# ERRATUM TO "THE VANISHING EULER CHARACTERISTIC OF AN ISOLATED DETERMINANTAL SINGULARITY", ISRAEL J. MATH. 197 (2013), 475–495

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In [4, Theorem A.5], we present the following result.

**Theorem A.5** (flawed version)**.** *Let $X \subset \mathbb{C}^n$ be a complex analytic manifold of dimension $d$ and let $M = X \cap B(0, \epsilon)$ for some $\epsilon > 0$. Let $f : X \to \mathbb{C}$ be a holomorphic Morse function without critical points on $X \cap S_\epsilon$. Then,*

$$\chi(f^{-1}(c) \cap M) = \chi(M) + (-1)^{d+1} \# \Sigma f|_M,$$

*where $c$ is a regular value of $f|_M$ and $\#\Sigma f|_M$ is the number of critical points of $f|_M$.*

We use [4, Theorem A.4] to show Theorem A.5. But [4, Theorem A.4] is wrong, we are grateful to Matthias Zach for providing us a counter-example (see fig. 1). Here $f$ is a linear projection which has no critical points on $X \cap S_\epsilon$. However, $f$ also has no critical points on $M$, but $M_a$ and $M_b$ are not homotopy equivalent. So $(M, f)$ cannot satisfy condition (C) of Palais-Smale.

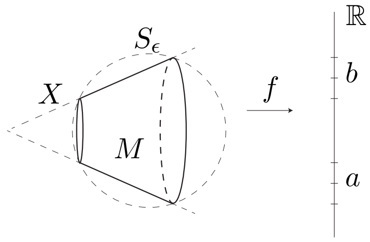


FIGURE 1

Therefore, we cannot ensure that Theorem A.5 is true unless we add the condition (C) of Palais-Smale. However, in [4], we present this result because it was necessary to use in a much less general framework. We just need it to be true when $X_s$ is a smoothing of an isolated singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ and $f$ is a Morsification of a function germ with isolated singularity $f_0 : (X, 0) \to \mathbb{C}$. Therefore, Theorem A.4 of [4] must be removed and Theorem A.5 must be replaced by Theorem A.5, which we show in this note.

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a germ of analytic variety with isolated singularity and $f : (X, 0) \to (\mathbb{C}, 0)$ a function on it, also with isolated singularity. A Morsification is a function $F : (\mathscr{X}, 0) \to (\mathbb{C}, 0)$ such that:

(1) $(\mathscr{X}, 0) \subset (\mathbb{C}^N \times \mathbb{C}, 0)$ is a smoothing of $(X, 0)$. This means that the projection $\pi : (\mathscr{X}, 0) \to (\mathbb{C}, 0)$ given by $\pi(x, s) = s$ is flat and that if we put $X_s := \pi^{-1}(s)$, then $X_0 = X$ and $X_s$ is smooth for $s \neq 0$.

(2) If $f_s : X_s \to \mathbb{C}$ is given by $f_s(x) = F(x, s)$, then $f_0 = f$ and $f_s$ is a Morse function for $s \neq 0$.

We fix the representative of $F : (\mathscr{X}, 0) \to (\mathbb{C}, 0)$ in the open set $B_\epsilon \times D_\beta$, where $B_\epsilon = \{x \in \mathbb{C}^N : \|x\| < \epsilon\}$, $D_\beta = \{z \in \mathbb{C} : |z| < \beta\}$ and $\epsilon, \beta > 0$ are small enough. Hence, $X_s$ is a closed analytic subset of $B_\epsilon$ and $f_s : X_s \to \mathbb{C}$ is a holomorphic function, for each $s \in D_\beta$.

