Sectional theory of a morphism

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Numerical invariants

- Topology
- Topological robotics
- Group theory
- Category theory

Sectional number

Related topics

Sectional category

- The *sectional category* or *Schwarz genus* of a map $f : X \to Y$, denoted by $\mathrm{secat}(f)$, is the minimal cardinality of open covers of Y, such that each element of the cover admits a continuous homotopy local section to f.

  This notion was first studied extensively by Schwarz for fibrations in [Švarc, A. S. *'The genus of a fiber space'*. (Russian) Dokl. Akad. Nauk SSSR (N.S.) 119 1958 219–222.] or [A. Schwarz, *'The genus of a fiber space'*, Amer. Math. Soc. Transl. Ser. 2 55(1966), 49 – 140.]

  Later by Berstein and Ganea for arbitrary maps in [I. Berstein, T. Ganea, *' The category of a map and a cohomology class'*, Fund. Math. 50 (1961/1962), 265–279.]

Relative sectional category

- In particular, the sectional category of the pullback $\varphi^*(p) : K \times_B E \to K$, where $p : E \to B$ is a fibration and $\varphi : K \to B$ is any map, was first studied by J. González et al in [González, Jesús, Mark Grant, and Lucile Vandembroucq. "Hopf invariants for sectional category with applications to topological robotics." The Quarterly Journal of Mathematics 70.4 (2019): 1209-1252.]

  This numerical invariant is denoted by $\mathrm{secat}_\varphi(p)$ and called *relative sectional category*.

Sectional number

- The standard *sectional number* of a map $f : X \to Y$, $sec(f)$, is the minimal cardinality of open covers of Y, such that each element of the cover admits a continuous local section to f,

  [P. Pavesic, *'Topological complexity of a map'*. arXiv preprint arXiv:1809.09021 (2019).]

  The sectional number of the pullback $\varphi^*(f) : K \times_B E \to K$, where $f : E \to B$ and $\varphi : K \to B$ are maps, is denoted by $\mathrm{sec}_\varphi(f)$ and called *relative sectional number*.

Sectional category vs sectional number

Note that $\mathrm{secat}(f) \leq \mathrm{sec}(f)$. By the homotopy lifting property, the equality holds when f is a fibration. The number $\mathrm{secat}(-)$ is a homotopy invariant, i.e, if $f \simeq g$ then $\mathrm{secat}(f) = \mathrm{secat}(g)$. However, $\mathrm{sec}(-)$ does not hold.

The sectional category $\mathrm{secat}(f) = \mathrm{secat}(\rho_f)$, where ρ_f is the fibration given by $\rho_f : E_f \to Y,\ (x, \gamma) \mapsto \gamma(1)$ with $E_f = \{(x, \gamma) \in X \times PY :\ f(x) = \gamma(0)\}$.

For a fibration $p : E \to B$ with fiber F, we have $\mathrm{secat}(p) \leq n$ if and only if there exists a cross-section for the fibration $p_n : E_n \to B$, where (E_n, B, F_n, p_n) is the sum of n copies of p.

Sectional category vs sectional number

The notion of sectional number is easy to apply in Robotics, specifically, to the inverse kynematic problem, see [T. Bajd, M. Mihelj, J. Lenarcic, A. Stanovnik and M. Munih, *'Robotics'*, International series on intelligent systems, control, and automation:Science and engineering. **43** (2010).]

The secat is the homotopy version of this invariant.

Bounds

Standard lower and upper bounds:

$$\operatorname{nil}\left(\operatorname{Ker}(f^* : h^*(Y) \to h^*(X))\right) \leq \operatorname{secat}(f) \leq \min\{\operatorname{sec}(f), \operatorname{cat}(Y)\},$$

where h^* is any multiplicative cohomology on HTop^2.

Recall that the relative cohomology product

$$\cup : h^*(X, A) \otimes h^*(X, B) \to h^*(X, A \cup B)$$

holds for subspaces $A, B \subset X$ that are excisive. In our case, A and B are open sets.

LS category

If $f : X \to Y$ is null homotopic, then $\mathrm{secat}(f) = \mathrm{cat}(Y)$. Recall that, the *Lusternik-Schnirelmann category* (LS category) of a topological space Y, denoted $\mathrm{cat}(Y)$, is the least integer m such that Y can be covered by m open sets, all of which are contractible within Y.

For any $(q-1)$-connected CW complex X,

$$\mathrm{cat}(X) \leq \frac{hd(X)}{q} + 1.$$

For a survey of LS category see [O. Cornea, G. Lupton, J. Oprea and D. Tanré, *'Lusternik-Schnirelmann Category'*, Mathematical Surveys and Monographs, 103 (American Mathematical Society, Providence, RI, 2003).]

