# ON THE UNIQUENESS OF SOLUTIONS FOR KAWAHARA TYPE EQUATIONS

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ABSTRACT. In this paper we extend to Kawahara type equations a uniqueness result obtained by C. E. Kenig, G. Ponce, and L. Vega for KdV type equations. We prove that, under certain decay's conditions, the null solution is the unique solution.

Keywords: Kawahara type equations; KdV type equations; uniqueness problems.

AMS Subject Classifications: 35A02; 35Q53; 35Q55.

## 1. Introduction

Consider the fifth order nonlinear differential equation

$$\partial_t u + \beta\partial_x^3 u - \gamma\partial_x^5 u = a(u)\partial_x u, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.1}$$

where $\beta, \gamma \geq 0$ are real constants such that $(\beta,\gamma) \neq (0,0)$ and $a$ is a real-valued continuous function satisfying, for some constants $\rho \geq 1$ integer and $M_1 > 0$ real,

$$|a(s)| \leq M_1(|s| + |s|^\rho). \tag{1.2}$$

In this class one finds a large set of models arising in both mathematical and physical settings. For a real constant $\sigma$ nonzero, if $a(u) = \sigma u$, the equation (1.1) corresponds to the Kawahara equation (KE), for $a(u) = \sigma u^2$, we have the modified Kawahara equation (mKE) and on a more general model, for $\rho_0 > 0$, $a(u) = \sigma u^{\rho_0}$ represents the generalized Kawahara equation (gKE). Several problems related to Waves, see [1, 8, 12, 14], Fluid Dynamics and Plasma Physics, see [10, 15], are represented by equations of Kawahara type.

In (1.1) when $\gamma = 0$ and $\beta = 1$, situation in which we get an equation of KdV type, Kenig, Ponce, and Vega [13] showed that if a solution has suitable linear exponential decay, then it is the null solution. More precisely, the result obtained by them is:

**Theorem 1.** *Let $\gamma = 0$ and $\beta = 1$ in* (1.1)*. There exist constants $\lambda_0 \geq 1$ and $C > 0$ such that, if, for $b \in \mathbb{R}$, a real solution $u \in C^1\left(\mathbb{R}, H^1(\mathbb{R})\right)$ satisfies*

$$\sup_{t\geq 0} \|u(t,\cdot)\|^2_{H^1(\mathbb{R})} \leq M_2,$$

$$\sup_{t\geq 0} \int_{\mathbb{R}} e^{|x-bt|}\, |u(t,x)|^2\, dx \leq M_3,$$

*and*

$$\sup_{t\geq 0}\int_{\mathbb{R}} e^{2\lambda|x-bt|}\left|u(t,x)\right|^2 dx<\infty,$$

*where*

$$\lambda \geq \max\left\{\lambda_0, CM_1^2\left[1+\left(M_2^{\frac{1}{2}}M_3^{\frac{1}{2}}+M_3\right)^{\frac{\rho}{2}}\right]\right\},$$

*then* $u\equiv 0$.

Theorem 1 is an extension to KdV type equations of a study done by Escauriaza, Kenig, Ponce, and Vega, see [2, 3, 4, 5, 6, 7], about unique continuation properties for the KdV and Schrödinger equations.

Our aim is to extend Theorem 1 for any $\beta,\gamma\geq 0$ such that $(\beta,\gamma)\neq(0,0)$. The case $\gamma>0$ proved to be more delicate, requiring additional hypotheses, as we will see in Section 2. Our result is:

**Theorem 2.** *There exist constants* $C_0\geq 1$ *and* $C_1>0$ *such that, for any* $C_2>0$ *fixed, if, for*

$$\lambda\geq\max\left\{C_0(1+\beta+\gamma+\beta\gamma),\frac{2^\rho C_1^\rho M_1^2(1+C_2^\rho)}{\min\{1,\beta+\gamma^2\}}\right\}\quad \text{and}\quad b\geq\lambda^5, \tag{1.3}$$

*a real solution* $u\in C^1\left(\mathbb{R},H^1(\mathbb{R})\right)$ *of (1.1) satisfies*

$$\sup_{t\in\mathbb{R}}\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}\leq M_2, \tag{1.4}$$

$$\sup_{t\in\mathbb{R}}\int_{\mathbb{R}} e^{|x-bt|}\left|u(t,x)\right|^2 dx\leq M_3, \tag{1.5}$$

*and*

$$\sup_{t\in\mathbb{R}}\int_{\mathbb{R}} e^{2\lambda|x-bt|}\left|\partial_x^k u(t,x)\right|^2 dx<\infty,\qquad k=0,1,\dots,5, \tag{1.6}$$

*for some* $M_2>0$ *and* $M_3>0$ *such that* $M_2^{\frac{1}{2}}M_3^{\frac{1}{2}}+M_3\leq C_2$, *then* $u\equiv 0$.

In the hypotheses of Theorem 1 the supremum were taken on $t\geq 0$. For Theorem 2 we took on $t\in\mathbb{R}$, the reason for this comes from the fact that, so far, we do not know a uniqueness theorem for Kawahara type equation, like in [13].

In the case $\gamma=0$ and $\beta=1$ in (1.1), Theorem 1 ensures that the result of Theorem 2 holds for any $b$ real. However, if $\gamma>0$ the condition $b\geq\lambda^5$ should be imposed for the validity of Theorem 2. To verify the necessity of the hypothesis (1.3), the solution model for (1.1) proposed by Jeffrey and Mohamad [9], can be used. Furthermore, yet in the case $\gamma=0$ and $\beta=1$, in (1.1), the Kato's theory [11], ensures that it is sufficient for the validity of Theorem 2 that, in addition to (1.4) and (1.5), (1.6) holds only for $k=0$.

The rest of this paper is organized as follows: In Section 2, we present counterexamples that show the necessity of the hypotheses on $\lambda$ and $b$ in Theorem 2, making accurate our results. In Section 3, we prove Theorem 2. In Section 4, we dedicate an appendix to the construction of the Carleman weight function that we use in the proof of Theorem 2.

## 2. Necessity of the hypotheses on $\lambda$ and $b$ in Theorem 2

In order to prove the necessity of the condition (1.3) in Theorem 2, if $\gamma > 0$, two examples using a solution model for the equation (1.1) proposed by Jeffrey and Mohamad in [9] will be presented. We will see that this solution, although being non-zero, satisfies the hypotheses of Theorem 2, except (1.3) for the parameter $\lambda$, in the first example, and $b$, in the second example.

Let $C_0$ and $C_1$ be the constants of Theorem 2. Fixed any $C_2 > 0$, take $\kappa > 0$ such that

$$\left(\frac{\kappa}{13}\right)^{\frac{1}{2}} > C_0, \tag{2.1}$$

and

$$C_0 \geq \frac{2C_1(1+C_2)}{\kappa}. \tag{2.2}$$

Let $\gamma_0 > 0$ be small enough such that

$$2C_1(1+C_2)\gamma_0 \leq C_0, \tag{2.3}$$

$$\kappa\gamma_0 + \gamma_0 + \kappa\gamma_0^2 \leq 1, \tag{2.4}$$

$$\frac{36\kappa^2\gamma_0}{169} \leq C_0^5, \tag{2.5}$$

and

$$M_2^{\frac{1}{2}} M_3^{\frac{1}{2}} + M_3 \leq C_2, \tag{2.6}$$

where

$$M_2 = \frac{11025\kappa^4\gamma_0 2^9}{114244\left(\frac{\kappa}{13}\right)^{\frac{1}{2}}} + \frac{44100\kappa^5\gamma_0 2^9}{1485172\left(\frac{\kappa}{13}\right)^{\frac{1}{2}}} \quad \text{and} \quad M_3 = \frac{11025\kappa^4\gamma_0 2^9}{28561\left[4\left(\frac{\kappa}{13}\right)^{\frac{1}{2}} - 1\right]}. \tag{2.7}$$

In these conditions, taking in (1.1) $\beta = \kappa\gamma_0$, $\gamma = \gamma_0$, and $a(s) = -\gamma_0^{\frac{1}{2}} s$, we obtain the equation

$$\partial_t u + \kappa\gamma_0 \partial_x^3 u - \gamma_0 \partial_x^5 u = -\gamma_0^{\frac{1}{2}} u \partial_x u, \tag{2.8}$$

which, according to Jeffrey and Mohamad [9], admits as solution

$$u(t,x) = \frac{105\kappa^2\gamma_0^{\frac{1}{2}}}{169} sech^4\left[\frac{1}{2}\left(\frac{\kappa}{13}\right)^{\frac{1}{2}}\left(x - \frac{36\kappa^2\gamma_0 t}{169}\right)\right]. \tag{2.9}$$

**Example 1.** *Take* $\lambda = \left(\frac{36\kappa^2\gamma_0}{169}\right)^{\frac{1}{5}}$ *and* $b = \frac{36\kappa^2\gamma_0}{169}$. *It follows from* (2.5) *that*

$$\lambda < \max\left\{C_0(1+\kappa\gamma_0+\gamma_0+\kappa\gamma_0^2), \frac{2C_1\gamma_0(1+C_2)}{\min\{1, \kappa\gamma_0+\gamma_0^2\}}\right\} \text{ and } b = \lambda^5.$$

*Considering the solution (2.9), observing that* $\operatorname{sech}(s) \leq \frac{2}{e^{|s|}}$, *for all* $s \in \mathbb{R}$, *we can use* (2.1) *and* (2.7) *to verify that*

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}} e^{|x-bt|}|u(t,x)|^2 dx \leq \frac{11025\kappa^4\gamma_0 2^8}{28561} \int_{\mathbb{R}} \frac{e^{|y|}}{e^{4\left(\frac{\kappa}{13}\right)^{\frac{1}{2}}|y|}} dy = M_3.$$

*On the other hand, since*

$$|\partial_x u(t,x)| \leq \frac{210\kappa^{\frac{5}{2}}\gamma_0^{\frac{1}{2}}}{169\sqrt{13}} \operatorname{sech}^4\left[\frac{1}{2}\left(\frac{\kappa}{13}\right)^{\frac{1}{2}}(x-bt)\right],$$

*we can use (2.7) to obtain*

$$\sup_{t\in\mathbb{R}} \|u(t,\cdot)\|^2_{H^1(\mathbb{R})} \leq M_2.$$

*Finally, to check the condition (1.6), observing that there exist constants* $\kappa_j > 0$, *for* $j = 1, \dots, 5$, *such that*

$$\left|\partial_x^j \left(\operatorname{sech}^4\left[\frac{1}{2}\left(\frac{\kappa}{13}\right)^{\frac{1}{2}}(x-bt)\right]\right)\right| \leq \kappa_j \operatorname{sech}^4\left[\frac{1}{2}\left(\frac{\kappa}{13}\right)^{\frac{1}{2}}(x-bt)\right], \quad j=1,\dots,5,$$

*from (2.1) and (2.5) we have (1.6).*

**Example 2.** *Take* $\lambda = C_0(1+\kappa\gamma_0+\gamma_0+\kappa\gamma_0^2)$ *and* $b = \frac{36\kappa^2\gamma_0}{169}$. *From* (2.2) *and* (2.3), *we have*

$$\lambda = \max\left\{C_0(1+\kappa\gamma_0+\gamma_0+\kappa\gamma_0^2), \frac{2C_1\gamma_0(1+C_2)}{\min\{1, \kappa\gamma_0+\gamma_0^2\}}\right\},$$

*and by* (2.5), *we conclude that* $b < \lambda^5$. *Furthermore, we can proceed as in Example 1 to verify that (2.9) satisfies* (1.4), (1.5), *and* (1.6).

