A HOPF LEMMA FOR HOLOMORPHIC FUNCTIONS IN HARDY SPACES AND APPLICATIONS TO CR MAPPINGS

S. BERHANU AND J. HOUNIE

ABSTRACT. We present results on a local version of Hopf's Lemma and its application to the unique continuation problem for CR mappings between hypersurfaces.

CONTENTS

INTRODUCTION

Let $\Delta^+ = \{z \in \mathbb{C} : |z| < 1, \mathfrak{Im}\, z > 0\}$ be the half-disc and $E = \{x : -1 < x < 1\}$ its diameter. Let f be a holomorphic function on Δ^+. If $f(z) = o(z^n)$ for all positive integers n, we will say that f vanishes to infinite order at 0. When f is also assumed to be continuous up to E, some conditions on the image $f(E)$ are known that guarantee that $f \equiv 0$ on Δ^+. Results in this direction have been

2010 *Mathematics Subject Classification.* Primary 32H12; 30E25, 35A02; Secondary 30H10, 32A10.

Key words and phrases. Unique continuation, Holomorphic Hopf Lemma.

Work supported in part by NSF DMS 1600024, CNPq (grant 303634/2014-6) and FAPESP (grant 2012/03168-7).

obtained in [ABR], [BR1], [BR2], [HKMP], [HK] and [L]. Such results may be viewed as a local version of the classical Hopf Lemma and they have important applications to the unique continuation problem for CR mappings between hypersurfaces. The best result on such a localized Hopf Lemma is due to H. Alexander ([A]) who proved that if f as above is continuous up to E and the image $f(E)$ is a non-spiraling set (see Section 3 for the definition), then $f \equiv 0$ whenever it vanishes to infinite order at 0. A good example of a non-spiraling set is the complement in $\mathbb{C}$ of a ray emanating from the origin. All the previous results on Hopf's Lemma contained in [ABR], [BR1], [BR2], [HKMP], [HK] and [L] are consequences of Alexander's theorem.

In this paper we are concerned with extending Alexander's theorem to the case in which f is not necessarily continuous up to E and discuss applications to unique continuation for CR mappings. One of our results states that if f is holomorphic in Δ^+, has no zeros in Δ^+ and the imaginary part of $\log f$ is bounded on E, then $f \equiv 0$ whenever it vanishes to infinite order at 0. This is in sharp contrast to a function like $f(z) = e^{-\frac{1}{\sqrt{-iz}}}$ (take the main branch of the square root defined on the complement of the positive real axis) which is C^∞ on $\overline{\Delta^+}$ and vanishes to infinite order at 0. In this example, the curve $f(E)$ winds around the origin an infinite number of times. The boundedness assumption on the imaginary part of $\log f$ is weaker than Alexander's non-spiraling condition. In Section 3 we give an example where the imaginary part of $\log f$ is bounded but the non-spiraling condition is not satisfied. We also present a result similar to Alexander's theorem for f which is not necessarily continuous up to E but belongs to a Hardy space. The final section of the paper is devoted to applications to unique continuation for CR mappings between hypersurfaces.

1. Alexander's theorem

The following is a result of Huang et al.

Theorem 1.1. (Huang et al. [HKMP]). *If $f = u + iv$ is holomorphic in Δ^+, continuous up to $(-1, 1) \subset \partial H^+$ and such that*

$$|v(x)| \leq |u(x)|, \quad -1 < x < 1, \tag{1.1}$$

and $|f(z)| = O(|z|^k)$ *as* $H^+ \ni z \to 0$, $k = 1, 2, 3, \dots$, *then* $f \equiv 0$.

In the same paper they asked whether (1.1) could be weakened to

$$|v(x)| \le C|u(x)|, \quad -1 < x < 1,\ C > 0 \tag{1.2}$$

in the theorem without changing the conclusion. The answer is yes and it follows from a result, proved later by Alexander [A], that may be stated as follows.

Theorem 1.2. (Alexander [A]). *Let* $f = u + iv$ *be holomorphic in* $\Delta^+ \doteq \{z = x + iy \in \mathbb{C} : x^2 + y^2 < 1,\ y > 0\}$ *and continuous up to* $E \doteq (-1, 1) \subset \partial\Delta^+$. *Suppose that there is a non-spiraling set* F *such that*

$$f(E) \subset F, \tag{1.3}$$

$f(0) = 0$, *and* f *is not identically zero. Then for some positive number* $\theta > 0$ *it follows that for every non-tangential approach angle* Γ *there exists a constant* $C = C(\Gamma) > 0$ *such that*

$$|f(z)| \ge C|z|^{\theta}, \quad z \in \Gamma. \tag{1.4}$$

Roughly speaking, condition (1.3) requires that for any connected component $I_j = (\alpha_j, \beta_j)$ of $E \setminus \{f(x) = 0\}$, the argument function $\arg f(t)$, $t \in I_j$, is bounded with a bound independent of j (the precise definition of non-spiraling set will be given below in Section 2) and this is the key property in the proof of Theorem 1.2. Hence, (1.4) implies that f cannot vanish of infinite order at the origin unless it is identically zero and we may state the following uniqueness consequence:

Corollary 1.1. *Let* $f = u + iv$ *be holomorphic in* $\Delta^+ \doteq \{z = x + iy \in \mathbb{C} : x^2 + y^2 < 1,\ y > 0\}$ *and continuous up to* $E \doteq (-1, 1) \subset \partial\Delta^+$. *Suppose that there is a non-spiraling set* F *such that* (1.3) *holds. If* $|f(z)| = O(|z|^k)$ *as* $H^+ \ni z \to 0$, $k = 1, 2, 3, \dots$, *then* $f \equiv 0$.

When (1.2) holds, the bounded argument condition is satisfied with the bound π because $f(I_j)$ is contained in a half-plane for any j proving Theorem 1.1.

In this paper we address the problem of obtaining similar uniqueness results when the boundary values of f are not necessarily continuous and handle the case of holomorphic functions in Hardy spaces.

2. Uniqueness for functions in Hardy space

Let $\Delta \doteq \{z = x + iy \in \mathbb{C} : x^2 + y^2 < 1\}$ and $f \in H^p(\Delta)$, $0 < p \leq \infty$. We are interested in a local uniqueness boundary property around some point $z_0 = e^{i\theta_0} \in \partial\Delta$. According to F. Riesz's factorization theorem ([Du, Theorem 2.5]), if $f \neq 0$ we may write $f(z) = F(z)B(z)$ where $B(z)$ is the Blaschke product determined by the zeros of $f(z)$, $F(z)$ has no zeros and $\|F\|_{H^p} = \|f\|_{H^p}$. Set

$$U(z) = \log F(z).$$

The boundary value of the harmonic function $\log|F(z)| = \mathfrak{Re}\, U$ is a measure that can be decomposed as $\log|f^*(e^{i\theta})| + \mu$ where f^* is defined a.e. by the radial limit $f^*(e^{i\theta}) = \lim_{r\nearrow 1} f(re^{i\theta})$, $\log|f^*| \in L^1(\partial\Delta)$ and μ is a singular measure on $\partial\Delta$. Denoting by $h^p(\partial\Delta)$ the local Hardy space of Goldberg [G] (which can be defined on any compact manifold), it is a known fact that $\mathfrak{Re}\, bU$ is in $h^p(\partial\Delta)$ for any $0 < p < 1$ and so is its Hilbert transform $\mathfrak{Im}\, bU$. Therefore, $bU \in h^p(\partial\Delta)$ and, since U is holomorphic, this implies that $U(z) \in H^p(\Delta)$ for all $0 < p < 1$.

