Neumann problems with resonance in the first eigenvalue

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Abstract

The aim of this work is to present results of existence of solutions for a class of superlinear asymmetric elliptic systems with resonance in the first eigenvalue. The asymmetry that we consider has linear behavior on $-\infty$ and superlinear on $+\infty$. To obtain these results we apply topological degree theory.

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1 Introduction

This work is dedicated to present results for a class of nonlinear elliptic systems with superlinear asymmetric nonlinearities and resonant in the first eigenvalue. Elliptic problems of type

$$-\Delta u = \lambda u + (u^+)^p + f(x),$$

where $\lambda \in \mathbb{R}$, $p > 1$ and f is a function, have been studied in several papers (see [3, 4, 5, 8, 17, 18] and their references). There has recently been increasing interest in study elliptic problems with asymmetric nonlinearities, we infer the reader to [2, 6, 9, 12, 13, 14, 15, 16], where other references can be found.

Denote by λ_1 the first eigenvalue of $-\Delta$ (with the proper condition on the border). The case $\lambda \neq \lambda_1$ can be classified as superlinear Ambrosetti-Prodi type problem. In this case the associated functional satisfies the (PS) compactness condition, see [2], and the problem can be studied by linking theorems. When $\lambda = \lambda_1$ the problem is called resonant and functional associated does not satisfy a condition of compactness.

The motivation for this work is the paper [5], in which the authors worked with the following resonant Dirichlet problems

$$\begin{cases} -\Delta u = \lambda_1 u + (u^+)^p + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta u = \lambda_1 u + (v^+)^p + f(x), & x \in \Omega, \\ -\Delta v = \lambda_1 v + (u^+)^q + g(x), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

In [5], the assumptions are $f \in L^r(\Omega)$, $r > N$, $1 < p < \dfrac{N+1}{N-1}$ and

$$\int_\Omega f(x)\varphi_1(x)dx < 0, \tag{1}$$

where $\varphi_1 > 0$ is the first eigenfunction of the Laplacian with Dirichlet condition. The authors have shown that the scalar equation has at least one solution in $W^{2,r}(\Omega) \cap H_0^1(\Omega)$. For the system, in addition to f and g satisfy hypothesis (1), the exponents p and q should satisfy

$$\frac{1}{p+1} + \frac{N-1}{N+1}\frac{1}{q+1} > \frac{N-1}{N+1}$$

and

$$\frac{1}{q+1}+\frac{N-1}{N+1}\frac{1}{p+1}>\frac{N-1}{N+1}.$$

In both cases, the authors used Brezis-Turner technique to get a priori bounds and degree theory to prove existence of solutions.

The Neumann scalar equation

$$\left\{\begin{array}{ll} -\Delta u=(u^+)^p+f(x), & x\in\Omega,\\ \dfrac{\partial u}{\partial\nu}=0, & x\in\partial\Omega, \end{array}\right. \tag{2}$$

with f satisfying (1), was studied in [18]. The author applied a continuation theorem due to Mawhin and others results due to Brezis and Strauss.

We will study the Hamiltonian system

$$\left\{\begin{array}{ll} -\Delta u=(v^+)^p+f(x), & \text{in }\ \Omega,\\ -\Delta v=(u^+)^q+g(x), & \text{in }\ \Omega,\\ \dfrac{\partial u}{\partial\nu}=\dfrac{\partial v}{\partial\nu}=0, & \text{on }\ \partial\Omega. \end{array}\right. \tag{3}$$

Here we also assume that f and g satisfy (1). Note that, with the Neumann conditions at the border, φ_1 is constant (so has signal set) and so we must to assume that f and g have strictly negative integral in Ω.

As well as in [5], we use topological methods for obtaining the results of existence of solutions. The theory of the degree of Leray-Schauder was an essential tool in this process. To obtain the essential *a priori* estimates for the solutions of system (3), we assume

$$1<p,q<\frac{N}{N-2}$$

and $f,g\in L^r(\Omega)$, for some $r>N/2$. This piece of work was inspired by paper [11] in which the authors study an equation of type

$$\left\{\begin{array}{ll} -\Delta u=g(u)+f(x), & x\in\Omega,\\ \dfrac{\partial u}{\partial\nu}=0, & x\in\partial\Omega, \end{array}\right.$$

where $g:\mathbb{R}\to\mathbb{R}$ behaves as $(u^+)^p$.

