GLOBAL L^q GEVREY FUNCTIONS, PALEY-WEINER THEOREMS, AND THE FBI TRANSFORM

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Abstract. The goal of this paper is to develop machinery to explore the global behavior of solutions to partial differential equations. To that end, we analyze the FBI and Fourier transforms as possible tools in this endeavor, particularly with respect to the global L^q Gevrey classes which were designed to capture exponential decay of the $\Box_b$-heat kernel on suitable CR manifolds [BR13]. In this paper, we characterize the global behavior and exponential decay of the FBI transform. Additionally, we show that the Fourier transform has significant problems as tool for global analysis in L^p spaces. This is most strongly demonstrated by our analysis of Salem's example in the context of global L^q Gevrey functions. For each $q > 2$, Salem constructed an example of a positive, compactly supported Radon measure that is supported on a set of Hausdorff dimension α, $0 < \alpha < 1$, and whose Fourier transform is an element of $L^q(\mathbb{R})$ when $q > 2/\alpha$. We show that his construction is in a global L^q Gevrey space. As a technical, yet useful result in its own right, we extend the inversion formula for the FBI transform to the elements in the dual space of the L^q Gevrey function classes.

1. Introduction

The goal of this paper is to explore the relationship of global L^q Gevrey classes with the FBI and Fourier transforms. In particular, we present a characterization in terms of the global behavior and exponential decay of the FBI transform. Our motivation is to build tools that enable an investigation of global properties of solutions to partial differential equations. For example, Boggess and Raich [BR13] used an early version of the global L^q Gevrey classes to capture a very particular type of exponential decay of the $\Box_b$-heat kernel on a class of unbounded CR manifolds of finite type.

One of the major focuses of this paper is to establish the deficiency of the Fourier transform as a tool for global analysis in L^p spaces and offer the Fourier-Bros-Iagolnitzer (FBI) transform as a replacement. The FBI transform is a nonlinear Fourier transform that has been used to characterize analyticity. See [BCH08, Chapter V.2] and [Chr97] for background. For $y \in \mathbb{R}^d$, set

$$\langle y \rangle = \sqrt{1 + |y|^2}$$

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and define the function $\alpha(x,\xi)$ and the form ω by

$$\begin{aligned}\omega &= dx_1 \wedge \cdots \wedge dx_d \wedge d(\xi_1 + ix_1\langle\xi\rangle) \wedge \cdots \wedge d(\xi_d + ix_d\langle\xi\rangle) \\ &= \alpha(x,\xi)\, dx^1 \wedge \cdots \wedge dx_d \wedge d\xi_1 \wedge \cdots \wedge d\xi_d\end{aligned}$$

where $\xi \in \Gamma$, and $\Gamma \subset \mathbb{C}^d$ is a conic neighborhood of $\mathbb{R}^d$ in which $\langle\cdot\rangle$ is a holomorphic function. For a function $u \in L^p(\mathbb{R}^d)$, define the *FBI transform* of u by

$$\mathcal{F}\, u(x,\xi) = \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi - \langle\xi\rangle(x-y)^2} u(y)\alpha(x-y,\xi)\, dy. \tag{1}$$

The FBI transform first appeared in the work of Bros and Iagolnitzer [BI73] to study local analyticity and later was shown to be the right tool to study microlocal (hypo) regularity among many function classes, including (real) analytic, Gevrey, Denjoy-Carleman, and C^∞ (see [BCT83, BH12, Bon77, Chr97, HM, Hör90, Sjö82]) and to study the local and microlocal regularity of CR functions, solutions of more general vector fields, and even first order nonlinear partial differential equations (see [AB12, AH10, AH15, Asa95, BT82, BP09, Ber10, BH12, Ber10, BX15, EG03, HT92, HM] and references therein). There are several very powerful and useful FBI inversion formulas, see for instance [BCT, BCH, BH, CT]. We use a version proved by Christ in [Chr97, Lemma 2.1]. Namely, for $u \in \mathcal{E}'(\mathbb{R}^d)$. define

$$u_\epsilon(x) = (2\pi)^{-d} \int_{|\xi|\leq\epsilon^{-1}} \mathcal{F}\, u(x,\xi) d\xi. \tag{2}$$

Christ proves that $u_\epsilon \to u$ in $\mathcal{E}'(\mathbb{R}^d)$ as $\epsilon \to 0$. In Theorem 2.1 below, we show that the inversion formula holds on a much wider class of (ultra)distributions.

To capture exponential decay of a function via a Fourier or FBI transform, one must recognize that these transforms trade decay for smoothness, and the function classes that capture exponential decay of the form $\exp(-a|t|^{1/\beta})$ for $\beta > 0$ are the global L^q-Gevrey classes.

Definition 1.1. Let $\Omega \subset \mathbb{R}^d$. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ of nonnegative integers, positive constants $A, \beta > 0$ and $1 \leq q \leq \infty$, define the seminorm $\varrho_{\alpha,A,\Omega,q,\beta} : C^\infty(\Omega) \to [0,\infty)$

$$\varrho_{\alpha,A,\Omega,q,\beta}(g) = \varrho_\alpha(g) = \frac{\|D^\alpha g\|_{L^q(\Omega)}}{A^{|\alpha|}|\alpha|^{|\alpha|\beta}}.$$

We suppress as many indices for the ϱ_α as possible.

Let $1 \leq q \leq \infty$ and $\beta \geq 0$. A function $g \in C^\infty(\Omega)$ is said to satisfy *global L^q Gevrey estimates of order β* if there exist constants $A, C > 0$ so that for every d-tuple of nonnegative integers α

$$\|D^\alpha g\|_{L^q(\Omega)} \leq CA^{|\alpha|}|\alpha|^{|\alpha|\beta},$$

that is, $\varrho_\alpha(g) \leq C$. We say that a function that satisfies global L^q Gevrey estimates of order β is a *global L^q Gevrey function of order β*. For a fixed $A > 0$, we set

$$\mathcal{G}^{q,\beta}_A(\Omega) = \left\{ g \in C^\infty(\Omega) : \{\varrho_{\alpha,A,\Omega,q,\beta}(g)\}_{|\alpha|\geq 0} \in \ell^q(\mathbb{Z}^d_{\geq 0}) \right\}$$

and

$$\mathcal{G}^{q,\beta}(\Omega) = \bigcup_{A>0} \mathcal{G}^{q,\beta}_A(\Omega).$$

It turns out that we can actually extend (1) to $\mathcal{G}^{q,\beta}(\mathbb{R}^d)'$, the dual space of $\mathcal{G}^{q,\beta}(\mathbb{R}^d)$ (see (6) below).

Global L^q Gevrey functions arose naturally in the work of Boggess and Raich [BR13] in the context of recovering exponential decay of the $\Box_b$-heat kernel on polynomial models in $\mathbb{C}^2$. The paradigm that underlies the function classes is that the Fourier transform interchanges smoothness and decay. Exponential decay, in this case, decay of the form $e^{-a|t|^{1/\beta}}$ for some $\beta \geq 1$ is very quantitative form of decay (see, for example, Proposition 2.3), so capturing this decay via the Fourier transform therefore requires very quantitative estimates of the smoothness of the transform. They were quickly led to the necessity of L^∞ estimates and sufficiency of the L^1 estimates. Adwan, Hoepfner and Raich [AHR17] continued the investigation for L^q estimates but were unable to characterize decay via the Fourier transform. We answer the question in this paper – the Fourier transform is a deficient tool for characterizing global L^q Gevrey function classes via exponential decay of functions. The difficulty is subtle because the very smooth L^q Gevrey functions may lack enough decay for the Fourier transform to even be a function. For example, we show the existence of global L^q Gevrey functions whose Fourier transforms are (of course) tempered distributions but fail to be L^1_{loc}. Similarly, exponentially decaying functions that are not smooth will have transforms that are smooth but may have so little decay that they fail to be in any L^q class.

Now that we have defined the fundamental objects with which we are working, we can list the major objectives of the paper. They are 1) To characterize the classes of global L^q Gevrey functions within the context of the FBI transform; 2) To explore the strengths and limitations of using the Fourier transform to analyze global L^q Gevrey functions; and 3) To present the construction of a positive, compactly supported Radon measure whose Fourier transform is a global L^q Gevrey function. Additionally, we prove a technical, but very useful result: the classical inversion formula for the FBI transform extends to $\mathcal{G}^{q,\beta}(\mathbb{R}^d)'$. The extension is significant because the L^q Gevrey function classes are *global* objects and our results are global whereas previous results were local. We also express the dual space from [AHR17] into a user-friendly form in Proposition 2.2.

The definition of the global L^q Gevrey classes in its present state was formulated by Adwan, Hoepfner, and Raich [AHR17] where the authors developed the function theory, presented examples, and used them to build almost analytic extensions and approximate solutions of first order linear partial differential equations.

The main positive result in this paper is that Fourier–Bros–Iagolnitzer transform or FBI transform is an extremely satisfying way of characterizing the global L^q Gevrey function classes.

Theorem 1.2. *Let $\beta > \frac{1}{2}$ and $A > 0$. Then there exist positive constants A_0, a, and c that do not depend on u, and a positive constant C so that for any multiindex $J \in \mathbb{N}_0^d$, and any r satisfying $q \leq r \leq \infty$, $\mathcal{F}u(\cdot,\xi) \in \mathcal{G}^{r,\beta}_{A_0}(\mathbb{R}^d)$ and satisfies the estimates*

$$\|D^J_x \mathcal{F}u(x,\xi)\|_{L^r(\mathbb{R}^d)} \leq CA_0^{|J|}|J|^{|J|\beta}e^{-\frac{1}{c}M(a|\xi|)} \tag{3}$$

for any $u \in \mathcal{G}^{q,\beta}_A(\mathbb{R}^d)$. Conversely, to each $A_0 > 0$, there exists $A = A(A_0)$ so that for any $u \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)'$ such that $\mathcal{F}u(\cdot,\xi) \in \mathcal{G}^{q,\beta}_{A_0}(\mathbb{R}^d)$ and (3) holds, then u is a function and $u \in \mathcal{G}^{q,\beta}_A(\mathbb{R}^d)$.

In Theorem 1.2, the function $M(t)$ is the *associated function* to the sequence $(j^{j\beta})_{j\in\mathbb{N}_0}$ and it is defined by

$$M(t) = \sup_j \log\Big(\frac{t^j}{j^{j\beta}}\Big). \tag{4}$$

Theorem 1.2 shows that the FBI transform is the right tool to study regularity on a global scale, that is, it can capture global behavior. The difference between the Fourier and FBI transforms is quite dramatic. For example, there is no result corresponding estimate like (3) for the Fourier transform. There are several striking features about (3). First, in the previous applications of the FBI transform, the analogous estimate is written with an L^∞ norm instead of L^q norm. It turns out that in this context, the L^q estimate is a better estimate than the L^∞ estimate. The second global feature of the estimate (3) is the presence of global L^q Gevrey type constants in $A_0^{|J|}|J|^{|J|\beta}$. It is because of Theorem 1.2 that the FBI transform has manifold applications to the global analysis of partial differential equations. Our exploration of this topic begins in [HR] in which we develop the theory of microglobal analysis, the microglobal wavefront set, the global characteristic set, and present characterizations of global functions and their duals in terms of boundary values/extensions in a wedge.

The next theorem describes the postive and negative results that we establish regarding the Fourier transform and exponential decay. Recall that the Fourier transform of a function f is given by $\hat f(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx$, and the inverse transform (when it exists) is given by $\check f(x) = \frac{1}{(2\pi)^d}\hat f(-\xi)$. Also, we denote the set of exponentially decaying functions by

$$\begin{aligned}\operatorname{Exp}_\beta(\mathbb{R}^d) = \{f:\mathbb{R}^d\to\mathbb{C} \text{ such that } |f(x)| \le Ce^{-a|x|^{1/\beta}} \\ \text{for some } C, a>0 \text{ and } |x|\ge R \text{ for some } R>0\}.\end{aligned} \tag{5}$$

Theorem 1.3. *1. If $1\le q\le\infty$, there exists a function $f_q(x)\in\operatorname{Exp}_\beta(\mathbb{R})$ with compact support so that $\hat f_q \notin L^q(\mathbb{R})$ and consequently not in $\mathcal{G}^{q,\beta}(\mathbb{R})$.*
2. For any $\beta\ge 0$, $\hat f\in\operatorname{Exp}_\beta(\mathbb{R}^d)$ for every $f\in\mathcal{G}^{1,\beta}(\mathbb{R}^d)$.
3. If $f\in\operatorname{Exp}_\beta(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, then $\hat f\in\mathcal{G}^{q,\beta}(\mathbb{R})$ for any $2\le q\le\infty$.
4. For $2\le q\le\infty$, there exists $\hat f_q\in\mathcal{G}^{q,\beta}(\mathbb{R}^d)$ so that $f\notin\operatorname{Exp}_\beta(\mathbb{R}^d)$. In fact, if $q>2$, there exist examples for which f is not a function.

For example, Salem showed that for $q>2$ and $0<\alpha<1$ satisfying $q>\frac{2}{\alpha}$, there exists a perfect set $K\subset[0,1]$ of Hausdorff dimension α and a measure μ supported on K so that $\hat\mu\in L^q(\mathbb{R})$ [Sal51]. Certainly, the compact support of K, and hence of μ, means that $\hat\mu$ will be an entire function described by the Paley-Weiner Theorem. What the Paley-Weiner Theorem does not specify, however, are any integrability conditions for $\hat\mu$, and that is the crux of the Salem example. It turns out that Salem's example is striking in the global L^q Gevrey context as well.