Our basic assumption on $f(z)$ is that for some $\varepsilon > 0$

$$\mathfrak{Im}\, bU \in L^\infty(\theta_0 - \varepsilon, \theta_0 + \varepsilon). \tag{2.1}$$

By the pseudo-local property of the Hilbert transform it follows that for some open interval I, $\theta_0 \in I \subset (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$, $\mathfrak{Re}\, bU \in bmo(I) \subset h^p(I)$, $0 < p < \infty$, in particular $bU \in h^1(I)$.

Suppose that $B(z)$ has a finite number of Blaschke factors, so $f(z) \neq 0$ if $z \in \Delta$ is close enough to z_0. After a conformal transformation of $\Delta \cap \{|z - z_0| < \varepsilon\}$, where $\varepsilon > 0$ is taken sufficiently small, we may assume that we are in the following situation: $f(z)$ is a holomorphic function in $\Delta^+ = \{z = x + iy \in \mathbb{C} : |z| < 1,\ y > 0\}$ satisfying:

- $f(z)$ does not vanish in Δ^+;
- $U(z) = \log f(z)$ has a boundary value bU such that $\mathfrak{Im}\, bU \in L^\infty(-1, 1)$;

Fix $0 < r_1 < r_2 < 1$, $\varepsilon > 0$, and consider the annular sector

$$R_\varepsilon = \{re^{i\theta} : \ r \in (r_1, r_2), \ \varepsilon < \theta < \pi - \varepsilon\},$$

with boundary ∂R_ε. For a.e. $r \in (0,1)$, the limits

$$\lim_{\varepsilon \searrow 0} \mathfrak{Im}\, U(re^{i\varepsilon}) \text{ and } \lim_{\varepsilon \searrow 0} \mathfrak{Im}\, U(re^{i(\pi-\varepsilon)})$$

exist simultaneously and are bounded by $\|\mathfrak{Im}\, bU\|_{L^\infty}$. We will always choose r_1 and r_2 satisfying this property which will be important in what follows.

Consider a circular arc $\Gamma_{r,\varepsilon} = \{re^{i\theta}, \ \varepsilon < \theta < \pi - \varepsilon\}$, $0 < r < 1$. Since the radial and non-tangential limits to the boundary value bU coincide a.e., the limit

$$I(r) \doteq \lim_{\varepsilon \searrow 0} \mathfrak{Im} \int_{\Gamma_{r,\varepsilon}} U'(z)\, dz = \mathfrak{Im}\, bU(-r) - \mathfrak{Im}\, bU(r) \quad \text{exists.} \tag{2.2}$$

This has the following geometric interpretation: the integral

$$\mathfrak{Im} \int_{\Gamma_{r,\varepsilon}} U'(z)\, dz = \mathfrak{Im} \int_{\Gamma_{r,\varepsilon}} \frac{f'(z)}{f(z)}\, dz = \mathfrak{Im} \int_{f \circ \Gamma_{r,\varepsilon}} \frac{1}{\zeta}\, d\zeta$$

measures the argument variation along the curve $f \circ \Gamma_{r,\varepsilon}$ and this variation remains bounded as $\varepsilon \searrow 0$, i.e., for almost every r, $f \circ \Gamma_{r,\varepsilon}$ does not wind around the origin infinitely many times. In particular,

$$|I(r)| \leq 2\, \|\mathfrak{Im}\, bU\|_{L^\infty}. \tag{2.3}$$

At this point, one may follow the arguments of [L, Lemma 1] that estimate the integral $\int_{r_1}^{r_2} I(r)\, r^{-1}\, dr$ by making use of the Cauchy-Riemann equations on the region R_ε. To do this, write $U = u + iv$, note that $u_r = v_\theta / r$ and consider the identity

$$\int_{R_\varepsilon} \frac{v_\theta}{r}\, dr d\theta = \int_{R_\varepsilon} u_r\, dr d\theta = \int_\varepsilon^{\pi-\varepsilon} (u(r_2 e^{i\theta}) - u(r_1 e^{i\theta}))\, d\theta.$$

The non-tangential limit $bU(x)$ exists for a.e. $x \in (-1,1)$, in particular, the continuous function $(0,\pi) \ni \theta \mapsto u(re^{i\theta})$ has limits as $\theta \searrow 0$ or $\theta \nearrow \pi$ for a.e. $r \in (0,1)$ and is bounded (with a bound that depends on r). Letting $\varepsilon \searrow 0$ we get

$$\int_{r_1}^{r_2} \frac{I(r)}{r}\, dr = \int_0^{\pi} u(r_2 e^{i\theta}) - u(r_1 e^{i\theta})\, d\theta$$

$$= c(r_2) - \int_0^{\pi} \log|f(r_1 e^{i\theta})|\, d\theta. \tag{2.4}$$

If we assume that for some $r_2 > 0$, $k \in \mathbb{Z}^+$ and $K > 0$ the estimate

$$\frac{|f(z)|}{|z|^k} \leq K, \quad z \in \Delta^+ \cap \{|z| < r_2\},$$

holds, then (2.3) and (2.4) give

$$2\,\|\mathfrak{Im}\, bU\|_{L^\infty} \log(r_2/r_1) \geq \int_{r_1}^{r_2} \frac{I(r)}{r}\, dr$$
$$\geq c(r_2) - \log K + k\log(1/r_1)$$

and letting $r_1 \searrow 0$ we conclude that $k \leq 2\,\|bU\|_{L^\infty}$. Hence, for $k > 2\,\|\mathfrak{Im}\, bU\|_{L^\infty}$ the quotient

$$\frac{|f(z)|}{|z|^k}$$

cannot remain bounded in $\Delta^+ \cap \{|z| < \delta\}$ for any $\delta > 0$. This can be restated as a uniqueness property by saying that if $f(z)$ vanishes to infinite order at $z = 0$ it must vanish identically.

3. A generalized weak holomorphic Hopf lemma

The arguments in the previous section show a boundary point uniqueness property for $f(z) = F(z)B(z) = e^{U(z)}B(z) \in H^p(\Delta)$ that we may state as

Theorem 3.1. *Assume that $f(z) = e^{U(z)}B(z) \in H^p(\Delta)$ and let $z_0 = e^{i\theta_0} \in \partial\Delta$. Assume that for some $\varepsilon > 0$*

$$f(z) \neq 0 \text{ for } z \in \Delta \cap \{|z - z_0| < \varepsilon\}; \tag{3.1}$$

$$\mathfrak{Im}\, bU \in L^\infty\big((\theta_0 - \varepsilon, \theta_0 + \varepsilon)\big). \tag{3.2}$$

Then for some $k \in \mathbb{N}$ we have

$$\limsup_{\Delta \ni z \to z_0} \frac{|f(z)|}{|z - z_0|^k} = \infty. \tag{3.3}$$

As it is customary, we identify $\partial\Delta = \{e^{i\theta}\}$ with the closed interval $[0, 2\pi]$ with $\{0\} \simeq \{2\pi\}$ and functions defined on $\partial\Delta$ are denoted as functions of the variable θ.

We now discuss condition (3.2) in more detail. As mentioned at the beginning of Section 2, $\mathfrak{Re}\, U = \log|f^*| + \mu$ where $\log|f^*| \in L^1(\partial\Delta)$ and μ is a singular measure. Suppose that instead of (3.2) we assume that

$$\mathfrak{Im}\, bU \in L^\infty\big((\theta_0 - \varepsilon, \theta_0) \cup (\theta_0, \theta_0 + \varepsilon)\big) \tag{$\bullet$}$$

which means that $\mathfrak{Im}\, bU = g + \nu$ with $g \in L^\infty\big((\theta_0 - \varepsilon, \theta_0 + \varepsilon)\big)$ and ν supported in $\{\theta_0\}$. We will see that (3.2) is implied by the seemingly weaker condition ($\bullet$). In other words, under this hypothesis, taking the weak limit of the restrictions of $U(z)$ to $|z| = r$ given by $U(re^{i\theta})$ and letting $r \nearrow 1$, the boundary value does not pick up a distribution supported in $\{\theta_0\}$.