2 Resonant system

In this section we will present our main result. Consider the following system

$$(4) \qquad \begin{cases} -\Delta u = (v^+)^p + f(x), & x \in \Omega, \\ -\Delta v = (u^+)^q + g(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

where

(H_1) $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 3$;

(H_2) $1 < p, q < \dfrac{N}{N-2}$;

(H_3) $f, g \in L^r(\Omega)$, for some $r > N/2$, and satisfies

$$(5) \qquad \int_\Omega f(x)dx < 0 \quad \text{and} \quad \int_\Omega g(x)dx < 0.$$

We prove the following theorem:

Theorem 1. *Assume that* (H_1), (H_2) *and* (H_3) *hold. Then there is at least one solution* $(u, v) \in H^1(\Omega) \times H^1(\Omega)$ *of system* (4).

We will use topological methods for finding a solution to system (4). For that, we need *a priori* estimates about possible solutions. If $f, g \equiv 0$ then system (4) admits only the trivial solution $(u, v) = (0, 0)$. Hypothesis (5) is a necessary condition for the existence of solutions (in this case is also a sufficient condition) as to the existence of *a priori* estimate.

For example, if (u, v) is a solution of (4), using the test-function $\varphi_1 = 1$, we obtain

$$\int_\Omega f(x)dx = -\int_\Omega (v^+)^p dx \leq 0 \quad \text{and} \quad \int_\Omega g(x)dx = -\int_\Omega (u^+)^q dx \leq 0,$$

and so system (4) has a solution with non-trivial positive part if and only if (5) occurs.

Remark 1. *From the regularity theory, all weak solutions of* (4) *belongs to* $W^{2,r}(\Omega) \times W^{2,r}(\Omega)$. *Since* $r > N/2$ *we have* $W^{2,r}(\Omega) \subset C^0(\overline{\Omega})$ *and, for all* $u \in W^{2,r}(\Omega)$, *the following inequality is true*

$$(6) \qquad \|u\|_{W^{2,r}} \geq C|u|_\infty.$$

3 *A priori* estimates

Since $\lambda_1 = 0$ and associated eigenfunctions are constant functions, given $u, v \in H^1(\Omega)$, we can decompose

$$u = \overline{u} + \tilde{u} \quad \text{and} \quad v = \overline{v} + \tilde{v} \tag{7}$$

where $\overline{u}, \overline{v} \in \mathbb{R}$ are constants and $\displaystyle\int_\Omega \tilde{u}(x)dx = \int_\Omega \tilde{v}(x)dx = 0$.

Integrating the equations of (4) using $\varphi_1 = 1$ as a test-function, we obtain

$$\int_\Omega (v^+)^p dx + \int_\Omega f(x)dx = 0 \quad \text{and} \quad \int_\Omega (u^+)^q dx + \int_\Omega g(x)dx = 0 \tag{8}$$

$$\Rightarrow |v^+|_p^p \le |f|_1 \quad \text{and} \quad |u^+|_q^q \le |g|_1. \tag{9}$$

Theorem 2. *Assume the same hypotheses of Theorem 1 and let be $(u, v) \in H^1(\Omega) \times H^1(\Omega)$ a solution of system (4). Then there are constants $\sigma, \sigma' < 1$ and a continuous function $\gamma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, depending only on p, q, σ, σ' and Ω, such that $\gamma(0, 0) = 0$ and*

$$|\tilde{u}|_\infty + |\tilde{v}|_\infty \le \gamma(|f|_r, |g|_r). \tag{10}$$

Proof. Since

$$p, q < \frac{N}{N-2} = \frac{N/2}{(N/2) - 1},$$

by continuity, there is $\overline{r}$ such that $r > \overline{r} > N/2$ and

$$p, q < \frac{\overline{r}}{\overline{r} - 1} \Rightarrow \sigma = \frac{p(\overline{r} - 1)}{\overline{r}} < 1 \quad \text{and} \quad \sigma' = \frac{q(\overline{r} - 1)}{\overline{r}} < 1.$$