## 3. Proof of Theorem 2

The proof of Theorem 2 is based on the method established by Kenig, Ponce, and Vega [13]. This method consists on the use of Carleman weights to construct two families of linear operators, $\{\mathcal{S}(t) : t\in\mathbb{R}\}$, with symmetric properties, and $\{\mathcal{A}(t) : t\in\mathbb{R}\}$, with anti-symmetric properties, which will be used to linearize the equation (1.1). Then a study of the properties of this operators and of its commutator together with $L^2$ conservation law enable us to prove Theorem 2.

Consider, to be chosen later, a constant $\lambda > 0$ and a function $\theta \in C^{10}(\mathbb{R}^2)$ such that $\partial^\alpha\theta$ is bounded if $|\alpha| > 0$. For a solution $u \in C^1(\mathbb{R}, H^1(\mathbb{R}))$ of (1.1), we take the function

$$f(t,x) = e^{\lambda\theta(t,x)}u(t,x). \tag{3.1}$$

Then, observing that

$$e^{\lambda\theta}(\partial_t + \beta\partial_x^3 - \gamma\partial_x^5)(e^{-\lambda\theta}f) = a(u)(\partial_x f - \lambda\partial_x\theta f),$$

and

$$e^{\lambda\theta}(\partial_t + \beta\partial_x^3 - \gamma\partial_x^5)(e^{-\lambda\theta}f) = -\lambda\partial_t\theta f + \partial_t f + \beta e^{\lambda\theta}\partial_x^3(e^{-\lambda\theta}f) - \gamma e^{\lambda\theta}\partial_x^5(e^{-\lambda\theta}f),$$

we can write

$$\partial_t f = -\left[-\lambda\partial_t\theta f + \beta e^{\lambda\theta}\partial_x^3(e^{-\lambda\theta}f) - \gamma e^{\lambda\theta}\partial_x^5(e^{-\lambda\theta}f)\right] + a(u)(\partial_x f - \lambda\partial_x\theta f). \tag{3.2}$$

Later, we will see that the condition (1.6) of Theorem 2 locates the solutions of (1.1), with respect to the spatial variable, in the Sobolev space $H^5(\mathbb{R})$. Next, we define the operators that will give a linear characterization for (1.1).

For $j \in \{1,2\}$, consider the operators

$$T_j : \mathbb{R} \longrightarrow \mathcal{L}\left(H^5(\mathbb{R}), L^2(\mathbb{R})\right), \qquad T_j(t)g = e^{\lambda\theta(t,x)}\partial_x^{2j+1}(e^{-\lambda\theta(t,x)}g), \tag{3.3}$$

and

$$T_j^* : \mathbb{R} \longrightarrow \mathcal{L}\left(H^5(\mathbb{R}), L^2(\mathbb{R})\right), \qquad T_j^*(t)g = -e^{-\lambda\theta(t,x)}\partial_x^{2j+1}(e^{\lambda\theta(t,x)}g), \tag{3.4}$$

where $\mathcal{L}\left(H^5(\mathbb{R}), L^2(\mathbb{R})\right)$ denotes the space of all bounded linear operators from $H^5(\mathbb{R})$ to $L^2(\mathbb{R})$. Integrating by parts and using Sobolev's embedding theorem, to conclude that $H^5(\mathbb{R}) \subset C_0^4(\mathbb{R})$, we have

$$\int_{\mathbb{R}} (T_j(t)g)(x)h(x)dx = \int_{\mathbb{R}} g(x)(T_j^*(t)h)(x)dx, \qquad \forall g, h \in H^5(\mathbb{R}). \tag{3.5}$$

Through the operators (3.3) and (3.4), we define

$$\begin{aligned} S_1 &= \frac{\beta(T_1 + T_1^*)}{2} \\ &= -3\beta\lambda\partial_x\theta\partial_x^2 - 3\beta\lambda\partial_x^2\theta\partial_x - \beta\lambda^3(\partial_x\theta)^3 - \beta\lambda\partial_x^3\theta, \end{aligned} \tag{3.6}$$

$$\begin{aligned} A_1 &= \frac{\beta(T_1 - T_1^*)}{2} \\ &= \beta\partial_x^3 + 3\beta\lambda^2(\partial_x\theta)^2\partial_x + 3\beta\lambda^2\partial_x\theta\partial_x^2\theta, \end{aligned} \tag{3.7}$$

$$\begin{aligned} S_2 &= -\lambda\partial_t\theta - \frac{\gamma(T_2 + T_2^*)}{2} \\ &= -\lambda\partial_t\theta + 5\gamma\lambda\partial_x\theta\partial_x^4 + 10\gamma\lambda\partial_x^2\theta\partial_x^3 + 10\gamma\lambda^3(\partial_x\theta)^3\partial_x^2 \\ &\quad +10\gamma\lambda\partial_x^3\theta\partial_x^2 + 30\gamma\lambda^3(\partial_x\theta)^2\partial_x^2\theta\partial_x + 5\gamma\lambda\partial_x^4\theta\partial_x \\ &\quad +\gamma\lambda^5(\partial_x\theta)^5 + 10\gamma\lambda^3(\partial_x\theta)^2\partial_x^3\theta + 15\gamma\lambda^3\partial_x\theta(\partial_x^2\theta)^2 + \gamma\lambda\partial_x^5\theta, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \widetilde{A_2} &= -\frac{\gamma(T_2 - T_2^*)}{2} \\ &= -\gamma\partial_x^5 - 10\gamma\lambda^2(\partial_x\theta)^2\partial_x^3 - 30\gamma\lambda^2\partial_x\theta\partial_x^2\theta\partial_x^2 - 5\gamma\lambda^4(\partial_x\theta)^4\partial_x \\ &\quad -20\gamma\lambda^2\partial_x\theta\partial_x^3\theta\partial_x - 15\gamma\lambda^2(\partial_x^2\theta)^2\partial_x - 10\gamma\lambda^4(\partial_x\theta)^3\partial_x^2\theta \\ &\quad -10\gamma\lambda^2\partial_x^2\theta\partial_x^3\theta - 5\gamma\lambda^2\partial_x\theta\partial_x^4\theta. \end{aligned} \tag{3.9}$$

Hence, taking

$$\mathcal{A}_2 = \partial_t + \widetilde{\mathcal{A}}_2, \qquad \mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2, \qquad \mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \qquad \widetilde{\mathcal{A}} = \mathcal{A}_1 + \widetilde{\mathcal{A}_2}, \tag{3.10}$$

we have

$$\mathcal{S}^* = \mathcal{S}, \qquad \mathcal{A}^* = -\mathcal{A}, \tag{3.11}$$

and

$$\begin{aligned} [\mathcal{S}, \mathcal{A}] &= \mathcal{S}\mathcal{A} - \mathcal{A}\mathcal{S} \\ &= [\mathcal{S}_1, \mathcal{A}_1] + [\mathcal{S}_1, \mathcal{A}_2] + [\mathcal{S}_2, \mathcal{A}_1] + [\mathcal{S}_2, \mathcal{A}_2], \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} [\mathcal{S}_1, \mathcal{A}_1] &= \partial_x^2(9\beta^2\lambda\partial_x^2\theta\partial_x^2) + \partial_x\left((6\beta^2\lambda\partial_x^4\theta - 18\beta^2\lambda^3(\partial_x\theta)^2\partial_x^2\theta)\partial_x\right) \\ &\quad - 3\beta^2\lambda^3(\partial_x^2\theta)^3 - 18\beta^2\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^3\theta - 3\beta^2\lambda^3(\partial_x\theta)^2\partial_x^4\theta \\ &\quad + \beta^2\lambda\partial_x^6\theta + 9\beta^2\lambda^5(\partial_x\theta)^4\partial_x^2\theta, \end{aligned} \tag{3.13}$$

