Applications of Spectral Sequences

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1. Introduction

A spectral sequence is an invariant (some poetic license needed here) whose output can often be highly ambiguous. Spectral sequences, like Marmite, are either loved or hated, and in the 1960’s, graduate courses about them were regularly inflicted on unwilling students in a variety of research areas.

Victor P. Snaith

It remains unknown why Jean Leray named his sequences of bigraded differential algebras spectral. Sources close to Leray are unsure, but the connection between a linear operator and its spectrum was certainly well known to the analyst Leray. I have an image of arranging, for a complex operator, the eigenspaces in a manner that simplifies and illuminates the behavior of the operator. And this image gives a filtration of the domain of the operator for the purpose of relating the spectrum to the operator.

I am certain that spectral sequences have become basic tools throughout mathematics because many mathematical objects are equipped with filtrations. Topology contributes many fundamental examples—CW-complexes and simplicial complexes are naturally filtered by skeleta. Rings are filtered by powers of an ideal, groups are filtered by normal series, etc., etc. It was the introduction of homological methods that required the construction of spectral sequences.

Is there an everyday image of a spectral sequence? I offer a simple one that I told my niece when she insisted on an explanation. A spectral sequence is like a book. You can open to its first page and read and recognize as familiar what you read there. However, in order to turn the pages you need to know a lot more. With each page you get more and more information about the target, the last page, which may be infinitely many pages away. Sometimes a lot of ingenuity is needed to turn a page, and maybe all you can see is a little corner of the next page. But, as is always true in mathematics, sometimes that is enough. Sometimes you work in reverse—you know the target and you want to describe the initial pages. Like a novel, there can be lots of relevant action.

These notes are intended to give an introduction to the construction and properties of spectral sequences in general, together with applications that depend upon these properties. I have chosen widely in the applications—from the applicable notion of persistence to topological assaults on questions in combinatorial geometry, with a stop in differential geometry on the way. However, this only scratches the surface. The term spectral sequence gets 4359 hits in the MathSciNet database, in areas from logic to mathematical physics. I hope you find the notes useful, and I thank you for the privilege of bringing them to you.
2. What is a spectral sequence?

A spectral sequence is like a long exact sequence, only more complicated.

J. Frank Adams

As mentioned in the introduction, a filtration is a natural situation in many mathematical situations. Let’s assume a space (or simplicial set, or an algebraic object) $X$ is filtered. For simplicity, let’s assume it is a finite filtration. We can display the filtration:

$$\emptyset = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{k-1} \subset F_k = X.$$ 

The algebraic topological invariants of $X$ may be organized by considering each pair of consecutive pieces of the filtration: For example, for homology with coefficients in a field $F$, we have the long exact sequence associated with each pair $(F_j, F_{j-1})$.

$$\cdots \to H_i(F_{j-1}; F) \to H_i(F_j; F) \to H_i(F_j, F_{j-1}; F) \to H_{i-1}(F_{j-1}; F) \to \cdots.$$ 

We can assemble the long exact sequence into two relevant structures: an exact couple and an unrolled exact couple. The first is a (graded) triangle:

$$
\begin{array}{ccc}
H_*(F_{j-1}; F) & \xrightarrow{i=\subset} & H_*(F_j; F) \\
\Bigg\downarrow{k=i} & \Bigg\downarrow{j} & \Bigg\downarrow{k} \\
H_*(F_j, F_{j-1}; F) & & \\
\end{array}
$$

The main properties of the triangle are:

1. the sequence $i, j, k$ is exact, that is, ker $j = \text{image } i$, ker $k = \text{image } j$, and ker $i = \text{image } k$.
2. If we denote $H_{p+q}(F_p; F) = D_{p,q}$ and $H_{p+q}(F_p, F_{p-1}; F) = E_{p,q}$, then

$$i: D_{p,q} \to D_{p+1,q-1}, \quad j: D_{p,q} \to E_{p,q}, \quad \text{and } k: E_{p,q} \to D_{p-1,q}.$$ 

We write that the bidegrees of the homomorphisms are given by bideg $i = (1, -1)$, bideg $j = (0, 0)$, and bideg $k = (-1, 0)$.

3. Notice that $d = j \circ k: E_{p,q} \to E_{p,q-1}$ is a differential. Checking $d \circ d = j \circ k \circ j \circ k = j \circ 0 \circ k = 0$.

We define $E'_{p,q} = H_{p,q}(E_*, d)$.

Let $D'_{p,q} = \text{image}(i: D_{p-1,q+1} \to D_{p,q})$. Then we can define $i' = i|_{i(D)}: D'_{p,q} \to D'_{p+1,q-1}$, $j': D'_{p,q} \to E'_{p,q}$ given by $j'(i(u)) = u + dE_{p,q+1}$, and $k': E'_{p,q} \to D'_{p-1,q}$ given by $k'(v + dE) = k(v)$. Massey proved a remarkable theorem about the collection $(D', E', i', j', k')$:

**Theorem 2.1.** If $(D, E, i, j, k)$ is an exact couple, then $(D', E', i', j', k')$ is also an exact couple.
It follows that $k$ case, ker $k$.

Observe that the differentials have bidegree that depends on $n$.

We call the exact couple $(D, E, i, j, k)$ the derived couple. This construction can be iterated to obtain a sequence of derived couples: $C^{(1)} = (D, E, i, j, k)$ and $C^{(n)} = (D^n, E^n, i^{(n)}, j^{(n)}, k^{(n)}) = (C^{(n-1)})'$.

Definition. The sequence of derived vector spaces $E_{x, *}$, together with the associated differentials $d^n = j^{(n)} \circ k^{(n)}$ is a spectral sequence.

Observe that the differentials have bidegree that depends on $n$. In the case of a space with filtration, $\emptyset = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{k-1} \subset F_k = X$, the bidegree of $j$ is $(0, 0)$ and bideg $k = (-1, 0)$. Thus bideg $d = (0, 0) + (-1, 0) = (-1, 0)$. For the derived couple, $v = i(u) \in D^2_{p,q} = i(D_{p-1,q+1})$ has $j'(v) = j(u) + d(E)$, and $j(u) \in E_{p-1,q+1}$. Hence bideg $j' = (-1, 1)$, and $j'(v) \in E_{p-1,q+1}^2$. Since $k'(e + d(E)) = k(e)$, bideg $k' = (-1, 0)$, and so $d^2 : E^2_{p,q} \to E^2_{p-2,q+1}$. More generally, a similar argument with induction shows that $d^n : E^n_{p,q} \to E^n_{p-1,q+r-1},$ and so has bidegree $(-r, r-1)$.