Using (8) and (9), we have

$$|\Delta \tilde{u}|_{\overline{r}} \le |(v^+)^p|_{\overline{r}} + |f|_{\overline{r}} \le |v^+|_\infty^\sigma |f|_1^{\frac{1}{\overline{r}}} + |f|_{\overline{r}}$$

and

$$|\Delta \tilde{v}|_{\overline{r}} \le |(u^+)^q|_{\overline{r}} + |g|_{\overline{r}} \le |u^+|_\infty^{\sigma'} |g|_1^{\frac{1}{\overline{r}}} + |g|_{\overline{r}}.$$

By Remark 1, u and v (also $\tilde{u}$ and $\tilde{v}$) satisfy inequality (6). Moreover, from [11] (p. 393), is valid the following inequalities

$$\|\tilde{u}\|_{W^{2,t}} \le C|\Delta u|_t, \quad \text{and} \quad \|\tilde{v}\|_{W^{2,t}} \le C|\Delta v|_t, \quad \text{for} \quad 1 < t < \infty. \tag{11}$$

Now, using inequalities (6) and (11), along with the fact that $\overline{r} > N/2$ and the results above, we get

$$|\tilde{u}|_\infty \le C\|\tilde{u}\|_{W^{2,\overline{r}}} \le C|\Delta u|_{\overline{r}} = C|\Delta \tilde{u}|_{\overline{r}} \le C|\overline{v}|^\sigma |f|_{\overline{r}}^{\frac{1}{\overline{r}}} + C|\tilde{v}|_\infty^\sigma |f|_{\overline{r}}^{\frac{1}{\overline{r}}} + C|f|_{\overline{r}}$$

and

$$|\tilde{v}|_\infty \le C\|\tilde{v}\|_{W^{2,\overline{r}}} \le C|\Delta v|_{\overline{r}} = C|\Delta \tilde{v}|_{\overline{r}} \le C|\overline{u}|^{\sigma'} |g|_{\overline{r}}^{\frac{1}{\overline{r}}} + C|\tilde{u}|_\infty^{\sigma'} |g|_{\overline{r}}^{\frac{1}{\overline{r}}} + C|g|_{\overline{r}}.$$

Hence, we use Young's inequality with σ, σ' and the immersion $L^r(\Omega) \hookrightarrow L^{\overline{r}}(\Omega)$, to get

$$(12) \qquad |\tilde{u}|_\infty + |\tilde{v}|_\infty \le \gamma_1(|f|_r + |g|_r) + \gamma_2(|g|_r)|\overline{u}|^{\sigma'} + \gamma_3(|f|_r)|\overline{v}|^\sigma,$$

where the functions $\gamma_i's$ are continuous of $\mathbb{R}^+$ in $\mathbb{R}^+$, depending only on Ω, σ and σ', such that $\gamma_i(0) = 0$, $i = 1, 2, 3$.

Claim 1. *Let $E \subset L^r(\Omega)$ be bounded and such that if $f \in E$, then $\int_\Omega f(x)dx < 0$. Then, there exists a constant $M > 0$ sufficiently large such that $\max\{|\overline{u}|, |\overline{v}|\} \le M$, for all solutions (u, v) of system (4) associated with a pair $(f, g) \in E$.*

Indeed, given $(f, g) \in E$, suppose there is a sequence of solutions $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ of (4) associated with (f, g) such that $\max\{|\overline{u}_k|, |\overline{v}_k|\} \to \infty$ when $k \to \infty$. Going to subsequences, if necessary, we have some cases to consider:

Case 1. $|\overline{u}_k| \le C$ and $|\overline{v}_k| \to \infty$.
Then

$$w_k = \frac{v_k}{|\overline{v}_k|} = \frac{\overline{v}_k}{|\overline{v}_k|} + \frac{\tilde{v}_k}{|\overline{v}_k|} \to \pm 1,$$

because $\tilde{v}_k/|\overline{v}_k| \to 0$, when $k \to \infty$. To see it, divide (12) by $|\overline{v}_k|$, use the fact that $|\overline{u}_k| \le C$ and the exponents on the right in (12) are smaller than 1.