Theorem 1.4. *Let $q>2$. There exists a positive Radon measure μ on $\mathbb{R}^d$ so that*
i. $\operatorname{supp}\mu\subset[0,1]^d$;
ii. The Hausdorff dimension of $\operatorname{supp}\mu$ *is strictly less than d;*
iii. $\hat\mu\in\mathcal{G}_1^{q,0}(\mathbb{R}^d)$. In particular, there exists a constant $C>0$ so that for any multiindex α,

$$\|D^\alpha\hat\mu\|_{L^q(\Omega)}\le C_A.$$

The outline of the paper is as follows: In Section 2, we define our notation and state the remainder of our results. In Section 3, we prove Theorem 1.2, which characterizes the global L^q Gevrey function classes in terms of the FBI transform. In Section 4, we prove Theorem 1.3, our result discussing the relationship of the Fourier transform with global L^q Gevrey functions and exponential decay. In Section 5, we prove Theorem 1.4, our analysis of Salem's example. The global FBI inversion formula is proved in Section 6. We conclude with several appendices – the first suggests that the failure of Hausdorff-Young is catastrophic for the Fourier transform to characterize exponential decay of functions via the global L^q Gevrey classes. The second appendix contains a computation bounding the size of derivatives of a Gaussian. The final appendix contains several technical computations that we use in the proof of Theorem 2.1.

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2. Definitions and Basic Results

2.1. **Definitions.** For a distribution $u \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)'$, define the *FBI transform* of u by

$$\mathcal{F}\, u(x,\xi) = \big\langle u, e^{i(x-\cdot)\cdot\xi - \langle\xi\rangle(x-\cdot)^2}\alpha(x-\cdot,\xi)\big\rangle \tag{6}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing of a function in $\mathcal{G}^{q,\beta}(\mathbb{R}^d)$ with an element in its dual. The function $\mathcal{F}\, u$ is well defined for $u \in \mathcal{G}^{q,\beta}_A(\mathbb{R}^d)'$ when $\beta > \frac{1}{2}$ since the Gaussian is in $\mathcal{G}^{q,\frac{1}{2}}(\mathbb{R}^d)$ [AHR17, §4.1]. For a sequence $M_j \geq 0$, define its associated function

$$M(t) = \sup_j \log\Big(\frac{M_0 t^j}{M_j}\Big). \tag{7}$$

For the remainder of this paper, set $M_0 = 1$ and $M_j = j^{j\beta}$. In this case, we use Proposition 2.3 and observe

$$e^{M(t)} = e^{\sup_{j\geq 0}\log(\frac{t^j}{j^{j\beta}})} = \sup_{j\geq 0} e^{\log(\frac{t^j}{j^{j\beta}})} = \sup_{j\geq 0}\frac{t^j}{j^{j\beta}} = \Big[\inf_{j\geq 0}\frac{j^{j\beta}}{t^j}\Big]^{-1} \sim e^{\frac{\beta}{e}t^{1/\beta}}.$$

2.2. **Inversion of the FBI Transform.** Our final theorem establishes the inversion formula for the FBI transform on $\mathcal{G}^{q,\beta}(\mathbb{R}^d)'$. While this is technical, it is of central importance to the theory. Moreover, the extension is nontrivial because we work on a global objects and not compactly supported ones.

Theorem 2.1. *Let $\beta > \frac{1}{2}$. Then (2) holds in $\mathcal{G}^{q,\beta}(\mathbb{R}^d)'$. Specifically, if $u \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)'$, then the limit defined in (2) converges in $\mathcal{G}^{q,\beta}(\mathbb{R}^d)'$.*

One reason that the dual spaces $\mathcal{G}^{q,\beta}(\mathbb{R}^d)'$ are so useful is the following characterization.

Proposition 2.2. *Fix $1 < q \leq \infty$ and let p be the dual exponent of q, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. The dual of $\mathcal{G}^{q,\beta}(\mathbb{R}^d)$, $\mathcal{G}^{q,\beta}(\mathbb{R}^d)'$, can be identified with the space*

$$\Bigg\{ f = \sum_{\gamma\in\mathbb{Z}^n_+} f^{(\gamma)}_\gamma \text{ in } \mathcal{G}^{q,\beta}(\mathbb{R}^d)';\ f_\gamma \in L^p(\mathbb{R}^d) \text{ and } \forall\, A > 0,\ \sum_{\gamma\in\mathbb{Z}^d_+} A^{|\gamma|}|\gamma|!^\beta \|f_\gamma\|_{L^p(\mathbb{R}^d)} < \infty \Bigg\}.$$

2.3. **The Fourier transform and exponential decay.** To put Theorem 1.3 in context, we end this section with a discussion of the Fourier transform and its relationship with exponential decay. That we can use the Fourier transform at all to capture exponential decay is a consequence of the following results. The first is from [BR13] and the latter from [AHR17].

Proposition 2.3. *Let $a, \beta > 0$ and $\gamma \in \mathbb{R}$. Then*

$$e^{-a|t|^{1/|\beta|}} = \inf_{r>0} \left\{ \left(\frac{r\beta}{ae}\right)^{r\beta} \frac{1}{|t|^r} \right\} \leq \inf_{\substack{m\in\mathbb{Z} \\ m\geq 0}} \left\{ \left(\frac{m\beta}{ae}\right)^{m\beta} \frac{1}{|t|^m} \right\} \leq e^{e\beta/2} e^{-a|t|^{1/|\beta|}}. \tag{8}$$

Theorem 2.4 (Theorem 3.4, [AHR17])**.** *Let $\varphi : \mathbb{R}^d \to \mathbb{C}$.*

(1) Suppose there exist constants $a, \beta > 0$ so that $|\varphi(t)| \leq Ce^{-a|t|^{1/\beta}}$. If $A = (\frac{\beta}{ae})^\beta$ and $A' > A$, then it follows that

$$\big||t|^\ell \varphi(t)\big| \leq CA^\ell \ell^{\ell\beta}$$

for all integers $\ell \geq 0$ and $\hat{\varphi} \in \mathcal{G}^{\infty,\beta}_{A'}(\mathbb{R}^d)$.

(2) Suppose that $\hat{\varphi} \in \mathcal{G}^{1,\beta}_{A}(\mathbb{R}^d)$. Then there exists $C > 0$ so that

$$|t^\ell \varphi(t)| \leq CA^\ell \ell^{\ell\beta}$$

for all integers $\ell \geq 0$, i.e.,

$$|\varphi(t)| \leq Ce^{-a|t|^{1/\beta}}$$

where $A = (\frac{\beta}{ae})^\beta$.

Note that Theorem 2.4 contains part 2 of Theorem 1.3. Also the previous two results describe the positive directions for the relationship between exponential decay and the Fourier transform. Now let us discuss the limitations and put Theorem 1.3 in context. The short answer is that the failure of the Hausdorff-Young inequality turns out to be catastrophic for proving a truly satisfying relationship with exponential decay and the Fourier transform. Essentially, the Hausdorff-Young inequality holds for the L^p function classes whose Fourier transforms are again functions.

Although the Fourier transforms of very smooth functions do not have to decay in the pointwise sense, they ought to exhibit decay, but Theorem 1.3 shows that using *function classes* is not sufficient. The challenge, of course, is to find a replacement because Theorem 1.4 shows that there exist some nasty objects (albeit with compact support!) whose Fourier transforms are sublime. Compact support is the most extreme version of decay, so even in this example, the heuristic that smooth objects should have (inverse) Fourier transforms with decay holds.

Let $g \in \mathcal{G}^{q,\beta}_A(\mathbb{R}^d)$. This means $\hat{g} \in \mathcal{S}'(\mathbb{R}^d)$. However, control on g and its derivatives should yield decay for $\hat{g}$. In particular, if $\psi \in \mathcal{S}(\mathbb{R}^d)$, then so is $x^\alpha \psi$. This means

$$\big|\langle \hat{g}, x^\alpha \psi\rangle\big| = \big|\langle g, D^\alpha \hat{\psi}\rangle\big| = \big|\langle D^\alpha g, \hat{\psi}\rangle\big| \leq \|D^\alpha g\|_{L^q(\mathbb{R}^d)} \|\hat{\psi}\|_{L^p(\mathbb{R}^d)} \leq CA^{|\alpha|}|\alpha|^{|\alpha|\beta} \|\hat{\psi}\|_{L^p(\mathbb{R}^d)}. \tag{9}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. As a consequence of this calculation and the fact that $\mathcal{F}$ is of type (p', q') where if $1 \leq p' \leq 2$ and $\frac{1}{p'} + \frac{1}{q'} = 1$, we see that the tempered distribution $x^\alpha \hat{g} \in \mathcal{S}'(\mathbb{R}^d)$ satisfies

$$\big|\langle x^\alpha \hat{g}, \psi\rangle\big| \leq CA^{|\alpha|}|\alpha|^{|\alpha|\beta} \|\hat{\psi}\|_{L^p(\mathbb{R}^d)} \leq CA^{|\alpha|}|\alpha|^{|\alpha|\beta} \|\psi\|_{L^q(\mathbb{R}^d)}$$

if $1 \le q \le 2$. Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^q(\mathbb{R}^d)$ for any $1 \le q < \infty$, it follows that if $1 \le q \le 2$ and $g \in \mathcal{G}_A^{q,\beta}(\mathbb{R}^d)$, then $x^\alpha \hat{g} \in L^p(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\|x^\alpha \hat{g}\|_{L^p(\mathbb{R}^d)} \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}. \tag{10}$$

In the reverse direction, let us now assume that (10) holds for some $1 \le p \le \infty$ and $\hat{g} \in \mathcal{S}'(\mathbb{R}^d)$. We would like to see if this is enough to establish that $g \in \mathcal{G}_{A'}^{q,\beta}(\mathbb{R}^d)$ for some $A' \ge A$. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\big|\langle D^\alpha g, \psi\rangle\big| = \big|\langle \widehat{D^\alpha g}, \check{\psi}\rangle\big| = \big|\langle x^\alpha \hat{g}, \check{\psi}\rangle\big| \le \|x^\alpha \hat{g}\|_{L^p(\mathbb{R}^d)}\|\check{\psi}\|_{L^q(\mathbb{R}^d)} \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}\|\hat{\psi}\|_{L^q(\mathbb{R}^d)}. \tag{11}$$

As above, we know that $\|\hat{\psi}\|_{L^q(\mathbb{R}^d)} \le \|\psi\|_{L^p(\mathbb{R}^d)}$ if $2 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consequently, for such q,

$$\big|\langle D^\alpha g, \psi\rangle\big| \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}\|\psi\|_{L^p(\mathbb{R}^d)}.$$

By the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$ for any p, $1 \le p < \infty$, it then follows that $D^\alpha g \in L^q(\mathbb{R}^d)$ and moreover

$$\|D^\alpha g\|_{L^q(\mathbb{R}^d)} \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}.$$

This proves that for any $A' > A$, $g \in \mathcal{G}_{A'}^{q,\beta}(\mathbb{R}^d)$. Summarizing our results, we have shown the following.

Proposition 2.5. *Let $A, \beta > 0$ and assume that $1 \le p, q \le \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.*

i. If $1 \le q \le 2$ and $g \in \mathcal{G}_A^{q,\beta}(\mathbb{R}^d)$, then for all multiindices α, $x^\alpha \hat{g} \in L^p(\mathbb{R}^d)$ and satisfies (10)*.*

ii. If $2 \le q \le \infty$ and $x^\alpha \hat{g} \in L^p(\mathbb{R}^d)$ and satisfies (10) *for all multiindices α, then $g \in \mathcal{G}_{A'}^{q,\beta}$ for any $A' > A$.*

By repeating the calculations above, replacing x^α and D^α with $\frac{x^\alpha}{A^{|\alpha|}|\alpha|^{|\alpha|\beta}}$ and $\frac{D^\alpha}{A^{|\alpha|}|\alpha|^{|\alpha|\beta}}$, respectively, and summing in α, we can establish the following corollary.

Corollary 2.6. *Let $A, \beta > 0$. A tempered distribution g is an element of $\mathcal{G}_A^{2,\beta}(\mathbb{R}^d)$ if and only if*

$$\frac{1}{A^{|\alpha|}|\alpha|^{|\alpha|\beta}}\|x^\alpha \hat{g}\|_{L^2(\mathbb{R}^d)} \in \ell^2(\mathbb{Z}^d_{\ge 0})$$

where the summation occurs over $\{\alpha \in \mathbb{Z}^d_{\ge 0}\}$.

We would like to further investigate equations (9) and (11). In light of Theorem A.2 and equations (31) and (32), there exists a sequence of functions $g_N \in \mathcal{G}_A^{q,\beta}(\mathbb{R}^d)$ for some fixed A so that $\|D^\alpha g_N\|_{L^q(\mathbb{R}^d)} \sim N^{1/q}$ and $\|\xi^\beta \hat{g}_N\|_{L^p(\mathbb{R}^d)} \sim N^{1/2}$. Consequently, regardless of our choice of α and β, it can only happen that

$$N^{1/2} \sim \|\xi^\beta \hat{g}_N\|_{L^p(\mathbb{R}^d)} \le C\|D^\alpha g_N\|_{L^q(\mathbb{R}^d)} \sim N^{1/q}$$

for all N if $1 < q \le 2$. Conversely, it can only happen that

$$N^{1/q} \sim \|D^\alpha g_N\|_{L^q(\mathbb{R}^d)} \le C\|\xi^\beta \hat{g}_N\|_{L^p(\mathbb{R}^d)} \sim N^{1/2}$$

if $2 \le q < \infty$. This means when we look at (9) and (11), there is no amount of manipulation of $\hat{\psi}$ by either differentiation or multiplication (but not both) which will improve Proposition 2.5.