How is all of this algebra related to $\mathbb{H}_i(X; \mathbb{F})$? In the case of the filtered space, since direct limits commute with homology, $\lim_{\to} D_{p,q} = H_{p+q}(X; \mathbb{F})$. For a general exact couple one can prove:

**Proposition 2.2.** There is an exact sequence

$$0 \to D_{p,*}/i(D_{p-1,*}) \to E_{p,*} \to (\ker i : D_{p-1,*} \to D_{p,*}) \to 0$$

**Proof.** This is just a version of the short exact sequence $0 \to \ker k \to E_{p,*} \to \operatorname{image} k \to 0$. In this case, $\ker k = \operatorname{image} j \cong D_{p,*}/\operatorname{image} i$. Furthermore, $\operatorname{image} k = \ker i$. $\square$

It follows that $E_{p,*}$ is determined by the sequence $\cdots \stackrel{i}{\to} D_{p-1,*} \stackrel{i}{\to} D_{p,*} \stackrel{i}{\to} D_{p+1,*} \stackrel{i}{\to} \cdots$. 

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The subsequent derived couples can also be related to the sequence of iterated images:

\[ \cdots \rightarrow \phi^{(r-1)}(D_{p-r+2,*}) \xrightarrow{i_j} \phi^{(r-1)}(D_{p-r+1,*}) \xrightarrow{i_j} \phi^{(r-1)}(D_{p-r,*}) \xrightarrow{i_j} \cdots, \]

and the induced couple:

\[ E^r_{p,*} \]

**Proposition 2.3.** There is a short exact sequence for all \( r \geq 1, \)

\[ 0 \rightarrow j(\ker \phi^{(r-1)}): D_{p,*} \rightarrow D_{p+r-1,*} \rightarrow k^{-1}(\text{image } \phi^{(r-1)}): D_{p-r,*} \rightarrow D_{p-1,*} \rightarrow E^r_{p,*} \rightarrow 0. \]

Let \( Z^r_p = k^{-1}(\text{image } \phi^{(r-1)}): D_{p-r,*} \rightarrow D_{p-1,*} \) and \( B^r_p = j(\ker \phi^{(r-1)}): D_{s,*} \rightarrow D_{s+r-1,*}. \) We let

\[ Z^\infty_p = \bigcap_s Z^r_p, \quad B^\infty_p = \bigcup_s B^r_p. \]

Then \( E^\infty_{p,*} = Z^\infty_{p,*}/B^\infty_{p,*} \) is the target of the spectral sequence in the following sense: Suppose \( X \) is a filtered space:

\[ \emptyset = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{k-1} \subset F_k = X. \]

Then we can associate to \( H_*(X; \mathbb{F}) \) the filtration given by

\[ F_*=H_*(X; \mathbb{F}) = \text{image } H_*(F_*/\mathbb{F}) \xrightarrow{\sim} H_*(X; \mathbb{F}). \]

Since \( F_{s-1} \subset F_{s} \) factoring the induced mappings gives \( F_{s-1} \subset \mathcal{F}_s. \) Homology respects direct limits, so \( H_*(X; \mathbb{F}) = \bigcup_s \mathcal{F}_s. \)

**Theorem 2.4.** For \( s > n, \) \( E^s_{p,*} \cong E^\infty_{p,*} \cong \mathcal{F}_p/\mathcal{F}_{p+1}. \)

**Proof.** The short exact sequence \( 0 \rightarrow F_{p-1} \rightarrow F_p \rightarrow F_p/F_{p-1} \rightarrow 0 \) may be replaced by \( 0 \rightarrow i(D_{p-1}/\ker i^{\infty} \rightarrow D_p/\ker i^{\infty} \rightarrow D_p/(\ker i^{\infty} + iD_{p-1}) \rightarrow 0 \) where by \( i^{\infty} \) we mean \( i^r \) for \( r > n. \)

By the previous proposition, we can relate \( E^r_p \) to \( Z^r_p = k^{-1}(\text{image } \phi^{(r-1)}): D_{p-r,*} \rightarrow D_{p-1,*} \) and \( B^r_p = j(\ker \phi^{(r-1)}): D_{p,*} \rightarrow D_{p+r-1,*}. \) First we refine the relation between \( E^r_p \) and image \( k \cap \text{image } i^{r-1}: \)

\[ k^{-1}(\text{image } i^{r-1}): D_{p-r} \rightarrow D_{p-1} \xrightarrow{k} \text{image } k \cap \text{image } i^{r-1} \rightarrow 0 \]
Lift an element in $E^r_p$ to $k^{-1}(\text{image } i^{r-1})$ and apply $k$. This mapping is well defined because $k \circ j = 0$. Now $\text{image } k = \ker i$, so we have an epimorphism $\tilde{k}: E^r_p \to \text{image } i^{r-1} \cap \ker i$.

Our goal is to prove that we have a short exact sequence of the form:

$$0 \to D_p/(\ker i^{r-1} \cap \ker i^{r-1}) \to E^r_p \to \text{image } i^{r-1} \cap \ker i \to 0.$$ 

The canonical quotients give rise to the following commutative diagram:

$$
\begin{array}{ccccccc}
0 & \to & i(D_{p-1}) + \ker i^{r-1} & \to & D_p & \to & D_p/(i(D_{p-1}) + \ker i^{r-1}) & \to & 0 \\
\downarrow j & & \downarrow j & & \downarrow j & & \downarrow j & \\
0 & \to & j(\ker i^{r-1}) & \to & \text{image } j & \to & \text{image } j/(\ker i^{r-1}) & \to & 0
\end{array}
$$

Since $\text{image } j = \ker k$, we obtain the mapping $\tilde{j}: D_p/(i(D_{p-1}) + \ker i^{r-1}) \to \ker k/\ker j$, induced by $\tilde{j}$. We know that $\tilde{j}$ is onto. Suppose $\tilde{j}([a]) = \tilde{j}([b])$ for $[a], [b] \in D_p/(i(D_{p-1}) + \ker i^{r-1})$. Then $ja - b = 0$, that is, $ja - b \in j(\ker i^{r-1})$, so $a - b \in \ker i^{r-1}$ or $a - b \in \ker j = \text{image } i$. In either case, $a - b \in i(D_{p-1}) + \ker i^{r-1}$ and $[a - b] = 0$. Hence $\tilde{j}$ is an isomorphism.