Lemma 3.1. *Let $U = u + iv \in H^p(\Delta)$ and assume that $bu = h + \mu$ with $h \in L^1(\partial\Delta)$ and μ a singular measure. Then*

$$bv \in L^\infty\big(\partial\Delta \setminus \{\theta_0\}\big) \Longrightarrow bv \in L^\infty(\partial\Delta). \tag{3.4}$$

Proof: Since bv is bounded in $\partial\Delta \setminus \{\theta_0\}$, bu is locally integrable in $\partial\Delta \setminus \{\theta_0\}$ by the pseudolocal property of the Hilbert transform. Hence, $bu = h + \mu$ with the measure μ concentrated on the set $\{\theta_0\}$.

Take $g \in L^\infty(\partial\Delta)$ such that $\nu \doteq bv - g$ is supported in $\{\theta_0\}$ and consider the holomorphic function

$$\tilde{U}(z) = \tilde{u}(z) + i\tilde{v}(z) = \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} g(e^{i\theta})\, d\theta, \quad z \in \Delta,$$

given by the holomorphic completion of Poisson's integral.

Then $b\tilde{u} \in bmo(\partial\Delta) \subset L^1(\Delta)$ and $b\tilde{v} = g$. It follows that $b(u - \tilde{u}) \in L^1\big(\partial\Delta \setminus \{\theta_0\}\big)$ and $b(v - \tilde{v})$ is supported in $\{\theta_0\}$. We conclude that there exists a polynomial $p(\zeta)$ without constant term such that

$$(U - \tilde{U})(z) = p(1/(z - z_0)), \quad z \in \Delta.$$

Since $b(U - \tilde{U}) \in L^1\big(\partial\Delta \setminus \{\theta_0\}\big)$, the degree of $p(\zeta)$ must be identically zero and so $bv = b\tilde{v} = \nu \in L^\infty(\partial\Delta)$ as we wished to prove. $\square$

This result may be localized as follows.

Lemma 3.2. *Let $U = u + iv$ be a holomorphic function on Δ with tempered growth at the boundary and assume that $bu = h + \mu$ with $h \in L^1(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ and μ a singular measure. Then*

$$bv \in L^\infty\big((\theta_0 - \varepsilon, \theta_0) \cup (\theta_0, \theta_0 + \varepsilon)\big) \Longleftrightarrow bv \in L^\infty(\theta_0 - \varepsilon, \theta_0 + \varepsilon). \tag{3.5}$$

Proof: We may find $g \in L^\infty(\partial\Delta \setminus \{\theta_0\})$ supported in $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ that coincides with v on a neighborhood of θ_0 and apply the complex form of Poisson's formula to g to get a holomorphic function $\tilde{U} = \tilde{u} + i\tilde{v}$ such that $b\tilde{u} \in L^1(\partial\Delta)$ and $b\tilde{v} = g$. Then $(U - \tilde{U})(z)$ is smooth up to the boundary for z close to z_0 and $\tilde{U}$ satisfies the hypothesis of Lemma 3.1 so (3.5) follows. ☐

Theorem 3.1 and Lemma 3.2 give

Corollary 3.1. *Assume that $f(z) = e^{U(z)}B(z) \in H^p(\Delta)$ and let $z_0 = e^{i\theta_0} \in \partial\Delta$. Assume that for some $\varepsilon > 0$*

$$f(z) \neq 0 \text{ for } z \in \Delta \cap \{|z - z_0| < \varepsilon\} \text{ and}$$
$$\mathfrak{Im}\, bU \in L^\infty\big((\theta_0 - \varepsilon, \theta_0) \cup (\theta_0, \theta_0 + \varepsilon)\big).$$

Then for some $k \in \mathbb{N}$ we have

$$\limsup_{\Delta \ni z \to z_0} \frac{|f(z)|}{|z - z_0|^k} = \infty.$$

3.1. **Connections with Alexander's theorem.** In order to compare Theorem 3.1 with Theorem 1.2 we now recall the notion of non-spiraling set defined in [A]. A closed set $F \subset \mathbb{C}$ is said to be *non-spiraling (at the origin)* if there exists a compact Jordan arc γ in $\mathbb{C}$ and an open neighborhood of the origin V such that

(a) γ has no point in common with F except for the origin which is an endpoint of γ and
(b) $V \setminus \gamma$ is connected and simply connected and there exists an analytic branch of the logarithm of z on $V \setminus \gamma$ with its imaginary part bounded above and below on $V \setminus \gamma$.

Although condition (3.2) in Theorem 3.1 and condition (1.3) in Theorem 1.2 do not look alike, we will see that the former is essentially a generalization of the latter that makes sense when the boundary values of $f(z)$ are not continuous. Note that if $bf \in L^p(\partial\Delta)$ is a non continuous function, requiring that an arc $I \subset \partial\Delta$ be mapped by bf into a non-spiraling set is of little value. Indeed, when bf is continuous, (1.3) imposes that the continuous arc $bf(I)$ does not cross the Jordan arc γ in the definition of non-spiraling set but that property looses its strength as soon as bf is allowed to be modified on a set of null measure. On the other hand, assume that $f(z)$ is holomorphic in Δ^+ and continuous on $\overline{\Delta^+}$ and $f(z)$ only vanishes at $z = 0$. This allows us to consider (a branch of) the function $U = \log f$. It follows from part (b) of the definition of non-spiraling set that after some shrinking (1.3) implies that $\mathfrak{Im}\, bU(x) \in L^\infty(-\varepsilon, \varepsilon)$. Hence, the latter condition (that corresponds to (3.2) in Theorem 3.1) may be regarded as a weaker form of the condition $f(E) \subset F$ in (1.3).

Observe that in Theorem 3.1 it is assumed that $f(z)$ does not vanish in Δ (by a localization argument it would have been enough to assume that the zero set of $f(z)$ in Δ lies at a positive distance from z_0) while this is not assumed in Alexander's theorem. However, as shown in [A] it is a non-trivial consequence of the non-spiraling condition that the interior zeros of $f(z)$ are bounded away from the boundary point under study. Considering that the situation in Alexander's theorem is precisely the absence of zeros and that for non continuous boundary values it does not look feasible to impose hypotheses on bf that would preclude the existence of interior zeros of f, it seems a reasonable assumption to assume the absence of zeros. Furthermore, under our assumptions, even for continuous bf we are able to rule out that f vanishes to infinite order at the relevant boundary point in cases that are not covered by Alexander's theorem because the spiraling condition is not satisfied (see Example 3.1 below).

Finally, we also point out that the conclusion (3.3) in Theorem 3.1, although clearly weaker than estimate (1.4) stated in Alexander's theorem, it is nevertheless strong enough to imply that if $f(z)$ vanishes to infinite order at the boundary point it must vanish identically (boundary uniqueness property).

Let us now compare conditions (1.3) and (3.2) in the specific case in which we know a priori that bf is continuous. Suppose that $f(z)$ is holomorphic in Δ^+ and continuous on $\overline{\Delta^+}$ and $f(z)$ only vanishes at $z = 0$. This allows us to consider (a branch of) the function $U = \log f$. Assume that $\mathfrak{Im}\, bU(x) \in L^\infty(-1, 1)$. Denoting by Δ_r^+ the half disk of radius $r > 0$, the assumption implies that $\mathfrak{Im}\, bU(x) \in L^\infty(\Delta_r^+)$ for $0 < r < 1$. Hence, we may find $h(z)$ holomorphic in Δ_r^+ and continuous up to the boundary such that $f(z) = h(z)^n$ for some $n \in \mathbb{N}$ and $h(-r, r)$ contained in a half-plane. Since a closed half-plane is obviously non-spiraling we may apply Alexander's theorem to $h(z)$ and conclude that on any sufficiently short acute truncated sector we have $|f(z)|^{1/n} = \sup |h(z)| \geq C|z|^m$ and therefore $|f(z)| \geq C^n|z|^{nm}$. We will give below an example of the situation just described in which $f(z)$ itself does not satisfy the non-spiraling condition. This will show that, for continuous boundary values bf, condition (1.3) is a strictly stronger hypothesis than condition (3.2) while both imply local estimates from below of the type $|f(z)| \geq C|z|^\theta$ on cones of aperture $< \pi$.