3. $\mathcal{G}^{q,\beta}(\mathbb{R}^d)$ and the FBI transform – Proof of Theorem 1.2

Proof of Theorem 1.2. We first assume that $|\xi| \geq 1$. In this case, $\langle\xi\rangle \leq \sqrt{2}|\xi|$. Next, observe that

$$d(\xi_j + ix_j\langle\xi\rangle) = d\xi_j + ix_j \sum_{k=1}^{d} \frac{\xi_k}{\langle\xi\rangle}\, d\xi_k$$

so that $\alpha(x,\xi)$ (which can, of course, be written as a determinant) is a sum of terms of the form $i^\ell x^\alpha\big(\frac{\xi}{\langle\xi\rangle}\big)^\gamma$ where $|\alpha| = |\gamma| = \ell$ and $0 \leq \ell \leq d$. Therefore, to attain the estimate (3), it suffices from the definition of the FBI transform in (6) to attain the same bound for

$$\begin{aligned} f_u(x,\xi) &= \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi - \langle\xi\rangle|x-y|^2}(x-y)^\alpha\left(\frac{\xi}{\langle\xi\rangle}\right)^\gamma u(y)\,dy \\ &= \int_{\mathbb{R}^d} e^{iy\cdot\xi-\langle\xi\rangle|y|^2} y^\alpha \left(\frac{\xi}{\langle\xi\rangle}\right)^\gamma u(x-y)\,dy. \end{aligned} \tag{12}$$

We now apply D^J to f_u and integrate by parts carefully. In particular, $\xi = (\xi_1,\dots,\xi_d)$ and

$$|\xi| \leq \big(d\max \xi_j^2\big)^{1/2} = d^{1/2}\max|\xi_j| \leq d^{1/2}|\xi|.$$

Without loss of generality, we may assume that $|\xi_1| = \max|\xi_j|$. Therefore,

$$\begin{aligned} D^J f_u(x,\xi) &= \int_{\mathbb{R}^d} e^{iy\cdot\xi-\langle\xi\rangle|y|^2} y^\alpha\left(\frac{\xi}{\langle\xi\rangle}\right)^\gamma D^J u(x-y)\,dy \\ &= \frac{1}{(-i\xi_1)^k}\left(\frac{\xi}{\langle\xi\rangle}\right)^\gamma \int_{\mathbb{R}^d} e^{iy\cdot\xi}\frac{\partial^k}{\partial y_1^k}\Big(e^{-\langle\xi\rangle|y|^2}y^\alpha D^J u(x-y)\Big)\,dy. \end{aligned}$$

By the Leibniz rule and the standard multiindex notation $\binom{k}{k_1,k_2,k_3} = \frac{k!}{k_1!k_2!k_3!}$, we compute

$$D^J f_u(x,\xi) = \frac{1}{(-i\xi_1)^k}\left(\frac{\xi}{\langle\xi\rangle}\right)^\gamma \sum_{\substack{k_1+k_2+k_3=k\\ k_2\leq\ell}} \binom{k}{k_1,k_2,k_3}\int_{\mathbb{R}^d} e^{iy\cdot\xi}\frac{\partial^{k_1}}{\partial y_1^{k_1}}e^{-\langle\xi\rangle|y|^2}\frac{\partial^{k_2}}{\partial y_1^{k_2}}y^\alpha\frac{\partial^{k_3}}{\partial y_1^{k_3}}D^J u(x-y)\,dy.$$

Taking the L^r norm of $D^J f_u(\cdot,\xi)$, we use Young's inequality with p satisfying $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}+1$ and estimate

$$\begin{aligned} \|D^J f_u(\cdot,\xi)\|_{L^r(\mathbb{R}^d)} &\leq CA^k\frac{1}{|\xi|^k}\sum_{\substack{k_1+k_2+k_3=k\\ k_2\leq\ell}}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left|\frac{\partial^{k_1}}{\partial y_1^{k_1}}e^{-\langle\xi\rangle|y|^2}|y|^{\ell-k_2}\right|\left|\frac{\partial^{k_3}}{\partial y_1^{k_3}}D^J u(x-y)\right|dy\right)^r dx\right)^{1/r} \\ &= CA^k\frac{1}{|\xi|^k}\sum_{\substack{k_1+k_2+k_3=k\\ k_2\leq\ell}}\left\|\left|\frac{\partial^{k_1}}{\partial y_1^{k_1}}e^{-\langle\xi\rangle|\cdot|^2}|\cdot|^{\ell-k_2}\right| * \left|\frac{\partial^{k_3}}{\partial y_1^{k_3}}D^J u\right|(x)\right\|_{L^r(\mathbb{R}^d)} \\ &\leq CA^k\frac{1}{|\xi|^k}\sum_{\substack{k_1+k_2+k_3=k\\ k_2\leq\ell}}\left\|\frac{\partial^{k_1}}{\partial y_1^{k_1}}e^{-\langle\xi\rangle|\cdot|^2}|\cdot|^{\ell-k_2}\right\|_{L^p(\mathbb{R}^d)}\left\|\frac{\partial^{k_3}}{\partial y_1^{k_3}}D^J u\right\|_{L^q(\mathbb{R}^d)} \end{aligned} \tag{13}$$

We separately analyze each piece of (13) and complete the estimate of $\|D^J f_u(\cdot,\xi)\|_{L^r(\mathbb{R}^d)}$ in (14) below. The constant A_0 may increase from line to line. We estimate the latter term of

(13) as follows:

$$\left\| \frac{\partial^{k_3}}{\partial y_1^{k_3}} D^J u \right\|_{L^q(\mathbb{R}^d)} \le CA^{|J|+k_3}(|J|+k_3)^{(|J|+k_3)\beta} \le CA_0^{|J|} A_0^{k_3} |J|^{|J|\beta} k_3^{k_3\beta}$$

since

$$(|J|+k_3)^{(|J|+k_3)} \le e^{|J|+k_3} \frac{(|J|+k_3)!}{|J|!k_3!} |J|!k_3! = e^{|J|+k_3} \binom{|J|+k_3}{|J|} |J|!k_3! \le (2e)^{|J|+k_3} |J|^{|J|} k_3^{k_3}.$$

For the former term of (13), we use Proposition B.1 and bound

$$\begin{aligned}
\left\| \frac{\partial^{k_1}}{\partial y_1^{k_1}} e^{-\langle\xi\rangle|\cdot|^2} |\cdot|^{\ell-k_2} \right\|_{L^p(\mathbb{R}^d)} &\le \left(\int_{\mathbb{R}^d} \left| \frac{\partial^{k_1}}{\partial y_1^{k_1}} e^{-\langle\xi\rangle|y|^2} \right|^p |y|^{p(\ell-k_2)}\, dy \right)^{1/p} \\
&\le CA_0^{k_1} \langle\xi\rangle^{k_1/2} k_1^{k_1/2} \left(\int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle\xi\rangle|y|^2} |y|^{p(\ell-k_2)}\, dy \right)^{1/p} \\
&= CA_0^{k_1} \langle\xi\rangle^{k_1/2} k_1^{k_1/2} \frac{1}{\langle\xi\rangle^{\frac{\ell-k_2}{2}+\frac{d}{2p}}} \left(\int_{\mathbb{R}^d} |t|^{p(\ell-k_2)} e^{-\frac{p}{2}|t|^2}\, dt \right)^{1/p} \\
&\le CA_0^{k_1} \langle\xi\rangle^{k_1/2} k_1^{k_1/2} \frac{1}{\langle\xi\rangle^{\frac{\ell-k_2}{2}+\frac{d}{2p}}}
\end{aligned}$$

where the constant does not depend on p since the integral is bounded uniformly in $p \in [1, q']$.

Next, we plug the previous two estimates into (13) and observe that there are at most d possibilities for k_2 and the worst estimate occurs when $k_2 = 0$ (and there fewer than $k^2 d$ terms in the sum). For the next estimation, we use the function $M(t)$ defined in (7) and the fact that $\beta > \frac{1}{2}$ so for any integer k_1, $k_1^{\frac{k_1}{2}} \le k_1^{k_1\beta} \le Cb^{k_1}(k_1!)^\beta$ for any $b > e$ and some constant $C = C(b)$. We now estimate

$$\begin{aligned}
\|D^J f_u(\cdot,\xi)\|_{L^r(\mathbb{R}^d)} &\le CA^k \sum_{\substack{k_1+k_2+k_3=k \\ k_2\le\ell}} \frac{1}{\langle\xi\rangle^{\frac{\ell-k_2}{2}+\frac{d}{2p}} |\xi|^k} A_0^{|J|+k_1+k_3} \langle\xi\rangle^{k_1/2} k_1^{k_1/2} |J|^{|J|\beta} k_3^{k_3\beta} \\
&\le CA^k \sum_{k_1+k_3=k} \frac{1}{\frac{d}{2p}|\xi|^k} A_0^{|J|+k} \langle\xi\rangle^{k_1/2} k_1^{k_1/2} |J|^{|J|\beta} k_3^{k_3\beta} \\
&\le CA_0^{k+|J|} |J|^{|J|\beta} \frac{1}{|\xi|^k} \sum_{k_1+k_3=k} \frac{|\xi|^{k_1/2}}{k_1!^{\frac{\beta}{2}}} k_1!^\beta k_3!^\beta \\
&\le CA_0^{k+|J|} |J|^{|J|\beta} \frac{1}{|\xi|^k} \sum_{k_1+k_3=k} \left(\sup_{k_1} \frac{|\xi|^{k_1}}{k_1!^\beta} \right)^{\frac{1}{2}} k!^\beta \\
&\le CA_0^{k+|J|} |J|^{|J|\beta} \frac{k!^\beta}{|\xi|^k} e^{\frac{1}{2}M(\xi)}.
\end{aligned}$$

This estimate holds for all $k \in \mathbb{N}$, so

$$\|D^J f_u(x,\xi)\|_{L^r(\mathbb{R}^d)} \le CA_0^{|J|} |J|^{|J|\beta} e^{\frac{1}{2}M(\xi)} \inf_k \frac{A_0^k (k!)^\beta}{|\xi|^k} = CA_0^{|J|} |J|^{|J|\beta} e^{\frac{1}{2}M(\xi) - M(\delta\xi)}, \quad \delta = 1/A_0.$$

Therefore, if $|\xi|$ is sufficiently large, there exists a constant $R > 0$, independent of x such that

$$\|D^J \mathcal{F} u(x,\xi)\|_{L^r(\mathbb{R}^d)} \le C A_0^{|J|} |J|^{|J|\beta} e^{-M(\delta\xi)/4}, \qquad (x,\xi) \in \mathbb{R}^d \times \{|\xi| \ge R\}. \tag{14}$$

For the case $|\xi| \le R$, we start with (12). Since $|\xi| \le R$ and $\langle\xi\rangle \ge 1$, we estimate in a straight forward fashion. With the substitution $t = \langle\xi\rangle^{1/2} y$ so that $\frac{1}{\langle\xi\rangle^{d/2}} dt = dy$, we see (using the fact that $\langle\xi\rangle \ge 1$)

$$|D_x^J f_u(x,\xi)| \le C \int_{\mathbb{R}^d} e^{-|y|^2} (1+|y|^2)^{d/2} |D_x^J u(x-y)|\, dy = C\big(|\cdot|^{|\alpha|} e^{-|\cdot|^2} * D^J u\big)(x)$$

We choose p so that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. Since $q \le r \le \infty$, p satisfies $1 \le p \le q'$ where q and q' are Hölder conjugates. By Young's inequality

$$\|D_x^J f_u(\cdot,\xi)\|_{L^r(\mathbb{R}^d)} \le \|(1+|\cdot|^2)^{d/2} e^{-|\cdot|^2}\|_{L^p(\mathbb{R}^d)} \|D^J u\|_{L^q(\mathbb{R}^d)}$$

from which (3) follows easily since $|\xi|$ is bounded and $u \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$.

For the proof of the converse, we use Theorem 2.1. Since $\mathcal{F} u$ satisfies (3), we can send $\epsilon \to 0$ as in (2) to conclude that $u(x)$ is a well defined function and apply the Dominated Convergence Theorem for the derivatives of u of any order and obtain

$$D^J u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} D_x^J \mathcal{F} u(x,\xi) d\xi.$$

Therefore, using Minkowski's inequality for integrals, we have

$$\|D^J u\|_{L^q(\mathbb{R}^d)} \le \int \left(\int |D_x^J \mathcal{F} u(x,\xi)|^q dx \right)^{\frac{1}{q}} d\xi \le C A_0^{|J|} |J|^{|J|\beta} \int e^{-\frac{1}{c} M(a|\xi|)} d\xi \le C A_0^{|J|} |J|^{|J|\beta}.$$

□

4. $\mathcal{G}^{q,\beta}(\mathbb{R}^d)$ and the Fourier transform – Proof of Theorem 1.3

Proof of Theorem 1.3. The $q = 1$ case is straightforward. In fact, let $f_1(x) = 1_{[-1,1]}(x)$. Then $\hat{f}_1(\xi) = \frac{2\sin\xi}{\xi}$. $\hat{f}_1 \notin L^1(\mathbb{R})$, so $\hat{f}_1 \notin \mathcal{G}^{q,\beta}(\mathbb{R})$. Also, f_1 has compact support, the most extreme version of possible decay!

We now prove 1) for $1 < q \le \infty$. Let $\chi \in C_c^\infty(\mathbb{R})$ so that $\chi_{[-1,1]} \equiv 1$, $0 \le \chi \le 1$, and $\operatorname{supp}\chi \subset [-2,2]$. For $0 < \alpha < 1$, set

$$g_\alpha(x) = \frac{1}{|x|^\alpha}\chi(x) = \frac{1}{|x|^\alpha} - \big(1-\chi(x)\big)\frac{1}{|x|^\alpha}.$$

Since $\frac{1}{|x|^\alpha}\chi(x) \in L^1(\mathbb{R})$, $\hat{g}_\alpha$ is a (continuous) function. Next, by homogeneity, $\widehat{\frac{1}{|x|^\alpha}} = \frac{c}{|\xi|^{1-\alpha}}$. These are both functions (away from $\xi = 0$) which means $\mathcal{F}\{(1-\chi(x))\frac{1}{|x|^\alpha}\}$ is also a function in ξ away from $\xi = 0$. Since $(1-\chi(x))\frac{1}{|x|^\alpha} \in C^\infty(\mathbb{R})$ and is bounded, $|\mathcal{F}\{(1-\chi(x))\frac{1}{|x|^\alpha}\}| \le C_k|\xi|^{-k}$ for any integer $k \ge 0$. Consequently, $\hat{g}_\alpha(\xi)$ decays like $|\xi|^{-(1-\alpha)}$, and the slowness of this decay means that there is no hope for $\widehat{g_\alpha}$ to be in $L^q(\mathbb{R})$ if $(1-\alpha)q < 1$, i.e., $q < \frac{1}{1-\alpha}$. By setting $f_q = g_\alpha$ for $q < \frac{1}{1-\alpha}$, 1) is proven.