$$
\begin{aligned}
[\mathcal{S}_1,\mathcal{A}_2] &= [\mathcal{S}_2,\mathcal{A}_1]\\
&= \partial_x^3\left(-15\beta\gamma\lambda\partial_x^2\theta\partial_x^3\right)+\partial_x^2\left((15\beta\gamma\lambda^3(\partial_x\theta)^2\partial_x^2\theta-20\beta\gamma\lambda\partial_x^4\theta)\partial_x^2\right)\\
&+\partial_x((60\beta\gamma\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^3\theta+15\beta\gamma\lambda^3(\partial_x\theta)^2\partial_x^4\theta+15\beta\gamma\lambda^5(\partial_x\theta)^4\partial_x^2\theta\\
&+3\beta\lambda\partial_{xt}^2\theta-8\beta\gamma\lambda\partial_x^6\theta-45\beta\gamma\lambda^3(\partial_x^2\theta)^3)\partial_x)+25\beta\gamma\lambda^3\partial_x\theta\partial_x^3\theta\partial_x^4\theta\\
&-\beta\gamma\lambda\partial_x^8\theta-60\beta\gamma\lambda^3\partial_x^2\theta(\partial_x^3\theta)^2-30\beta\gamma\lambda^3(\partial_x^2\theta)^2\partial_x^4\theta\\
&-5\beta\gamma\lambda^5(\partial_x\theta)^4\partial_x^4\theta-15\beta\gamma\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^5\theta+2\beta\gamma\lambda^3(\partial_x\theta)^2\partial_x^6\theta\\
&-15\beta\gamma\lambda^5(\partial_x\theta)^2(\partial_x^2\theta)^3-30\beta\gamma\lambda^5(\partial_x\theta)^3\partial_x^2\theta\partial_x^3\theta\\
&-15\beta\gamma\lambda^7(\partial_x\theta)^6\partial_x^2\theta+\beta\lambda\partial_{x^3t}^4\theta+3\beta\lambda^3(\partial_x\theta)^2\partial_{xt}^2\theta,
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
[\mathcal{S}_2,\mathcal{A}_2] &= \partial_x^4\left(25\gamma^2\lambda\partial_x^2\theta\partial_x^4\right)+\partial_x^3\left((50\gamma^2\lambda\partial_x^4\theta-100\gamma^2\lambda^3(\partial_x\theta)^2\partial_x^2\theta)\partial_x^3\right)\\
&+\partial_x^2((35\gamma^2\lambda\partial_x^6\theta-10\gamma\lambda\partial_{xt}^2\theta+150\gamma^2\lambda^5(\partial_x\theta)^4\partial_x^2\theta\\
&+450\gamma^2\lambda^3(\partial_x^2\theta)^3-100\gamma^2\lambda^3(\partial_x\theta)^2\partial_x^4\theta-300\gamma^2\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^3\theta)\partial_x^2)\\
&+\partial_x((10\gamma^2\lambda\partial_x^8\theta-100\gamma^2\lambda^7(\partial_x\theta)^6\partial_x^2\theta-10\gamma\lambda\partial_{x^3t}^4\theta-60\gamma\lambda^3(\partial_x\theta)^2\partial_{xt}^2\theta\\
&+400\gamma^2\lambda^5(\partial_x\theta)^3\partial_x^2\theta\partial_x^3\theta-300\gamma^2\lambda^5(\partial_x\theta)^2(\partial_x^2\theta)^3+50\gamma^2\lambda^5(\partial_x\theta)^4\partial_x^4\theta\\
&-140\gamma^2\lambda^3(\partial_x\theta)^2\partial_x^6\theta+750\gamma^2\lambda^3(\partial_x^2\theta)^2\partial_x^4\theta+1400\gamma^2\lambda^3\partial_x^2\theta(\partial_x^3\theta)^2\\
&+700\gamma^2\lambda^3(\partial_x^2\theta)^2\partial_x^4\theta-200\gamma^2\lambda^3\partial_x\theta\partial_x^3\theta\partial_x^4\theta+100\gamma^2\lambda^3(\partial_x\theta)^2\partial_x^6\theta\\
&-200\gamma^2\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^5\theta-500\gamma^2\lambda^3(\partial_x^2\theta)^2\partial_x^4\theta)\partial_x)+25\gamma^2\lambda^9(\partial_x\theta)^8\partial_x^2\theta\\
&-150\gamma^2\lambda^7(\partial_x\theta)^4(\partial_x^2\theta)^3-100\gamma^2\lambda^7(\partial_x\theta)^5\partial_x^2\theta\partial_x^3\theta+\gamma^2\lambda\partial_x^{10}\theta\\
&+\lambda\partial_t^2\theta-2\gamma\lambda\partial_{x^5t}^6\theta-255\gamma^2\lambda^5(\partial_x^2\theta)^5-10\gamma\lambda^5(\partial_x\theta)^4\partial_{xt}^2\theta\\
&+1300\gamma^2\lambda^3(\partial_x^3\theta)^2\partial_x^4\theta+725\gamma^2\lambda^3\partial_x^2\theta(\partial_x^4\theta)^2\\
&+1000\gamma^2\lambda^3\partial_x^2\theta\partial_x^3\theta\partial_x^5\theta-50\gamma^2\lambda^3\partial_x\theta\partial_x^4\theta\partial_x^5\theta-60\gamma\lambda^3\partial_x\theta\partial_x^2\theta\partial_{x^2t}^3\theta\\
&-30\gamma\lambda^3\partial_{xt}^2\theta(\partial_x^2\theta)^2-20\gamma\lambda^3(\partial_x\theta)^2\partial_{x^3t}^4\theta-40\gamma\lambda^3\partial_x\theta\partial_{xt}^2\theta\partial_x^3\theta\\
&-800\gamma^2\lambda^5(\partial_x\theta)^2\partial_x^2\theta(\partial_x^3\theta)^2-2100\gamma^2\lambda^5\partial_x\theta(\partial_x^2\theta)^3\partial_x^3\theta\\
&-60\gamma^2\lambda^3\partial_x\theta\partial_x^3\theta\partial_x^6\theta+10\gamma^2\lambda^5(\partial_x\theta)^4\partial_x^6\theta+130\gamma^2\lambda^3(\partial_x^2\theta)^2\partial_x^6\theta\\
&-40\gamma^2\lambda^3\partial_x\theta\partial_x^2\theta\partial_x^7\theta+100\gamma^2\lambda^5(\partial_x\theta)^3\partial_x^3\theta\partial_x^4\theta+100\gamma^2\lambda^5(\partial_x\theta)^3\partial_x^2\theta\partial_x^5\theta\\
&-5\gamma^2\lambda^3(\partial_x\theta)^2\partial_x^8\theta-300\gamma^2\lambda^5(\partial_x\theta)^2(\partial_x^2\theta)^2\partial_x^4\theta.
\end{aligned}
\tag{3.15}
$$

From (3.10), we can rewrite the equation (3.2) as

$$\partial_t f = -(\mathcal{S}+\widetilde{\mathcal{A}})f + F, \tag{3.16}$$

where $F = a(u)(\partial_x f - \lambda\partial_x\theta f)$. Then, by observing that $\mathcal{A} = \partial_t + \widetilde{\mathcal{A}}$, we have

$$F = \mathcal{S}f + \mathcal{A}f, \tag{3.17}$$

and

$$[\mathcal{S},\mathcal{A}] = [\mathcal{S},\widetilde{\mathcal{A}}] - [\partial_t,\mathcal{S}]. \tag{3.18}$$

As in [13], it follows from (3.16) that

$$\frac{d}{dt}\langle f,f\rangle = -2\langle \mathcal{S}f,f\rangle + 2\langle F,f\rangle, \tag{3.19}$$

and

$$\frac{d}{dt}\langle \mathcal{S}f, f\rangle = -\langle [\mathcal{S}, \mathcal{A}]f, f\rangle - 2\langle \mathcal{S}f, \mathcal{S}f\rangle + 2\langle F, \mathcal{S}f\rangle . \tag{3.20}$$

Here $\langle \cdot, \cdot\rangle$ means the usual extension to $H^s \times H^{-s}$ of $\langle \cdot, \cdot\rangle$ in $L^2 \times L^2$.

Now, we will choose our function $\theta$. Let $\varphi_0 \in C^{10}(\mathbb{R})$ be a function even, non-negative, decreasing for $r > \frac{3}{2}$, with the following properties:

$$\inf_{|r|\geq 1} |\varphi_0'(r)| > 0, \tag{3.21}$$

$$|\varphi_0'(r)| \leq 3, \qquad \forall r \geq 0, \tag{3.22}$$

$$\varphi_0''(r) = 1, \qquad \forall |r| \leq \frac{3}{2}, \tag{3.23}$$

and

$$0 < \varphi_0''(r) \leq 1, \qquad \forall r \geq 0. \tag{3.24}$$

In addition, we also require that there exists a constant $\widetilde{C_1} > 0$ such that

$$e^{\varphi_0(r)} \leq \widetilde{C_1} e^{3r}, \qquad \forall r \geq 0, \tag{3.25}$$

$$e^{\frac{-\varphi_0(r)}{6}} \leq \widetilde{C_1}\varphi_0''(r), \qquad \forall r \geq 0, \tag{3.26}$$

and

$$\left|\frac{d^k \varphi_0(r)}{dr^k}\right| \leq \widetilde{C_1}\varphi_0''(r), \qquad \forall r \geq 0, \qquad k = 3, 4, 5, 6, 7, 8, 9, 10. \tag{3.27}$$

The existence of $\varphi_0$ will be discussed in the Appendix. Then, for a constant $b > 0$, to be chosen in Claim 2, we take $\varphi = \frac{\varphi_0}{3}$ and $\theta(t, x) = \varphi(x - bt)$.

The proof of Theorem 2 will result from the next four claims. Recall that for a solution $u$ of (1.1), we take the function $f = e^{\lambda\varphi(x-bt)}u$ given in (3.1).

**Claim 1.** If for some positive constants $\lambda$ and $b$, the solution $u$ of (1.1) satisfies (1.6), then:

*(i)* $u(t, \cdot), f(t, \cdot) \in H^5(\mathbb{R}), \quad \forall t \in \mathbb{R}$.

*(ii)* $\|u(t, \cdot)\|_{L^2(\mathbb{R})} = \|u(0, \cdot)\|_{L^2(\mathbb{R})}, \quad \forall t \in \mathbb{R}$.

**Proof.** Fixed $k \in \{0, 1, 2, 3, 4, 5\}$, by using (3.25) we obtain

$$\int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} \left|\partial_x^k u(t, x)\right|^2 dx \leq \widetilde{C_1}^{\frac{2\lambda}{3}} \int_{\mathbb{R}} e^{2\lambda|x-bt|} \left|\partial_x^k u(t, x)\right|^2 dx. \tag{3.28}$$

Since $\lambda > 0$ and $\varphi$ is non-negative, it follows from (1.6) that $u(t, \cdot) \in H^5(\mathbb{R})$. On the other hand, due the derivatives of $\varphi$ are bounded, we conclude that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} \left|\partial_x^k \left(e^{\lambda\varphi(x-bt)}u(t, x)\right)\right|^2 dx \leq C \sum_{j=0}^{k} \int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} \left|\partial_x^j u(t, x)\right|^2 dx.$$

Hence, from (1.6) and (3.28), we have $f(t, \cdot) \in H^5(\mathbb{R})$.

To obtain $(ii)$, we multiply the equation (1.1) by $u$ to get

$$\partial_t\left(\frac{u^2}{2}\right)+\beta\left[\partial_x\left(u\partial_x^2u\right)-\frac{\partial_x\left((\partial_xu)^2\right)}{2}\right] \tag{3.29}$$
$$-\gamma\left[\partial_x\left(u\partial_x^4u\right)-\partial_x\left(\partial_xu\partial_x^3u\right)+\frac{\partial_x\left((\partial_x^2u)^2\right)}{2}\right]=\partial_x\left(G(u)\right),$$

where

$$G(z)=\int_0^z a(w)wdw.$$

Then, from (1.2), we have

$$|G(u)|\leq M_1(|u|^3+|u|^{\rho+2}),$$

and therefore, since $u(t,\cdot)\in C_0^4(\mathbb{R})$, by Sobolev's embedding theorem,

$$\lim_{x\to\pm\infty}G(u(t,x))=0.$$

Thus, integrating (3.29) in the spatial variable, we have

$$\frac{d}{dt}\int_{\mathbb{R}}|u(t,x)|^2\,dx=0,\qquad\forall t\in\mathbb{R}.\ \square$$

**Claim 2.** There exists a constant $C_0\geq 1$ such that if, for $\lambda\geq C_0(1+\beta+\gamma+\beta\gamma)$ and $b\geq\lambda^5$, the solution $u$ satisfies (1.6), then

$$\langle[S,A]f,f\rangle\geq\min\left\{1,\beta+\gamma^2\right\}\int_{\mathbb{R}}\varphi_0''(x-bt)\left[\lambda^3f^2+\lambda^2(\partial_xf)^2\right]dx.$$