Example 3.1: Set $a = \cos(\pi/4) = \sqrt{2}/2$ and consider the sequence of points

$$A_k = a_k = a^k, \quad B_k = a_{k+1} + ia_{k+1} = A_k e^{i\pi/4}, \quad k = 0, 1, 2 \ldots$$

Define a Jordan arc by joining each point A_k to B_k by the circular arc $\{a_k e^{i\theta} : \; 0 \leq \theta \leq \pi/4\}$ and each point B_k to A_{k+1} by the vertical segment $[B_k, A_{k+1}]$. This arc is contained in the first quadrant and joins $z = 1$ to $z = 0$. To obtain a closed Jordan curve γ, join the origin to $z = 1$ by some Jordan arc contained in the fourth quadrant (e.g., take the arc of parabola $x + ix(x - 1)$, $0 < x < 1$). Now let $h(z)$ a Riemann mapping of Δ^+ to the region D bounded by γ. By Carathéodory's theorem, $h(z)$ extends to a homeomorphism from $\overline{\Delta^+}$ onto $\overline{D}$. Assume without loss of generality that $h(0) = 0$. Since the image of $h(z)$ is contained in a half-plane it satisfies the hypothesis of Alexander's theorem and in particular the variation of the argument of $h(z)$ is bounded. If $f(z) = h(z)^n$, the variation of the angle of $f(z)$ will be also bounded for any fixed $n \in \mathbb{N}$. On the other hand, if $n > 8$ each one of the circular arcs $\{a_k e^{i\theta} :\ 0 \le \theta \le \pi/4\}$ will be mapped by $z \mapsto z^n$ onto the closed circle $|z| = a_k$. If follows that for any $r > 0$, $f(-r, r)$ will contain infinitely many closed circles of arbitrary small radii and $f(-r, r)$ cannot be contained in the complement of some curve stemming from the origin as in the definition of non-spiraling set. Thus, $f(z)$ does not satisfy the hypothesis of Alexander's theorem although its argument function is bounded and therefore it satisfies estimates like $|f(z)| \ge C|z|^m$ on acute sectors.

4. A limiting non-spiraling condition

In this section we introduce a non-spiraling condition quite similar to Alexander's non-spiraling condition but it makes sense and can be defined for the larger class of functions in Hardy space. It will be convenient to work in the setup of the half-disk $\Delta_r^+ \doteq \{z = x + iy \in \mathbb{C} : x^2 + y^2 < r,\ y > 0\}$ and denote $E_r \doteq (-r, r) \subset \partial\Delta^+$ as before. Given $f(z)$ in $H^p(\Delta_r^+)$ define

$$S_f = \{w \in \mathbb{C} :\ \exists\ (z_k) \in \Delta_r^+,\ z_k \to x \in [-r, r] \text{ such that } f(z_k) \to w\}.$$

Note that when $f(z)$ is continuous up to $[-r, r]$ we have $S = f\big([-r, r]\big)$.

Theorem 4.1. *Let $f = u + iv \in H^p(\Delta_r^+)$, $0 < p \le \infty$. Suppose that there is a non-spiraling set F such that*

$$S_f \subset F, \tag{4.1}$$

the non-tangential limit n.t. $\lim_{z\to 0} f(z) = 0$ *and f is not identically zero. Then there is $\theta > 0$ such that for every non-tangential approach angle Γ there exists a $C > 0$ with*

$$|f(z)| \ge C|z|^\theta, \quad z \in \Gamma. \tag{4.2}$$

Remark 4.1: It will be shown in the course of the proof that condition (4.1) implies that, after shrinking $r > 0$, $f(z)$ does not vanish in Δ_r^+ and $\mathfrak{Im}\,\log|f(z)|$ is bounded, so condition (4.1) implies the hypotheses (3.1) and (3.2) assumed in Theorem 3.1. On the other hand, the conclusion

(4.2) is stronger than (3.3). Furthermore, the geometric meaning of condition (4.1) makes it easier to check it in the applications.

PROOF: We will follow Alexander and first show that for s small enough, f has no zeros in Δ_s^+. By shrinking r if necessary, we may assume that f is continuous on $T_r = \{z \in \partial\Delta_r^+, |z| = r\}$ and never zero there. Since the set S_f and $f(T_r)$ are closed, near the origin, after shrinking γ, $\gamma \setminus \{0\}$ is disjoint from the set $A = f(T_r) \cup S_f$. After shrinking γ, we may assume that $\gamma \setminus \{0\}$ is contained in a component Ω of $\mathbb{C} \setminus A$. We will next show that $f : \Delta_r^+ \cap f^{-1}(\Omega) \to \Omega$ is a proper map. Suppose a sequence $z_j \in \Delta_r^+ \cap f^{-1}(\Omega)$ converges to $z_0 \in \partial(\Delta_r^+ \cap f^{-1}(\Omega))$ and $f(z_j)$ converges to w_0. We need to show that $w_0 \notin \Omega$. Assume $w_0 \in \Omega$. If $z_0 \in \Delta_r^+$ then $f(z_j) \to f(z_0)$ and so $f(z_0) = w_0$ implying that $z_0 \in \Delta_r^+ \cap f^{-1}(\Omega)$, contradicting the fact that $z_0 \in \partial(\Delta_r^+ \cap f^{-1}(\Omega))$. If $z_0 \in T_r$, then $f(z_0) = w_0 \in f(T_r) \cap \Omega$ contradicting the disjointness of Ω and $f(T_r)$. Finally, $z_0 \in [-r, r]$ implies that $w_0 \in S_f \cap \Omega$ which is also impossible since S_f and Ω are disjoint. Thus $w_0 \notin \Omega$ and so $f : \Delta_r^+ \cap f^{-1}(\Omega) \to \Omega$ is proper.

This map is a branched covering of Ω of multiplicity m and we may assume that m is minimal and that f has no zeros in Δ_r^+. If $m = 0$, then $f(\Delta_r^+)$ will be disjoint from $\gamma \setminus \{0\}$ and therefore, $\log f$ will be defined on Δ_r^+ with a bounded imaginary part and we will be done. Assume therefore that $m \geq 1$ and that γ contains no critical values of f on Δ_r^+. Let σ be one of the m pre images of $\gamma \setminus \{0\}$ and K its cluster set. Since γ is disjoint from $f(T_r)$, the set $K \subset [-r, r]$.