We now prove 3). Recall that $\mathrm{Exp}_\beta(\mathbb{R}^d)$ was defined in (5). Since $f \in \mathrm{Exp}_\beta(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, there exist constants $C, a > 0$ so that $|f(x)| \le Ce^{-a|x|^{1/\beta}}$. An application of Hausdorff-Young shows that

$$|\langle D^\alpha \hat{f}, \varphi\rangle| = |\langle f, \xi^\alpha \hat{\varphi}\rangle| \le \|\xi^\alpha f\|_{L^1(\mathbb{R}^d)} \|\hat{\varphi}\|_{L^\infty(\mathbb{R}^d)} \le \|\xi^\alpha f\|_{L^1(\mathbb{R}^d)} \|\varphi\|_{L^1(\mathbb{R}^d)},$$

and

$$\int_{\mathbb{R}^d} |x|^{|\alpha|} |f(x)|\, dx \le C \int_{\mathbb{R}^d} |x|^{|\alpha|} e^{-a|x|^{1/\beta}}\, dx = C \int_0^\infty t^{|\alpha|+d-1} e^{-at^{1/\beta}}\, dt$$

We make a change of variables $s = at^{1/\beta}$ so that $t = (s/a)^\beta$ and $\frac{\beta}{a} s^{\beta-1} ds = dt$ and observe

$$\int_{\mathbb{R}^d} |x|^{|\alpha|} |f(x)|\, dx \le CA^{|\alpha|} \int_0^\infty s^{\beta(|\alpha|+d-1)} e^{-s}\, ds = CA^{|\alpha|} \Gamma(\beta(|\alpha| + d - 1) + 1)$$

by increasing C and A. To show that $D^\alpha \hat{f} \in \mathcal{G}^{\infty,\beta}(\mathbb{R}^d)$, it follows that by increasing A and C,

$$CA^{|\alpha|}\Gamma(\beta(|\alpha| + d - 1) + 1) \le \tilde{C}\tilde{A}^{|\alpha|} |\alpha|^{|\alpha|\beta}.$$

This inequality follows from a combination of Stirling's formula and the argument that for any $b > e$, there exists $c = c(b)$ so that

$$(s + 1)^{s+1} = s^s \Big(1 + \frac{1}{s}\Big)^s (s + 1) \le cb^s s^s$$

since linear terms increase at a subgeometric rate.

A similar computation with Plancherel's formula replacing Hausdorff-Young shows that

$$|\langle D^\alpha \hat{f}, \varphi\rangle| \le \|x^\alpha f\|_{L^2(\mathbb{R}^d)} \|\varphi\|_{L^2(\mathbb{R}^d)}$$

and $\|x^\alpha f\|_{L^2(\mathbb{R}^d)} \le CA^{|\alpha|}|\alpha|^{|\alpha|\beta}$. This means $\hat{f} \in \mathcal{G}^{2,\beta}(\mathbb{R}^d)$. Interpolating between the L^2 and L^∞ inequalities yields the result that $\hat{f} \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ for any $2 \le q \le \infty$.

Proof of 4). The theorem conclusion mentions two (counter)examples – one where f is a function and one where it is not. The function counterexample and its proof is the content of [AHR17, §3.3]. The remaining example is Salem's example and it is the content of Section 5. □

5. Salem's example – the proof of Theorem 1.4

By taking products, it suffices to prove the result for $d = 1$.

The measure μ that we use was first constructed by Salem [Sal51], though we follow Donoghue [Don69] and (in notation) Folland [Fol99]. We only sketch the construction – please see either of the references for details.

5.1. **Salem's constructions.**

Definition 5.1. Given a set of numbers $\{a_1, \dots, a_N\} \subset \mathbb{R}$, we say they are *independent* if they are linearly independent over $\mathbb{Z}$.

Definition 5.2. Let $r > 0$. A *similitude with scaling factor* r is a map $S : \mathbb{R}^d \to \mathbb{R}^d$ of the form $S(x) = rT(x) + b$ where T is an orthogonal transformation and $b \in \mathbb{R}^d$ is fixed. If $S = (S_1, \dots, S_m)$ is a family of similitudes with common scaling factor $r < 1$ and $E \subset \mathbb{R}^d$, then define $S(E) = \bigcup_{j=1}^m S_j(E)$.

Let $q > 2$ and choose α so that $0 < \alpha < 1$ and $q > \frac{2}{\alpha}$. Let $N \in \mathbb{N}$ be an integer (selected later) and choose η so that $N\eta^\alpha = 1$. Fix an independent set $\{a_1, \dots, a_N\}$ with $0 \le a_1 < a_2 < \cdots < a_N \le 1$ so that $a_j + \eta < a_{j+1}$. To maximize the flexibility of the construction, we introduce the probability space (Ω, ω) defined as follows: let $I_j = [\eta(1 - \frac{1}{(j+1)^2}), \eta]$ and $\Omega = \prod_{j=1}^\infty I_j$. On Ω, the probability measure $d\omega$ is defined as follows: if $E = \{\{x_j\} : b_{n_j} < x_{n_j} < c_{n_j}, j = 1, \dots, k\}$, then

$$\omega(E) = \frac{1}{\eta^k} \prod_{j=1}^k (c_{n_j} - b_{n_j})(j+1)^2 = \prod_{j=1}^k \frac{c_{n_j} - b_{n_j}}{|I_j|}. \tag{15}$$

It follows that $\omega(\Omega) = 1$ and Ω is indeed a probability space. A point $t \in \Omega$ is a sequence $t = (\eta_1, \eta_2, \dots)$ where $\eta_j \in I_j$.

For each $t \in \Omega$, Salem builds a Cantor set K_t as follows. If S^n is the family of similitudes $S^n = (S_1^n, \dots, S_N^n)$ so that $S_j^n(x) = \eta_n x + a_j$, then recursively define $K_1 = S^1([0,1])$, $K_2 = S^2 S^1([0,1])$, and, in general $K_{j+1} = S^{j+1} K_j$. For example, the set K_2 is a union of N^2 intervals of the form $[a_i + a_j\eta_1, a_i + a_j\eta_1 + \eta_1\eta_2]$. In this way, $K_n = S^n \circ \cdots \circ S^1([0,1])$. The resulting set

$$K = K_t = \bigcap_{n=1}^\infty K_n$$

is a perfect and compact set of Hausdorff dimension α. On each K_t, there is a positive Radon measure μ_t of finite mass so that

$$\hat{\mu}_t(\xi) = \prod_{n=0}^\infty p(\xi\eta_1 \cdots \eta_n)$$

where

$$p(\xi) = \frac{1}{N} \sum_{k=1}^N e^{-i\xi a_k}$$

and with the understanding that the $n = 0$ entry in the product is $p(\xi)$.

5.2. **Modified goal.** To prove Theorem 1.4, it suffices to show that there exists $C > 0$ so that

$$\int_\Omega \int_{\mathbb{R}} |\hat{\mu}_t^{(k)}(\xi)|^q \, d\xi \, d\omega(t) \le C^q. \tag{16}$$

Indeed, once (16) is proven, it must be the case that there exists $t \in \Omega$ so that $\int_{\mathbb{R}} |\hat{\mu}_t^{(k)}(\xi)|^q \, d\xi \le C^q$ for every k. There are countably many k and uncountably many t which must satisfy $\int_{\mathbb{R}} |\hat{\mu}_t^{(k)}(\xi)|^q \, d\xi \le C^q$ for each k. That $t \in \Omega$ exists is a simple consequence of the pigeon hole principle.

5.3. **Setting up the computations.** We use the following notation. Define the vector $P_t(\xi) = (p(\xi), p(\xi\eta_1), p(\xi\eta_1\eta_2), \dots)$. For $\alpha = (\alpha_0, \alpha_1, \dots,)$ with $\alpha \in \{0, 1, \dots\}^{\{0,1,\dots\}}$, set

$$D^\alpha P_t(\xi) = \prod_{j=0}^\infty \frac{d^{\alpha_j}}{d\xi^{\alpha_j}} \{p(\xi\eta_1 \cdots \eta_j)\} = \prod_{j=0}^\infty (\eta_1 \cdots \eta_j)^{\alpha_j} p^{(\alpha_j)}(\xi\eta_1 \cdots \eta_j). \tag{17}$$

Returning to (16), we apply Tonelli's Theorem and see the object to estimate is

$$\int_{\mathbb{R}} \int_{\Omega} |\hat{\mu}_t^{(k)}(\xi)|^q \, d\omega(t) \, d\xi.$$

Set

$$H_k(\xi) = \int_{\Omega} |\hat{\mu}_t^{(k)}(\xi)|^q \, d\omega(t).$$

An induction argument shows that

$$\hat{\mu}_t^{(k)}(\xi) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha P_t(\xi).$$

To see this, observe that the $k = 0$ case is immediate. Assume the result holds for k. Then if $e_\ell = (0, \dots, 0, 1, 0, \dots)$ is the vector with 1 in the ℓth position and is 0 otherwise,

$$\hat{\mu}_t^{(k+1)}(\xi) = \frac{d}{d\xi} \hat{\mu}_t^{(k)}(\xi) = \frac{d}{d\xi} \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha P_t(\xi) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \sum_{\ell=0}^{\infty} D^{\alpha+e_\ell} P_t(\xi).$$

Let $|\beta| = k + 1$. Suppose that $\beta_{j_1}, \dots, \beta_{j_n} \neq 0$ and $\beta_\ell = 0$ for $\ell \notin \{j_1, \dots, j_n\}$. Now, $\beta = \alpha + e_\ell$ if and only if $\alpha = (\beta_0, \beta_1, \dots, \beta_\ell - 1, \beta_{\ell+1}, \dots)$. Therefore, the coefficient of $D^\beta P_t$ is

$$k! \Big(\frac{1}{(\beta_{j_1} - 1)! \beta_{j_2}! \cdots \beta_{j_n}!} + \cdots + \frac{1}{\beta_{j_1}! \cdots \beta_{j_{n-1}}! (\beta_{j_n} - 1)!} \Big) = \frac{(k+1)!}{\beta!}.$$

We will need the following consequence of Jensen's inequality.

Lemma 5.3. *Let (X, dP) be a probability space. Suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is a concave, strictly increasing function. Then*

$$\int_X \psi(f) \, dP \leq \psi\Big(\int_X f \, dP \Big).$$

Proof. Since ψ is concave and strictly increasing, it has an inverse ϕ that is convex and (strictly) increasing. By Jensen's inequality,

$$\varphi\Big(\int_X \psi(f) \, dP \Big) \leq \int_X \varphi(\psi(f)) \, dP = \int_X f \, dP.$$

Applying ψ to this inequality finishes the proof. □

We will also need the following variant of a lemma by Salem (see [Don69, p.266]).

Lemma 5.4. *Let $\{a_1, \cdots, a_N\}$ be independent and the function $p : \mathbb{R} \to \mathbb{C}$ be defined by*

$$p(\xi) = \frac{1}{N} \sum_{j=1}^{N} e^{-i a_j \xi}.$$

There exists a constant T_0 that depends on the collection $\{a_1, \dots, a_N\}$ such that for all $b \in \mathbb{R}$, $\ell \in \mathbb{N}_0$, and $T \geq T_0$,

$$\frac{1}{T} \int_b^{b+T} \prod_{m=1}^{\ell} \Big| \{ p^{(j_m)} \overline{p^{(n_m)}} \}(\xi) \Big| \, d\xi \leq (1 - \eta)^{j_1 + \cdots + j_m + n_1 + \cdots + n_m} \Big(\frac{2\ell}{N} \Big)^\ell.$$

Proof. Since $\frac{1}{T}\,d\xi$ is a probability measure on $[b, b+T]$, we may use Lemma 5.3 to estimate

$$\frac{1}{T}\int_b^{b+T}\prod_{m=1}^{\ell}\Big|\{p^{(j_m)}\overline{p^{(n_m)}}\}(\xi)\Big|\,d\xi \le \Big(\frac{1}{T}\int_b^{b+T}\prod_{m=1}^{\ell}\Big|\{p^{(j_m)}\overline{p^{(n_m)}}\}(\xi)\Big|^2\,d\xi\Big)^{1/2}.$$

Let $j_{\ell+m} = n_m$, so that we may estimate

$$\frac{1}{T}\int_b^{b+T}\big|p^{(j_1)}\cdots p^{(j_{2\ell})}\big|^2\,d\xi.$$

We compute

$$\begin{aligned}
\big|p^{(j_1)}\cdots p^{(j_{2\ell})}\big|^2 =& \frac{1}{N^{4\ell}}\Big|\sum_{k_1,\dots,k_{2\ell}=1}^{N} a_{k_1}^{j_1}\cdots a_{k_{2\ell}}^{j_{2\ell}}e^{-i(a_{k_1}+\cdots+a_{k_{2\ell}})}\Big|^2\\
=& \frac{1}{N^{4\ell}}\sum_{\substack{k_1,\dots,k_{2\ell}=1\\ n_1,\dots,n_{2\ell}=1\\ \{k_1,\dots,k_{2\ell}\}=\{n_1,\dots,n_{2\ell}\}\text{ as sets}}}^{N} a_{k_1}^{j_1}\cdots a_{k_{2\ell}}^{j_{2\ell}}a_{n_1}^{j_1}\cdots a_{n_{2\ell}}^{j_{2\ell}}\\
&+\frac{1}{N^{4\ell}}\sum_{\substack{k_1,\dots,k_{2\ell}=1\\ n_1,\dots,n_{2\ell}=1\\ \{k_1,\dots,k_{2\ell}\}\neq\{n_1,\dots,n_{2\ell}\}\text{ as sets}}}^{N} a_{k_1}^{j_1}\cdots a_{k_{2\ell}}^{j_{2\ell}}a_{n_1}^{j_1}\cdots a_{n_{2\ell}}^{j_{2\ell}}e^{-i\xi(a_{k_1}+\cdots+a_{k_{2\ell}}-a_{n_1}-\cdots-a_{n_{2\ell}})}
\end{aligned}\tag{18}$$