Now, if there is a subsequence $\overline{v}_k \to -\infty$ then $w_k \to -1$. So, $v_k \to -\infty$ and $v_k^+ \equiv 0$ for k sufficiently large. This is an absurd, because for all k, by (8), we have

$$\int_\Omega (v_k^+)^p dx + \int_\Omega f(x)dx = 0$$

and f satisfies $\int_\Omega f(x)dx < 0$.

In the other hand, if there is a subsequence $\overline{v}_k \to +\infty$, using the same ideas above, we have $(v_k)^+ \to \infty$ for k sufficiently large and, by Fatou's Lemma

$$\liminf_{k\to\infty} \int_\Omega (v_k^+)^p dx = +\infty.$$

But, by (9), we have

$$|v_k^+|_p^p \leq |f|_1.$$

As Ω is bounded, we know that $L^r(\Omega) \hookrightarrow L^1(\Omega)$. And, as E is a bounded subset of $L^r(\Omega)$, we get a contradiction, because

$$|v_k^+|_p^p \leq |f|_1 \leq C_1|f|_r \leq C, \quad \forall k.$$

Case 2. $|\overline{v}_k| \leq C$ and $|\overline{u}_k| \to \infty$.
Analogous to Case 1.

Case 3. $|\overline{u}_k|, |\overline{v}_k| \to \infty$.
Some situations may occur:

3.1. $\overline{u}_k, \overline{v}_k \to -\infty$ or $\overline{u}_k, \overline{v}_k \to +\infty$.

Consider

$$w_k = \frac{u_k + v_k}{|\overline{u}_k + \overline{v}_k|} = \frac{\overline{u}_k + \overline{v}_k}{|\overline{u}_k + \overline{v}_k|} + \frac{\tilde{u}_k + \tilde{v}_k}{|\overline{u}_k + \overline{v}_k|}.$$

Note that $|\overline{u}_k + \overline{v}_k| \to +\infty$, when $k \to \infty$. As in previous cases, using (12), the second parcel of the sum above tends to zero. This implies $w_k \to \pm 1$, thus, for k sufficiently large, or $u_k + v_k >> 1$ or $u_k + v_k << -1$. Thus, we can conclude that at least one of the sequences $\{u_k\}$, $\{v_k\}$ is unlimited (tending to $\pm\infty$), and we obtain similar contradictions as in cases 1 and 2 using (8) and (9).

3.2. $\overline{u}_k \to -\infty$ and $\overline{v}_k \to +\infty$ (or, similarly, $\overline{u}_k \to +\infty$ and $\overline{v}_k \to -\infty$).

In this case consider (for k sufficiently large)

$$\begin{aligned} w_k = \frac{v_k - u_k}{|\overline{v}_k - \overline{u}_k|} &= \frac{\overline{v}_k + \tilde{v}_k - \overline{u}_k - \tilde{u}_k}{\overline{v}_k - \overline{u}_k} \\ &= \frac{\overline{v}_k - \overline{u}_k}{\overline{v}_k - \overline{u}_k} + \frac{\tilde{v}_k - \tilde{u}_k}{\overline{v}_k - \overline{u}_k} \\ &= 1 + \frac{\tilde{v}_k - \tilde{u}_k}{\overline{v}_k - \overline{u}_k}. \end{aligned}$$

For k sufficiently large

$$\left|\frac{\tilde{v}_k - \tilde{u}_k}{\overline{v}_k - \overline{u}_k}\right|_\infty \leq \frac{|\tilde{u}_k|_\infty + |\tilde{v}_k|_\infty}{\overline{v}_k - \overline{u}_k},$$

so, by (12)

$$\frac{\tilde{v}_k - \tilde{u}_k}{\overline{v}_k - \overline{u}_k} \to 0 \quad \text{when} \quad k \to \infty.$$

Hence, $v_k - u_k \to +\infty$ and, as previously, using (8) and (9), we have reached a contradiction. Therefore, the claim is proved.

