \tag{3.7}$$

and

$$A^j(r^{-1}) = 3^2 \cdot 5^2 \cdots (1 + 2(j-1))^2 r^{-2j-1}. \tag{3.8}$$

Since

$$\begin{aligned}2^j j! &\le 2\cdot 4\cdots(2+2(j-1))^2 \le 3\cdot 5\cdots(3+2(j-1))\\ &\le 2\cdot 4\cdots(4+2(j-1))^2 = 2^{j+1}(j+1)!\end{aligned}$$

it follows that

$$\frac{2^{2j}(j!)^2}{r^{2j+3}} \le A^j(r^{-3}) \le \frac{2^{2j+2}((j+1)!)^2}{r^{2j+3}}. \tag{3.9}$$

Similarily,

$$\frac{2^{2j-2}((j-1)!)^2}{r^{2j+1}} \le A^j(r^{-1}) \le \frac{2^{2j}(j!)^2}{r^{2j+1}}. \tag{3.10}$$

Recalling that

$$A^jQ = A^j(r^{-3}) - \frac{1}{2}A^j(r^{-1}) + A^jf(r),$$

(3.6), (3.7), and (3.8) imply that for some constant $C > 0$ independent of j,

$$|A^jQ - A^j(r^{-3})| \le Cr^2|A^j(r^{-3})|. \tag{3.11}$$

We can now estimate, for each $k \ge 1$,

$$D^{2k}Q = \sum_{i=0}^{k}(-1)^i m_i A^i Q.$$

It is well known that each

$$m_i = \frac{k!}{i!(k-i)!}.$$

Let k be an even integer. Then we can write

$$D^{2k}Q = Q+(m_2A^2Q-m_1AQ)+(m_4A^4Q-m_3A^3Q)+\cdots(m_kA^kQ-m_{k-1}A^{k-1}Q). \tag{3.12}$$

Because of (3.11), A^jQ can be approximated by $A^j(r^{-3})$ and the resulting error will be inconsequential as we will see at the end. Thus $D^{2k}Q$ is comparable to

$$\begin{aligned}D^{2k}Q \simeq Q &+ \big(m_2A^2(r^{-3}) - m_1A(r^{-3})\big)\\ &+\cdots+\big(m_{k-2}A^{k-2}(r^{-3}) - m_{k-3}A^{k-3}(r^{-3})\big)\\ &+\big(m_kA^k(r^{-3}) - m_{k-1}A^{k-1}(r^{-3})\big).\end{aligned} \tag{3.13}$$

We claim that for each i,

$$m_i(A^i(r^{-3}) - m_{i-1}A^{i-1}(r^{-3}) > 0. \tag{3.14}$$

To see this, observe that the coefficient of the power r^{-2i-3} of $m_iA^i(r^{-3})$ is given by

$$\frac{k!}{i!(k-i)!}3^2\cdot 5^2\cdots(3+2(i-1))^2$$

while that of $m_{i-1}A^{i-1}(r^{-3})$ is

$$\frac{k!}{(i-1)!(k-i+1)!}3^2 \cdot 5^2 \cdots (3+2(i-2))^2.$$

Since $\frac{k!}{(i-1)!(k-i+1)!} < \frac{k!}{i!(k-i)!}$ and r^{-1} has a higher power in $A^i(r^{-3})$ than in $A^{i-1}(r^{-3})$, (3.14) follows. From (3.13) and (3.14), we conclude that

$$D^{2k}Q > m_k A^k(r^{-3}) - m_{k-1}A^{k-1}(r^{-3}). \tag{3.15}$$

Note that since $Q|_{y_3=0} = \frac{1}{r^3} + f(r) > 0$ near the origin, we can neglect it. Therefore, there is a constant $C > 0$ independent of k such that

$$D^{2k}Q > C\frac{(k!)^2 2^{2k}}{r^{2k}}.$$

It is now clear why each term A^jQ can be replaced with $A^j(r^{-3})$. The rest of the argument is as before.

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