**Theorem A.5** (corrected version)**.** *Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a germ of analytic variety with isolated singularity, with $d = \dim(X, 0)$. Let $f : (X, 0) \to \mathbb{C}$ be a function with isolated singularity and $F : (\mathscr{X}, 0) \to (\mathbb{C}, 0)$ be a Morsification of $f$. There exist small enough real numbers $0 < \beta \ll \delta \ll \epsilon \ll 1$ such that*

$$\chi(f_s^{-1}(c)) = \chi(X_s) + (-1)^{d+1} \# \Sigma f_s,$$

*for any $c \in D_\delta$ a regular value of $f_s$ and $s \in D_\beta \setminus \{0\}$.*

The proof of the theorem is based on Morse theory. We will use it in two steps of the proof, one for the function $G_s : X_s \to [0, +\infty)$, where $G_s = |f_s|^2$ and another one for $g_s : f_s^{-1}(\overline{D}_\delta) \to [-\delta, \delta]$, where $g_s$ is the real part of $f_s$ and $\delta > 0$ is small enough. In the first case, $X_s$ is not compact, so we need to control the critical points at infinity (in the sense of [2, 10.8]). These are the critical points of the restriction of the function to the boundary $\overline{X} \cap S_\epsilon$. In the second case, $f_s^{-1}(\overline{D}_\delta)$ is a closed submanifold with boundary of $X_s$. Besides the critical points at infinity, we also need to consider the critical points of the boundary itself, that is, the critical points of the restriction to $f_s^{-1}(\partial D_\delta)$. We present a pair of lemmas which will be useful to deal with these points.

Assume that $f : M \to \mathbb{R}$ is a smooth function, where $M$ is a smooth manifold with boundary. Let $p \in \partial M$ be a regular point of $f$. Then, $p$ is a critical point of the restriction $f|_{\partial M} : \partial M \to \mathbb{R}$ if and only if the gradient of $f$ at $p$ is collinear with the normal vector to the boundary of $M$ at $p$ (with respect to some Riemannian metric). We recall that $p$ is called an outward (resp. inward) boundary critical point if the gradient of $f$ at $p$ points outward (resp. inward).

Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a real analytic variety with isolated singularity and let $g\colon (X, 0) \to (\mathbb{R}, 0)$ be an analytic function with isolated singularity. We denote by $(\Sigma, 0)$ the analytic variety given by the set of points $x$ where the gradients of $g$ and $\rho$ are collinear, where $\rho : (X, 0) \to \mathbb{R}$ is the function $\rho(x) = \|x\|^2$. Assume $(X, 0) = V(\phi_1, \dots, \phi_r)$ for some analytic functions $\phi_i : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. We also suppose that $g$ is the restriction of some analytic function $\overline{g}\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. Then $(\Sigma, 0)$ is given by the zeros of the minors of the matrix $(\nabla \overline{g}, x, \nabla \phi_1, \dots, \nabla \phi_r)$ of order $n - d + 2$, where $d = \dim(X, 0)$. Moreover, if $\epsilon > 0$ is a Milnor radius for $(X, 0)$ and we fix a representative $g : X \to \mathbb{R}$ in the open ball $B_\epsilon$, the critical points at infinity of $g$ are exactly the points in $\Sigma \cap S_\epsilon$.

**Lemma 0.1.** *Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a real analytic variety with isolated singularity and let $g\colon (X, 0) \to (\mathbb{R}, 0)$ be an analytic function with isolated singularity. There exists $\epsilon_0 > 0$ such that for all $\epsilon$ with $0 < \epsilon < \epsilon_0$ and for all $x \in \Sigma \cap S_\epsilon$, $g(x) \neq 0$ and if $g(x) > 0$ (resp. $g(x) < 0$), then $x$ is an outward (resp. inward) boundary critical point.*

*Proof.* We first show that $(\Sigma \cap g^{-1}(0), 0) = (\{0\}, 0)$. If not, by the curve selection lemma, there would be an analytic arc $\gamma : [0, \eta) \to \mathbb{R}^n$ such that $\gamma(0) = 0$ and $\gamma(0, \eta) \subset \Sigma \cap g^{-1}(0) \setminus \{0\}$.