**Proof.** For any $a_1(t),\ a_2(t)\geq 0$, we will denote $a_1(t)\lesssim a_2(t)$, if there exists a constant $C>0$, independent of $t$, such that $a_1(t)\leq Ca_2(t)$, for all $t\in\mathbb{R}$. Let $r=x-bt$, where $b\geq\lambda^5$, from (3.12), (3.13), (3.14), and (3.15) we have

$$\langle[\mathcal{S},\mathcal{A}]f,f\rangle=\langle([\mathcal{S}_1,\mathcal{A}_1]+2[\mathcal{S}_1,\mathcal{A}_2]+[\mathcal{S}_2,\mathcal{A}_2])f,f\rangle\,,$$

where,

$$\langle[\mathcal{S}_1,\mathcal{A}_1]f,f\rangle=\underbrace{9\beta^2\lambda\int_{\mathbb{R}}\varphi''(r)(\partial_x^2f)^2dx}_{(1)}-\underbrace{6\beta^2\lambda\int_{\mathbb{R}}\varphi^{(IV)}(r)(\partial_xf)^2dx}_{(2)}$$
$$\underbrace{+18\beta^2\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi''(r)(\partial_xf)^2dx}_{(3)}-\underbrace{3\beta^2\lambda^3\int_{\mathbb{R}}\varphi''(r)^3f^2dx}_{(4)}$$
$$\underbrace{-18\beta^2\lambda^3\int_{\mathbb{R}}\varphi'(r)\varphi''(r)\varphi'''(r)f^2dx}_{(5)}-\underbrace{3\beta^2\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi^{(IV)}(r)f^2dx}_{(6)}$$
$$\underbrace{+\beta^2\lambda\int_{\mathbb{R}}\varphi^{(VI)}(r)f^2dx}_{(7)}+\underbrace{9\beta^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^4\varphi''(r)f^2dx}_{(8)},$$

$$
\begin{aligned}
2\langle[\mathcal{S}_1,\mathcal{A}_2])f,f\rangle =& \underbrace{30\beta\gamma\lambda\int_{\mathbb{R}}\varphi''(r)(\partial_x^3 f)^2dx}_{(1)} + \underbrace{30\beta\gamma\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi''(r)(\partial_x^2 f)^2dx}_{(2)}\\
&-\underbrace{40\beta\gamma\lambda\int_{\mathbb{R}}\varphi^{(IV)}(r)(\partial_x^2 f)^2dx}_{(3)} - \underbrace{120\beta\gamma\lambda^3\int_{\mathbb{R}}\varphi'(r)\varphi''(r)\varphi'''(r)(\partial_x f)^2dx}_{(4)}\\
&-\underbrace{30\beta\gamma\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi^{(IV)}(r)(\partial_x f)^2dx}_{(5)} - \underbrace{30\beta\gamma\lambda^5\int_{\mathbb{R}}\varphi'(r)^4\varphi''(r)(\partial_x f)^2dx}_{(6)}\\
&+\underbrace{6\beta\lambda b\int_{\mathbb{R}}\varphi''(r)(\partial_x f)^2dx}_{(7)} + \underbrace{16\beta\gamma\lambda\int_{\mathbb{R}}\varphi^{(VI)}(r)(\partial_x f)^2dx}_{(8)}\\
&+\underbrace{90\beta\gamma\lambda^3\int_{\mathbb{R}}\varphi''(r)^3(\partial_x f)^2dx}_{(9)} + \underbrace{50\beta\gamma\lambda^3\int_{\mathbb{R}}\varphi'(r)\varphi'''(r)\varphi^{(IV)}(r)f^2dx}_{(10)}\\
&-\underbrace{2\beta\gamma\lambda\int_{\mathbb{R}}\varphi^{(VIII)}(r)f^2dx}_{(11)} - \underbrace{120\beta\gamma\lambda^3\int_{\mathbb{R}}\varphi''(r)\varphi'''(r)^2f^2dx}_{(12)}\\
&-\underbrace{60\beta\gamma\lambda^3\int_{\mathbb{R}}\varphi''(r)^2\varphi^{(IV)}(r)f^2dx}_{(13)} - \underbrace{10\beta\gamma\lambda^5\int_{\mathbb{R}}\varphi'(r)^4\varphi^{(IV)}(r)f^2dx}_{(14)}\\
&-\underbrace{30\beta\gamma\lambda^3\int_{\mathbb{R}}\varphi'(r)\varphi''(r)\varphi^{(V)}(r)f^2dx}_{(15)} + \underbrace{4\beta\gamma\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi^{(VI)}(r)f^2dx}_{(16)}\\
&-\underbrace{30\beta\gamma\lambda^5\int_{\mathbb{R}}\varphi'(r)^2\varphi''(r)^3f^2dx}_{(17)} - \underbrace{60\beta\gamma\lambda^5\int_{\mathbb{R}}\varphi'(r)^3\varphi''(r)\varphi'''(r)f^2dx}_{(18)}\\
&\underbrace{-30\beta\gamma\lambda^7\int_{\mathbb{R}}\varphi'(r)^6\varphi''(r)f^2dx}_{(19)} - \underbrace{2\beta\lambda b\int_{\mathbb{R}}\varphi^{(IV)}(r)f^2dx}_{(20)}\\
&-\underbrace{6\beta\lambda^3 b\int_{\mathbb{R}}\varphi'(r)^2\varphi''(r)f^2dx}_{(21)},
\end{aligned}
$$

$$
\begin{aligned}
\langle[\mathcal{S}_2,\mathcal{A}_2]f,f\rangle =& \underbrace{25\gamma^2\lambda\int_{\mathbb{R}}\varphi''(r)(\partial_x^4 f)^2dx}_{(1)} - \underbrace{50\gamma^2\lambda\int_{\mathbb{R}}\varphi^{(IV)}(r)(\partial_x^3 f)^2dx}_{(2)}\\
&\underbrace{+100\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi''(r)(\partial_x^3 f)^2dx}_{(3)} + \underbrace{35\gamma^2\lambda\int_{\mathbb{R}}\varphi^{(VI)}(r)(\partial_x^2 f)^2dx}_{(4)}
\end{aligned}
$$

$$\underbrace{+10\gamma\lambda b\int_{\mathbb{R}}\varphi''(r)(\partial_x^2 f)^2dx}_{(5)}+\underbrace{150\gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^4\varphi''(r)(\partial_x^2 f)^2dx}_{(6)}$$

$$\underbrace{+450\gamma^2\lambda^3\int_{\mathbb{R}}\varphi''(r)^3(\partial_x^2 f)^2dx}_{(7)}-\underbrace{100\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi^{(IV)}(r)(\partial_x^2 f)^2dx}_{(8)}$$

$$\underbrace{-300\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)\varphi''(r)\varphi'''(r)(\partial_x^2 f)^2dx}_{(9)}-\underbrace{10\gamma^2\lambda\int_{\mathbb{R}}\varphi^{(VIII)}(r)(\partial_x f)^2dx}_{(10)}$$

$$\underbrace{+100\gamma^2\lambda^7\int_{\mathbb{R}}\varphi'(r)^6\varphi''(r)(\partial_x f)^2dx}_{(11)}-\underbrace{10\gamma\lambda b\int_{\mathbb{R}}\varphi^{(IV)}(r)(\partial_x f)^2dx}_{(12)}$$

$$\underbrace{-60\gamma\lambda^3 b\int_{\mathbb{R}}\varphi'(r)^2\varphi''(r)(\partial_x f)^2dx}_{(13)}-\underbrace{400\gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^3\varphi''(r)\varphi'''(r)(\partial_x f)^2dx}_{(14)}$$

$$\underbrace{+300\gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^2\varphi''(r)^3(\partial_x f)^2dx}_{(15)}-\underbrace{50\gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^4\varphi^{(IV)}(r)(\partial_x f)^2dx}_{(16)}$$

$$\underbrace{+140\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi^{(VI)}(r)(\partial_x f)^2dx}_{(17)}-\underbrace{750\gamma^2\lambda^3\int_{\mathbb{R}}\varphi''(r)^2\varphi^{(IV)}(r)(\partial_x f)^2dx}_{(18)}$$

$$\underbrace{-1400\gamma^2\lambda^3\int_{\mathbb{R}}\varphi''(r)\varphi'''(r)^2(\partial_x f)^2dx}_{(19)}-\underbrace{700\gamma^2\lambda^3\int_{\mathbb{R}}\varphi''(r)^2\varphi^{(IV)}(r)(\partial_x f)^2dx}_{(20)}$$

$$\underbrace{+200\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)\varphi'''(r)\varphi^{(IV)}(r)(\partial_x f)^2dx}_{(21)}-\underbrace{100\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi^{(VI)}(r)(\partial_x f)^2dx}_{(22)}$$

$$\underbrace{+200\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)\varphi''(r)\varphi^{(V)}(r)(\partial_x f)^2dx}_{(23)}+\underbrace{500\gamma^2\lambda^3\int_{\mathbb{R}}\varphi''(r)^2\varphi^{(IV)}(r)(\partial_x f)^2dx}_{(24)}$$

$$\underbrace{+25\gamma^2\lambda^9\int_{\mathbb{R}}\varphi'(r)^8\varphi''(r)f^2dx}_{(25)}-\underbrace{150\gamma^2\lambda^7\int_{\mathbb{R}}\varphi'(r)^4\varphi''(r)^3f^2dx}_{(26)}$$

$$\underbrace{-100\gamma^2\lambda^7\int_{\mathbb{R}}\varphi'(r)^5\varphi''(r)\varphi'''(r)f^2dx}_{(27)}+\underbrace{\gamma^2\lambda\int_{\mathbb{R}}\varphi^{(X)}(r)f^2dx}_{(28)}$$

$$\underbrace{+\lambda b^2\int_{\mathbb{R}}\varphi''(r)f^2dx}_{(29)}+\underbrace{2\gamma\lambda b\int_{\mathbb{R}}\varphi^{(VI)}(r)f^2dx}_{(30)}-\underbrace{255\gamma^2\lambda^5\int_{\mathbb{R}}\varphi''(r)^5f^2dx}_{(31)}$$

$$\underbrace{+10\gamma\lambda^5 b\int_{\mathbb{R}}\varphi'(r)^4\varphi''(r)f^2dx}_{(32)}+\underbrace{1300\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'''(r)^2\varphi^{(IV)}(r)f^2dx}_{(33)}$$

$$\underbrace{+725\gamma^2\lambda^3\int_{\mathbb{R}}\varphi''(r)\varphi^{(IV)}(r)^2f^2dx}_{(34)}+\underbrace{1000\gamma^2\lambda^3\int_{\mathbb{R}}\varphi''(r)\varphi'''(r)\varphi^{(V)}(r)f^2dx}_{(35)}$$