We show next that if $z_n \in \sigma$ is a sequence converging to a point in K, then $f(z_n)$ converges to 0. It suffices to show that every subsequence $\{f(z_{n_k})\}$ has a further subsequence that converges to 0. Since the sequence $f(z_{n_k})$ is bounded, it has a subsequence that converges to a point w which is in $S_f \cap \overline{\gamma}$ and hence $w = 0$. The set K is connected. Suppose an interval $[a, b] \subset K$ with $a < b$. Let $c \in (a, b)$ such that f has a nonzero non tangential limit L at c. Since the points a and b are cluster points of σ, there is a sequence $\{c_n\} \subset \sigma \cap \{z : \mathfrak{Re}\, z = c\}$ such that $c_n \to c$. But then $f(c_n)$ converges to $L \neq 0$, which is a contradiction. Therefore, K is a point. The rest of the proof is the same as that of Alexander. $\square$

In the next section it will be convenient to have a version of Theorem 4.1 in which role of the half-disk $\Delta_r^+ \doteq \{z = x + iy \in \mathbb{C} : x^2 + y^2 < r,\ y > 0\}$ is played by the disk $\Delta_r \doteq \{z = x + iy \in \mathbb{C} : x^2 + y^2 < r\}$ and $E_r \doteq \{re^{i\theta} :\ -\pi/2 < \theta < \pi/2\} \subset \partial\Delta_r$. Given a function $f(z) \in H^p(\Delta_r)$ we set

$$S_f = \{w \in \mathbb{C} :\ \exists\ (z_k) \in \Delta_r,\ z_k \to z \in E_r \text{ such that } f(z_k) \to w\}.$$

We then have

Corollary 4.1. *Let $f = u + iv \in H^p(\Delta_r)$, $0 < p \leq \infty$. Suppose that there is a non-spiraling set F such that*

$$S_f \subset F, \tag{4.3}$$

the non-tangential limit n.t. $\lim_{z\to r} f(z) = 0$ and f is not identically zero. Then for every non-tangential approach angle Γ with vertex at $z = r$ there exists a $C > 0$ and $\theta > 0$ such that

$$|f(z)| \geq C|z - r|^\theta, \quad z \in \Gamma.$$

Hence, if (4.3) holds and

$$|f(z)| = O(|z - r|^k), \quad z \in \Gamma, \tag{4.4}$$

for some cone Γ with vertex at $z = r$ and every $k \in \mathbb{N}$, then $f \equiv 0$.

We will next present an example where f is not continuous on $\overline{\Delta_r}$ but it satisfies the hypotheses in the corollary.

Example 4.1: Let $D = \{z \in \mathbb{C} :\ |z| < 1\}$. Let $B(z)$ be a Blaschke product on D whose zeros accumulate on a dense set $T \subset \partial D$. For some $0 < a < \frac{1}{2}$ consider the function $f(z) = (z + 1)^a(B(z) + 1)$ for some branch of z^a in D. Then $f(z) \to 0$ as $z \to -1$ and the set $f(D)$ intersects a sector about the negative x- axis only at the origin. We therefore have many choices of F and γ and the set S_f satisfies the condition in the corollary. The function f is discontinuous at every point on ∂D except at $z = -1$.

5. Examples and applications

A well known problem asks when a CR mapping between two hypersurfaces has the unique continuation property. The problem is open even for mappings between two real analytic hypersurfaces of finite type. The papers [HK], [HKMP], [ABR], [BR1], [BR2], [BL], [A], [P], and [E] contain several results on this problem. Here we will indicate that the main results in all these works follow easily from Alexander's theorem. We will also present two new applications of this theorem. We begin by recalling the main results in the aforementioned papers. Let M be a totally real submanifold of $\mathbb{C}^n$ of class $C^{1,\alpha}$ $(0 < \alpha < 1)$ of dimension n, $0 \in M$. We may assume that in a neighborhood Ω of the origin in $\mathbb{C}^n$, M is given by

$$M = \{w \in \mathbb{C}^n : \mathfrak{Im}\, w = \varphi(\mathfrak{Re}\, w)\},\ \varphi \in C^{1,\alpha}(U),$$

where $U \subset \mathbb{R}^n$ is a neighborhood of 0, φ is real and $\varphi(0) = \varphi'(0) = 0$. A wedge $\mathcal{W}$ with edge M is a set of the form

$$\mathcal{W} = \{w \in \Omega : \mathfrak{Im}\, w - \varphi(\mathfrak{Re}\, w) \in \Gamma\},$$

where $\Gamma \subset \mathbb{R}^n$ is a convex open cone. The main result in [ABR] is as follows:

Theorem 5.1 ([ABR, Theorem 2])**.** *Let $\mathcal{W}$ and M be as above, with M of class C^2 and $h : \overline{\mathcal{W}} \to \mathbb{C}^k$ continuous, holomorphic on $\mathcal{W}$ satisfying:*

(1) for every $N \in \mathbb{N}$, there exists C_N such that $|h(w)| \leq C_N |w|^N$ and

(2) $h(M) \subset M'$, with M' a totally real submanifold of $\mathbb{C}^k$ of class C^2.

Then $h \equiv 0$ in the connected component of 0 in $\overline{\mathcal{W}}$.

We will present a short proof of a strengthened version which can be stated as follows:

Theorem 5.2. *Let $\mathcal{W}$ and M be as above, M of class $C^{1,\alpha}$ and $h : \mathcal{W} \to \mathbb{C}^k$ holomorphic satisfying:*

(1) for every $N \in \mathbb{N}$, there exists C_N such that $|h(w)| \leq C_N |w|^N$ and

(2) whenever $w_j \in \mathcal{W}$ is a sequence converging to a point in M and $h(w_j)$ converges to some $z \in \mathbb{C}^k$, then $z \in M'$, with M' a totally real submanifold of $\mathbb{C}^k$ of class C^1.

Then $h \equiv 0$ in the connected component of 0 in $\overline{\mathcal{W}}$.

Proof: Let $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Since M is of class $C^{1,\alpha}$, by [BER, Theorem 7.4.12], there is a family of analytic discs $A : \overline{\Delta} \to \overline{\mathcal{W}}$ such that the interiors $A(\Delta)$ fill an open set in $\mathcal{W}$ and the sets $\gamma = \{A(e^{i\theta}) : |\theta| < \frac{\pi}{2}\}$ lie in M, with $A(1) = 0$. We can get holomorphic coordinates $z = x + iy = (z', z'') \in \mathbb{C}^r \times \mathbb{C}^{k-r}$ with $z' = x' + iy'$ such that M' near 0 is given locally by $\{y' = \psi_1(x'),\, z'' = \psi_2(x')\}$ where $\psi_j(0) = 0, d\psi_j(0) = 0,\ j = 1, 2$. Write $h = (h_1, \ldots, h_k)$ and, as in Example 3 in [A], let $F(z) = \sum_{j=1}^{k} {z_j}^2$ so $F \circ h = \sum_{j=1}^{k} h_j^2$. Then after shrinking M' near the origin, $F(M')$ lies in the closed right half plane which is a non-spiraling set.

Set $f \doteq F \circ h \circ A \in H^\infty(\Delta)$, consider a sequence $\zeta_\ell \in \Delta$ such that $A(\zeta_\ell)$ converges to some point in M and suppose that $f(\zeta_\ell) \doteq F \circ h \circ A(\zeta_\ell)$ converges to a complex number $\tilde{\zeta}$. Then since h is bounded on $\mathcal{W}$, a subsequence $h(A(\zeta_{j_m}))$ converges to some $z \in \mathbb{C}^k$. Since $A(\zeta_{j_m})$ converges to a point in M it follows from assumption (2) that $z \in M'$. Hence, $\tilde{\zeta} = F(z)$ is in the right half plane. Therefore, the set S_f (defined right before Corollary 4.1) is contained in a non-spiraling set and an application of Corollary 4.1 to f shows that $f = F \circ h \circ A \equiv 0$.