Since $\gamma$ is not constant and after shrinking $\eta$ if necessary, we can assume that $\gamma(t) \neq 0$ and $\gamma'(t) \neq 0$, for all $t \in (0, \eta)$. In fact, we also assume that for each $i = 1, \dots, n$, either $\gamma_i = 0$ or $\gamma_i(t) \neq 0$ and $\gamma_i'(t) \neq 0$, for all $0 < t < \eta$. Since $\gamma_i$ is either constant, strictly increasing or strictly decreasing and $\gamma_i(0) = 0$, it follows that $\gamma_i(t)$ and $\gamma_i'(t)$ must have the same sign along $(0, \eta)$. In particular, we have

$$\langle \gamma(t), \gamma'(t) \rangle = \sum_{i=1}^{n} \gamma_i(t)\gamma_i'(t) > 0, \quad \forall t \in (0, \eta).$$

Since $\gamma(t) \in \Sigma$, we have for all $0 < t < \eta$

$$\nabla \overline{g}(\gamma(t)) = \lambda_0(t)\gamma(t) + \sum_{i=1}^{r} \lambda_i(t) \nabla \phi_i(\gamma(t)),$$

for some $\lambda_i(t) \in \mathbb{R}$, $i = 0, \dots, r$. Note that the fact that $g$ has isolated singularity implies that $\lambda_0(t) \neq 0$.

On the other hand, $g(\gamma(t)) = 0$, for all $0 < t < \eta$, so

$$0 = (g \circ \gamma)'(t) = (\overline{g} \circ \gamma)'(t) = \langle \nabla \overline{g}(t), \gamma'(t) \rangle = \lambda_0(t) \langle \gamma(t), \gamma'(t) \rangle,$$

which gives a contradiction. Hence, we have shown the first part of the lemma.

Observe that a similar argument proves also the second part of the lemma. In fact, if we take now an analytic arc $\gamma : [0, \eta) \to \mathbb{R}^n$ such that $\gamma(0) = 0$ and $\gamma(0, \eta) \subset \Sigma \setminus \{0\}$, we know that $g(\gamma(t)) \neq 0$, for all $0 < t < \eta$. After shrinking $\eta$ if necessary, we can assume that $(g \circ \gamma)'(t) \neq 0$ and $\langle \gamma(t), \gamma'(t) \rangle > 0$, for all $0 < t < \eta$. Since $g(\gamma(0)) = 0$, the sign of $g(\gamma(t))$ coincides with the sign of its derivative:

$$(g \circ \gamma)'(t) = (\overline{g} \circ \gamma)'(t) = \langle \nabla \overline{g}(t), \gamma'(t) \rangle = \lambda_0(t) \langle \gamma(t), \gamma'(t) \rangle.$$

If $g(\gamma(t)) > 0$ (resp. $g(\gamma(t)) < 0$), then $\lambda_0(t) > 0$ (resp. $\lambda_0(t) < 0$) and thus, $\gamma(t)$ is an outward (resp. inward) boundary critical point. □

Let $X$ be a complex analytic manifold and $f : X \to \mathbb{C}$ be a holomorphic function. Let $g$ be the real part of $f$ and $G = |f|^2$. We denote by $\widetilde{\Sigma}$ the subset of points $x \in X$ where $\nabla g(x)$ and $\nabla G(x)$ are collinear. Assume $\delta^2 > 0$ is a regular value of $G$ and consider the restriction

$$g : G^{-1}[0, \delta^2] = f^{-1}(\overline{D}_\delta) \longrightarrow [-\delta, \delta].$$

Then, $\widetilde{\Sigma} \cap G^{-1}(\delta^2)$ is equal to the set of boundary critical points of $g : f^{-1}(\overline{D}_\delta) \to [-\delta, \delta]$.

**Lemma 0.2.** *With the above notation, we have:*

(1) *If $f(x) \neq 0$, then $f$ is regular at $x$ if and only if $G$ is regular at $x$.*
(2) *For all $x \in \widetilde{\Sigma} \cap G^{-1}(\delta^2)$, $g(x) \neq 0$ and if $g(x) > 0$ (resp. $g(x) < 0$), then $x$ is an outward (resp. inward) boundary critical point.*

*Proof.* By taking local coordinates in $X$, we can assume that $X$ is an open subset of $\mathbb{C}^d$. For all $i = 1, \dots, d$, we have $G_{z_i} = \overline{f} f_{z_i}$ and $G_{\overline{z}_i} = f \overline{f}_{\overline{z}_i}$, so item (1) is obvious.