$$\underbrace{-50\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)\varphi^{(IV)}(r)\varphi^{(V)}(r)f^2dx}_{(36)}+\underbrace{60\gamma\lambda^3 b\int_{\mathbb{R}}\varphi'(r)\varphi''(r)\varphi'''(r)f^2dx}_{(37)}$$

$$\underbrace{+30\gamma\lambda^3 b\int_{\mathbb{R}}\varphi''(r)^3f^2dx}_{(38)}+\underbrace{20\gamma\lambda^3 b\int_{\mathbb{R}}\varphi'(r)^2\varphi^{(IV)}(r)f^2dx}_{(39)}$$

$$\underbrace{+40\gamma\lambda^3 b\int_{\mathbb{R}}\varphi'(r)\varphi''(r)\varphi'''(r)f^2dx}_{(40)}-\underbrace{800\gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^2\varphi''(r)\varphi'''(r)^2f^2dx}_{(41)}$$

$$\underbrace{-2100\gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)\varphi''(r)^3\varphi'''(r)f^2dx}_{(42)}-\underbrace{60\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)\varphi'''(r)\varphi^{(VI)}(r)f^2dx}_{(43)}$$

$$\underbrace{+10\gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^4\varphi^{(VI)}(r)f^2dx}_{(44)}+\underbrace{130\gamma^2\lambda^3\int_{\mathbb{R}}\varphi''(r)^2\varphi^{(VI)}(r)f^2dx}_{(45)}$$

$$\underbrace{-40\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)\varphi''(r)\varphi^{(VII)}(r)f^2dx}_{(46)}+\underbrace{100\gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^3\varphi'''(r)\varphi^{(IV)}(r)f^2dx}_{(47)}$$

$$\underbrace{+100\gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^3\varphi''(r)\varphi^{(V)}(r)f^2dx}_{(48)}-\underbrace{5\gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi^{(VIII)}(r)f^2dx}_{(49)}$$

$$\underbrace{-300\gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^2\varphi''(r)^2\varphi^{(IV)}(r)f^2dx}_{(50)}.$$

First, we will analyse $\langle[\mathcal{S}_2,\mathcal{A}_2]f,f\rangle$. For $|r|\geq 1$, it follows from (3.21) that

$$\gamma^2\lambda^2\int_{|r|\geq 1}\varphi_0''(r)(\partial_x f)^2dx\lesssim\gamma^2\lambda^2\int_{|r|\geq 1}\varphi'(r)^6\varphi''(r)(\partial_x f)^2dx,$$

and therefore, there exists a constant $C_0>1$ such that for $\lambda\geq C_0$ holds

$$\gamma^2\lambda^2\int_{|r|\geq 1}\varphi_0''(r)(\partial_x f)^2dx\leq\gamma^2\lambda^7\int_{\mathbb{R}}\varphi'(r)^6\varphi''(r)(\partial_x f)^2dx. \tag{3.30}$$

For $|r|\leq 1$, we take a cut-off function $\eta\in C_0^2(\mathbb{R})$ satisfying

$$0\leq\eta(s)\leq 1,\ \forall s\in\mathbb{R},\qquad \eta(s)=1,\ \forall|s|\leq 1\quad\text{and}\quad\eta(s)=0,\ \forall|s|\geq\frac{3}{2}.$$

Thus, as $\varphi''(r) = \frac{1}{3}$, for all $|r| \leq \frac{3}{2}$, from (3.23), we get

$$\int_{\mathbb{R}} \eta(r)(\partial_x f)^2 dx = \frac{9}{2} \int_{\mathbb{R}} \varphi'(r)^2 \partial_x^2 \left(\eta(r)(\partial_x f)^2\right) dx \tag{3.31}$$
$$= \frac{9}{2} \int_{\mathbb{R}} \varphi'(r)^2 \left[\eta''(r)(\partial_x f)^2 + 4\eta'(r)\partial_x f \partial_x^2 f + 2\eta(r)(\partial_x^2 f)^2 + 2\eta(r)\partial_x f \partial_x^3 f\right] dx.$$

On the other hand, we have

$$\left|\int_{\mathbb{R}} \varphi'(r)^2 \eta'(r) \partial_x f \partial_x^2 f dx\right| \leq \int_{1\leq|r|\leq\frac{3}{2}} \left(\varphi'(r)^2 |\eta'(r)|^{\frac{1}{2}} |\partial_x f|\right) \left(|\eta'(r)|^{\frac{1}{2}} |\partial_x^2 f|\right) dx$$
$$\lesssim \int_{1\leq|r|\leq\frac{3}{2}} \varphi'(r)^4 |\eta'(r)| (\partial_x f)^2 dx + \int_{1\leq|r|\leq\frac{3}{2}} |\eta'(r)| (\partial_x^2 f)^2 dx$$
$$\lesssim \int_{\mathbb{R}} \varphi'(r)^2 \varphi''(r)^3 (\partial_x f)^2 dx + \int_{\mathbb{R}} \varphi''(r)(\partial_x^2 f)^2 dx. \tag{3.32}$$

and likewise,

$$\left|\int_{\mathbb{R}} \varphi'(r)^2 \eta''(r)(\partial_x f)^2 dx\right| \lesssim \int_{\mathbb{R}} \varphi'(r)^2 \varphi''(r)^3 (\partial_x f)^2 dx, \tag{3.33}$$

$$\left|\int_{\mathbb{R}} \varphi'(r)^2 \eta(r)(\partial_x^2 f)^2 dx\right| \lesssim \int_{\mathbb{R}} \varphi''(r)(\partial_x^2 f)^2 dx, \tag{3.34}$$

and

$$\left|\int_{\mathbb{R}} \varphi'(r)^2 \eta(r) \partial_x f \partial_x^3 f dx\right| \lesssim \int_{\mathbb{R}} \varphi'(r)^2 \varphi''(r)^3 (\partial_x f)^2 dx$$
$$+ \int_{\mathbb{R}} \varphi'(r)^2 \varphi''(r)(\partial_x^3 f)^2 dx. \tag{3.35}$$

Therefore, from (3.31), (3.32), (3.33), (3.34), and (3.35), we conclude that

$$\gamma^2\lambda^2 \int_{|r|\leq 1} \varphi_0''(r)(\partial_x f)^2 dx \leq \gamma^2\lambda^2 \int_{\mathbb{R}} \eta(r)(\partial_x f)^2 dx$$
$$\lesssim \gamma^2\lambda^2 \int_{\mathbb{R}} \varphi'(r)^2 \varphi''(r)^3 (\partial_x f)^2 dx + \gamma^2\lambda^2 \int_{\mathbb{R}} \varphi''(r)(\partial_x^2 f)^2 dx$$
$$+ \gamma^2\lambda^2 \int_{\mathbb{R}} \varphi'(r)^2 \varphi''(r)(\partial_x^3 f)^2 dx. \tag{3.36}$$

Thus, increasing $C_0$ if necessary and taking $\lambda \geq C_0(1+\gamma)$, from (3.30) and (3.36), we have

$$\gamma^2\lambda^2 \int_{\mathbb{R}} \varphi_0''(r)(\partial_x f)^2 dx \leq \gamma^2\lambda^3 \int_{\mathbb{R}} \varphi'(r)^2 \varphi''(r)(\partial_x^3 f)^2 dx + \gamma\lambda b \int_{\mathbb{R}} \varphi''(r)(\partial_x^2 f)^2 dx$$
$$+ \gamma^2\lambda^7 \int_{\mathbb{R}} \varphi'(r)^6 \varphi''(r)(\partial_x f)^2 dx + \gamma^2\lambda^5 \int_{\mathbb{R}} \varphi'(r)^2 \varphi''(r)^3 (\partial_x f)^2 dx. \tag{3.37}$$

In this way, we see that it is possible to use the terms (3), (5), (11), and (15) of $\langle[\mathcal{S}_2, \mathcal{A}_2]f, f\rangle$ to obtain an upper bound for $\gamma^2\lambda^2 \int_{\mathbb{R}} \varphi_0''(r)(\partial_x f)^2 dx$.

Now, we will seek an upper bound for $\lambda^3 \int_{\mathbb{R}} \varphi_0''(r) f^2 dx$. We begin by using the identity

$$(\partial_x f)^2 = \frac{1}{2}\partial_x^2(f^2) - f\partial_x^2 f, \tag{3.38}$$

to obtain

$$
\begin{aligned}
|(13)| &= 60\gamma\lambda^3 b\int_{\mathbb{R}} \varphi'(r)^2\varphi''(r)(\partial_x f)^2 dx\\
&= 30\gamma\lambda^3 b\int_{\mathbb{R}} \varphi'(r)^2\varphi''(r)\partial_x^2(f^2)dx - 60\gamma\lambda^3 b\int_{\mathbb{R}} \varphi'(r)^2\varphi''(r)f\partial_x^2 f dx.
\end{aligned}
$$

From (3.22), we have

$$
\begin{aligned}
&\left|60\gamma\lambda^3 b\int_{\mathbb{R}} \varphi'(r)^2\varphi''(r)f\partial_x^2 f dx\right| \leq 60\gamma\lambda^3 b\int_{\mathbb{R}} \left(\varphi''(r)^{\frac{1}{2}}|f|\right)\left(\varphi'(r)^2\varphi''(r)^{\frac{1}{2}}\left|\partial_x^2 f\right|\right)dx\\
&\leq 30\gamma\lambda^3 b\left[\frac{b}{75\gamma\lambda^2}\int_{\mathbb{R}} \varphi''(r)f^2dx + \frac{75\gamma\lambda^2}{b}\int_{\mathbb{R}} \varphi'(r)^4\varphi''(r)(\partial_x^2 f)^2dx\right]\\
&\leq 0,4\lambda b^2\int_{\mathbb{R}} \varphi''(r)f^2dx + 2250\gamma^2\lambda^5\int_{\mathbb{R}} \varphi''(r)(\partial_x^2 f)^2dx. \qquad (3.39)
\end{aligned}
$$

On the other hand, integrating by parts,

$$
\begin{aligned}
30\gamma\lambda^3 b&\int_{\mathbb{R}} \varphi'(r)^2\varphi''(r)\partial_x^2(f^2)dx = 30\gamma\lambda^3 b\int_{\mathbb{R}} \partial_x^2\left(\varphi'(r)^2\varphi''(r)\right)f^2dx\\
&= 30\gamma\lambda^3 b\int_{\mathbb{R}} \left[\varphi'(r)^2\varphi^{(IV)}(r) + 6\varphi'(r)\varphi''(r)\varphi'''(r) + 2\varphi''(r)^3\right]f^2dx, \qquad (3.40)
\end{aligned}
$$

we can use (3.27) to conclude that

$$
\left|30\gamma\lambda^3 b\int_{\mathbb{R}} \varphi'(r)^2\varphi^{(IV)}(r)f^2dx\right| \lesssim \gamma\lambda^3 b\int_{\mathbb{R}} \varphi''(r)f^2dx,
$$

$$
\left|180\gamma\lambda^3 b\int_{\mathbb{R}} \varphi'(r)\varphi''(r)\varphi'''(r)f^2dx\right| \lesssim \gamma\lambda^3 b\int_{\mathbb{R}} \varphi''(r)f^2dx,
$$

and

$$
\left|60\gamma\lambda^3 b\int_{\mathbb{R}} \varphi''(r)^3 f^2dx\right| \lesssim \gamma\lambda^3 b\int_{\mathbb{R}} \varphi''(r)f^2dx.
$$