Putting all the pieces together, we get the commutative diagram:

$$
\begin{array}{ccccccc}
0 & \to & \ker k & \to & k^{-1}(\text{image } i^{r-1}) & \to & \text{image } i^{r-1} \cap \text{image } k & \to & 0 \\
\downarrow j & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 & \to & \ker k/\ker j(\ker i^{r-1}) & \to & k^{-1}(\text{image } i^{r-1})/j(\ker i^{r-1}) & \to & \text{image } i^{r-1} \cap \text{image } k & \to & 0 \\
\downarrow j & & \cong & & \cong & & \cong & \\
0 & \to & D_p/(i(D_{p-1}) + \ker i^{r-1}) & \to & E^r_p & \to & k^{-1}(\text{image } i^{r-1}) \cap \text{image } k & \to & 0
\end{array}
$$

To finish the proof, remember that $F_p = \text{image } i^{r-1}D_p \to D_{p+r} = H_*(X; \mathbb{F})$ for $r$ large enough. For such $r$, image $i^{r-1}D_{p-r} \to D_{p-1}$ is $\{0\}$, making $E^r_p$ isomorphic to $D_p/(i(D_{p-1}) + \ker i^{r-1})$. Finally, image $i^{r-1} \cong D_p/\ker i^{r-1}$, and the initial short exact sequences prove the theorem.

Among the simplest examples of a filtered complex is the cellular filtration of a regular finite CW complex $X$ by skeleton $X^{(p)}$. The vector spaces $E_p = H_*(X^{(p)}, X^{(p-1)}; \mathbb{F})$ are concentrated in dimension $p$ and have a basis of the $p$-cells of $X$. One can show that the first differential $d^1$ is the cellular boundary homomorphism and so $E^2 \cong E^\infty$ is the cellular homology of $X$. Since we began with the ordinary singular homology, this proves the isomorphism between singular and cellular homology of $X$.

The next simplest example is the Bockstein spectral sequence where the exact couple is built from the short exact sequence of coefficients: $0 \to \mathbb{Z} \xrightarrow{x_p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$. On homology we get an exact couple with $E = H_*(X; \mathbb{F}_p)$ and $D = H_*(X; \mathbb{Z})$ where the mappings $i$ are induced by multiplication.
by \( p \) on the chain complex. The subtleties of this spectral sequence when \( X \) is an H-space were developed by Browder [7].

### 3. Persistence

*Persistent homology is an algebraic method for measuring topological features of shapes and of functions. Small size features are often categorized as noise . . . But noise is in the eye of the beholder . . .*

**Edelsbrunner and Harer**

We saw how the cellular filtration of a finite CW-complex can be used to prove the isomorphism between cellular homology and singular homology of CW-complexes. However, a filtration may be more general and the same arguments lead to a spectral sequence whose properties may unlock some of the important features of the filtration.

A motivating example is found in Morse theory. Suppose \( M \) is a compact orientable manifold and \( f: M \to \mathbb{R} \) a Morse function. We can choose values \( a_0 < a_1 < \cdots < a_N \) for which the critical values of \( f \), \( \{b_1, b_2, \ldots, b_N\} \), satisfy \( a_{i-1} < b_i < a_i \) for all \( i \). We denote by \( M^{a_i} = f^{-1}((-\infty, a_i)) \).

This gives the filtration

\[
\emptyset = M^{a_0} \subset M^{a_1} \subset \cdots \subset M^{a_{N-1}} \subset M^{a_N} = M.
\]

We can assume that the critical points are isolated and occur at distinct critical values. The homology exact couple may be unrolled:

\[
\cdots \to H_*(M^{a_{p-1}}; \mathbb{F}) \xrightarrow{i} H_*(M^{a_p}; \mathbb{F}) \xrightarrow{i} H_*(M^{a_{p+1}}; \mathbb{F}) \to \cdots
\]

By the theorems of Morse theory, \( H_*(M^{a_p}, M^{a_{p-1}}; \mathbb{F}) \) is a copy of \( \mathbb{F} \) in the degree that corresponds to the index of the critical point. This fact has various consequences, such as the Morse inequalities [36]. By analogy with cellular theory, we get a chain complex of vector spaces, generated by critical points and their indices, that may be used to compute \( H_*(M; \mathbb{F}) \) from the exact couple.

Because there may be differentials in the spectral sequence (not every Morse function is perfect), there may be classes \( \gamma \in H_p(M^{a_i}; \mathbb{F}) \) that do not persist to \( H_p(M; \mathbb{F}) \). If \( i^k(\gamma) \) is nonzero in \( H_p(M^{a_{i+k}}; \mathbb{F}) \) and \( i^k(\gamma) \notin i^{k+1}(H_p(M^{a_{i-1}}; \mathbb{F})) \), that is, \( \gamma \) is not in the image of the inclusion of a stage before \( a_i \), then we say that the persistence of \( \gamma \) is \( \geq k + 1 \). The least value of \( k \) for which \( \gamma \) joins the image of \( H_p(M^{a_{i-1}}; \mathbb{F}) \) is the persistence of \( \gamma \). If the Morse function values are important we say that \( \gamma \) has persistence \( a_{i+k} - a_i \). If \( m = \min\{k \mid \gamma \in H_p(M^{a_k}; \mathbb{F})\} \), then we say that \( \gamma \) is born at \( M^{a_m} \). If it persists to \( M^{a_i} \), we say that \( \gamma \) dies entering \( M^{a_i} \).
In fact, this terminology extends to any filtered situation.

**Definition.** The \( p \)-th persistent homology of a filtration \( \emptyset = K_{-1} \subset K_0 \subset K_1 \subset \cdots \subset K_N = K \) is defined as \( H^p_{i,j} = \text{image} (H_p(K_i; F) \xrightarrow{i^{(j-i+1)}} H_p(K_j; F)) \) for \( 0 \leq i \leq j \leq N \). The dimension of \( H^p_{i,j} \) is the \( p \)-persistent Betti number \( \beta^p_{i,j} \).