Consider now a sequence $\zeta_\ell = r_\ell e^{i\theta} \in \Delta$, $r_\ell \nearrow 1$, such that $h \circ A(r_\ell e^{i\theta})$ converges to some w. Then since $w \in M'$, $F(w) = 0$, and $\{F(z) = 0\} \cap M' = \{0\}$ because $|y| < |x|$ on $M' \setminus \{0\}$, we conclude that $w = 0$. Thus the non tangential limit of $h \circ A$ is zero a.e. which in turn implies that $h \circ A \equiv 0$. Since this holds for any one of the maps A in the family we conclude that $h \equiv 0$. $\square$

The main result in the work [BL] is the following theorem:

Theorem 5.3 ([BL, Theorem 2]). *Suppose that M_1 and M_2 are C^∞ hypersurfaces in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, and suppose there is a point $z_0 \in M_1$ and a ball B centered at z_0 such that $B \setminus M_1$ consists of exactly two connected components, B^+ and B^-. Suppose that there is* *a holomorphic mapping f defined on B^+, which extends continuously to M_1 such that the extension maps M_1 into M_2. If M_2 is Levi flat, then $\mathcal{N} \circ f$ extends C^∞ smoothly up to M_1 near z_0 for any flattened complex normal coordinate $\mathcal{N}$ for M_2. Furthermore, if f does not map B^+ into M_2, then $\mathcal{N} \circ f$ cannot be flat at z_0, and hence, f cannot be flat at z_0.*

The preceding result was generalized in the papers [E] and [P] as follows:

Theorem 5.4 ([P, Theorem 3])**.** *Let M_1 and M_2 be smooth hypersurfaces in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, and let $f : M_1 \to M_2$ be a continuous CR mapping that extends holomorphically to one side of M_1, say M_1^-. If f vanishes to infinite order at $p \in M_1$ and M_2 contains a complex hypersurface through $f(p)$, then $f(M_1 \cup M_1^-) \subset M_2$.*

Once again, Alexander's theorem can be applied to give a short proof of this result. Indeed, we may assume that near the origin, the complex hypersurface $\{(z', 0) : z' \in \mathbb{C}^{n-1}\} \subset M_2$ and that near the origin, $M_2 = \{(z', x_n + i\varphi(z', x_n)\}$ where φ is real-valued, $\varphi(0) = 0, d\varphi(0) = 0$, and $\varphi(z', 0) \equiv 0$. Then $\varphi(z', x_n) = o(x_n)$, and so if we set $f = (f_1, \dots, f_{n-1}, g)$, after shrinking M_1 about the origin, the set $g(M_1)$ is non-spiraling. The theorem then follows as before by considering $g \circ A$ where $A : \overline{\Delta} \to \overline{M_1^-}$ is an analytic disc.

We present here two applications inspired by Example 2 in Alexander's paper. Alexander's examples considered holomorphic mappings $f : \Delta_r^+ \to \mathbb{C}^n$. Here we will consider CR mappings $f : M_1 \to M_2$ between CR manifolds.

Theorem 5.5. *Let $M_1 \subseteq \mathbb{C}^{n+d}$ be a smooth CR manifold of CR dimension n and $M_2 \subseteq \mathbb{C}^m$ a hypersurface. Suppose $f : M_1 \to M_2$ is a continuous CR mapping which extends to a holomorphic map into a wedge $\mathcal{W}$ with edge M_1 at $0 \in M_1$. Assume $f(0) = 0$ and f vanishes to infinite order at 0 in the sense that $|f(z)| = O(|z|^k), k = 1, 2, \dots$. Suppose there is a complex hypersurface Σ that locally intersects M_2 only at 0. Then $f \equiv 0$.*

PROOF: We may assume that near the origin, $M_1 = \{(z, s + i\varphi(z,s))\}$ where $(z,s) \in \mathbb{C}^n \times \mathbb{R}^d$, $\varphi = (\varphi_1, \ldots, \varphi_d), \varphi(0,0) = 0$, and $d\varphi(0,0) = 0$. By Theorem 7.4.12 in [BER], there is a family of analytic discs $A_{(x,s,v)}$ partially attached to M_1 whose interiors fill an open subset of the wedge $\mathcal{W}$. In fact, the discs are attached to the maximally real submanifold $X = \{(x, s + i\varphi(x,s))\}$ and their "partial boundaries" fill a neighborhood of the origin in X. Let r be a defining function for M_2 near the origin. We may assume that $r(z', z_m) = y_m - \psi(z', x_m)$, $z = (z', z_m) \in \mathbb{C}^{m-1} \times C$, $z_m = x_m + \sqrt{-1}y_m$. We may also assume that $d\psi(0,0) = 0$. Let $h(z)$ be a holomorphic defining function for Σ near 0. Since $M_2 \cap \Sigma = \{0\}$, and $\Sigma \setminus \{0\}$ is connected, without loss of generality, we may assume that

$$r(z) \geq 0 \quad \text{for all}\, z \in \Sigma. \tag{5.1}$$

Since the restriction of r to Σ has a local minimum at the origin, $v(r) = 0$ for all tangent vectors v to Σ at 0. From the form of r we also know that $D_{x_j} r(0) = D_{y_j} r(0) = 0$ for $j = 1, \ldots, m-1$, $D_{x_m} r(0) = 0$, and $D_{y_m} r(0) \neq 0$. It follows that $\frac{\partial h}{\partial z_m}(0) \neq 0$. Therefore, we may assume that $h(z', z_m) = z_m - g(z')$ for some holomorphic function g. From the Taylor expansion of r at the origin, we have:

(5.2)

$$\begin{aligned} &r(z', z_m) \\ &= r(z', g(z')) + \mathfrak{Re}\,(r_{z_m}(z', g(z'))(z_m - g(z')) + O(|z_m - g(z')|^2) \\ &= r(z', g(z')) - \frac{1}{2}\mathfrak{Im}\,(z_m - g(z')) + O(|z'||z_m - g(z')|) \end{aligned}$$

(5.3)

$$+ O(|z_m - g(z')|^2).$$

In particular, at a point $z = (z', z_m) \in M_2$, $r(z', z_m) = 0$ while from (5.1) $r(z', g(z')) \geq 0$. Therefore for points $z \in M_2$, (5.2) implies that

$$\mathfrak{Im}\,(z_m - g(z')) + O(|z'||z_m - g(z')| + O(|z_m - g(z')|^2) \geq 0. \tag{5.4}$$

Thus, when $z \in M_2$ is in a neighborhood of the origin, $\sqrt{-1}(z_m - g(z')) \notin (0, \infty)$. That is, if $P(z) = \sqrt{-1}(z_m - g(z'))$, after shrinking M_2 around the origin, the set $P(M_2)$ is a non-spiraling set in $\mathbb{C}$. Consider now one of the partially attached discs A mentioned above with $A(1) =$

0. We have $P(f(A(w))$ is holomorphic in the interior D near $1 \in \partial D$ and for w on the boundary of the disc D near 1, $P(f(A(w))$ lies in a non-spiraling set. By Alexander's theorem, either

(1) $|P(f(A(w))| \geq C|w|^\theta$ nontangentially for some $C, \theta > 0$, or
(2) $P(f(A(w)) \equiv 0$.

If (1) holds, then for any positive integer k, since $P(0) = 0$,

$$|A(w)|^k \geq |f(A(w))| \geq |P(f(A(w))| \geq C|w|^\theta$$

which is impossible. Therefore, $P(f(A(w)) \equiv 0$ and so $f(A(w)) \in \Sigma$ for all w. Since M_2 meets Σ only at the origin, $f(A(w)) \equiv 0$. The images $A(w)$ for w in the interior of D fill an open set in the wedge $\mathcal{W}$ which we may assume is connected. We conclude that $f \equiv 0$. $\square$

REMARK 5.1: We remark that the hypersurfaces called *positive* in [HKMP] and used in Example 2 of Alexander satisfy the condition on M_2 in the theorem. The theorem can be considered a result on unique continuation for continuous CR mappings.