To estimate the first sum in the right-hand side of (18), we note that for each 2ℓ-tuple $(a_{k_1},\dots,a_{k_{2\ell}})$, there exists an N-tuple α so that $\alpha_1+\cdots+\alpha_N = 2\ell$ and

$$a_{k_1}+\cdots+a_{k_{2\ell}} = \alpha_1 a_1+\cdots+\alpha_N a_N$$

and there are $\frac{(2\ell)!}{\alpha!}$ distinct reorderings of $(a_{k_1},\dots a_{k_{2\ell}})$. Also, $a_r \le 1-\eta$ for each $r = 1,\dots N$ so that

$$\begin{aligned}
\frac{1}{T}\int_b^{b+T}\frac{1}{N^{4\ell}}\sum_{\substack{k_1,\dots,k_{2\ell}=1\\ n_1,\dots,n_{2\ell}=1\\ \{k_1,\dots,k_{2\ell}\}=\{n_1,\dots,n_{2\ell}\}\text{ as sets}}}^{N} a_{k_1}^{j_1}\cdots a_{k_{2\ell}}^{j_{2\ell}}a_{n_1}^{j_1}\cdots a_{n_{2\ell}}^{j_{2\ell}}\,d\xi &\le \frac{1}{N^{4\ell}}(1-\eta)^{2j_1+\cdots+2j_{2\ell}}\sum_{|\alpha|=2\ell}\Big(\frac{(2\ell)!}{\alpha!}\Big)^2\\
&\le \frac{1}{N^{4\ell}}(1-\eta)^{2j_1+\cdots+2j_{2\ell}}(2\ell)!\sum_{|\alpha|=2\ell}\frac{(2\ell)!}{\alpha!}\\
&= \frac{1}{N^{4\ell}}(1-\eta)^{2j_1+\cdots+2j_{2\ell}}(2\ell)!N^{2\ell}\\
&= \frac{(2\ell)!}{N^{2\ell}}(1-\eta)^{2j_1+\cdots+2j_{2\ell}} < \Big(\frac{2\ell}{N}\Big)^{2\ell}(1-\eta)^{2j_1+\cdots+2j_{2\ell}}.
\end{aligned}$$

For the second sum in the right-hand side of (18), the independence of $\{a_1,\dots,a_N\}$ establishes that

$$\min\{a_{k_1}+\cdots+a_{k_{2\ell}}-a_{n_1}-\cdots-a_{n_{2\ell}} : 1\le k_m, n_m\le N\}\ge c>0$$

where $c = c(a_1, \ldots, a_N, N, \ell)$. Thus, for any $\delta > 0$, there exists $T_0 = T_0(a_1, \ldots, a_N, \ell)$ so that if $T \geq T_0$, then

$$\begin{aligned}
&\left| \frac{1}{T} \int_b^{b+T} \frac{1}{N^{4\ell}} \sum_{\substack{k_1,\ldots,k_{2\ell}=1 \\ n_1,\ldots,n_{2\ell}=1 \\ \{k_1,\ldots,k_{2\ell}\} \neq \{n_1,\ldots,n_{2\ell}\} \text{ as sets}}}^{N} a_{k_1}^{j_1} \cdots a_{k_{2\ell}}^{j_{2\ell}} a_{n_1}^{j_1} \cdots a_{n_{2\ell}}^{j_{2\ell}} e^{-i\xi(a_{k_1}+\cdots+a_{k_{2\ell}}-a_{n_1}-\cdots-a_{n_{2\ell}})}\, d\xi \right| \\
&\quad \leq \frac{1}{T} \frac{1}{N^{4\ell}} (1-\eta)^{2j_1+\cdots+2j_{2\ell}} \sum_{\substack{k_1,\ldots,k_{2\ell}=1 \\ n_1,\ldots,n_{2\ell}=1 \\ \{k_1,\ldots,k_{2\ell}\} \neq \{n_1,\ldots,n_{2\ell}\} \text{ as sets}}}^{N} \left| \int_b^{b+T} e^{-i\xi(a_{k_1}+\cdots+a_{k_{2\ell}}-a_{n_1}-\cdots-a_{n_{2\ell}})}\, d\xi \right| \\
&\quad \leq C \frac{1}{T} \frac{(2\ell)!}{N^{2\ell}} (1-\eta)^{2j_1+\cdots+2j_{2\ell}} \frac{1}{c} \; < \; \delta.
\end{aligned}$$

By taking δ suitably small, the proof is complete. ◻

5.4. **The L^∞ estimate.** The L^∞-estimate is contained in the following proposition.

Proposition 5.5. *For every $k \in \mathbb{N}_0$ and $\xi \in \mathbb{R}$, we have*

$$\left| \hat{\mu}_t^{(k)}(\xi) \right| \leq 1.$$

Proof. Observe that

$$\begin{aligned}
|\hat{\mu}_t^{(k)}(\xi)| &= \Big| \sum_{|\alpha|=k} \frac{k!}{\alpha!} \prod_{j=0}^{\infty} (\eta_1 \cdots \eta_j)^{\alpha_j} p^{(\alpha_j)}(\xi \eta_1 \cdots \eta_j) \Big| \\
&\leq \sum_{|\alpha|=k} \frac{k!}{\alpha!} \prod_{j=0}^{\infty} \eta^{j\alpha_j} (1-\eta)^{\alpha_j} = (1-\eta)^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} \prod_{j=0}^{\infty} \eta^{j\alpha_j}
\end{aligned}$$

The key to analyzing the sum is to decompose it into pieces we can understand. Each $\alpha = (\alpha_0, \alpha_1, \ldots)$ can be expressed $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n, 0, \ldots)$ where $\alpha_n > 0$ and $\alpha_1 + \cdots + \alpha_n = k$. Thus, we can break down the sum into pieces according the final nonzero component. In the subsequent estimate, we identify $(\alpha_1, \ldots, \alpha_n, 0, \ldots)$ with (γ, m) where

$\gamma = (\gamma_1, \dots, \gamma_{n-1}) = (\alpha_1, \dots, \alpha_{n-1})$ and $1 \leq m = \alpha_n \leq k$. Consequently, $|\gamma| = k - m$, and

$$
\begin{aligned}
&(1-\eta)^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} \prod_{j=0}^{\infty} \eta^{j\alpha_j} \\
&= (1-\eta)^k \Big(1 + \sum_{n=1}^{\infty} \sum_{m=1}^{k} \frac{k!}{m!(k-m)!} \eta^{nm} \Big[\sum_{\substack{|\gamma|=k-m \\ \gamma=(\gamma_0,\dots,\gamma_{n-1})}} \frac{(k-m)!}{\gamma!} \eta^{\gamma_1} \cdots \eta^{(n-1)\gamma_{n-1}} \Big] \Big) \\
&= (1-\eta)^k \Big(1 + \sum_{n=1}^{\infty} \sum_{m=1}^{k} \frac{k!}{m!(k-m)!} \eta^{nm} (1 + \eta + \eta^2 + \cdots + \eta^{n-1})^{k-m} \Big) \\
&= (1-\eta)^k \Big(1 + \sum_{n=1}^{\infty} \sum_{m=1}^{k} \binom{k}{m} \eta^{nm} \Big(\frac{1-\eta^n}{1-\eta} \Big)^{k-m} \Big) \\
&= (1-\eta)^k + (1-\eta)^k \sum_{n=1}^{\infty} \Big[\Big(\eta^n + \frac{1-\eta^n}{1-\eta} \Big)^k - \Big(\frac{1-\eta^n}{1-\eta} \Big)^k \Big] \\
&= (1-\eta)^k + \sum_{n=1}^{\infty} \Big[\big(1-\eta^{n+1}\big)^k - \big(1-\eta^n\big)^k \Big]
\end{aligned}
$$

The final sum telescopes and therefore can be computed explicitly. We are therefore able to compute

$$
(1-\eta)^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} \prod_{j=0}^{\infty} \eta^{j\alpha_j} \leq (1-\eta)^k + 1 - (1-\eta)^k = 1. \tag{19}
$$

□

5.5. **The L^q bound.** We know from Proposition 5.5 and the definition of H_k that

$$
|H_k(\xi)| \leq 1
$$

since $d\omega$ is a probability measure. We write $q = \frac{2\ell}{q_1}$ where $\ell \in \mathbb{N}$ and $q_1 \geq 1$.

The boundedness of the L^q norms will follow by establishing decay for $H_k(\xi)$ at ∞. By Lemma 5.3,

$$
H_k(\xi) = \int_\Omega \big|\hat{\mu}_t^{(k)}(\xi)\big|^{2\ell/q_1} \, d\omega(t) \leq \Big(\int_\Omega \big|\hat{\mu}_t^{(k)}(\xi)\big|^{2\ell} \, d\omega(t) \Big)^{1/q_1},
$$

and we start by estimating

$$
G_k(\xi) = \int_\Omega \big|\hat{\mu}_t^{(k)}(\xi)\big|^{2\ell} \, d\omega(t).
$$

We compute

$$
|\hat{\mu}_t^{(k)}(\xi)|^{2\ell} = \hat{\mu}_t^{(k)}(\xi)^\ell \overline{\hat{\mu}_t^{(k)}(\xi)^\ell},
$$

and

$$
\hat{\mu}_t^{(k)}(\xi)^\ell = \Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha P_t \Big)^\ell = \sum_{|\alpha^1|=\cdots=|\alpha^\ell|=k} \frac{(k!)^\ell}{\alpha^1! \cdots \alpha^\ell!} D^{\alpha^1} P_t \cdots D^{\alpha^\ell} P_t.
$$

This means

$$|\hat{\mu}_t^{(k)}(\xi)|^{2\ell} = \sum_{\substack{|\alpha^1|=\cdots=|\alpha^\ell|=k\\|\beta^1|=\cdots=|\beta^\ell|=k}} \frac{(k!)^{2\ell}}{\alpha^1!\cdots\alpha^\ell!\beta^1!\cdots\beta^\ell!} D^{\alpha^1}P_t\cdots D^{\alpha^\ell}P_t\cdot\overline{D^{\beta^1}P_t\cdots D^{\beta^\ell}P_t}$$

Picking out just one term and using (17), we estimate

$$\begin{aligned}\left|D^{\alpha^1}P_t\cdots D^{\alpha^\ell}P_t\overline{D^{\beta^1}P_t\cdots D^{\beta^\ell}P_t}\right| &= \Big|\prod_{j=0}^{\infty}(\eta_1\ldots\eta_j)^{\alpha_j^1+\cdots+\alpha_j^\ell+\beta_j^1+\cdots+\beta_j^\ell}\prod_{m=1}^{\ell}\{p^{(\alpha_j^m)}\overline{p^{(\beta_j^m)}}\}(\xi\eta_1\ldots\eta_j)\Big|\\ &\le \Big(\prod_{j=1}^{\infty}\eta^{j(\alpha_j^1+\cdots+\alpha_j^\ell+\beta_j^1+\cdots+\beta_j^\ell)}\Big)\prod_{j=0}^{\infty}\prod_{m=1}^{\ell}\left|\{p^{(\alpha_j^m)}\overline{p^{(\beta_j^m)}}\}(\xi\eta_1\ldots\eta_j)\right|\end{aligned}$$

so that

$$G_k(\xi) \le \sum_{\substack{|\alpha^1|=\cdots=|\alpha^\ell|=k\\|\beta^1|=\cdots=|\beta^\ell|=k}} \frac{(k!)^{2\ell}\prod_{j=1}^{\infty}\eta^{j(\alpha_j^1+\cdots+\alpha_j^\ell+\beta_j^1+\cdots+\beta_j^\ell)}}{\alpha^1!\cdots\alpha^\ell!\beta^1!\cdots\beta^\ell!}\int_\Omega\prod_{j=0}^{\infty}\prod_{m=1}^{\ell}\left|\{p^{(\alpha_j^m)}\overline{p^{(\beta_j^m)}}\}(\xi\eta_1\ldots\eta_j)\right|d\omega(t).$$

It is always the case that $|p^{(\alpha_j^m)}(\xi)| \le (1-\eta)^{\alpha_j^m}$. Consequently, for any integer $M = M(\xi)$

$$\begin{aligned}G_k(\xi) \le \sum_{\substack{|\alpha^1|=\cdots=|\alpha^\ell|=k\\|\beta^1|=\cdots=|\beta^\ell|=k}} &\frac{(k!)^{2\ell}\prod_{j=1}^{\infty}\eta^{j(\alpha_j^1+\cdots+\alpha_j^\ell+\beta_j^1+\cdots+\beta_j^\ell)}}{\alpha^1!\cdots\alpha^\ell!\beta^1!\cdots\beta^\ell!}\prod_{\substack{j=M+1\\ \text{and } j=0}}^{\infty}(1-\eta)^{\alpha_j^1+\cdots+\alpha_j^\ell+\beta_j^1+\cdots\beta_j^\ell}\times\\ &\times\int_\Omega\prod_{j=0}^{M}\prod_{m=1}^{\ell}\left|\{p^{(\alpha_j^m)}\overline{p^{(\beta_j^m)}}\}(\xi\eta_1\ldots\eta_j)\right|d\omega(t).\end{aligned}$$

We will now estimate

$$I_k(\xi) = \int_\Omega\prod_{j=0}^{M}\prod_{m=1}^{\ell}\left|\{p^{(\alpha_j^m)}\overline{p^{(\beta_j^m)}}\}(\xi\eta_1\ldots\eta_j)\right|d\omega(t).$$

Using the definition of Ω and ω in (15), we observe that

$$\begin{aligned}I_k(\xi) =& \eta^{-(M-1)}\prod_{n=1}^{M-1}(n+1)^2\int_{I_1}\cdots\int_{I_{M-1}}\prod_{j=0}^{M-1}\prod_{m=1}^{\ell}\left|\{p^{(\alpha_j^m)}\overline{p^{(\beta_j^m)}}\}(\xi\eta_1\ldots\eta_j)\right|\times\\ &\times\frac{(M+1)^2}{\eta}\int_{I_M}\prod_{m=1}^{\ell}\left|\{p^{(\alpha_M^m)}\overline{p^{(\beta_M^m)}}\}(\xi\eta_1\ldots\eta_M)\right|d\eta_M\cdots d\eta_1\end{aligned}$$