Let \( \mu^i_{p,j} \) denote the number of independent \( p \)-classes born at \( K_i \) that die entering \( K_j \). It follows from the description and the picture above that, for \( i < j \),

\[
\mu^i_{p,j} = (\beta^p_{i,j} - 1) - (\beta^p_{i-1,j} - 1).
\]

The persistence of a class \( \gamma \in H^p(K_i; F) \) is \( j - i \) if \( \gamma \) dies entering \( H^p(K_j; F) \). The connection between spectral sequences and persistence lies in the sequence of homomorphisms that are induced by inclusion, \( H^p(K_s; F) \xrightarrow{i} H^p(K_{s+1}; F) \), that is, the sequence along the top of an unrolled exact couple. From the algebra of derived exact couples, we prove the following result of [1]:

**Proposition 3.1.** Let \( F \) denote a filtration \( \emptyset = K_{-1} \subset K_0 \subset K_1 \subset \cdots \subset K_{n-1} \subset K_N = K \) with each \( K_i \) a locally finite space. Then for each \( r > 0 \) and \( k, l \geq 0 \),

\[
\dim E^r_{k,l} = (\beta^k_{k+l} - \beta^k_{k+l-1}) + (\beta^{k-r,k} - \beta^{k-r,k-1}).
\]

**Proof.** The exact couple associated to this filtration has

\[
D^1_{s,t} = H_{s+t}(K_s; F) \text{ and } E^1_{s,t} = H_{s+t}(K_s, K_{s-1}; F).
\]

The mapping \( i: D^1_{s,t} \to D^1_{s+1,t-1} \) is the mapping that defines persistence, and so we find that \( D^r_{s,t} = \text{image} (i^r(i^{r-1}): H_{s+t}(K_{s-r+1}; F) \to H_{s+t}(K_s; F)) \). But this is what we have denoted by \( H^r_{s-t+1,s} \), one of the persistence vector spaces associated to the filtration.

Suppose \( V_0 \xrightarrow{a} V_1 \xrightarrow{b} V_2 \xrightarrow{c} V_3 \xrightarrow{d} V_4 \) is an exact sequence of finite dimensional vector spaces and linear maps. By an elementary linear algebra argument the reader can prove that

\[
\dim V_2 = (\dim V_1 - \dim(\text{image } a)) + (\dim V_3 - \dim(\text{image } d)).
\]
We apply this to the $r$th derived couple, where we have the exact sequence:

$$D^r_{s-1,t+1} \xrightarrow{\partial} D^r_{s,t} \xrightarrow{j^{(r)}} E^r_{s-r+1,t+r-1} \xrightarrow{k^{(r)}} D^r_{s-r,t+r-1} \xrightarrow{i} D^r_{s-1,t+1}.$$ 

Furthermore, $i(D^r_{s-1,t+1}) = i(H^s_{s+t} - 2s-1) = H^s_{s+t} - 2s$. 

Putting these facts together, we have

$$\dim E^r_{s-r+1,t+r-1} = \dim H^s_{s+t} - r + 1, s - \dim i(H^s_{s+t} - 2s-1))$$

$$+ (\dim H^s_{s+t-1} - s - \dim i(H^s_{s+t-1} - 1))$$

$$= \beta^s_{s+t} - \beta^s_{s+t-1} + (\beta^s_{s+t-1} - 1)$$

To complete the proof, let $k = s-r+1$ and $l = t+r-1$ and substitute the appropriate expressions in $k$ and $l$. □

The proposition leads to a computation of the persistence multiplicities $\mu_{p}^{i,j}$. The proof follows by careful bookkeeping (see [1]).

**Corollary 3.2.** The following relation holds for $r > 0$ and $n \geq 0$:

$$\sum_{k+l=n} \dim E^r_{k,l} = \beta_n(K) + \sum_{j \geq r} (\mu_{n}^{i,j} + \mu_{n-1}^{i,j}).$$

A remarkable application of these ideas involves multidimensional data clouds. Statisticians look for the manner in which a cloud of data breaks into pieces, separated by hyperplanes. Think of this as computing $H_0$ applied to the cloud, equipped with a loose sense of close. Given the higher dimensionality of the data, it is possible to ask if there are topological features of the cloud, higher connectivities, that is, $H_p$ for $p \geq 1$. However, a data cloud is a set of discrete points floating in a metric space $\mathbb{R}^d$. How can we associate a filtered topological space to the data? It turns out that the solution was introduced by L. Vietoris in the 1920’s in the first paper [41] in which homology groups appear.

**Definition.** Given a collection of points $\mathcal{U} = \{p_i\} \in \mathbb{R}^d$ and a real number $\epsilon > 0$, let $B_i = B_\epsilon(p_i)$, be the $\epsilon$-open ball centered at $p_i$. The **Vietoris-Rips complex** is the abstract simplicial set with vertex set $\mathcal{U}$ and $q$-simplices given by sets $\{p_{i_0}, p_{i_1}, \ldots, p_{i_q}\}$ whenever $B_{i_0} \cap B_{i_1} \cap \cdots \cap B_{i_q} \neq \emptyset$.

If we denote the Vietoris-Rips complex by $\text{VR}(\mathcal{U}, \epsilon)$, then there is finite sequence of values of $\epsilon$, $\epsilon_0 < \epsilon_1 < \cdots < \epsilon_N$ for which the topology of $\text{VR}(\mathcal{U}, \epsilon_{i-1})$ may differ from $\text{VR}(\mathcal{U}, \epsilon_i)$ at a single point between $\epsilon_{i-1}$ and $\epsilon_i$. Because inclusion of subsets respects the simplicial structure, we obtain a filtration

$$\mathcal{U} = \text{VR}(\mathcal{U}, \epsilon_0) \subset \text{VR}(\mathcal{U}, \epsilon_1) \subset \cdots \subset \text{VR}(\mathcal{U}, \epsilon_{N-1}) \subset \text{VR}(\mathcal{U}, \epsilon_N) \cong \Delta^{|U-1|}.$$ 

In the diagram below, the persistence data for homology classes are pictured as barcodes, where a line segment has length $\epsilon_j - \epsilon_i$ associated to a class of persistence $j - i$.

The homology of a Vietoris-Rips complex for a particular $\epsilon$ encodes holes in the data cloud. The barcodes describe the scales over which such holes persist. Since large enough $\epsilon$ renders the data
cloud into a contractible simplex, it is the set of barcodes that carry the topological information about the data cloud, refined by the spread of scales for each class born. Noise in the data may result in short-lived holes, and this idea gives a topological notion of noise. Long term persistence is long-lived topology—topological information.

Here is an example of a Vietoris-Rips complex prepared by R. Ghrist [24].