Equivariant sectional category

- The *equivariant sectional category* of a G-equivariant map $f : X \to Y$, denoted by $\mathrm{secat}_G(f)$, is the minimal cardinality of invariant open covers of Y, such that each element of the cover admits a G-homotopy section to f.

  This notion was first studied in [H. Colman and M. Grant, *'Equivariant topological complexity'*, Algebr. Geom. Topol. 12 (2012) 2299– 2316.]

Equivariant sectional number

- The *equivariant sectional number* of a G-equivariant map $f : X \to Y$, denoted by $\mathrm{sec}_G(f)$, is the minimal cardinality of invariant open covers of Y, such that each element of the cover admits a G-equivariant section to f.

Topological complexity

If $e_2^X : PX \to X \times X$ is the path fibration given by $e_2^X(\gamma) = (\gamma(0), \gamma(1))$, then $\mathrm{secat}(e_2^X) = \mathrm{TC}(X)$ is the *topological complexity* of the topological space X.

This notion was first studied by Michael farber in [M. Farber, *'Topological complexity of motion planning'*, Discrete and Computational Geometry. **29** (2003), no. 2, 211–221.]

Higher topological complexity

For $n \geq 2$, consider the evaluation fibration $e_n^X : PX \to X^n$ given by

$$e_n^X(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{n-1}\right), \ldots, \gamma\left(\frac{n-2}{n-1}\right), \gamma(1)\right),$$

then $\mathrm{secat}(e_n^X) = \mathrm{TC}_n(X)$ is the *higher topological complexity* of X.

This notion was presented by Rudyak in [Y. Rudyak, *'On higher analogs of topological complexity'*, Topology and its Applications, Elsevier. **157** (2010), no. 5, 916–920.]

Topological complexity of a map

For a map $f : X \to Y$, consider the application $e_f : PX \to X \times Y$ given by

$$e_f(\gamma) = (\gamma(0), f(\gamma(1))),$$

then $\mathrm{sec}(e_f) = \mathrm{TC}(f)$ is the *topological complexity* of f.

Note that $\mathrm{TC}(1_X) = \mathrm{TC}(X)$.

[P. Pavesic, *'Topological complexity of a map'*. arXiv preprint arXiv:1809.09021 (2019).]

Topological complexity of a map

For any map $f : X \to Y$,

$$\mathrm{TC}(f) \geq \max\{\mathrm{cat}(Y), \mathrm{sec}(f)\}.$$

If f admits a cross-section, then

$$\mathrm{TC}(Y) \leq \mathrm{TC}(f) \leq \mathrm{TC}(X).$$

If f is a fibration, then

$$\mathrm{TC}(f) \leq \mathrm{TC}(Y).$$

Example

For the polynomial map $f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = x^2$, we have $secat(f) = 1$, $\mathsf{sec}_{\mathsf{op}}(f) = \infty$ and $\mathsf{TC}(f) = \infty$.

The covering number

- The *covering number* of a group G is defined as the cardinality of the smallest covering of G, and it is denoted by $\sigma(G)$. Here, a *covering* of a group G is a family of proper subgroups of G whose union equals G.

  [S. Haber and A. Rosenfield, *'Groups as unions of proper subgroups'*, Amer. Math. Monthly 66, 491-–494 (1959).]

The categorical LS category

- The *categorical Lusternik-Schnirelmann category* of a small category $\mathcal{C}$, denoted $\mathrm{ccat}(\mathcal{C})$, is the least integer m such that $\mathcal{C}$ can be covered by m subcategories $\mathcal{C}_i$, all of which are contractible within $\mathcal{C}$ and $\{\mathcal{C}_i\}$ is a geometric cover of $\mathcal{C}$, i.e., $\{B\mathcal{C}_i\}$ covers $B\mathcal{C}$.

  It is easy to see that $\mathrm{cat}(B\mathcal{C}) \leq \mathrm{ccat}(\mathcal{C})$.

  [Tanaka, Kohei. "Lusternik–Schnirelmann category for categories and classifying spaces." Topology and its Applications 239 (2018): 65-80.]

Sectional number

Let $\mathcal{C}$ be a category with a notion of covering. For example,
$\mathcal{C} =$ Top the category of topological spaces and open coverings.
The category of groups Grp and coverings with proper subgroups.

Recall that an suboject of an object $Y \in \mathcal{C}$ is a equivalence class $[incl : U \to Y]$ of monomorphisms with target Y under the relation $\sim$: for $u : T \to Y$ and $v : S \to Y$ monomorphisms,

$u \sim v$ iff there exists an isomorphism $\varphi : T \to S$ such that $u = v \circ \varphi$.