By Claim 1, there exists $M > 0$ sufficiently large such that $\max\{|\overline{u}|, |\overline{v}|\} \leq M$, then

$$\begin{aligned} |\tilde{u}|_\infty + |\tilde{v}|_\infty &\leq \gamma_1(|f|_r + |g|_r) + \gamma_2(|g|_r)|\overline{u}|^{\sigma'} + \gamma_3(|f|_r)|\overline{v}|^{\sigma} \\ &\leq \gamma_1(|f|_r + |g|_r) + \gamma_2(|g|_r)M^{\sigma'} + \gamma_3(|f|_r)|M^{\sigma} \\ &= \gamma(|f|_r, |g|_r), \end{aligned}$$

where $\gamma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous depending only on Ω, σ and σ', and $\gamma(0) = 0$, and the theorem is proved. □

4 Proof of Theorem 1

Here we will use some ideas of [5] along with the results in [1, Section 3.4]. The main idea is to linearize system (4), show that all solutions of that system are non-degenerate and show that there is no characteristic values of the linearization between 0 and 1. The number of characteristic values in this range we call β. Finally, using the homotopic invariance of the degree and the relationship between the degree, the index of a non-degenerate solution and β, we conclude that (4) admits at least one solution.

We start proving that solutions (u, v) of (4) are non-degenerate and $\beta = 0$, as long f, g are "*sufficiently small*" and satisty hypothesis (H_3).

The linearization of (4), in some solution (u_0, v_0), is given by

$$\text{(13)} \qquad \begin{cases} -\Delta w = p(v_0^+)^{p-1} z, & x \in \Omega, \\ -\Delta z = q(u_0^+)^{q-1} w, & x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Both proofs that solutions are non-degenerate as the calculation of β will be consequences of following lemmas.

Lemma 1. *There exists $\epsilon > 0$ such that whatever $a, b \in L^\infty(\Omega)$, $c \in \mathbb{R}$ and $t \in [0,1]$ with $a, b \geq 0$ a.e., a, b not identical null, $|a|_\infty < \epsilon$, $|b|_\infty < \epsilon$ and $0 < c < \epsilon$, the system*

$$\left\{\begin{array}{ll} -\Delta w = ta(x)z + (1-t)cz, & x \in \Omega, \\ -\Delta z = tb(x)w + (1-t)cw, & x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \end{array}\right. \tag{14}$$

has only the trivial solution $w = z = 0$.

Proof. Suppose, by contradiction, that for all $\epsilon_n = 1/n$, there are sequences of functions not identical zero satisfying

$$0 \leq a_n(x), b_n(x) \leq \frac{1}{n},$$

and sequences of real numbers $t_n \in [0,1]$, $c_n \in (0, 1/n]$ such that (14) has non-trivial solution (w_n, z_n) for such choices of corresponding coefficients.

Consider the sequences

$$\tilde{w}_n = \frac{w_n}{|w_n|_2} \quad \text{and} \quad \tilde{z}_n = \frac{z_n}{|z_n|_2}.$$

We have two possibilities for $|z_n|_2/|w_n|_2$ as to their boundness.

(i) $|z_n|_2/|w_n|_2$ is bounded.