Theorem 5.6. *Let M_1 be a smooth hypersurface in $\mathbb{C}^m$, M_2 a real analytic hypersurface in $\mathbb{C}^n$ and let $f : M_1 \to M_2$ be a continuous CR mapping that extends holomorphically to one side of M_1, say M_1^-. Assume that there is a holomorphic function h defined in a neighborhood Ω of the origin in $\mathbb{C}^n$ such that $h(0) = 0$ and $h(M_2 \cap \Omega)$ does not contain a neighborhood of the origin. If f vanishes to infinite order at $0 \in M_1$, then $f(M_1 \cup M_1^-) \subset \{h = 0\}$ (and thus f is a degenerate map in some sense). If in addition, $M_2 \cap \{h = 0\} = \{0\}$, then $f \equiv 0$.*

PROOF: Let T be a semi-analytic neighborhood of 0 in M_2 such that $\overline{T}$ is compact and $\overline{T} \subset \Omega$. The set $h(\overline{T})$ is a subanalytic set in $\mathbb{C}$. Therefore, the complement $\mathbb{C} \backslash h(\overline{T})$ is also subanalytic (see for example, p. 446 in [S]). Since 0 is in the closure of $\mathbb{C} \setminus h(\overline{T})$, the subanalyticity of $\mathbb{C} \setminus h(\overline{T})$ implies that there is a real analytic curve $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(0) = 0$ and $\gamma(t) \notin h(\overline{T})$ when $0 < t \leq 1$. It follows that after shrinking M_1 about the origin, the image $h(f(M_1))$ is contained in a non spiraling set. By composing $h \circ f$ with partially attached analytic discs A and using Alexander's theorem, we arrive at the conclusions of the theorem. $\square$

Let $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^N$ be smooth hypersurfaces each containing the origin. After holomorphic, linear coordinates, near the origin, we can write

$$M = \{(z', x_n + i\varphi_1(z', x_n) : z' \in \mathbb{C}^{n-1}\}$$

and likewise,

$$M' = \{(w', s_n + i\varphi_2(w', s_n) : w' \in \mathbb{C}^{N-1}\}$$

with $\varphi_j(0,0) = 0$, $d\varphi_j(0,0) = 0$, $j = 1,2$ and the φ_j real-valued. If $H : M \to M'$ is a mapping, and M and M' are as given as above, $H(0) = 0$, we write $H = (f, g) = (f_1, \ldots, f_{N-1}, g)$ and refer to g as the transversal component of H. In the following theorem from [E], M, M' and H are as above.

Theorem 5.7. (Theorem 2.1 in [E]). *Suppose $H : M \to M'$ is a CR mapping that extends holomorphically to a side M^- of M and the complex hypersurface $\Sigma = \{(w', 0)\} \subset M'$, $0 \in \Sigma$ (that is, M' is not minimal at the origin). If the transversal component g is not identically zero, then g is not flat at 0. (Thus if H is flat at 0, then $g \equiv 0$).*

This theorem is the same as Theorem 5.4 (Theorem 3 in [P]) because the identical vanishing of g is equivalent to the condition that $H(M) \subset M'$. Note that the proof of Theorem 5.4 actually showed that $f(M_1 \cup M_1^-) \subset S$ where S is the complex hypersurface through $f(p)$.

In general, under the hypotheses of the preceding theorem, we can not conclude that H itself vanishes identically. Indeed, let M' be as in the theorem and M a hypersurface which has a CR map $f : M \to \mathbb{C}^{N-1}$ that extends holomorphically to a side, is flat at 0, but nonconstant. Such an f exists if say M is strictly pseudoconvex at the origin or is real analytic and Levi flat in a neighborhood of the origin. Then the map $H = (f, 0)$ is a CR map from M to M' that is flat at 0, but it is not identically zero.

The following class of examples show that in some cases, the conclusion in the theorem above may also be reached when M' contains a complex manifold through the origin but M' is minimal at the origin.

EXAMPLE 5.1: Let $M' \subset \mathbb{C}^N$ be the hypersurface defined by $t_N = |w_{N-1}|^2 + s_N h(w', s_N)$, $w_j = s_j + \sqrt{-1}t_j$, $w' = (w_1, \dots, w_{N-1})$, where h is a real-valued smooth function. M' is minimal at the origin since its Levi form has a nonzero eigenvalue. The complex manifold $\Sigma = \{(w_1, \dots, w_{N-2}, 0, 0)\}$ is a subset of M'. Let M be a CR submanifold of $\mathbb{C}^n$ of any codimension, $0 \in M$. Suppose $H : M \to M'$ is a CR mapping that extends holomorphically to a wedge $\mathcal{W}$, $H = (f, g)$ with g its transversal component. If H is flat at 0, then $H(\mathcal{W}) \subset \Sigma$, in particular, $g \equiv 0$. To see this, observe that the image of M' under the coordinate function w_N intersects the set $\{iy : y < 0\}$ only at the origin and hence this image lies in a non spiraling set. Therefore, arguing as before, we conclude that if H is flat at the origin, $H(M) \subset M' \cap \{w : w_N = 0\} = \Sigma$.

In [HKMP] the authors defined a hypersurface $M \subset \mathbb{C}^n$ to be positive at a point $p \in M$ if there is a holomorphic change of coordinates mapping p to 0 and in the new coordinates $(z', z_n) = (z', x_n + iy_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$, M is given by $\{y_n = \rho(z', x_n, y_n)\}$ with $\rho(0) = 0, d\rho(0) = 0$ and $\rho(z', x_n + iy_n) > 0$ when $z' \neq 0$. This condition is equivalent to finding holomorphic coordinates vanishing at p such that in the new coordinates, M is given by $\{y_n = \varphi(z', x_n)\}$ with $\varphi(0) = 0, d\varphi(0) = 0$ and $\varphi(z', x_n) > 0$ when $z' \neq 0$. Note that the complex hypersurface $\Sigma = \{z_n = 0\}$ intersects M only at 0 and so the hypothesis on M_2 in Theorem 5.5 is satisfied whenever M_2 is positive at 0. In particular, when M is positive at a point $p \in M$, there are holomorphic coordinates in which M satisfies:

(1) $M \subset \{y_n \geq 0\}$, and
(2) there is a complex hypersurface Σ that intersects M only at 0.

In order to show that our Theorem 5.5 is more general than the result in [HKMP], we need to find an example of a hypersurface M which is not positive at a point $p \in M$ but there is a complex hypersurface Σ that intersects M only at p. Note that M satisfies property (1) at a point p if and only if there is a holomorphic function f defined on a neighborhood of p such that $df(p) \neq 0$ and $|f|$ has a local weak maximum at p, that is, $|f(q)| \leq |f(p)|$ for $q \in M$ near p. We thus have three conditions:

(A) M is positive at p;
(B) M satisfies (1) and (2) at p; and
(C) there is a complex hypersurface that meets M only at p.

Clearly (A) implies (B) and (B) implies (C).

EXAMPLE 5.2: We will show that an example in the work [B] satisfies condition (C) but not (A). The example (page 144 in [B]) is a real analytic hypersurface $M \subset \mathbb{C}^2$ defined by

$$r = s + 100(|z|^{10} + |z|^2\mathrm{Re}(z^8)) + t|z|^6 + t^2|z|^2$$

where $(z, w), w = s + it$ denote the coordinates in $\mathbb{C}^2$. It is shown in [B] that near $p = (0, 0)$, M is part of the boundary of a smoothly bounded domain U which is strongly pseudoconvex everywhere except at $(0, 0)$ and $r < 0$ on U. It is also shown that for some appropriate complex number β, $\Sigma = \{(z, w) : w = \beta z^{16}\}$ meets $\overline{U}$ only at $(0, 0)$ and so condition (C) is satisfied by M. Bloom proved that there is no function $f \in C^{13}(\overline{U})$, holomorphic on U which peaks at $(0, 0)$, that is, $|f(q)| < |f(0, 0)|$ for all $q \in \overline{U}$, $q \neq (0, 0)$. We will show that there is no smooth (in fact no C^{13}) CR function on M defined near $(0, 0)$ that peaks at $(0, 0)$ on M. Since M is strongly pseudoconvex away from $(0, 0)$, it is minimal there and hence any CR function on M defined near $(0, 0)$ extends to a holomorphic function on a side of M. Since U is pseudoconvex, by the main result in [BF], this extension holds on the side $r < 0$. Thus, for every neighborhood M_1 of $(0, 0)$ in M, there is a ball B centered at the origin such that each CR function on M_1 extends holomorphically to $U_1 = U \cap B$. It follows that if $g \in C^0(M_1)$ is a CR function and if G denotes its holomorphic extension, then the image $G(U_1) \subset g(M_1)$. Indeed, if say $q \in U_1$ and $G(q) \notin g(M_1)$, then the function $\frac{1}{g(z)-G(q)}$ would be CR on M_1 with no holomorphic extension to U_1. Suppose now h is a smooth CR function on a neighborhood M_3 of the origin in M satisfying $|h(q)| < |h(0, 0)|$ for all $q \in M_3$, $q \neq (0, 0)$. By our arguments, this implies that there is a neighborhood V of the origin in $\mathbb{C}^2$ and a holomorphic function (the extension of h) H on