To see item (2), we observe that $g = (f + \overline{f})/2$, hence $g_{z_i} = \frac{1}{2} f_{z_i}$ and $g_{\overline{z}_i} = \frac{1}{2} \overline{f}_{\overline{z}_i}$, for all $i = 1, \dots, d$. If $x \in \widetilde{\Sigma} \cap G^{-1}(\delta^2)$, then $x$ is a regular point of $G$ and $G(x) = \delta^2 > 0$. Thus, $x$ is also a regular point of $f$ and $g$, hence $G_{z_i}(x) = \lambda g_{z_i}(x)$ for some $\lambda \neq 0$. Hence,

$$\overline{f}(x) f_{z_i}(x) = \lambda \frac{1}{2} f_{z_i}(x), \forall i = 1, \dots, d.$$

This implies $f(x) = \lambda/2 \in \mathbb{R}$, so $g(x) = \lambda/2 \neq 0$ and if $g(x) > 0$ (resp. $g(x) < 0$), then $x$ is an outward (resp. inward) boundary critical point. □

*Proof of Theorem A.5.* We can see $X_s$ as the germ of a real analytic variety of dimension $2d$ in $\mathbb{R}^{2N}$. We write $f_s(x) = g_s(x) + i h_s(x)$ and $G_s(x) = |f_s(x)|^2$, for each $x \in X_s$.

We use [4, Lemma A.6], which is unaffected by the mistake, then $g_0$ has isolated singularity. By Lemma 0.1 and after shrinking $\epsilon$ if necessary, we have that for all $x \in \Sigma \cap S_\epsilon$, $g_0(x) \neq 0$ and if $g_0(x) > 0$ (resp. $g_0(x) < 0$), then $x$ is an outward (resp. inward) boundary critical point. We also assume that $\epsilon$ is small enough in such a way that $(g_0)|_{\overline{X} \cap S_\epsilon}$ has only isolated critical points.

Let $\eta > 0$ such that $|g_0(x)| > \eta$, for all $x \in \Sigma(g_0|_{\overline{X} \cap S_\epsilon})$. Take also $\alpha, \delta > 0$ such that $0 < \alpha < \delta < \eta$ and the closed disk $\overline{D}_\delta$ is contained in the image $f(X)$ and $\delta^2$ is a regular value of $G$. Finally, by continuity, we can choose $\beta > 0$ small enough, such that:

(1) $\overline{D}_\delta \subset f_s(X_s)$ and $\delta^2$ is a regular value of $G_s$,
(2) $|g_s(x)| > \eta$, for all $x \in \Sigma(g_s|_{\overline{X}_s \cap S_\epsilon})$,
(3) $|g_s(x)| < \alpha$, for all $x \in \Sigma(g_s)$,

for all $0 < |s| < \beta$.

We apply Morse theory to the function $g_s : f_s^{-1}(\overline{D}_\delta) \to [-\delta, \delta]$. Observe that this is a non-proper Morse function, so we have to use stratified Morse theory in the sense of [2, 10.8]. Let $b_1, b_2, \in \mathbb{R}$ such that $-\delta < b_1 < -\alpha$ and $\alpha < b_2 < \delta$ (see fig. 2). Then,

$$f_s^{-1}(\overline{D}_\delta) = g_s^{-1}[-\delta, \delta] = g_s^{-1}[-\delta, b_1] \cup g_s^{-1}(b_1, b_2] \cup g_s^{-1}(b_2, \delta].$$

By conditions (2) and (3), $g_s^{-1}(b_2, \delta]$ does not contain critical points at infinity nor interior critical points. Also, by Lemma 0.2, all the boundary critical points are outward. It follows from [2, 10.8] or [1, Theorem 4.1] that the homotopy type of $g_s^{-1}[-\delta, b]$, with $b_2 \leq b \leq \delta$, does not change when passing through these critical values, that is,

$$f_s^{-1}(\overline{D}_\delta) \simeq g_s^{-1}[-\delta, b_1] \cup g_s^{-1}(b_1, b_2].$$

Also by [2, 10.8], since $g_s^{-1}(b_1, b_2]$ contains $\#\Sigma f_s$ Morse critical points of $g_s$ with index $d$, we have

$$f_s^{-1}(\overline{D}_\delta) \simeq g_s^{-1}[-\delta, b_1] \text{ with } \#\Sigma f_s \text{ cells of dimension } d \text{ attached.}$$