The idea of persistence was introduced by Edelsbrunner, Letscher, and Zomorodian [16] to study metric features of models for biomolecules (alpha shapes) that are derived from coverings by balls. Persistence in higher dimensions was refined in the work of Carlsson and Zomorodian ([9], [45]). A striking application is to natural image statistics: a digital camera image may be considered as a vector in a very high dimensional vector space. How do we distinguish from an image and a random pixel array? Apparently humans are good at making the distinction. Is there something about the data generated by an image that makes it different from noise? Working with a large set of image data, Carlsson and his collaborators were able to show that the persistent homology indicates an underlying Klein bottle to which the data cloud from images lies close. de Silva and Ghrist [13] have applied persistent homology to the problem of coverage by networks of sensors. Holes indicate a connection failure in the network. I wonder what we would find if the Sloane Digital Sky Survey were subjected to analysis for persistence and barcodes. A fingerprint of the universe?

In the context of spectral sequences, computations require good algorithms and these have been developed by practitioners and their teams. Since the $E^\infty$-page of a typical sequence of Vietoris-Rips
complexes is trivial, all of the action in the spectral sequence has an interpretation in terms of persistence. The algorithms are based on integral linear algebra and the Smith matrix theorem.

Other spectral sequences that arise as exact couples admit an interpretation of persistence in terms of the birth of a class (its filtration level) and when it dies (differentials). The study of such notions, for example for the EHP sequence, gives a more nuanced view of homotopy theory phenomena.

4. Extra structures

It is no exaggeration to say that the calculational utility of spectral sequences largely stems from such multiplicative structure.

J. Peter May [32]

Cohomology is a functor into the category of graded algebras and this extra ring structure is the source of the great power of this functor. One of the most useful properties a spectral sequence can have is that enjoy the structure of an algebra on each page, and that the differential satisfies the Leibniz rule. Usually we abbreviate these conditions by speaking of a differential graded algebra (DGA).

Establishing a product structure can be challenging, especially if the spectral sequence arises from an exact couple. Massey [30] stated conditions on an exact couple that lead to the each page being a DGA. Following [32], it is clearer to express the conditions more generally, in terms of pairings.

Given three exact couples, \( C_1 = (D_1, E_1, i_1, j_1, k_1), C_2 = (D_2, E_2, i_2, j_2, k_2), \) and \( C = (D, E, i, j, k) \), a pairing,

\[ \phi: E_1 \otimes E_2 \to E, \quad \text{with } \phi(u \otimes v) \text{ denoted } uv, \]

satisfies condition \( \mu_n \) if for any \( x \in E_1, y \in E_2, a \in D_1, \) and \( b \in D_2 \) for which \( k_1(x) = i_1^n(a) \) and \( k_2(y) = i_2^n(b) \), there exists \( c \in D \) such that

\[ k(xy) = i^n(c) \text{ and } j(c) = j_1(a)y + (-1)^{\deg x}xj_2(b). \]

When \( n = 0 \), we set \( i_1^0 = \text{id}, i_2^0 = \text{id} \), and the choices become \( a = k_1(x), b = k_2(y), \) and \( c = k(xy) \), so that

\[ d(xy) = jk(xy) = j_1 k_1(x)y + (-1)^{\deg x}xj_2k_2(y) = d_1(x)y + (-1)^{\deg x}xd_2(y). \]

That is, the differentials (graded) commute with the pairing. Then \( \phi \) is a morphism of chain complexes and it induces a mapping of chain complexes \( \phi': E'_1 \otimes E'_2 \to E' \).

We want a DGA structure on every page of the spectral sequence. We say that the pairing \( \phi \) satisfies condition \( \mu \) if \( \phi \) satisfies \( \mu_n \) for all \( n \geq 0 \).

**Proposition 4.1.** [32] Suppose that \( \phi \) satisfies \( \mu_0 \). Then \( \phi \) satisfies \( \mu_n \) if and only if \( \phi' \) satisfies \( \mu_{n-1} \).
The cohomology spectral sequence associated to this filtration can be analyzed carefully to prove $E_{\infty}$ given by skeleta $\emptyset$. Another general approach to products may be found in [15].

Condition $\mu$ may be difficult to ascertain in general conditions where a specific argument for the particular setting is called for. Another general approach to products may be found in [15].

The most celebrated spectral sequence is the Leray-Serre spectral sequence [39] for a fibration $p: E \to B$ with connected fiber $F$. Assume that $B$ is a finite CW-complex with cellular filtration given by skeleton $\emptyset = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(n-1)} \subset X^{(n)} = B$. This filtration induces a filtration of the total space $E$ by taking inverse images of the skeleton:

$$\emptyset = J^{(-1)} \subset J^{(0)} \supset p^{-1}(X^{(0)}) \supset \cdots \supset J^{(k)} = p^{-1}(X^{(k)}) \supset \cdots \supset J^{(n)} = E.$$ 

The cohomology spectral sequence associated to this filtration can be analyzed carefully to prove

**Theorem 4.3.** For a fibration $p: E \to B$ with connected fiber $F$, there is a first quadrant spectral sequence with

$$E_{2}^{p,q} \cong H^{p}(B; \mathcal{H}^{q}(B; \mathbb{F})), $$

where $\mathcal{H}^{*}(F; \mathbb{F})$ denoted the system of local coefficients for the fundamental group $\pi_{1}(B)$ action on the fiber $F$. The spectral sequence is a spectral sequence of algebras and the $E_{\infty}$-page is an associated graded algebra of $H^{*}(E; \mathbb{F})$. 

**Proof.** Suppose that $\phi$ satisfies $\mu$ and that $x' \in E'_{1}$, $y' \in E'_{2}$, $a' \in D'_{1}$, $b' \in D'_{2}$ and that $k'_{1}(x') = i_{1}^{*}n^{-1}(a')$ and $k'_{2}(y') = i_{2}^{*}n'(b')$. If $x' = \bar{x}$, $y' = \bar{y}$, $a' = i_{1}(a)$ and $b' = i_{2}(b)$, then it follows that $k_{1}(x) = i_{1}^{*}(a)$ and $k_{2}(y) = i_{2}^{*}(b)$. From the assumption, there exists $c \in D$ with $k(xy) = i^{*}(c)$ and $j(c) = j_{1}(a)y + (-1)^{\text{deg}x}xj_{2}(b)$. Take $c' = i(c)$. Then

$$k'(xy) = k'(x) = i^{*}n^{-1}(c')$$

and $j'(c') = j_{1}(a')y' + (-1)^{\text{deg}x'}x'j_{2}(b')$.

The converse follows similarly. \(\square\),

**Corollary 4.2.** If $\phi$ satisfies $\mu$, then so does $\phi'$, and all subsequent pairings of derived couples. Furthermore, $\phi^{r+1}: E_{1}^{r+1} \otimes E_{2}^{r+1} \to E^{r+1}$ is determined by

$$H(\phi^{r}): H_{*}(E_{1}^{r}, d'_{r}) \otimes H_{*}(E_{2}^{r}, d'_{r}) \to H_{*}(E_{1}^{r} \otimes E_{2}^{r}) \to H_{*}(E^{r}, d'_{r})$$

via the K"{u}nneth map and $H_{*}(\phi^{r})$. 