$V \cap U$, $H \in C^\infty(\overline{V \cap U})$ that peaks in $V \cap U$ at $(0,0)$, contradicting the result in [B]. We observe next that M is not positive at $(0,0)$. Indeed, if in some holomorphic coordinates, $M = \{(z, s + i\phi(z,s))\}$ near the origin with ϕ smooth, real-valued, $\phi(0,0) = 0$ and $\phi(z,s) > 0$ for $z \neq 0$, then for $\epsilon > 0$, the smooth CR function $1/\big(s + i(\phi(z,s) + \epsilon)\big)$ will peak at $(0,0)$.

We will next give a class of examples of smooth hypersurfaces which satisfy Condition (C) but not Condition (A).

Example 5.3: Let $M \subset \mathbb{C}^2$ be a smooth hypersurface given by

$$M = \{(z,w) \in \mathbb{C}^2 : t = A(z) + B(z)s\}$$

where $z = x + iy, w = s + it$, $A(z)$ and $B(z)$ smooth functions. We choose $B(z)$ that is flat at the origin and $B(z) > 0$ for $z \neq 0$. Set $A(z) = |z|B(z)$. Then $A(z)$ is also a smooth function. The complex line $\Sigma = \{(z,0)\}$ meets M only at $(0,0)$ and so M satisfies condition C. Suppose it satisfies condition A at the origin. Then there exists a local biholomorphic map $H(z,w) = (z', w')$, $H(0,0) = (0,0)$, such that near the origin, $H(M)$ is given by

$$\mathfrak{Im}\, w' = \psi(z', w'), \text{ with } \psi(0,0) = 0,\, d\psi(0,0) = 0$$

and $\psi(z', w') > 0$ when $z' \neq 0$. Let $H^{-1}(z', w') = (f(z', w'), g(z', w'))$. Since the complex tangent spaces of M and $H(M)$ at the origin are generated respectively by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}$, $g_{z'}(0,0) = 0$, $f_{z'}(0,0) \neq 0$ and $g_{w'}(0,0) \neq 0$. It follows that

$$f(z', w') = f_{z'}(0,0)z' + h_1(z', w') \text{ and } g(z', 0) = O(|z'|^2)$$

for some holomorphic function h_1 satisfying $h_1(z', w') = O(|z'|^2 + |w'|)$. Since M is defined by $r(z,w) = \mathfrak{Im}\, w - A(z) - B(z)\mathfrak{Re}\, w$, $H(M)$ is defined near the origin by

$$
\begin{aligned}
&\rho(z',w') = r(H^{-1}(z',w')) \\
&= \mathfrak{Im}\, g(z',w') - A(f(z',w')) - B(f(z',w'))\mathfrak{Re}\, g(z',w') \\
&= \mathfrak{Im}\, g_{w'}(0,0)(s'+it') + h_2(z',w') - A(f(z',w')) \\
&\qquad\qquad\qquad - B(f(z',w'))\mathfrak{Re}\, g(z',w') \\
&= at' + bs' + h_2(z',w') - A(f(z',w')) - B(f(z',w'))\mathfrak{Re}\, g(z',w') \\
&\qquad\qquad (\text{with } g_{w'}(0,0) = a+ib) \\
&= a\left(t' + \frac{b}{a}s' + \frac{h_2(z',w')}{a} - \frac{A(f(z',w'))}{a} - \frac{B(f(z',w'))\mathfrak{Re}\, g(z',w')}{a}\right)
\end{aligned}
$$

where $h_2(z',w')$ is a pluriharmonic function. By assumption,

$$\psi(z',w') = -\frac{b}{a}s' - \frac{h_2(z',w')}{a} + \frac{A(f(z',w'))}{a} + \frac{B(f(z',w'))\mathfrak{Re}\, g(z',w')}{a}$$

satisfies $d\psi(0,0) = 0$ and $\psi(z',w') > 0$ for $z' \neq 0$. It follows that $b = 0$ and in a neighborhood of the origin, either

$$h_2(z',w') \geq A(f(z',w')) + B(f(z',w'))\mathfrak{Re}\, g(z',w')$$

or

$$h_2(z',w') \leq A(f(z',w')) + B(f(z',w'))\mathfrak{Re}\, g(z',w').$$

If the first case holds, then

$$
\begin{aligned}
h_2(z',0) &\geq A(f(z',0)) + B(f(z',0))\mathfrak{Re}\, g(z',0) \\
&= B(f(z',0))(|f(z',0)| + \mathfrak{Re}\, g(z',0)) \geq 0,
\end{aligned}
$$

$h_2(z',0)$ is harmonic, and $h_2(0,0) = 0$. By the minimum principle, $h_2(z',0) \equiv 0$. But this contradicts the fact that $B(f(z',0))(|f(z',0)| + \mathfrak{Re}\, g(z',0)) > 0$ for z' small, $z' \neq 0$. The second case implies that for each integer n, there is a constant C_n such that

$$h_2(z',w') \leq C_n(|z'| + |w'|)^n.$$

This implies that $h_2(z',w') \leq 0$ in a neighborhood of the origin. Otherwise, consider a component V of $\{h_2 > 0\}$ that contains the origin in its closure. This is a semi-analytic set and by the curve selection lemma [Lo, p. 94] there is an analytic curve $\gamma(s) : (-\varepsilon,\varepsilon) \to \mathbb{R}^4 \simeq \mathbb{C}^2$ such that $\gamma(0) = 0$ and $\gamma(s) \in V$ for $0 < s < \varepsilon$. Hence, after shrinking $\varepsilon > 0$ if necessary, the analytic function $\psi(s) = h_2(\gamma(s))$, $-\varepsilon < s < \varepsilon$, satisfies $\psi(s) > cs^k$ for $0 < s < \varepsilon$, $c > 0$ and some $k \in \mathbb{N}$. On the other hand, $\psi(s) \le C_n|\gamma(s)|^n \le C'_n s^{mn}$ for some $m \in \mathbb{N}$ and all $n \in \mathbb{N}$, which is a contradiction.

Thus $h_2(z', w') \le 0$ on a neighborhood of the origin. Since $h_2(0,0) = 0$, by the maximum principle, $h_2(z', w') \equiv 0$. This implies that $0 \le A(f(z', w')) + B(f(z', w'))\mathfrak{Re}\, g(z', w')$, that is, $0 \le A(z) + B(z)s$ which is impossible from the definitions of $A(z)$ and $B(z)$. We conclude that M does not satisfy condition (A).

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Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

E-mail address: berhanu@temple.edu

Departamento de Matemática, UFSCAR, 13.565-905, SÃO CARLOS, SP, BRASIL

E-mail address: hounie@dm.ufscar.br