Make the change of variables $s = |\xi|\eta_1\cdots\eta_{M-1}\eta_M$ so that if $T = \frac{\eta|\xi|\eta_1\cdots\eta_{M-1}}{(M+1)^2}$ the last integral becomes

$$\frac{(M+1)^2}{\eta}\int_{I_M}\prod_{m=1}^{\ell}\left|\{p^{(\alpha_M^m)}\overline{p^{(\beta_M^m)}}\}(\xi\eta_1\ldots\eta_M)\right|d\eta_M = \frac{1}{T}\int_b^{b+T}\prod_{m=1}^{\ell}\left|\{p^{(\alpha_M^m)}\overline{p^{(\beta_M^m)}}\}(s)\right|ds$$

for some $b \in \mathbb{R}$. To use Lemma 5.4, we require $T \ge T_0$. In this case, Lemma 5.4 guarantees that the integral is bounded by $(1-\eta)^{\alpha_M^1+\cdots\alpha_M^\ell+\beta_M^1+\cdots+\beta_M^\ell}Q$ where $Q = (\frac{2\ell}{N})^\ell$. We will therefore need $M = M(\xi)$ to satisfy

$$T_0 \leq \frac{1}{(M+1)^2}\eta|\xi|\eta_1\cdots\eta_{M-1} = \frac{\eta^M|\xi|}{(M+1)^2}\prod_{j=1}^{M-1}\Big(1-\frac{1}{(j+1)^2}\Big). \tag{20}$$

Additionally, since $\eta_j < 1$, we can integrate in η_{M-1} in the same way as as we did for η_M because $\frac{1}{M^2}\eta|\xi|\eta_1\cdots\eta_{M-2} \geq \frac{1}{(M+1)^2}\eta|\xi|\eta_1\cdots\eta_{M-1}$. In fact, we can repeat the integration to integrate in $d\eta_{M-1}\cdots d\eta_1$. The bound will then be

$$|I_k(\xi)| \leq Q^{M(\xi)}\prod_{j=1}^{M(\xi)}(1-\eta)^{\alpha_j^1+\cdots+\alpha_j^\ell+\beta_j^1+\cdots+\beta_j^\ell}.$$

Let $P = \prod_{j=1}^{\infty}(1-\frac{1}{(j+1)^2})$. It follows from (20) that it is sufficient to require $M = M(\xi)$ to satisfy

$$\big(M(\xi)+1\big)^2\frac{T_0}{P} \leq \eta^{M(\xi)}|\xi|$$

or, equivalently,

$$2\log(M(\xi)+1) + \log\frac{T_0}{P} \leq M(\xi)\log\eta + \log|\xi|. \tag{21}$$

The other requirement for $M(\xi)$ is that we want $Q^{M(\xi)} \leq |\xi|^{-(1+\epsilon)q_1}$ for some $\epsilon > 0$. This requires $Q < 1$ and

$$M(\xi)\log Q \leq -(1+\epsilon)q_1\log|\xi|. \tag{22}$$

For a moment, let us ignore the fact that $M(\xi)$ must be an integer and play with (21) and (22). If $M(\xi)$ is no longer required to be an integer, assume $M(\xi) = \mu\log|\xi|$ for some $\mu > 0$. Rearranging our inequalities, we see that (22) becomes

$$\mu \geq \frac{(1+\epsilon)q_1}{|\log Q|}. \tag{23}$$

In a similar manner, (21) becomes

$$\mu\Big[|\log\eta| + 2\frac{\log(M(\xi)+1)}{M(\xi)}\Big] + \frac{1}{\log|\xi|}\log\frac{T_0}{P} \leq 1. \tag{24}$$

With $|\xi|$ sufficiently large (which does not depend on k), we see that it is enough to require $\mu < \frac{1}{|\log\eta|}$. Putting our inequalities together, we want

$$\frac{(1+\epsilon)q_1}{|\log Q|} < \mu < \frac{1}{|\log\eta|}.$$

We therefore require $\log Q < q_1\log\eta$ (recall that $0 < Q, \eta < 1$). Our setup is that $q = \frac{2\ell}{q_1} > \frac{2}{\alpha}$ and $N\eta^\alpha = 1$. Consequently, $\log\eta = -\frac{1}{\alpha}\log N$ and $\ell > \frac{q_1}{\alpha}$. Therefore, since $Q = (\frac{2\ell}{N})^\ell$, the inequality $\log Q < q_1\log\eta$ is equivalent to

$$\ell\log(2\ell) < \Big(\ell - \frac{q_1}{\alpha}\Big)\log N. \tag{25}$$

By choosing N large, ($N > (qq_1)^{\frac{q}{q-\frac{2}{\alpha}}}$), (25) follows. Additionally, by taking $|\xi|$ large enough and possibly further increasing N, we can also find an interval $[\mu', \mu'']$ which satisfy (23) and (24). Since $M(\xi) = \mu \log|\xi|$, as ξ increases, there is always an integer $M(\xi) \in [\mu' \log|\xi|, \mu'' \log|\xi|]$.

We have now established two bounds for I_k. The first is the simple L^∞ bound. Namely,

$$I_k(\xi) \leq \prod_{j=1}^{M(\xi)} (1-\eta)^{\alpha_j^1+\cdots+\alpha_j^\ell+\beta_j^1+\cdots+\beta_j^\ell}.$$

The second bound is the decay bound. Specifically, there exists $R > 0$ depending on N, ϵ, and δ so that if $|\xi| \geq R$, then

$$I_k(\xi) \leq \left(\prod_{j=1}^{M(\xi)} (1-\eta)^{\alpha_j^1+\cdots+\alpha_j^\ell+\beta_j^1+\cdots+\beta_j^\ell} \right) |\xi|^{-q_1(1+\epsilon)}.$$

We now wish to take our bounds for I_k and plug them into G_k. The decay bound does not affect the combinatorics. First assume that $|\xi| \leq R$. Recall that $|\alpha^m| = |\beta^m| = k$, $m = 1, \dots, \ell$ so that

$$\begin{aligned} G_k(\xi) &\leq \sum_{\substack{|\alpha^1|=\cdots=|\alpha^\ell|=k \\ |\beta^1|=\cdots=|\beta^\ell|=k}} \frac{(k!)^{2\ell} \prod_{j=1}^\infty \eta^{j(\alpha_j^1+\cdots+\alpha_j^\ell+\beta_j^1+\cdots+\beta_j^\ell)}}{\alpha^1! \cdots \alpha^\ell! \beta^1! \cdots \beta^\ell!} (1-\eta)^{2\ell k} \\ &= \left| \sum_{|\alpha^1|=\cdots=|\alpha^\ell|=k} \frac{(k!)^\ell \prod_{j=1}^\infty \eta^{j(\alpha_j^1+\cdots+\alpha_j^\ell)}}{\alpha^1! \cdots \alpha^\ell!} (1-\eta)^{\ell k} \right|^2 \end{aligned}$$

This last quantity, fortunately, is exactly what we bounded in (19). Consequently, we have the bound

$$G_k(\xi) \leq \begin{cases} 1 & |\xi| \leq R \\ |\xi|^{-q_1(1+\epsilon)} & |\xi| \geq R. \end{cases}$$

Our estimate on $G_k(\xi)$ shows that the function $H_k(\xi) = (G_k(\xi))^{1/q_1}$ is integrable since

$$H_k(\xi) \leq \begin{cases} 1 & |\xi| \leq R \\ |\xi|^{-(1+\epsilon)} & |\xi| \geq R. \end{cases}$$

therefore

$$\int_{\mathbb{R}} \int_\Omega |\hat{\mu}_t^{(k)}(\xi)|^q \, d\omega(t) \, d\xi = \int_{\mathbb{R}} H_k(\xi) \, d\xi \leq C.$$

6. The inversion formula for the FBI transform – the proof of Theorem 2.1

6.1. Proof of Proposition 2.2.

Proof of Proposition 2.2. By [AHR17, Proposition 2.17], $\mathcal{G}^{q,\beta}(\mathbb{R}^d)' = \bigcap_{A>0} \mathcal{G}_A^{q,\beta}(\mathbb{R}^d)'$, so it suffices to prove the proposition for an arbitrary $A > 0$. It is certainly the case that if

$f = \sum_{\gamma\in\mathbb{Z}^n_+} f^{(\gamma)}_\gamma$ for f as in the statement of the proposition, then $f \in \mathcal{G}^{q,\beta}_A(\mathbb{R}^d)'$ as

$$|\langle f, \varphi\rangle| \le \sum_{\gamma\in\mathbb{Z}^d_+} |\langle f^{(\gamma)}_\gamma, \varphi\rangle| = \sum_{\gamma\in\mathbb{Z}^d_+} |\langle f_\gamma, \varphi^{(\gamma)}\rangle| \le \sum_{\gamma\in\mathbb{Z}^d_+} \|f_\gamma\|_{L^p(\mathbb{R}^d)} C A^{|\gamma|}|\gamma|^{|\gamma|\beta} < \infty$$

The converse follows from two facts. The first is the observation in [AHR17, Lemma 2.18] that if $1 \le q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\ell^q(L^q(\Omega))' = \ell^p(L^p(\Omega))$, and if $q = \infty$, then $\ell^\infty(L^\infty(\Omega))' = (\ell^\infty)'(L^\infty(\Omega)')$. The second fact is that the map $P_A : \mathcal{G}^{q,\beta}_A(\mathbb{R}^d) \to \ell^q(L^q(\mathbb{R}^d))$ defined by the inclusion map

$$P_A(\varphi) = \{\varphi^{(\gamma)}/(A^{|\gamma|}|\gamma|^{|\gamma|\beta})\}$$

allows us to transfer problem to $\ell^p(L^p(\mathbb{R}^d))$ (for a Sobolev space version of this argument, see [AF03, Chapter 3]). The space $\mathcal{G}^{q,\beta}_A(\mathbb{R}^d)$ is closed, hence P_A is an isometry of $\mathcal{G}^{q,\beta}_A(\mathbb{R}^d)$ into a subspace S of $\ell^q(L^q(\mathbb{R}^d))$. Although S is a subspace of $\ell^q(L^q(\mathbb{R}^d))$, any element $u \in S'$ extends to an element $U \in \ell^q(L^q(\mathbb{R}^d))'$ by the Hahn-Banach Theorem. If $1 \le q < \infty$, then such a U will be a vector $U = (U_\gamma)$ so that $\sum_{\gamma\in\mathbb{Z}^d_+} \|U_\gamma\|_{L^p(\mathbb{R}^d)} < \infty$. We would like to compute the dual of P_A. Let $U \in S'$. Then

$$\langle U, P_A\varphi\rangle = \sum_{\gamma\in\mathbb{Z}^d_+} \int_{\mathbb{R}^d} U_\gamma \overline{\frac{\varphi^{(\gamma)}}{A^{|\gamma|}|\gamma|^{|\gamma|\beta}}}\, dV = \sum_{\gamma\in\mathbb{Z}^d_+} \int_{\mathbb{R}^d} \frac{(-1)^\gamma U^{(\gamma)}_\gamma}{A^{|\gamma|}|\gamma|^{|\gamma|\beta}} \overline{\varphi}\, dV = \langle P_A' U, \varphi\rangle$$

and we see

$$P_A' U = \sum_{\gamma\in\mathbb{Z}^d_+} \frac{(-1)^\gamma}{A^{|\gamma|}|\gamma|^{|\gamma|\beta}} U^{(\gamma)}_\gamma.$$

The dual map gives an isomorphism between S' and $\mathcal{G}^{q,\beta}_A(\mathbb{R}^d)'$, finishing the proof. (To see this, $\mathcal{G}^{q,\beta}_A(\mathbb{R}^d)'$ and S' are isomorphic as if $\Lambda \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)'$, then $\Lambda \circ P_A^{-1} \in S'$ and if $U \in S'$, then $U \circ P_A \in \mathcal{G}^{q,\beta}_A(\mathbb{R}^d)'$. But we have already seen that integration by parts establishes the equality $P_A' U = U \circ P_A$). ☐

6.2. **Proof of Theorem 2.1.**

Proof of Theorem 2.1. Suppose that $u \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)'$ and $\beta > 1/2$. We want to prove that

$$u_\epsilon(x) = (2\pi)^{-d} \int_{|\xi|\le\epsilon^{-1}} \mathcal{F}\, u(x,\xi) d\xi \longrightarrow u(x) \tag{26}$$

in $\mathcal{G}^{q,\beta}(\mathbb{R}^d)'$ as $\epsilon \to 0$.

Assume first that $u \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$. In this case we will show that $u_\epsilon \to u$ in $u \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ as $\epsilon \to 0$. In fact, introducing a partition of unity in $\mathcal{G}^{q,\beta}(\mathbb{R}^d)$ we may assume that u is supported in a ball $B(x_0, r)$ with r small to be chosen. By Lemma C.1 and (36), in particular, we can write

$$u(x) = (2\pi)^{-d} \lim_{\epsilon\to 0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x') e^{i(x-x')\cdot\xi} e^{-\epsilon^2\xi^2}\, dx'\, d\xi \tag{27}$$

where the convergence is in $\mathcal{G}^{q,\beta}(\mathbb{R}^d)$.

