HOMOTOPY ALGEBRA, OPERADS AND SPECTRA

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Every topological abelian group is a grouplike E_∞-space, i.e. is a topological space that is an algebras over an E_∞-operad $\mathcal{E}$ that admits inverses up to homotopy. One of the advantages of working in the category of E_∞-spaces is that these objects are invariant under homotopy, i.e. a space homotopy equivalent to a grouplike E_∞-space is also a grouplike E_∞-space (This is not true for abelian groups). The category $\mathbb{S} - \mathtt{Mod}$ of $\mathbb{S}$-modules is a category of spectra that admits a smash product $\wedge_{\mathscr{L}}$ constructed using the E_∞-operad $\mathscr{L}$ of linear isometries which makes it a monoidal model category [1]. There is an equivalence between the homotopy category of grouplike E_∞-spaces $\mathcal{H}o\mathcal{E}[\mathtt{Top}]_{\mathrm{grp}}$ and the homotopy category of connective $\mathbb{S}$-modules $\mathcal{H}o\mathbb{S} - \mathtt{Mod}_{\mathrm{con}}$. This result is known as the recognition principle [2], and it provides a powerful analogy between abelian groups and spectra. Spectra are ubiquitous since every additive reduced generalized cohomology theory is representable by a spectrum.

In the same way that commutative rings are commutative monoids in the monoidal category of abelian groups equipped with the tensor product $\otimes$ we can define commutative ring spectra as commutative monoids in $\mathbb{S}-\mathtt{Mod}$. If the E_∞-operad $\mathcal{E}$ of the recognition principle is chosen such that $\mathscr{L}$ acts on $\mathcal{E}$ we can define the category of $(\mathcal{E},\mathscr{L})$-spaces as spaces equipped with compatible algebra structures over $\mathcal{E}$ and $\mathscr{L}$ (this compatibility condition is a generalization of the distributivity law of multiplication over addition in rings). The recognition principle then induces an equivalence between the homotopy categories of ringlike $(\mathcal{E},\mathscr{L})$-spaces $\mathcal{H}o(\mathcal{E},\mathscr{L})[\mathtt{Top}]_{\mathrm{ring}}$ and the homotopy category of connective commutative ring spectra $\mathcal{H}o\mathtt{CRingSp}_{\mathtt{con}}$ [3, 4]. Multiplicative cohomology theories are represented by ring spectra.

For all $R \in \mathtt{CRingSp}$ there is a category $R - \mathtt{Mod}$ of R-modules and a smash product $\wedge_R$ of R-modules. Commutative monoids in $R - \mathtt{Mod}$ are commutative algebra spectra over R [1]. In this talk I propose a definition of actions between relative operads such that a natural relative version $\mathscr{L}^{\rightarrow}$ of the linear isometries operad acts on the infinite dimensional relative Fulton-MacPherson operad $\mathcal{FM}^{\rightarrow}_{\infty}$ of compactifications of configuration spaces. My recent work on a relative version of the recognition principle [5] suggests there is an equivalence between the homotopy category of algebralike $(\mathcal{FM}^{\rightarrow}_{\infty},\mathscr{L}^{\rightarrow})$-spaces $\mathcal{H}o(\mathcal{FM}^{\rightarrow}_{\infty},\mathscr{L}^{\rightarrow})[\mathtt{Top}]_{\mathrm{alg}}$ and the homotopy category of connective commutative algebra spectra $\mathcal{H}o\mathtt{AlgSp}_{\mathtt{con}}$.

References

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