Therefore, increasing $C_0$ if necessary and taking $\lambda \geq C_0(1+\gamma)$, from (3.40), we obtain

$$
\left|30\gamma\lambda^3 b\int_{\mathbb{R}} \varphi'(r)^2\varphi''(r)\partial_x^2(f^2)dx\right| \leq 0,1\lambda b^2\int_{\mathbb{R}} \varphi''(r)f^2dx, \qquad (3.41)
$$

and

$$
2250\gamma^2\lambda^5\int_{\mathbb{R}} \varphi''(r)(\partial_x^2 f)^2dx \leq \gamma\lambda b\int_{\mathbb{R}} \varphi''(r)(\partial_x^2 f)^2dx. \qquad (3.42)
$$

Concluding from (3.39), (3.41), and (3.42) that

$$
|(13)| \leq 0,5\lambda b^2\int_{\mathbb{R}} \varphi''(r)f^2dx + \gamma\lambda b\int_{\mathbb{R}} \varphi''(r)(\partial_x^2 f)^2dx. \qquad (3.43)
$$

By using the identity (3.38), we obtain

$$
\begin{aligned}
(12) &= -10\gamma\lambda b\int_{\mathbb{R}} \varphi^{(IV)}(r)(\partial_x f)^2dx\\
&= -5\gamma\lambda b\int_{\mathbb{R}} \varphi^{(IV)}(r)\partial_x^2(f^2)dx + 10\gamma\lambda b\int_{\mathbb{R}} \varphi^{(IV)}(r)f\partial_x^2 f dx,
\end{aligned}
$$

then, integration by parts and (3.27) yield

$$
\left|5\gamma\lambda b\int_{\mathbb{R}} \varphi^{(IV)}(r)\partial_x^2(f^2)dx\right| \leq 5\gamma\lambda b\int_{\mathbb{R}} |\varphi^{(VI)}(r)|f^2dx \lesssim \gamma\lambda b\int_{\mathbb{R}} \varphi''(r)f^2dx,
$$

and

$$\left|10\gamma\lambda b\int_{\mathbb{R}}\varphi^{(IV)}(r)f\partial_x^2 f dx\right| \leq 5\gamma\lambda b\left[\lambda\int_{\mathbb{R}}\left|\varphi^{(IV)}(r)\right|f^2dx+\frac{1}{\lambda}\int_{\mathbb{R}}\left|\varphi^{(IV)}(r)\right|(\partial_x^2 f)^2dx\right]$$
$$\lesssim \gamma\lambda b\left[\lambda\int_{\mathbb{R}}\varphi''(r)f^2dx+\frac{1}{\lambda}\int_{\mathbb{R}}\varphi''(r)(\partial_x^2 f)^2dx\right].$$

Therefore, increasing $C_0$ if necessary and taking $\lambda \geq C_0(1+\gamma)$, we have

$$|(12)| \leq 0,1\lambda b^2\int_{\mathbb{R}}\varphi''(r)f^2dx+\gamma\lambda b\int_{\mathbb{R}}\varphi^{''}(r)(\partial_x^2 f)^2dx. \tag{3.44}$$

To estimate the other terms of $\langle[\mathcal{S}_2,\mathcal{A}_2]f,f\rangle$, we can increase $C_0$, if necessary, to obtain the estimates

$$|(2)| \leq \gamma^2\lambda^3\int_{\mathbb{R}}\varphi'(r)^2\varphi''(r)(\partial_x^3 f)^2dx,$$

$$|(4)+(8)+(9)| \leq \gamma^2\lambda^5\int_{\mathbb{R}}\varphi'(r)^4\varphi''(r)(\partial_x^2 f)^2dx,$$

$$|(10)+(14)+(16)+...+(24)| \leq \gamma^2\lambda^7\int_{\mathbb{R}}\varphi'(r)^6\varphi''(r)(\partial_x f)^2dx,$$

and

$$|(26)+...+(28)+(30)+(31)+(33)+...+(37)+(39)+...+(50)| \leq 0,1\lambda b^2\int_{\mathbb{R}}\varphi''(r)f^2dx.$$

By these estimates, in addition to (3.43) and (3.44), we verified that the terms (3), (5), (6), (11), and (29) dominate the non-positive terms of $\langle[\mathcal{S}_2,\mathcal{A}_2]f,f\rangle$. Furthermore, also using (3.37), for all $\lambda \geq C_0(1+\gamma)$ and $b \geq \lambda^5$, we have

$$\begin{aligned}\langle[\mathcal{S}_2,\mathcal{A}_2]f,f\rangle &= (1)+...+(50)\\ &\geq 0,3\lambda b^2\int_{\mathbb{R}}\varphi''(r)f^2dx+\gamma^2\lambda^2\int_{\mathbb{R}}\varphi_0''(r)(\partial_x f)^2dx.\end{aligned} \tag{3.45}$$

For the terms of $2\left\langle([\mathcal{S}_1,\mathcal{A}_2]f,f\right\rangle$, we can increase $C_0$ if necessary and take $\lambda \geq C_0(1+\beta+\gamma+\beta\gamma)$ to obtain

$$(2)-|(3)| \geq 0,$$

$$(7)-|(4)+(5)+(6)+(8)| \geq 3\beta\lambda b\int_{\mathbb{R}}\varphi''(r)(\partial_x f)^2dx \geq \beta\lambda^2\int_{\mathbb{R}}\varphi_0''(r)(\partial_x f)^2dx,$$

and

$$0,1\lambda b^2\int_{\mathbb{R}}\varphi''(r)f^2dx \geq |(10)+...+(21)|.$$

Therefore, from (3.45), for all $\lambda \geq C_0(1+\beta+\gamma+\beta\gamma)$ and $b \geq \lambda^5$, we have

$$\begin{aligned}\langle(2[\mathcal{S}_1,\mathcal{A}_2]+[\mathcal{S}_2,\mathcal{A}_2])f,f\rangle &\geq 0,2\lambda b^2\int_{\mathbb{R}}\varphi''(r)f^2dx\\ &\quad+(\beta+\gamma^2)\lambda^2\int_{\mathbb{R}}\varphi_0''(r)(\partial_x f)^2dx.\end{aligned} \tag{3.46}$$

Finally, to complete the proof of claim 2, we need to estimate the portion corresponding to $\langle[\mathcal{S}_1,\mathcal{A}_1]f,f\rangle$. In fact, we can increase $C_0$, if necessary, to obtain

$$(3)-|(2)| \geq 0,$$

and

$$0,1\lambda b^2 \int_{\mathbb{R}} \varphi''(r) f^2 dx \geq \ |(4) + ... + (7)|.$$

Then, from (3.46), for all $\lambda \geq C_0(1+\beta+\gamma+\beta\gamma)$ and $b \geq \lambda^5$, we conclude that

$$\begin{aligned} \langle([\mathcal{S},\mathcal{A}])f,f\rangle &= \langle([\mathcal{S}_1,\mathcal{A}_1]+2[\mathcal{S}_1,\mathcal{A}_2]+[\mathcal{S}_2,\mathcal{A}_2])f,f\rangle \\ &\geq \lambda^3 \int_{\mathbb{R}} \varphi_0''(r) f^2 dx + (\beta+\gamma^2)\lambda^2 \int_{\mathbb{R}} \varphi_0''(r)(\partial_x f)^2 dx \\ &\geq \min\left\{1,\beta+\gamma^2\right\} \int_{\mathbb{R}} \varphi_0''(x-bt)\left[\lambda^3 f^2 + \lambda^2(\partial_x f)^2\right] dx. \quad \square \end{aligned}$$

Now, we will consider from here that for a constant $C_2>0$ fixed and

$$\lambda \geq \max\left\{C_0(1+\beta+\gamma+\beta\gamma), \frac{2^\rho C_1^\rho M_1^2(1+C_2^\rho)}{\min\left\{1,\beta+\gamma^2\right\}}\right\} \qquad \text{and} \qquad b \geq \lambda^5, \quad (3.47)$$

where $C_0$ is the constant of Claim 2 and $C_1 = \max\left\{1, 2^8\widetilde{C_1}^2\right\}$ with $\widetilde{C_1}$ given by (3.27), the solution $u$ of (1.1) satisfies (1.4), (1.5), and (1.6) for some $M_2>0$ and $M_3>0$ such that $M_2^{\frac{1}{2}}M_3^{\frac{1}{2}} + M_3 \leq C_2$.

**Claim 3.** There exists a constant $C>0$ which depends on $M_1$, $M_2$, $M_3$ and (1.6) such that

$$|\langle a(u)\left(\partial_x f - \lambda\varphi'(\cdot - bt)f\right), f\rangle| \leq C.$$

**Proof.** It follows from Sobolev's embedding theorem and (1.4) that

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq \widetilde{C} M_2^{\frac{1}{2}},$$

for some universal constant $\widetilde{C}$. On the other hand, since $u = e^{-\lambda\varphi(x-bt)}f$, we have

$$e^{\lambda\varphi(x-bt)}\partial_x u = \partial_x f - \lambda\varphi'(x-bt)f,$$

and therefore

$$\int_{\mathbb{R}} a(u)\left(\partial_x f - \lambda\varphi'(x-bt)f\right) f dx = \int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} a(u) u \partial_x u dx.$$

Integrating by parts, we obtain that

$$\int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} a(u) u \partial_x u dx = -\int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} 2\lambda\varphi'(x-bt) u^2 \alpha(u) dx,$$

where

$$\alpha(s) = \begin{cases} \dfrac{1}{s^2}\displaystyle\int_0^s a(s')s'ds', & s \neq 0, \\ 0, & s = 0. \end{cases}$$

Then, by using (1.5) and the boundedness of the functions $\alpha(u)$ and $\varphi'$, we conclude that there exists a constant $C>0$, which depend on $M_1$, $M_2$, $M_3$ and (1.6), such that

$$\left|\int_{\mathbb{R}} e^{2\lambda\varphi(x-bt)} 2\lambda\varphi'(x-bt) u^2 \alpha(u) dx\right| \leq C. \quad \square$$