For small ϵ we change the contour of integration to

$$(x', \xi) \mapsto (y, \eta) := \Gamma(x', \xi) = (x', \xi + i\langle\xi\rangle(x - x'))$$

The integrand is holomorphic with respect to ξ and decays rapidly, as $|\Gamma(x',\xi,t)| \to \infty$, where

$$\Gamma(x',\xi,t) := (1-t)(x',\xi) + t\Gamma(x',\xi), \qquad 0 \le t \le 1$$

if $|x-x'|$ is sufficiently small. At this point we choose $r = \frac{1}{2\sqrt{2}}$ and let x be in a small neighborhood of $B(x_0,r)$, say $B(x_0,2r)$. We then obtain

$$\mathrm{Re}(-\epsilon^2\Gamma(x',\xi,t)) = -\epsilon^2\xi^2 + \epsilon^2 t^2\langle\xi\rangle^2(x-x')^2 \le -\frac{1}{2}\epsilon^2\xi^2$$

for $|x-x'| \le 1/\sqrt{2} := 2r$. Since $dx' \wedge d\xi = \alpha(y,\eta)dy \wedge d\eta$ we have

$$\begin{aligned} u(x) &= (2\pi)^{-d}\lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(y)e^{i(x-y)\cdot\eta-\langle\eta\rangle(x-y)^2}\alpha(x-y,\eta)e^{-\epsilon^2(\eta+i\langle\eta\rangle(x-y))^2}\,dy\,d\eta \\ &= (2\pi)^{-d}\lim_{\epsilon\to 0}\left\langle u(y), \int_{\mathbb{R}^d} e^{i(x-y)\cdot\eta-\langle\eta\rangle(x-y)^2}\alpha(x-y,\eta)e^{-\epsilon^2(\eta+i\langle\eta\rangle(x-y))^2}\,d\eta\right\rangle \end{aligned} \tag{28}$$

We now want to send $\epsilon \to 0$. For this we integrate by parts as in the proof of Theorem 2.2 (also see Lemma C.2) and compute

$$\begin{aligned} &\Big|\int_{\mathbb{R}^d} u(y)e^{i(x-y)\cdot\eta}e^{-\langle\eta\rangle(x-y)^2}(x-y)^\alpha e^{-\epsilon^2(\eta+i\langle\eta\rangle(x-y))^2}\,dy\Big| \\ &\quad = \Big|\int_{\mathbb{R}^d} u(x-y)e^{iy\cdot\eta}e^{-\langle\eta\rangle y^2}y^\alpha e^{-\epsilon^2(\eta+i\langle\eta\rangle y)^2}\,dy\Big| \\ &\quad = \frac{A^{k+1}}{|\eta|^k}\sum_{\substack{k_1+k_2+k_3+k_4=k \\ k_3\le\ell}}\int_{\mathbb{R}^d}|\partial_{y_1}^{k_1}u(x-y)|\langle\eta\rangle^{\frac{|k_2|}{2}}|k_2|!^{\frac{1}{2}}e^{-\frac{1}{2}\langle\eta\rangle y^2}|y|^{|\alpha-k_3|}\,|\partial_{y_1}^{k_4}\{e^{-\epsilon^2(\eta+i\langle\eta\rangle y)^2}\}|\,dy \\ &\quad \le \frac{A^{k+1}}{|\eta|^k}\sum_{\substack{k_1+k_2+k_3+k_4=k \\ k_3\le\ell}}(k_2!)^\beta e^{\frac{1}{2}M(|\eta|)}\int_{\mathbb{R}^d}|\partial_{y_1}^{k_1}u(x-y)|e^{-\frac{1}{2}\langle\eta\rangle y^2}|y|^{|\alpha-k_3|}|k_4|!e^{-\frac{1}{4}\epsilon^2\eta^2}\,dy \\ &\quad \le \frac{A^{k+1}}{|\eta|^k}e^{\frac{1}{2}M(|\eta|)}\sum_{\substack{k_1+k_2+k_3+k_4=k \\ k_3\le\ell}}(k_2+k_4)!^\beta\|\partial_{y_1}^{k_1}u\|_{L^q}\Big(\int_{\mathbb{R}^d} e^{-\frac{p}{2}\langle\eta\rangle y^2}|y|^{p|\alpha-k_3|}\,dy\Big)^{1/p} \\ &\quad \le C\frac{A^k k^\beta}{|\eta|^k}e^{\frac{1}{2}M(|\eta|)}. \end{aligned}$$

These inequalities hold for every $k \in \mathbb{N}$ and the argument leading to (14) show that for every $(y,\eta) \in \mathbb{R}^d \times \{|\eta| \ge C\}$

$$\Big|\int_{\mathbb{R}^d} u(x-y)e^{iy\cdot\eta}e^{-\langle\eta\rangle y^2}\alpha(y,\eta)e^{-\epsilon^2(\eta+i\langle\eta\rangle y)^2}\,dy\Big| \le Ce^{-M(\delta|\eta|)/4}. \tag{29}$$

Similar estimates hold for $|\eta| \le C$. Therefore, for $u \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ supported in balls $B(x_0,r)$, the FBI inversion formula (26) is valid for every $x \in B(x_0,2r)$ and, moreover, the convergence is in $L^q(B(x_0,2r))$ as a consequence of Minkowski's inequality.

Using the arguments from the proof of Theorem 1.2, we see that the analogous estimates of (29) are true for any derivative of $u(x)$, $x \in B(x_0,2r)$, when $u \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ with support contained in $B(x_0, r)$, that is

$$\Big| \int_{\mathbb{R}^d} \partial^J u(x-y) e^{iy\cdot\eta} e^{-\langle\eta\rangle y^2} y^\alpha e^{-\epsilon^2(\eta+i\langle\eta\rangle y)^2}\, dy \Big| \le C A^{|J|} |J|^{|J|\beta} e^{-M(\delta|\eta|)/4} \tag{30}$$

thus $u_\epsilon \to u$ in $\mathcal{G}^{q,\beta}(B(x_0, 2r))$ as $\epsilon \to 0$.

Now we use an idea of Christ [Chr97] and the Identity Theorem for real analytic functions: specifically, if we define $v(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathcal{F}(u)(x,\xi)\, d\xi$, we see that $v \equiv u \equiv 0$ in $B(x_0, 2r) \setminus B(x_0, r)$, but v is real analytic outside the support of u, that is $v \equiv u$ and the formula holds everywhere. This allow us to conclude that the formula holds for every $x \in \mathbb{R}^d$ and the convergence is in the topology of $\mathcal{G}^{q,\beta}(\mathbb{R}^d)$.

Assume now that $u \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)'$. Given $\psi \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ we can write u as in Proposition 2.2 and compute

$$\begin{aligned}
\langle u_\epsilon, \psi\rangle &= \Big\langle (2\pi)^{-d} \int_{|\xi|\le\epsilon^{-1}} \mathcal{F}\, u(x,\xi)\, d\xi, \psi(x) \Big\rangle \\
&= \int_{\mathbb{R}^d} (2\pi)^{-d} \int_{|\xi|\le\epsilon^{-1}} \mathcal{F}\, u(x,\xi)\, d\xi\, \psi(x)\, dx \\
&= \int_{\mathbb{R}^d} (2\pi)^{-d} \int_{|\xi|\le\epsilon^{-1}} \big\langle u, e^{i(x-\cdot)\cdot\xi - \langle\xi\rangle(x-\cdot)^2} \alpha(x-\cdot,\xi) \big\rangle\, d\xi\, \psi(x)\, dx \\
&= \int_{\mathbb{R}^d} (2\pi)^{-d} \int_{|\xi|\le\epsilon^{-1}} \sum_\gamma (-1)^{|\gamma|} \int_{\mathbb{R}^d} u_\gamma(x') \partial^\gamma_{x'} \Big\{ e^{i(x-x')\cdot\xi - \langle\xi\rangle(x-x')^2} \alpha(x-x',\xi) \Big\} dx'\, d\xi\, \psi(x)\, dx \\
&= \int_{\mathbb{R}^d} (2\pi)^{-d} \int_{|\xi|\le\epsilon^{-1}} \sum_\gamma (-1)^{|\gamma|} \int_{\mathbb{R}^d} u_\gamma(x-x') \partial^\gamma_{x'} \Big\{ e^{ix'\cdot\xi - \langle\xi\rangle(x')^2} \alpha(x',\xi) \Big\} dx'\, d\xi\, \psi(x)\, dx.
\end{aligned}$$

For each γ, the function $u_\gamma(x-x') \partial^\gamma_{x'} \Big\{ e^{ix'\cdot\xi - \langle\xi\rangle(x')^2} \alpha(x',\xi) \Big\} \psi(x)$ is integrable in (x, x', ξ) for $|\xi| \le \epsilon^{-1}$ and dominated by a integrable function which is summable in γ.

In fact,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{|\xi|\le\epsilon^{-1}} |u_\gamma(x-x') \partial^\gamma_{x'} \Big\{ e^{ix'\cdot\xi - \langle\xi\rangle(x')^2} \alpha(x',\xi) \Big\} \psi(x)|\, d\xi\, dx\, dx' \\
&\qquad \le C^{|\gamma|+1} |\gamma|! \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{|\xi|\le\epsilon^{-1}} |u_\gamma(x-x')| \langle\xi\rangle^{|\gamma|} p(|x'|) e^{-\frac{1}{2}|x'|^2} |\psi(x)|\, d\xi\, dx\, dx' \\
&\qquad \le C^{|\gamma|+1} |\gamma|! \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u_\gamma(x-x')| p(|x'|) e^{-\frac{1}{2}|x'|^2} |\psi(x)|\, dx\, dx' \\
&\qquad \le C^{|\gamma|+1} |\gamma|! \|u_\gamma\|_{L^p} \|\psi\|_{L^q} \int_{\mathbb{R}^d} p(|x'|) e^{-\frac{1}{2}|x'|^2}\, dx' \\
&\qquad \le C^{|\gamma|+1} |\gamma|! \|u_\gamma\|_{L^p}
\end{aligned}$$

here $p(|x'|)$ is a positive polynomial in $|x'|$ that dominates all derivatives of $\alpha(x',\xi)$ in x' (of which there are finitely many).

We can therefore apply Fubini's Theorem and Dominated Convergence Theorem in the above expression to write

$$\begin{aligned}\langle u_\epsilon, \psi\rangle &= \int_{\mathbb{R}^d} \sum_\gamma (-1)^{|\gamma|} u_\gamma(-x') \partial_{x'}^\gamma \Big\{ (2\pi)^{-d} \int_{|\xi|\le \epsilon^{-1}} \int_{\mathbb{R}^d} e^{i(-x+x')\cdot\xi - \langle\xi\rangle(-x+x')^2} \alpha(-x+x',\xi)\psi(-x)dx \Big\} d\xi\, dx' \\ &= \int_{\mathbb{R}^d} \sum_\gamma (-1)^{|\gamma|} u_\gamma(-x') \partial_{x'}^\gamma \Big\{ (2\pi)^{-d} \int_{|\xi|\le\epsilon^{-1}} \mathcal{F}\,\check{\psi}(x',\xi) \Big\} d\xi\, dx' \\ &= \Big\langle u(-x'), (2\pi)^{-d} \int_{|\xi|\le\epsilon^{-1}} \mathcal{F}\,\check{\psi}(x',\xi)d\xi \Big\rangle \\ &\longrightarrow \langle u(-x'), \psi(-x')\rangle = \langle u, \psi\rangle\,. \end{aligned}$$

as we wished to prove. □

Appendix A. Notes on Hausdorff-Young

Let us recall why the Hausdorff-Young inequality only holds

$$\|f\|_{L^q(\mathbb{R}^d)} \le C\|\hat{f}\|_{L^p(\mathbb{R}^d)}$$

if $\frac{1}{q} + \frac{1}{p} = 1$ and $q \ge 2$. We follow standard ideas.

Recall Khinchin's inequality.

Proposition A.1. *Let $f_1, \dots, f_N : \mathbb{R}^d \to \mathbb{C}$ be a collection of functions, and let $\epsilon_1, \dots, \epsilon_N$ be a collection of i.i.d. random variables so that $P(\epsilon_n = 1) = P(\epsilon_n = -1) = \frac{1}{2}$. If $1 < q < \infty$, then*

$$\mathbb{E}\Big(\Big\|\sum_{k=1}^N \epsilon_k f_k\Big\|_{L^q(\mathbb{R}^d)}^q\Big) \sim \Big\|\Big(\sum_{k=1}^N |f_k|^2\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^d)}^q$$

where $\sim$ is independent of N and $\{f_k\}$ but depends on q.

Theorem A.2. *Let $2 < q < \infty$. There exists a sequence $g_N \in \mathcal{G}_A^{q,\beta}(\mathbb{R}^d)$ so that*

$$\|D^\alpha g_N\|_{L^q(\mathbb{R}^d)} \sim N^{1/q}$$

where $\sim$ does not depend on N but does depend on α or q and

$$\|\widehat{D^\alpha g_N}\|_{L^p(\mathbb{R}^d)}^p \sim N^{p/2}$$

where $\sim$ does not depend on N but does depend on p and α.

Proof. Fix $1 < q < \infty$, and let $\psi \in \mathcal{G}_A^{q,\beta}(\mathbb{R}^d) \cap C_c^\infty(B(0, \frac{1}{2}))$. Let $\{\epsilon_k\}$ be a collection of i.i.d. random variables satisfying $P(\epsilon_n = 1) = P(\epsilon_n = -1) = \frac{1}{2}$. Set

$$g_N(x) = \sum_{k=1}^N \epsilon_k \psi(x - ke_1)$$

where $e_1 = (1, \dots, 0)$. Then

$$D^\alpha g_N(x) = \sum_{k=1}^n \epsilon_k D^\alpha \psi(x - ke_1)$$

and the mutually disjoint supports of $\{\psi(\cdot - ke_1)\}$ means that

$$\|D^\alpha g_N\|_{L^q(\mathbb{R}^d)} = \Big\| \sum_{k=1}^N \epsilon_k D^\alpha \psi(x - ke_1) \Big\|_{L^q(\mathbb{R}^d)} = N^{1/q} \|D^\alpha \psi\|_{L^q(\mathbb{R}^d)}. \tag{31}$$

Next,

$$\mathcal{F}\left(D^\alpha \psi(x - ke_1)\right)(\xi) = (i\xi)^\alpha e^{-ik\xi_1} \hat{\psi}(\xi)$$

which means

$$\widehat{D^\alpha g_N}(\xi) = (i\xi)^\alpha \hat{\psi}(\xi) \sum_{k=1}^N \epsilon_k e^{-ik\xi_1}.$$

By Khinchin's inequality,

$$\begin{aligned} \mathbb{E}(\|\widehat{D^\alpha g_N}\|^p_{L^p(\mathbb{R}^d)}) &\sim \Big\| \Big(\sum_{k=1}^N |(i\xi)^\alpha \hat{\psi}(\xi) e^{-ik\xi_1}|^2 \Big)^{1/2} \|^p_{L^p(\mathbb{R}^d)} \\ &= \Big\| \Big(\sum_{k=1}^N |\xi^\alpha \hat{\psi}(\xi)|^2 \Big)^{1/2} \|^p_{L^p(\mathbb{R}^d)} = N^{p/2} \|\xi^\alpha \hat{\psi}\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

As a consequence of this (near) equality, there must exist a choice of signs for $\epsilon_1, \dots, \epsilon_N$ so that

$$\|\widehat{D^\alpha g_N}\|^p_{L^p(\mathbb{R}^d)} \geq c N^{p/2} \|\xi^\alpha \hat{\psi}\|_{L^p(\mathbb{R}^d)} \tag{32}$$

where c does not depend on α, ψ, or N. □

Appendix B. Derivatives of Gaussians

Proposition B.1. *Let $s > 0$ and $x, y \in \mathbb{R}$. Then there exists constants $C, A > 0$ so that for any integer $n \in \mathbb{N}$,*

$$\left| \frac{d^n}{dy^n} e^{-s(x-y)^2} \right| \leq C A^n e^{-\frac{s}{2}(x-y)^2} s^{\frac{n}{2}} n^{\frac{n}{2}}.$$

From [GR94, (0.430)], it follows that

$$\frac{d^n}{dy^n} e^{-ah(y)} = \sum_{\mathfrak{p}(n,m)} \frac{n!(-1)^m a^m}{i!j!d! \cdots k!} e^{-ah(y)} \Big(\frac{h'}{1!}\Big)^i \Big(\frac{h''}{2!}\Big)^j \Big(\frac{h'''}{3!}\Big)^d \cdots \Big(\frac{h^{(\ell)}}{\ell!}\Big)^k$$

where the set $\mathfrak{p}(n,m)$ contains tuples of integers $(i, j, d, \dots, k)$ such that $m = i+j+d+\dots+k$ and $i + 2j + 3d + \dots + \ell k = n$. We use $h(y) = s(x-y)^2$ for a fixed $s \in \mathbb{R}$ and $x \in \mathbb{R}$. In this case, $h'(y) = -2s(x-y)$ and $h''(y) = 2s$. Consequently,

$$\frac{d^n}{dy^n} e^{-s(x-y)^2} = e^{-s(x-y)^2} \sum_{\substack{j+k=m \\ j+2k=n}} \frac{n!(-1)^m s^m}{j!k!} \big(-2(x-y) \big)^j.$$

To estimate, we break the number of derivatives in the even and odd cases.