In this case, consider

$$\left\{\begin{array}{ll} -\Delta \tilde{w}_n = (t_n a_n(x) + (1-t_n)c_n)\dfrac{z_n}{|w_n|_2}, & x \in \Omega \\ \dfrac{\partial \tilde{w}_n}{\partial \nu} = 0, & x \in \partial\Omega. \end{array}\right. \tag{15}$$

Multiplying (15) for $\tilde{w}_n$, we obtain

$$-\Delta \tilde{w}_n . \tilde{w}_n = (t_n a_n(x) + (1-t_n)c_n)\frac{z_n}{|w_n|_2} . \tilde{w}_n.$$

Integrating

$$\begin{aligned} \int_\Omega |\nabla \tilde{w}_n|^2 dx &= \int_\Omega (t_n a_n(x) + (1-t_n)c_n)\frac{z_n}{|w_n|_2} dx \\ &\leq \frac{1}{n} \int_\Omega \frac{z_n}{|w_n|_2} \tilde{w}_n dx \\ &\leq \frac{1}{n} \frac{|z_n|_2}{|w_n|_2} . |\tilde{w}_n|_2 \leq \frac{1}{n} \frac{|z_n|_2}{|w_n|_2}. \end{aligned}$$

Then, $|\nabla \tilde{w}_n|_2 \to 0$ and, therefore, $\{\tilde{w}_n\}$ is bounded in $H^1(\Omega)$. Hence, there exists $\tilde{w} \in H^1(\Omega)$ such that, going to a subsequence, if necessary, we can assume that

$$\begin{aligned} \tilde{w}_n &\rightharpoonup \tilde{w}, \text{ in } H^1(\Omega) \text{ and} \\ \tilde{w}_n &\to \tilde{w}, \text{ in } L^2(\Omega). \end{aligned}$$

It follows that $|\tilde{w}|_2 = 1$.

Passing the limit (in the weak sense) in (15), we have

$$\begin{cases} -\Delta \tilde{w} = 0, & x \in \Omega \\ \dfrac{\partial \tilde{w}}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Therefore, $\tilde{w} = \pm\varphi_1 = \pm|\Omega|^{-\frac{1}{2}}$ and $\tilde{w}_n \to \pm\varphi_1$ in $H^1(\Omega)$ (strong convergence), because $|\nabla(\tilde{w}_n - (\pm\varphi_1))|_2 = |\nabla \tilde{w}_n|_2 \to 0$ and $\tilde{w}_n \to \pm\varphi_1$ in $L^2(\Omega)$, as we saw above. It also occurs that $\tilde{w}_n \to \pm\varphi_1$ in $C^0(\overline{\Omega})$. This is a consequence of the theory of elliptic equations applied regularly to the problem

$$\begin{cases} -\Delta(\tilde{w}_n \pm \varphi_1) = (t_n a_n(x) + (1-t_n)c_n)\dfrac{z_n}{|w_n|_2}, & x \in \Omega, \\ \dfrac{\partial(\tilde{w}_n \pm \varphi_1)}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Hence, $\tilde{w}_n$ (and therefore w_n) have defined signal for n sufficiently large.

By the other side, testing the second equation of system (14) with $\varphi_1 = |\Omega|^{-\frac{1}{2}}$, we deduce that

$$\int_\Omega (t_n b_n(x) + (1-t_n)c_n) w_n dx = 0,$$

what is a contradiction because w_n has defined signal and the term that is multiplying is non-negative and non-trivial.

(ii) $|z_n|_2/|w_n|_2 \to \infty$.

In this case, the sequence $|w_n|_2/|z_n|_2$ is bounded and we have the same situation of case (i) working now with the second equation of (14). □

Lemma 2. *Fix $\lambda_1 = 0 < c < \lambda_2$. Then the eingevalue problem with parameter μ*

$$\begin{cases} -\Delta w = \mu c z, & x \in \Omega, \\ -\Delta z = \mu c w, & x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{16}$$

has a single eingevalue μ in the range $[0,1]$, given by

$$\mu_1 = 0.$$

Proof. Write $w = \sum t_i\varphi_i$ and $z = \sum s_i\varphi_i$. Testing the first equation of system (16) with φ_j we obtain

$$-\Delta\left(\sum t_i\varphi_i\right)\varphi_j = \mu c\left(\sum s_i\varphi_i\right)\varphi_j \Rightarrow t_j\lambda_j = \mu c s_j$$
$$\Rightarrow \lambda_j t_j - \mu c s_j = 0$$

and, analogously, by the second equation, we get

$$\mu c t_j - \lambda_j s_j = 0.$$

So, considering the system with these two equations in order to find the eingefunctions of eingevalue problem (16), we must have

$$\det\begin{bmatrix} \lambda_j & -\mu c \\ \mu c & -\lambda_j \end{bmatrix} = 0.$$