The situation of interest has $C_{1} = C_{2} = C$ with condition $\mu$. Then each $(E_{r}, d'_{r})$ is a DGA, where the graded algebra is a bigraded algebra in this case, namely, $\phi^{r}: E_{p,q}^{r} \otimes E_{m,n}^{r} \to E_{m+p,n+q}^{r}$. The simplest example is the Bockstein spectral sequence in cohomology. The cup product on integral cohomology $H^{*}(X)$ has condition $\mu$ and descends mod $p$ to the cup product on $H^{*}(X; \mathbb{F}_{p})$ to make the Bockstein spectral sequence a spectral sequence of algebras.

Similarly, a filtered complex $X$, with $\emptyset = F_{-1} \subset F_{0} \subset F_{1} \subset \cdots \subset F_{n-1} \subset F_{n} = X$ with $D^{p,q} = H^{p+q}(F_{p}; \mathbb{F})$ and $E^{p,q} = H^{p+q}(F_{p}, F_{p-1}; \mathbb{F})$ can be shown to enjoy condition $\mu$. Notice the change to superscripts. We lower the index $(E_{r}, d_{r})$ for the successive pages of the spectral sequence. In this case, the spectral sequence is a spectral sequence of algebras and the $E_{\infty}$-page is an associated graded algebra associated to the filtration of $H^{*}(X; \mathbb{F})$ given by $F^{p} = \ker \text{inc}: H^{*}(X; \mathbb{F}) \to H^{*}(F_{p}; \mathbb{F})$. 

Condition $\mu$ may be difficult to ascertain in general conditions where a specific argument for the particular setting is called for. Another general approach to products may be found in [15].
In the case that $\pi_1(B) = \{0\}$, the system of local coefficients is constant and $E_2^{p,q} \cong H^p(B; H^q(F; \mathbb{F})) \cong H^p(B; \mathbb{F}) \otimes H^q(F; \mathbb{F})$.

We will discuss local coefficients later.

5. A geometric application

...the second author... wishes to acknowledge the motivation provided by conversations with Gromoll in 1967 who pointed out the surprising fact that the available techniques of algebraic topology loop spaces, spectral sequences, and so forth seemed inadequate to handle the "rational problem" of calculating the Betti numbers of the space of all closed curves on a manifold.

DENNIS SULLIVAN IN [42]

Since Serre’s thesis, spectral sequences have been useful to questions in differential geometry. By comparing the based loop space $\Omega(M, x_0)$ with the space of paths joining points $p$ and $q$ in a manifold $M$, $\Omega(M, p, q)$, Serre showed that there are infinitely many geodesics joining $p$ to $q$ when $M$ is a path-connected, complete Riemannian manifold for which $\check{H}_*(X) \neq \{0\}$. This result follows from Morse theory.

Morse [37] asked a more difficult question:

Given a closed, compact, Riemannian manifold $M$, how many closed geodesics lie on $M$?

Happily there is a Morse theory approach to this problem, obtained by extending Morse theory to infinite complexes. Let $\Lambda M$ denote the free loop space on $M$,

$$\Lambda M = \{ \lambda : S^1 \to M \mid \lambda \text{ is smooth.} \}.$$  

There is an energy functional on $\Lambda M$ given by $E: \Lambda M \to \mathbb{R}$, $E(\lambda) = \frac{1}{2} \int_{S^1} \langle \lambda', \lambda' \rangle \, ds$. The critical points of this functional are the closed geodesics on $M$. However, critical points are not isolated, and a closed geodesic may be iterated, $\gamma_n(z) = \gamma(z^n)$ to give a family of closed geodesics, all with the same image. Many researchers worked to overcome these details, and the most refined tool to study this problem is the following

**Theorem of Gromoll and Meyer** [26]. If $M$ is a compact, closed manifold of dimension $\geq 2$, and the set

$$\{ \dim_\mathbb{F} H_i(\Lambda M; \mathbb{F}) \mid i = 0, 1, 2, \ldots \}$$

is unbounded, then infinitely many closed geodesics lie on $M$ in any Riemannian metric.
The theorem makes the computation of the homology of the free loop space of manifolds an important problem. There is a fibration:

$$\Omega M \rightarrow \Lambda M$$

The associated Leray-Serre spectral sequence for $M$ simply-connected has $E_2^{p,q} \cong H^p(M; \mathbb{F}) \otimes H^q(\Omega M; \mathbb{F})$ with $E_\infty^{p,q}$ associated to $H^{p+q}(\Lambda M; \mathbb{F})$.

This setup seems ready for computation. However, recall that the path-loop fibration, $\Omega M \hookrightarrow PM \rightarrow M$ has the same $E_2$-page and $H^*(PM; \mathbb{F}) \cong \mathbb{F}$. Hence, the target $H^*(\Lambda M; \mathbb{F})$ lies somewhere between $H^*(M; \mathbb{F}) \otimes H^*(\Omega M; \mathbb{F})$ and $\mathbb{F}$. More can be deduced from the fibration. In particular, it has a section $s: M \rightarrow \Lambda M$, given by sending $m \in M$ to the constant loop at $m$ in $M$. A section implies that no differentials in the spectral sequence can land in $E_2^{0,*}$ this is significant, but it it insufficient to predict much further information.

What would be nice is a condition on a manifold that guarantees the existence of infinitely many closed geodesics. The spheres and projective spaces show us that the best result algebraic topology can provide is a condition that implies the assumption for the Gromoll-Meyer theorem. This is the point of one of outstanding achievements of rational homotopy theory in the 1970’s.

**Theorem of Sullivan and Vigué-Poirrier** [42]. If $X$ is a finite CW-complex and the cohomology algebra $H^*(X; \mathbb{Q})$ requires at least two generators as an algebra, then the set $\{\dim_\mathbb{Q} H_i(\Lambda M; \mathbb{Q}) \mid i = 0, 1, 2, \ldots\}$ is unbounded.