Hence,

$$\chi(f_s^{-1}(\overline{D}_\delta)) = \chi(g_s^{-1}[-\delta, b_1]) + (-1)^d \#\Sigma f_s.$$

We observe that $g_s^{-1}[-\delta, b_1]$ is a locally closed subset of $\mathbb{C}^N$, $f_s$ is a submersion on $g_s^{-1}[-\delta, b_1]$ and the restriction of $f_s$ to the closure of $g_s^{-1}[-\delta, b_1]$ is a proper map. Then, by the Thom first isotopy lemma, $f_s$ is a fibration on $g_s^{-1}[-\delta, b_1]$. Therefore,

$$\chi(g_s^{-1}[-\delta, b_1]) = \chi(f_s^{-1}(b_1))\chi(\overline{D}_\delta \cap ([-\delta, b_1] \times \mathbb{R})) = \chi(f_s^{-1}(b_1)).$$

Since $f_s$ is a fibration on the subset of $D_\delta$ of its regular values, the homotopy type of $f_s^{-1}(c)$ is independent of the regular value $c$. Thus,

$$\chi(f_s^{-1}(c)) = \chi(f_s^{-1}(\overline{D}_\delta)) + (-1)^{d+1} \# \Sigma f_s.$$

It only remains to show that $\chi(f_s^{-1}(\overline{D}_\delta)) = \chi(X_s)$. To see this, we apply again Morse theory to the function $G_s : X_s \to [0, +\infty)$. For $b_3 \in \mathbb{R}$ big enough, we have

$$X_s = G_s^{-1}[0, b_3] = G_s^{-1}[0, \delta^2] \cup G_s^{-1}[\delta^2, b_3].$$

By Lemma 0.1, all the critical points at infinity in $G_s^{-1}[\delta^2, b_3]$ are outward. Again, by [2, 10.8],

$$X_s \simeq G_s^{-1}[0, \delta^2] = f_s^{-1}(\overline{D}_\delta).$$

□

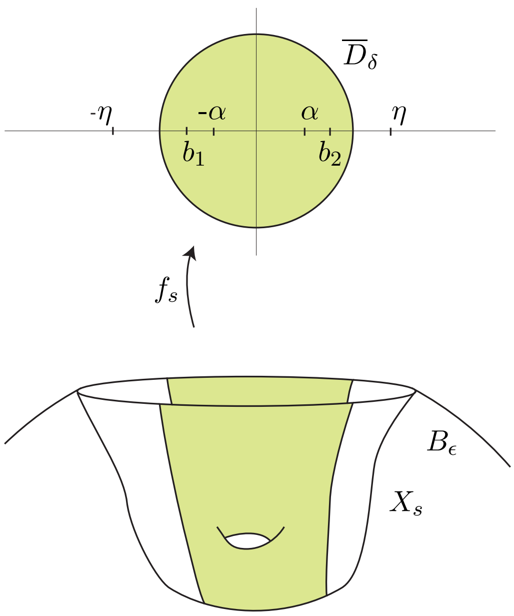


FIGURE 2

**Remark 0.3.** When $X = \mathbb{C}^n$, we get another proof of the following well know formula for the of Milnor number of a function (see [3, Theorem 7.2]): let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with isolated singularity, then

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f)},$$

where $\mathcal{O}_n$ is the local ring of homolorphic function germs in $(\mathbb{C}^n, 0)$ and $J(f)$ is the ideal generated by the partial derivatives of $f$. In fact, by definition

$\mu(f)$ is the number of $(n-1)$-spheres in the Milnor fibre $f^{-1}(c)$. Since $f^{-1}(c)$ has the homotopy type of a wedge of $(n-1)$-spheres, by Theorem A.5 we have:

$$\mu(f) = (-1)^{n-1}(\chi(f^{-1}(c)) - 1) = \#\Sigma f_s,$$

where $f_s : B_\epsilon \to \mathbb{C}$ is a Morsification of $f$. But the number $\#\Sigma f_s$ is equal to the local degree of the gradient $\nabla f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ which is equal to $\dim_{\mathbb{C}} \mathcal{O}_n/J(f)$.

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