**Claim 4.** The following inequality holds

$$\|a(u)\,(\partial_x f - \lambda\varphi'(\cdot - bt)f)\|^2_{L^2(\mathbb{R})} \le \frac{\min\left\{1,\beta+\gamma^2\right\}}{2}\int_{\mathbb{R}} \varphi_0''(x-bt)\left[\lambda^3 f^2 + \lambda(\partial_x f)^2\right]dx.$$

**Proof.** By Cauchy-Schwarz inequality, we have

$$\begin{aligned} e^{\frac{\varphi(x-bt)}{2}}u^2(t,x) &\le \int_{\mathbb{R}}\left|\frac{d}{dy}\left(e^{\frac{\varphi(y-bt)}{2}}u^2(t,y)\right)\right|dy \\ &\le 2\,\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}\left\|e^{\frac{\varphi(\cdot-bt)}{2}}u(t,\cdot)\right\|_{L^2(\mathbb{R})} + \left\|e^{\frac{\varphi(\cdot-bt)}{4}}u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{3.48}$$

Then, from (1.5), (1.6), and (3.25), we observe that

$$\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \le M_2^{\frac{1}{2}},$$

$$\left\|e^{\frac{\varphi(\cdot-bt)}{2}}u(t,\cdot)\right\|_{L^2(\mathbb{R})} \le \widetilde{C_1}^{\frac{1}{6}}M_3^{\frac{1}{2}},$$

and

$$\left\|e^{\frac{\varphi(\cdot-bt)}{4}}u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le \widetilde{C_1}^{\frac{1}{6}}M_3.$$

Therefore, from (3.48), we have

$$e^{\frac{\varphi(x-bt)}{2}}u^2(t,x) \le 2\widetilde{C_1}^{\frac{1}{6}}(M_2^{\frac{1}{2}}M_3^{\frac{1}{2}} + M_3) \le 2\widetilde{C_1}^{\frac{1}{6}}C_2,$$

which together with (1.2) and (3.26) yield

$$\begin{aligned} |a(u)|^2 &\le M_1^2(|u| + |u|^\rho)^2 \\ &\le 2M_1^2\left(|u|^2 e^{\frac{\varphi(x-bt)}{2}} + |u|^{2\rho}e^{\frac{\rho\varphi(x-bt)}{2}}\right)e^{\frac{-\varphi(x-bt)}{2}} \\ &\le 2M_1^2(2\widetilde{C_1}^{\frac{1}{6}}C_2 + 2^\rho\widetilde{C_1}^{\frac{\rho}{6}}C_2^\rho)\widetilde{C_1}\varphi_0''(x-bt). \end{aligned}$$

So, without loss of generality, assuming that $2\widetilde{C_1}^{\frac{1}{6}} \ge 1$, we have

$$|a(u)|^2 \le 2^{\rho+2}M_1^2\widetilde{C_1}^{\frac{\rho}{6}+1}(1+C_2^\rho)\varphi_0''(x-bt). \tag{3.49}$$

Thus, by observing that

$$\begin{aligned} |\partial_x f - \lambda\varphi'(x-bt)f|^2 &\le 2(\partial_x f)^2 + 2\lambda^2\sup_{r\in\mathbb{R}}\varphi'(r)^2 f^2 \\ &\le 2[(\partial_x f)^2 + \lambda^2 f^2], \end{aligned} \tag{3.50}$$

from (3.49) and (3.50), we obtain

$$\begin{aligned} &\|a(u)\,(\partial_x f - \lambda\varphi'(\cdot-bt)f)\|^2_{L^2(\mathbb{R})} \\ &\qquad \le \frac{2^{\rho+3}\widetilde{C_1}^{\frac{\rho}{6}+1}M_1^2(1+C_2^\rho)}{\lambda}\int_{\mathbb{R}}\varphi_0''(x-bt)\left[\lambda^3 f^2 + \lambda(\partial_x f)^2\right]dx. \end{aligned}$$

The conclusion of the proof follows from (3.47) to get

$$\frac{2^{\rho+3}\widetilde{C_1}^{\frac{\rho}{6}+1}M_1^2(1+C_2^\rho)}{\lambda} \le \frac{2^{\rho-1}C_1^\rho M_1^2(1+C_2^\rho)}{\lambda} \le \frac{\min\left\{1,\beta+\gamma^2\right\}}{2}. \qquad \square$$

Now, from Claims 1 - 4, we will show that Theorem 2 follows. We will begin obtaining the following estimate

$$\begin{aligned}&\langle[\mathcal{S},\mathcal{A}]f,f\rangle+2\langle\mathcal{S}f,\mathcal{S}f\rangle-2\langle F,\mathcal{S}f\rangle\\&\geq\langle\mathcal{S}f,\mathcal{S}f\rangle+\frac{\min\left\{1,\beta+\gamma^2\right\}}{2}\int_{\mathbb{R}}\varphi_0''(x-bt)\left[\lambda^3f^2+\lambda^2(\partial_xf)^2\right]dx.\end{aligned}\tag{3.51}$$

In fact, from (3.20), we have

$$\begin{aligned}\langle[\mathcal{S},\mathcal{A}]f,f\rangle+2\langle\mathcal{S}f,\mathcal{S}f\rangle-2\langle F,\mathcal{S}f\rangle&=-\frac{d}{dt}\langle\mathcal{S}f,f\rangle\\&=-\langle\partial_t\mathcal{S}f,f\rangle-\langle\mathcal{S}f,\partial_tf\rangle.\end{aligned}\tag{3.52}$$

On the other hand (3.16), (3.17), (3.18), Claim 2, and Claim 4 yield

$$\begin{aligned}&\langle\partial_t\mathcal{S}f,f\rangle+\langle\mathcal{S}f,\partial_tf\rangle=\langle\partial_t\mathcal{S}f,f\rangle+\left\langle\mathcal{S}f,-(\mathcal{S}+\widetilde{\mathcal{A}})f+F\right\rangle\\&=\langle[\partial_t,\mathcal{S}]f+S\partial_tf,f\rangle-\langle\mathcal{S}f,\mathcal{S}f\rangle-\left\langle\mathcal{S}f,\widetilde{\mathcal{A}}f\right\rangle+\langle\mathcal{S}f,F\rangle\\&=\langle[\partial_t,\mathcal{S}]f+S\partial_tf,f\rangle-\langle\mathcal{S}f,\mathcal{S}f\rangle-\left\langle\mathcal{S}f,\widetilde{\mathcal{A}}f\right\rangle+\langle F-\mathcal{A}f,F\rangle\\&=\langle[\partial_t,\mathcal{S}]f,f\rangle-\langle\mathcal{S}f,\mathcal{S}f\rangle-2\left\langle\mathcal{S}f,\widetilde{\mathcal{A}}f\right\rangle-\langle\mathcal{A}f,\mathcal{A}f\rangle+\langle F,F\rangle\\&=-\langle[\mathcal{S},\mathcal{A}]f,f\rangle-\langle\mathcal{S}f,\mathcal{S}f\rangle-\langle\mathcal{A}f,\mathcal{A}f\rangle+\langle F,F\rangle\\&\leq-\frac{\min\left\{1,\beta+\gamma^2\right\}}{2}\int_{\mathbb{R}}\varphi_0''(x-bt)\left[\lambda^3f^2+\lambda^2(\partial_xf)^2\right]dx-\langle\mathcal{S}f,\mathcal{S}f\rangle.\end{aligned}$$

Hence, from (3.52), we obtain (3.51).

The next step is to use the inequality (3.51) to obtain a uniform estimate for $\langle Sf,Sf\rangle$. We begin by observing that fixed a function $\eta:[a,b]\to\mathbb{R}$ of class $C^2$, we can integrate by parts and use (3.19) and (3.20) to conclude that

$$\begin{aligned}\int_a^b\eta'(t)\langle\mathcal{S}f,f\rangle dt=-\frac{1}{2}\left(\eta'\langle f,f\rangle\right)_a^b+\int_a^b\eta'(t)\langle F,f\rangle dt\\+\frac{1}{2}\int_a^b\eta''(t)\langle f,f\rangle dt,\end{aligned}\tag{3.53}$$

and

$$\begin{aligned}\int_a^b\eta'(t)\langle\mathcal{S}f,f\rangle dt=\left(\eta\langle\mathcal{S}f,f\rangle\right)_a^b+\int_a^b\eta(t)\langle[\mathcal{S},\mathcal{A}]f,f\rangle dt\\+2\int_a^b\eta(t)\langle\mathcal{S}f,\mathcal{S}f\rangle dt-2\int_a^b\eta(t)\langle F,\mathcal{S}f\rangle dt.\end{aligned}\tag{3.54}$$

Then, for each $n\in\mathbb{N}$, defining the functions

$$\eta_n:[n-\frac{1}{2},n+\frac{1}{2}]\to\mathbb{R},\qquad\eta_n(t)=\sin\pi(t-n+\frac{1}{2}),$$

from (3.51) and (3.54) we have

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n'(t)\langle\mathcal{S}f(t,\cdot),f(t,\cdot)\rangle dt\geq\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\eta_n(t)\|\mathcal{S}f(t,\cdot)\|_{L^2(\mathbb{R})}^2dt.\tag{3.55}$$

On the other hand $\eta_n$, $\eta_n'$, and $\eta_n''$ can be bounded by a constant that does not depend on $n$. Therefore, by using (1.6), (3.25), and Claim 3, we conclude that the right-hand side of (3.53) is bounded above by a constant $C_3 > 0$ independent of $n$. Thus, from (3.55), we obtain

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \eta_n(t) \left\| \mathcal{S} f(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 dt \leq C_3, \qquad \forall n \in \mathbb{N}.$$

Hence there exists a sequence of times $T_n \in [n, n + \frac{1}{4}]$ such that

$$\frac{\eta_n(T_n)}{4} \left\| \mathcal{S} f(T_n, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \int_n^{n+\frac{1}{4}} \eta_n(t) \left\| \mathcal{S} f(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 dt \leq C_3, \qquad \forall n \in \mathbb{N},$$

and observing that

$$\eta_n(T_n) \geq \frac{\sqrt{2}}{2}, \qquad \forall n \in \mathbb{N},$$

we obtain

$$\left\| \mathcal{S} f(T_n, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \frac{8C_3}{\sqrt{2}}, \qquad \forall n \in \mathbb{N}. \tag{3.56}$$