What provides a double sequence of eigenvalues given by

$$\mu_j = \pm\frac{\lambda_j}{c}.$$

Therefore, as $\lambda_1 = 0$ and $0 < c < \lambda_2$, it follows that

$$0 = \mu_1 < 1 < \mu_2 = \frac{\lambda_2}{c}$$

and the result is proved. □

Conclusion of the proof of Theorem 1

First, let us introduce the following formulation via fixed-point theory. Let $T_{(f,g)} : C^0(\Omega)\times C^0(\Omega) \to C^0(\Omega)\times C^0(\Omega)$ be the map:

$$T_{(f,g)}(u,v) = ((-\Delta + I)^{-1}((v^+)^p + u + f), (-\Delta + I)^{-1}((u^+)^q + v + g)).$$

It follows that $T_{(f,g)}$ is continuous, compact and $T_{(f,g)}(u,v) = (u,v)$ if, and only if, (u,v) is a solution of (4).

Define f_1, g_1 as $f_1 = -(\eta\varphi_1)^p$ and $g_1 = -(\xi\varphi_1)^q$, with $\varphi_1 = |\Omega|^{-\frac{1}{2}}$ and $\xi, \eta > 0$. It is easy to check that $(u, v) = (\xi\varphi_1, \eta\varphi_1)$ is a solution of (4) with (f_1, g_1). Moreover, choosing ξ and η sufficiently small, we can assume that

$$|p(v^+)^{p-1}|_\infty, |q(u^+)^{q-1}|_\infty < \epsilon,$$

where ϵ is given by Lemma 1 and (u, v) is any arbitrary solution of (4) with (f_1, g_1). We can conclude that (u, v) is non-degenerate and, for some $R_{(u,v)}$ small enough,

$$\deg(Id - T_{(f_1,g_1)}, B_{[C^0(\Omega)]^2}((u, v), R_{(u,v)}), 0) = (-1)^{\beta(u,v)},$$

where $\beta(u, v)$ is the number of characteristic values of the problem

$$\left\{\begin{array}{ll} -\Delta w = \mu(p(v^+)^{p-1})z, & x \in \Omega, \\ -\Delta z = \mu(q(u^+)^{q-1})w, & x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \end{array}\right. \tag{17}$$

between 0 e 1. Furthermore, $\beta(u, v)$ can be calculated by the homotopy given in (14) and using Lemma 2 we deduce that $\beta(u, v) = 0$. Then, for R large enough,

$$\deg(Id - T_{(f_1,g_1)}, B_{[C^0(\Omega)]^2}(0, R), 0) = \sum(-1)^0 = \sum 1 \neq 0,$$

since that all solutions associated to (f_1, g_1) are non-degenerated, and so in a finite number in $B_{[C^0(\Omega)]^2}(0, R)$.

Now, consider the following homotopy

$$\left\{\begin{array}{ll} -\Delta u = (v^+)^p + (1-\tau)f + \tau f_1, & x \in \Omega \\ -\Delta v = (u^+)^q + (1-\tau)g + \tau g_1, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{array}\right. \tag{18}$$

with $0 \leq \tau \leq 1$. As we are considering a finite subset $\{f, f_1, g, g_1\}$ of $L^r(\Omega)$ (so, bounded), we can apply the *a priori* estimate from Theorem 2 (along Claim 1) to conclude that all solutions of system (18) are uniformly bounded in $C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ by some constant. Hence, for $R > 0$ sufficiently large,

$$\deg(Id - T_{(f,g)}, B_{[C^0(\Omega)]^2}(0, R), 0) = \deg(Id - T_{(f_1,g_1)}, B_{[C^0(\Omega)]^2}(0, R), 0),$$

which concludes the proof of Theorem 1.

□

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