Sectional number

Let $f : X \to Y$ be a morphism, $u : T \to Y$ and $v : S \to Y$ be monomorphisms such that $u \sim v$, we have: there exists a morphism $t : T \to X$ with $f \circ t = u$ if and only if there exists a morphism $s : S \to X$ with $f \circ s = v$.

The *sectional number* of a morphism $f : X \to Y$, denoted by $\mathrm{sec}(f)$, is the least integer m such that the target Y can be covered with m subobjects U_i such that, for any $i = 1, 2, \ldots, m$, there exists a morphism $s_i : U_i \to X$ with $f \circ s_i = incl_{U_i}$. If no such m exists, we set $\mathrm{sec}(f) = \infty$. The covering $\{U_i\}$ is called *categorical*.

Examples

- In the category Top with the notion of open covering, we have the sectional number of a map.
- In HTop with the notion of open covering, we have the sectional category of a map.
- In Top_G with the notion of invariant open covering, we have the equivariant sectional number of an equivariant map.
- In HTop_G with the notion of invariant open covering, we have the equivariant sectional category of an equivariant map.

Examples

- In Grp with the notion of covering with proper subgroups, we have the *sectional number of a homomorphism*. Note that, if $h : G \to K$ is an isomorphism, then $\mathrm{sec}(h) = \sigma(K) = \sigma(G)$. The study of this invariant is a work in progress joint with Jesús González.
- In the category of categories with the notion of geometric covering, we have the *sectional number of a functor*.

Central problems

Among the central problems in sectional theory are:

- the computation of the sectional number $\mathrm{sec}(f)$,
- the realization of $\mathrm{sec}(f)$, i.e., finding a *categorical* covering $\{s_i : U_i \to X\}_{i=1}^{m}$ with $m = \mathrm{sec}(f)$.

Central problems

Generally, the determination of $\mathrm{sec}(-)$ is very difficult. In the categories Top, HTop, Top_G and HTop_G, it can be estimated from above by homotopical methods, and from below by cohomological methods. This problem has extensively been developed.

Central problems

But the realization problem, specifically, the concrete implementations of the multitasking motion planning problem (i.e., the realization of $\mathrm{sec}(e_n^X) = \mathrm{TC}_n(X)$) are still non-existent, and in fact the works:

- C. A. I. Zapata and J. Gonzalez, *Multitasking collision-free optimal motion planning algorithms in Euclidean spaces*, to appear in DMAA,
  Our algorithms are motivated by those presented by Mas-Ku and Torres-Giese (as streamlined by Farber).
- C. A. I. Zapata and J. Gonzalez, *Sequential collision-free optimal motion planning algorithms in punctured Euclidean spaces*, to appear in BAustMS,
- C. A. I. Zapata, *Collision-free spatial motion of rigid bodies via topological complexity*, arXiv preprint arXiv:1911.03859 (2019).

take the first step in this last direction (producing explicit algorithms.)

Related topics

- The sectional category $\mathrm{secat}(e_2^{\mathbb{RP}^m}) = \mathrm{TC}(\mathbb{RP}^m)$ is related with the immersion problem of $\mathbb{RP}^m$.

  The topological complexity $\mathrm{TC}(\mathbb{RP}^m)$ for any $m \neq 1, 3, 7$, coincides with the smallest integer k such that the projective space $\mathbb{RP}^m$ admits an immersion into $\mathbb{R}^{k-1}$.

  [M. Farber, S. Tabachnikov and S. Yuzvinsky, *'Topological robotics: motion planning in projective spaces'*, International Mathematics Research Notices, Hindawi Publishing Corporation, n. 34 (2003), 1853–1870.]

Related topics

- In Top, the sectional number $\mathrm{sec}(\pi^X_{k,r})$ is related with the fixed point property of X, where $\pi^X_{k,r}\colon F(X,k) \to F(X,r)$ is given by $\pi^X_{k,r}(x_1,\ldots,x_r,\ldots,x_k) = (x_1,\ldots,x_r)$.

  Let X be a Hausdorff space. The space X has the FPP if and only if $sec(\pi^X_{2,1}) = 2$.

  [Zapata, Cesar A. Ipanaque, and Jesús González. "Sectional category and The Fixed Point Property." arXiv preprint arXiv:1912.03448 (2019).]

Related topics

- More general, in Top, the relative sectional number $\sec_g(\pi^Y_{k,r})$ is related with the coincidence property of $(X, Y; g)$.

  $(X, Y; g)$ has *the coincidence property* (CP) if for every continuous map $f : X \to Y$ there is a point x of X such that $f(x) = g(x)$.

  Let X be a Hausdorff space. $(X, Y; g)$ has the CP if and only if $sec_g(\pi^Y_{2,1}) = 2$.