Case 1: $n = 2\ell$. Observe that $j + k = m$ and $k + m = 2\ell$. Since $j, k \geq 0$, it follows that $\ell \leq m \leq 2\ell$. Indeed, if $m \leq \ell - 1$, then $k \leq \ell - 1$, and therefore $k + m \leq 2\ell - 2 < 2\ell$. With m and 2ℓ fixed, it is easy to see that $k = 2\ell - m$ and $j = m - k = 2(m - \ell)$. This means

$$\frac{d^{2\ell}}{dy^{2\ell}} e^{-s(x-y)^2} = e^{-s(x-y)^2} \sum_{m=\ell}^{2\ell} \binom{m}{2\ell - m} \frac{(2\ell)!}{m!} (-1)^m s^m \big(2(x-y)\big)^{2(m-\ell)}$$

Next, calculus shows that

$$\max_{y \in \mathbb{R}} \left\{ \left| y^j e^{-\frac{s}{2} y^2} \right| \right\} = \left(\frac{j}{se} \right)^{\frac{j}{2}} \tag{33}$$

which means that with $j = 2(m - \ell)$,

$$\begin{aligned}
\left| \frac{d^{2\ell}}{dy^{2\ell}} e^{-s(x-y)^2} \right| &\leq e^{-\frac{s}{2}(x-y)^2} \sum_{m=\ell}^{2\ell} \binom{m}{2\ell - m} \frac{(2\ell)!}{m!} s^m 2^{2(m-\ell)} \sup_{y \in \mathbb{R}} \left\{ |x - y|^{2(m-\ell)} e^{-\frac{s}{2}(x-y)^2} \right\} \\
&\leq A^\ell e^{-\frac{s}{2}(x-y)^2} s^\ell \sum_{m=\ell}^{2\ell} \frac{(2\ell)!}{m!} (m - \ell)! \\
&= A^\ell e^{-\frac{s}{2}(x-y)^2} s^\ell \sum_{m=\ell}^{2\ell} \frac{(2\ell)!}{\ell!} \frac{1}{\binom{m}{\ell}} \\
&\leq A^\ell e^{-\frac{s}{2}(x-y)^2} s^\ell \frac{(2\ell)^{2\ell}}{\ell^\ell} \leq A^\ell e^{-\frac{s}{2}(x-y)^2} s^{\frac{n}{2}} n^{\frac{n}{2}}
\end{aligned}$$

Case 2: $n = 2\ell + 1$. In this case, observe that $j + k = m$ and $k + m = 2\ell + 1$. Since $j, k \geq 0$, it follows that $\ell + 1 \leq m \leq 2\ell + 1$. Indeed, if $m \leq \ell$, then $k \leq \ell$, and therefore $k + m \leq 2\ell < 2\ell + 1$. With m and $2\ell + 1$ fixed, it is easy to see that $k = 2\ell + 1 - m$ and $j = m - k = 2(m - \ell) - 1$. This means

$$\frac{d^{2\ell+1}}{dy^{2\ell+1}} e^{-s(x-y)^2} = e^{-s(x-y)^2} \sum_{m=\ell+1}^{2\ell+1} \binom{m}{2\ell + 1 - m} \frac{(2\ell+1)!}{m!} (-1)^{m-1} s^m \big(2(x-y)\big)^{2(m-\ell)-1}$$

With $j = 2(m - \ell) - 1$ in (33), a similar argument to the j even case yields

$$\left| \frac{d^{2\ell+1}}{dy^{2\ell+1}} e^{-s(x-y)^2} \right| \leq A^\ell e^{-\frac{s}{2}(x-y)^2} s^{\frac{n}{2}} n^{\frac{n}{2}}.$$

Appendix C. Auxiliary results for the proof of Theorem 2.1

Lemma C.1. *Let $\psi \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative function so that $\int_{\mathbb{R}^d} \phi(x)\,dx = 1$. Define $\phi_\epsilon(x) = \epsilon^{-d} \phi(\frac{x}{\epsilon})$. Then $\psi * \phi_\epsilon \to u$ in $\mathcal{G}^{q,\beta}(\mathbb{R}^d)$ as $\epsilon \to 0$.*

Proof. Since $\int_{\mathbb{R}^d} \phi(x) dx = 1$ we have

$$\begin{aligned}
\Big(\int_{\mathbb{R}^d} |\partial^\alpha (\phi_\epsilon * \psi(x) - \psi(x))|^q \,dx \Big)^{1/q} &\leq \int_{\mathbb{R}^d} \phi(y) \cdot \|(\partial_x^\alpha \psi)(\cdot - \epsilon y) - (\partial_x^\alpha \psi)(\cdot)\|_{L^q} \,dy \\
&\leq C \|\partial_x^\alpha \psi\|_{L^q}.
\end{aligned} \tag{34}$$

Given $\delta > 0$ choose $N \in \mathbb{N}$ so that

$$\left\{ \sum_{|\alpha| \geq N} \left(\frac{\|\partial_x^\alpha \psi\|_{L^q}}{A^{|\alpha|} M_{|\alpha|}} \right)^q \right\}^{1/q} \leq \frac{\delta}{2}$$

Then for each $|\alpha| < N$ there exists a constant $\epsilon_\alpha > 0$ such that

$$\|(\partial_x^\alpha \psi)(\cdot - \epsilon y) - (\partial_x^\alpha \psi)(\cdot)\|_{L^q} \leq \frac{\delta A^{|\alpha|}}{2^{d|\alpha|+1}}, \qquad \text{for all } \epsilon < \epsilon_\alpha.$$

For $\tilde{\epsilon} := \min\{\epsilon_\alpha : |\alpha| < N\}$ and $\epsilon < \tilde{\epsilon}$ it follows that

$$\begin{aligned} \sum_{\alpha} \left(\frac{\|\partial^\alpha (\phi_\epsilon * \psi - \psi)\|_{L^q}}{A^{|\alpha|} M_{|\alpha|}} \right)^q &= \sum_{|\alpha| \geq N} \left(\frac{\|\partial_x^\alpha \psi\|_{L^q}}{A^{|\alpha|} M_{|\alpha|}} \right)^q + \sum_{|\alpha| < N} \left(\frac{\|\partial_x^\alpha \psi\|_{L^q}}{A^{|\alpha|} M_{|\alpha|}} \right)^q \\ &\leq \frac{\delta^q}{2^q} + \sum_{|\alpha| < N} \frac{\delta^q}{2^{(d|\alpha|+1)q}} \\ &\leq \delta^q \end{aligned} \tag{35}$$

proving the lemma. ☐

For $\psi \in \mathcal{G}^{q,\beta}(\mathbb{R}^d)$, if we take $0 \leq \chi(x) := 2^{-n}(\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$. Then $\int_{\mathbb{R}^d} \chi(x) dx = 1$, $\sigma(\xi) := (2\pi)^{-d} \hat{\chi}(\xi) = e^{-|\xi|^2} \in \mathcal{G}^{q,\frac{1}{2}}(\mathbb{R}^d)$ and

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x') e^{i(x-x')\cdot\xi} e^{-\epsilon^2 |\xi|^2} \, dx' \, d\xi &= \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(\epsilon\xi) \hat{\psi}(\xi) \, d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat{\chi}(\epsilon\xi) \hat{\psi}(\xi) \, d\xi \\ &= (\chi_\epsilon * \psi)(x) \longrightarrow \psi(x) \end{aligned} \tag{36}$$

in $\mathcal{G}^{q,\beta}(\mathbb{R}^d)$ as $\epsilon \to 0$.

Lemma C.2. *There exist positive constants c_1, c_2 and C such that, for every $\alpha \in \mathbb{N}_0^d$, we have*

$$\left| \partial_y^\alpha \left\{ e^{-\epsilon^2 (\eta + i\langle\eta\rangle y)^2} \right\} \right| \leq C^{|\alpha|} |\alpha|! e^{-\frac{1}{4}\epsilon^2 |\eta|^2}, \qquad \textit{for all } |y| < c_1, \ |\eta| \geq c_2.$$

The notation y^2 is shorthand for $y^2 = y \cdot y$

Proof. First we note that

$$-\epsilon^2 (\eta + i\langle\eta\rangle y)^2 = -\epsilon^2 (\eta^2 - \langle\eta\rangle^2 y^2 + 2i\langle\eta\rangle \eta \cdot y)$$

thus, for $|y| \leq 1/2$,

$$\mathrm{Re}\{-\epsilon^2 (\eta + i\langle\eta\rangle y)^2\} = -\epsilon^2 (\eta^2 - \langle\eta\rangle^2 y^2) \leq -\tfrac{1}{2}\epsilon^2 \eta^2.$$

Also, if $\alpha \in \mathbb{N}_0^d$, then

$$\partial_y^\alpha \{-\epsilon^2 (\eta + i\langle\eta\rangle y)^2\} = \begin{cases} 0, & \text{if } |\alpha| \geq 3, \\ -\epsilon^2 (-2\langle\eta\rangle^2 y_j + 2i\langle\eta\rangle \eta_j) & \text{if } \alpha = e_j, \\ 2\epsilon^2 \langle\eta\rangle^2 & \text{if } \alpha = 2e_j, \\ 0, & \text{if } \alpha = e_j + e_\ell, j \neq \ell. \end{cases}$$

In all cases, we obtain

$$\left|\partial_y^\alpha\{-\epsilon^2(\eta+i\langle\eta\rangle y)^2\}\right| \le 4\epsilon^2\langle\eta\rangle^2, \quad |y|<1. \tag{37}$$

From [GR94, (0.430)] (or, more accurately, its higher dimension version, see [CS96, Corollary 2.10]), we have

$$\partial_y^\alpha\left\{e^{-\epsilon^2(\eta+i\langle\eta\rangle y)^2}\right\} \;=\; \sum_{r=1}^{|\alpha|} e^{-\epsilon^2(\eta+i\langle\eta\rangle y)^2}\sum_{\mathfrak{p}(\alpha,r)}\alpha!\prod_{j=1}^{|\alpha|}\frac{[\partial_y^{\alpha_j}\{-\epsilon^2(\eta+i\langle\eta\rangle y)^2\}]^{k_j}}{k_j!(\alpha_j!)^{k_j}}.$$

It follows from (37) that,

$$\left|\partial_y^\alpha\left\{e^{-\epsilon^2(\eta+i\langle\eta\rangle y)^2}\right\}\right| \;\le\; \sum_{r=1}^{|\alpha|} e^{\mathrm{Re}\{-\epsilon^2(\eta+i\langle\eta\rangle y)^2\}}\sum_{\mathfrak{p}(\alpha,r)}|\alpha|!\prod_{j=1}^{|\alpha|}\frac{(4\epsilon^2\langle\eta\rangle^2)^{k_j}}{k_j!(\alpha_j!)^{k_j}}$$

where $\sum_{j=1}^{|\alpha|}k_j=r$ and since $(a+b)!\le 2^{a+b}a!b!$ (for all $a,b\in\mathbb{N}$) we have

$$\prod_{j=1}^{|\alpha|}\alpha_j!^{k_j} \ge \prod_{j=1}^{|\alpha|}(2^{-|\alpha_j|}|\alpha_j|!)^{k_j} \ge 4^{-|\alpha|}|\alpha|!. \tag{38}$$

Using the properties of $\mathfrak{p}$, we obtain

$$\begin{aligned}\left|\partial_y^\alpha\left\{e^{-\epsilon^2(\eta+i\langle\eta\rangle y)^2}\right\}\right| &\le e^{-\frac{1}{2}\epsilon^2\eta^2}\sum_{r=1}^{|\alpha|}\sum_{\mathfrak{p}(\alpha,r)}|\alpha|!\frac{4^{|\alpha|+r}(\epsilon^2\langle\eta\rangle^2)^r}{|\alpha|!}\prod_{j=1}^{|\alpha|}\frac{1}{k_j!}\\ &\le C^{|\alpha|}(\epsilon^2\langle\eta\rangle^2)^{|\alpha|}e^{-\frac{1}{2}\epsilon^2\eta^2}\sum_{r=1}^{|\alpha|}\frac{r!}{r!}\sum_{\mathfrak{p}(\alpha,r)}\prod_{j=1}^{|\alpha|}\frac{1}{k_j!}\\ &\le C^{|\alpha|}(\epsilon^2\langle\eta\rangle^2)^{|\alpha|}e^{-\frac{1}{2}\epsilon^2\eta^2}\\ &\le C^{|\alpha|}\left(\frac{|\alpha|}{e}\right)^{|\alpha|}e^{-\frac{1}{4}\epsilon^2\eta^2},\end{aligned}$$

as we wished to prove. □

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