In spite of its weak conditions, the Sullivan–Vigué-Poirrier theorem leaves out many important manifolds. The simplest complicated example is the Stiefel manifold of 2-frames in $\mathbb{R}^{2n+1}$. There is a fibration $S^{2n-1} \hookrightarrow V_2(\mathbb{R}^{2n+1}) \rightarrow S^{2n}$, whose Leray-Serre spectral sequence is determined by a single differential $d_{2n}: E_{2n-1} \rightarrow E_{2n,0}$. This differential can be detected in the Gysin sequence for this sphere bundle over a sphere and, over the integers, it is given by multiplication by 2 (the Euler characteristic of $S^{2n}$). For fields of characteristic not equal to 2, $H^*(V_2(\mathbb{R}^{2n+1}); \mathbb{F}) \cong H^*(S^{2n-1}; \mathbb{F})$. However, over $\mathbb{F}_2$, the differential is zero and $H^*(V_2(\mathbb{R}^{2n+1}); \mathbb{F}) \cong \mathbb{F}_2[x_{2n}, y_{2n-1}]/(x_{2n}^2, y_{2n-1}^2)$. Thus the manifold satisfies having at least two algebra generators over one field, $\mathbb{F}_2$.

A theorem of Borel implies that $H_*(\Omega V_2(\mathbb{R}^{2n-1}; \mathbb{F}_2) \cong \mathbb{F}_2[u_{2n-1}, v_{2n-2}]$. It follows that the set $\{\dim_{\mathbb{F}_2} H_i(\Omega V_2(\mathbb{R}^{2k+1}); \mathbb{F}_2) \mid i = 0, 1, 2, \ldots\}$ is unbounded. And so it suffices to compute the differentials in this spectral sequence to determine $H_*(\Lambda V_2(\mathbb{R}^{2n+1}); \mathbb{F}_2)$ and settle the closed geodesics problem.

There are ad hoc arguments to make this calculation, however, recent further structure on the homology of the free loop space $\Lambda M$ provides better tools to work with. The idea is due to M. Chas and D. Sullivan [10]: Suppose $\alpha: \Delta^p \rightarrow \Lambda M$ and $\beta: \Delta^q \rightarrow \Lambda M$ are singular simplices in $\Lambda M$. Take the composite

$$\Delta^p \times \Delta^q \rightarrow \Lambda M \times \Lambda M \xrightarrow{\ev_1 \times \ev_1} M \times M$$

and suppose that it is transverse to the diagonal.
At each point where \( ev_1 \circ \alpha \) meets \( ev_1 \circ \beta \) you have two loops at \( \alpha(1) = \beta(1) \). Form the loop product there. Since the transverse intersection of a \( p \)-chain and a \( q \)-chain in a \( d \)-dimensional manifold has dimension \( p + q - d \), this construction gives a chain

\[
\alpha \circ \beta \in C_{p+q-d}(\Lambda M).
\]

If \( M \) has dimension \( d \), then define the string topology of \( M \) to be

\[
\mathbb{H}_*(\Lambda M) = H_{*,+d}(\Lambda M; \mathbb{F}).
\]

**Theorem 5.1.** The chain map \( C_p(\Lambda M) \otimes C_q(\Lambda M) \to C_{p+q-d}(\Lambda M) \) induces an associative, commutative algebra structure on \( \mathbb{H}_*(\Lambda M) \).

Is \( \mathbb{H}_*(\Lambda M) \) a homotopy invariant? Cohen and Jones [11] have proved that if we denote by \( \tilde{H}^\Sigma ) \) a homotopy invariant? Cohen and Jones [11] have proved that if we denote by \( \tilde{H}^*(\Lambda M) = ev_1^*(\tilde{H}^*(\tilde{M})) \) and let \( M^{-TM} \) denote the associated Thom spectrum, then the pullback \( (\Lambda M)^{-TM} = ev_1^*(\tilde{M}) \) satisfies the following properties:

1) \( (\Lambda M)^{-TM} \) is a homotopy commutative ring spectrum with unit.

2) The product on \( (\Lambda M)^{-TM} \) realizes \( \circ \) after applying the Thom isomorphism

\[
H_q((\Lambda M)^{-TM}) \cong H_{q+d}(\Lambda M) = \mathbb{H}_q(\Lambda M).
\]

For the closed geodesics problem there is an associated spectral sequence, called the CJY spectral sequence:

**Theorem 5.2.** [12] If \( M \) is an oriented, simply-connected manifold, then there is a 2nd quadrant spectral sequence of algebras \( \{ (E_{p,q}^r; d^r); p \leq 0, q \geq 0 \} \) such that

1) \( E_{*,*}^r \) is a bigraded algebra with \( d^r : E_{*,*}^r \to E_{*,r+1}^r \), a derivation for each \( r \geq 1 \).

2) The spectral sequence converges to \( \mathbb{H}_*(\Lambda M) \) as an algebra.

3) For \( m, n \geq 0 \), \( E_{m,n}^2 \cong H^m(M; H_n(\Omega M)) \) as algebras, with the product on \( H^*(M) \) given by the cup product, and the product on \( H_*(\Omega M) \) given by the Pontryagin product.

4) The spectral sequence is natural with respect to smooth maps.

With this tool, we can apply a classical argument: Consider the CJY spectral sequence for the mod 2 string topology of \( V_2(\mathbb{F}_2[\mathbb{F}_2]) \), \( \mathbb{H}_*(\Lambda V_2(\mathbb{F}_2[\mathbb{F}_2]) \), Since \( H_*(\Omega V_2(\mathbb{F}_2[\mathbb{F}_2]) \), we can consider the differentials on each sub-polynomial algebra \( \mathbb{F}_2[u_{2n-1}, v_{2n-2}] \).

Notice that \( d^r(x^2) = 0 \) because the algebra is commutative and \( d^r \) is a derivation. Thus, we can apply successive differentials that are zero on successive squares, and this procedure leaves a polynomial algebra on a pair of generators of the form \( u^{2^j} \) and \( v^{2^k} \). But a polynomial algebra on two generators has unbounded dimensions. Thus, so does \( \mathbb{H}_*(\Lambda V_2(\mathbb{F}_2[\mathbb{F}_2]) \).

This argument generalizes considerably. First off, we can ask if the condition that \( H^*(M; \mathbb{F}_p) \) requires at least two algebra generators obtains enough growth in the homology of the based loop space \( H_*(\Omega M; \mathbb{F}_p) \).
Theorem 5.3. [34] If $X$ is a finite CW-complex and the cohomology algebra $\hat{H}^\ast(X; \mathbb{F}_p)$ requires at least two generators as an algebra, then the set $\{\dim_{\mathbb{F}_p} H_i(\Omega M; \mathbb{F}_p) \mid i = 0, 1, 2, \ldots\}$ is unbounded.