The last step is to obtain a global space-time estimate for $u$ and use the $L^2$ conservation law to conclude that the solution $u$ vanishes identically. For this, let $\eta : [0, \infty) \to \mathbb{R}$ a function of class $C^2$ positive such that $\eta(0) = 0$ and $\eta(t) = 1$, if $t \geq 1$. By this choice of $\eta$, working in the interval $[0, T_n]$, we can use (1.6), (3.25), and Claim 3 to see that the right-hand side of (3.53) is bounded by a constant $C_4$ independent of $n$. For the right-hand side of (3.54), we use (3.51) to conclude that

$$\int_0^{T_n} \eta'(t) \left\langle \mathcal{S} f(t, \cdot), f(t, \cdot) \right\rangle dt \geq \left\langle \mathcal{S} f(T_n, \cdot), f(T_n, \cdot) \right\rangle$$
$$+ \frac{\min\left\{1, \beta + \gamma^2\right\}}{2} \int_0^{T_n} \eta \int_{\mathbb{R}} \varphi_0''(x - bt) \left[\lambda^3 f^2 + \lambda^2 (\partial_x f)^2\right] dx dt.$$

Thus,

$$C_4 - \left\langle \mathcal{S} f(T_n, \cdot), f(T_n, \cdot) \right\rangle \geq \frac{\min\left\{1, \beta + \gamma^2\right\}}{2} \int_0^{T_n} \eta \int_{\mathbb{R}} \varphi_0''(x - bt) \left[\lambda^3 f^2 + \lambda^2 (\partial_x f)^2\right] dx dt.$$

It follows from Cauchy-Schwarz inequality, (1.6), (3.25), and (3.56) that

$$\sup_{n \in \mathbb{N}} \left| \left\langle \mathcal{S} f(T_n, \cdot), f(T_n, \cdot) \right\rangle \right| < \infty,$$

therefore, there exists a constant $C_5$ independent of $n$ such that

$$\int_0^{T_n} \eta(t) \int_{\mathbb{R}} \varphi_0''(x - bt) \left[\lambda^3 f^2 + \lambda^2 (\partial_x f)^2\right] dx dt \leq C_5, \qquad \forall n \in \mathbb{N}.$$

By letting $T_n \longrightarrow \infty$, we conclude that

$$\int_1^{\infty} \int_{\mathbb{R}} \varphi_0''(x - bt) \left[\lambda^3 f^2 + \lambda^2 (\partial_x f)^2\right] dx dt < \infty,$$

and as a consequence

$$\int_1^{\infty} \int_{\mathbb{R}} \varphi_0''(x - bt) e^{\lambda \varphi_0(x - bt)} \left| u(t, x) \right|^2 dx dt < \infty.$$

Since $\lambda \geq 1$, it follows from (3.26) that

$$\inf_{r \in \mathbb{R}} \varphi_0''(r) e^{\lambda \varphi_0(r)} > 0.$$

Thus,

$$\int_1^\infty \int_{\mathbb{R}} |u(t,x)|^2\, dxdt < \infty,$$

so the $L^2$ conservation law proposed by Claim 1 implies that $u \equiv 0$. $\square$

## 4. Appendix

Fixed $\epsilon \in (0, \frac{1}{8})$, consider the following function

$$\phi_0(r) = \begin{cases} 1, & r \in [0, \frac{3}{2} + 2\epsilon], \\ ar + b, & r \in [\frac{3}{2} + 2\epsilon, 2 - 2\epsilon], \\ \frac{\ln 2}{4r(\ln r)^2}, & r \in [2 - 2\epsilon, +\infty), \end{cases} \tag{4.1}$$

where $a, b \in \mathbb{R}$ such that

$$\begin{cases} a(\frac{3}{2} + 2\epsilon) + b = 1, \\ a(2 - 2\epsilon) + b = \frac{\ln 2}{4(2-2\epsilon)[\ln(2-2\epsilon)]^2}, \end{cases} \tag{4.2}$$

that is, $\phi_0 \in C([0, +\infty))$.

By setting $\phi(r) = \int_0^r \int_0^s \phi_0(t)dtds$, we have

$$\phi(r) = \begin{cases} \frac{r^2}{2}, & r \in [0, \frac{3}{2} + 2\epsilon], \\ \frac{ar^3}{6} + \frac{br^2}{2} + c_1 r + c_2, & r \in [\frac{3}{2} + 2\epsilon, 2 - 2\epsilon], \\ -\int_{2-2\epsilon}^r \frac{\ln 2}{4\ln s} ds + c_3 r + c_4, & r \in [2 - 2\epsilon, +\infty), \end{cases} \tag{4.3}$$

where the constants $c_1$, $c_2$, $c_3$, and $c_4$ are

$$c_1 = -\frac{a}{2}(\frac{3}{2} + 2\epsilon)^2 - b(\frac{3}{2} + 2\epsilon) + \frac{3}{2} + 2\epsilon,$$

$$c_2 = -\frac{a}{6}(\frac{3}{2} + 2\epsilon)^3 - \frac{b}{2}(\frac{3}{2} + 2\epsilon)^2 + \frac{1}{2}(\frac{3}{2} + 2\epsilon)^2 - c_1(\frac{3}{2} + 2\epsilon),$$

$$c_3 = \frac{a}{2}(2 - 2\epsilon)^2 + b(2 - 2\epsilon) - \frac{a}{2}(\frac{3}{2} + 2\epsilon)^2 - b(\frac{3}{2} + 2\epsilon) + \frac{3}{2} + 2\epsilon + \frac{\ln 2}{4\ln(2 - 2\epsilon)},$$

and

$$c_4 = \frac{a}{6}(2 - 2\epsilon)^3 + \frac{b}{2}(2 - 2\epsilon)^2 + c_1(2 - 2\epsilon) + c_2 - c_3(2 - 2\epsilon).$$

By taking $\epsilon = 0$, we obtain

$$c_3 = 2 + \frac{1}{32\ln 2} \qquad \text{and} \qquad |\phi'(r)| \le 2 + \frac{1}{32\ln 2}, \quad \forall r \ge 0,$$

then, by continuity we can take a $\epsilon > 0$ small enough such that

$$2 < c_3 < 3 \qquad \text{and} \qquad |\phi'(r)| < 3, \quad \forall r \ge 0. \tag{4.4}$$

Hence, defining $\phi(r) = \phi(-r)$, for all $r < 0$, we conclude that $\phi \in C^2(\mathbb{R})$ is even and non-negative. On the other hand, by taking $\psi_\epsilon \in C_0^{10}(\mathbb{R})$ such that

$$\psi_\epsilon \ge 0, \quad \psi_\epsilon(x) = \psi_\epsilon(-x), \quad \text{supp}(\psi_\epsilon) \subset [-\epsilon, \epsilon] \quad \text{and} \quad \int_{\mathbb{R}} \psi_\epsilon(x)dx = 1,$$

we have $\varphi_0 = \phi * \psi_\epsilon \in C^{10}(\mathbb{R})$ also even and non-negative. Furthermore, from (4.1), we observe that $\phi'' > 0$ and therefore, $\phi'$ is a increasing function, thus for $r \ge 1$ and $y \in [-\epsilon, \epsilon]$, we have $r - y \ge 1 - \epsilon$ and

$$\varphi_0'(r) = \phi' * \psi_\epsilon(r) = \int_{-\epsilon}^{\epsilon} \phi'(r - y)\psi_\epsilon(y)dy \ge \phi'(1 - \epsilon) = 1 - \epsilon.$$

On the other hand, for $r \leq -1$ and $y \in [-\epsilon, \epsilon]$, it follows that $r - y \leq -1 + \epsilon$ and

$$\varphi_0'(r) = \phi' * \psi_\epsilon(r) = \int_{-\epsilon}^{\epsilon} \phi'(r-y)\psi_\epsilon(y)dy \leq \phi'(-1+\epsilon) = -1+\epsilon,$$

whence we conclude that $\inf_{|r|\geq 1} |\varphi_0'(r)| > 0$. From (4.4) we have $|\varphi_0'(r)| \leq 3$, for all $r \geq 0$. The properties $(iii)$, $(iv)$, and $(v)$ required for $\varphi_0$ are obtained by the properties of $\phi_0$. To obtain (3.25) we observe that, for $r \geq 2$ and $y \in [-\epsilon, \epsilon]$, it follows that $r - y > 2 - 2\epsilon$, thus from (4.3) and (4.4), we have

$$\varphi_0(r) = \phi * \psi_\epsilon(r) = \int_{-\epsilon}^{\epsilon} \phi(r-y)\psi_\epsilon(y)dy \leq \int_{-\epsilon}^{\epsilon} (3r + c_4)\psi_\epsilon(y)dy = 3r + c_4,$$

and therefore $e^{\varphi_0(r)} \leq e^{c_4} e^{3r}$, for all $r \geq 2$. For (3.25) is sufficient to obtain that

$$\lim_{r\to+\infty} \varphi_0''(r) e^{\frac{\varphi_0(r)}{6}} = +\infty.$$

In fact, if $r \geq 2$ and $y \in [-\epsilon, \epsilon]$, then

$$\begin{aligned}\varphi_0''(r) = \phi'' * \psi_\epsilon(r) &= \int_{-\epsilon}^{\epsilon} \frac{\ln 2}{4(r-y)[\ln(r-y)]^2}\psi_\epsilon(y)dy \\ &\geq \frac{\ln 2}{4(r+\epsilon)[\ln(r+\epsilon)]^2},\end{aligned} \tag{4.5}$$

moreover, remembering that $c_3 > 2$, by (4.4), we have

$$\varphi_0(r) \geq \int_{-\epsilon}^{\epsilon} \left[ -\int_{2-2\epsilon}^{r-y} \frac{\ln 2}{4\ln s} ds + 2(r-y) + c_4 \right] \psi_\epsilon(y)dy. \tag{4.6}$$

Thus, as

$$2r - \int_{2-2\epsilon}^{r} \frac{\ln 2}{4\ln s} ds \geq r, \qquad \forall r \geq 2 - 2\epsilon,$$

we can use (4.6) to obtain

$$\varphi_0(r) \geq r + c_4, \qquad \forall r \geq 2. \tag{4.7}$$

Therefore, from (4.5) and (4.7), we have

$$\varphi_0''(r) e^{\frac{\varphi_0(r)}{6}} \geq \frac{e^{(r+c_4)/6}\ln 2}{4(r+\epsilon)[\ln{(r+\epsilon)}]^2}, \qquad \forall r \geq 2,$$

and hence

$$\lim_{r\to+\infty} \varphi_0''(r) e^{\frac{\varphi_0(r)}{6}} = +\infty.$$

Finally, to obtain (3.27) is sufficient to observe that there exists a constant $\widetilde{C_1} > 0$ such that

$$|\phi_0^{(k-2)}(r)| \leq \widetilde{C_1}\phi_0(r), \qquad \forall r \geq 2, \quad \forall k \in \{3, \dots, 10\},$$

and furthermore

$$\varphi_0^{(k)}(r) = \int_{-\epsilon}^{\epsilon} \phi_0^{(k-2)}(r-y)\psi_\epsilon(y)dy, \qquad \forall r \geq 2, \quad \forall k \in \{3, \dots, 10\}.$$


## Acknowledgement

Marcos A. de Farias was supported by a grant from CNPq, Brazil.

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