This theorem generalizes to $\mathbb{F}_p$ coefficients a theorem of Sullivan [40] for rational coefficients. The proof uses the Bockstein spectral sequence for $H_\ast(\Omega M; \mathbb{F}_p)$ together with Sullivan’s result as input.

In recent work with John Jones, we have been able to extend the computation of $H_\ast(\Omega M; \mathbb{F}_p)$ to other manifolds. The key is to understand $H_\ast(\Omega M; \mathbb{F}_p)$. For this, we have the fundamental work of Y. Félix, S. Halperin, J.-M. Lemaire, and J.-C. Thomas [22]. They have introduced a dichotomy

Definition. A manifold $M$ is elliptic mod $p$ if there is an integer $N = N(p)$ and a constant $C = C(p)$ such that

$$\dim_{\mathbb{F}_p} H_r(\Omega M; \mathbb{F}_p) \leq Cr^N, \quad r = 1, 2, \ldots$$

A manifold $M$ is hyperbolic mod $p$ if there is a constant $K > 1$ such that

$$\sum_{i=0}^{n} \dim_{\mathbb{F}_p} H_i(\Omega M; \mathbb{F}_p) \geq K^{\sqrt{n}}, \text{ for } n \text{ large enough.}$$

For simply-connected elliptic manifolds, the following theorem holds:

Theorem 5.4. [23] If $M$ is an elliptic manifold, then $H_\ast(\Omega M; \mathbb{F}_p)$ is an elliptic Hopf algebra, and so it is a finitely generated module over a central sub-Hopf algebra which is a polynomial algebra in finitely many indeterminates.

That is, the Hopf algebra $G = H_\ast(\Omega M; \mathbb{F}_p)$, being elliptic, may be written as a $K$-module as $K \otimes G/K$ where $K$ is a polynomial algebra and $G/K$ is finite dimensional.

The growth of dimensions for $H_\ast(\Omega M; \mathbb{F}_p)$, when $H^\ast(M; \mathbb{F}_p)$ requires at least two algebra generators, forces the polynomial algebra $K$ to have at least two generators. The classical argument illustrated for $V_2(\mathbb{R}^{2n+1}; \mathbb{F}_2)$ above produces the growth in $H_\ast(\Lambda M; \mathbb{F}_p)$ needed to deduce infinitely many closed geodesics on $M$.

At the moment, we have not fully understood the case of hyperbolic manifolds and their mod $p$ string topology.
6. Fundamental group actions

The $E_2$-page of the Leray-Serre spectral sequence is given in terms of the cohomology of the base space with coefficients in a local system determined by the cohomology of the fibre and the fundamental group of the base space. The situation is simplest when the local system is trivial, but this is not often the case when studying the fibrations that arise in equivariant topology.

The isomorphisms in §4 between $E_1^{p,q} = H^{p+q}(J(p), J(p-1); \mathbb{F}) \cong C^p(B(p)) \otimes H^q(F; \mathbb{F})$ are rather subtle. They are expressed in terms of local system of coefficients.

In order to give a proper definition of a local system of coefficients we need a proper definition of a fibration. Suppose $p: E \to B$ is an onto continuous function. We can associate to $p$ the pullback diagram, where $B^I = \{ \lambda: [0,1] \to B \mid \lambda \text{ continuous}\}$:

$$
\begin{array}{ccc}
\Omega_p & \longrightarrow & B^I \\
\downarrow \quad \quad \quad \quad \downarrow ev_0 \\
E & \longrightarrow & B
\end{array}
$$

that is, $\Omega_p = \{(\lambda, x) \in B^I \times E \mid \lambda(0) = p(x)\}$. There is a canonical mapping $\tilde{p}: E^I \to \Omega_p$ given by $\tilde{p}(\lambda) = (p \circ \lambda, \lambda(0))$.

The mapping $p$ is a fibration (in the sense of Hurewicz) if there is a lifting function $\Lambda: \Omega_p \to E^I$ for which $\tilde{p} \circ \Lambda = \text{id}_{\Omega_p}$.

For a choice of base point $b_0 \in B$, let $F = p^{-1}(b_0)$ denote the fibre over $b_0$ and $\Omega B = \Omega(B, b_0)$, the based loop space at $b_0$. Then $\Omega B \times F \subset \Omega_p$. If we apply $\Lambda$ to this subspace of $\Omega_p$ and follow on with evaluation at 1, we get a mapping $ev_1 \circ \Lambda|_{\Omega B \times F}: \Omega B \times F \to E$. However, because $\tilde{p} \circ \Lambda = \text{id}_{\Omega_p}$ the reader can check that $\Lambda(\omega, x)(1) \in F$.

Let $h(\omega, x) = \Lambda(\omega, x)(1)$ for $\omega \in \Omega B$ and $x \in F$, and let $h_\alpha(x) = h(\omega, x)$ denote the function $F \to F$ determined by $\lambda$. The functions $h_\alpha: F \to F$ enjoy the following properties:

**Lemma 6.1.** If $\alpha$ and $\beta$ in $\Omega B$ are homotopic, then $h_\alpha$ and $h_\beta$ are homotopic. If $\alpha * \beta$ denotes loop multiplication, then $h_{\alpha * \beta}$ and $h_\beta \circ h_\alpha$ are homotopic. If $c_{b_0}$ is the constant loop at $b_0$, then $h_{c_{b_0}}$ is homotopic to the constant mapping.

Thus, up to homotopy, the action of $\Omega B$ on $F$ depends only on the path components of loops in $B$. This fact reduces to an action of $\pi_1(B) = \pi_0(\Omega B)$ on $F$. Let $p: \pi_1(B) \to \text{GL}(H_*(F; \mathbb{F}))$ denote the representation $\rho([\omega]) = h_\omega$.

If we denote by $D_{r,s} = C_{r+s}(p^{-1}(\Delta(B, b_0)^{(r)}))$, where $\Delta(B, b_0)^{(r)}$ denotes the $r$-skeleton of the simplicial set of singular simplices on $B$ with all vertices at $b_0$, then there is a well-defined mapping

$$
\phi: D_{r,s} \to C_r(B; C_s(F; \mathbb{F})), \quad \text{given by } \phi(au) = (a(\partial_{s+1})^r(u)) \otimes (\partial_h)^r p_*(u).
$$

With these data, we can state the full blown version of the Leray-Serre spectral sequence:
Theorem 6.2. The mapping $\phi$ induces a homomorphism $\phi_0: E^0_{r,s} \to C_r(B; C_s(F; F))$, where $E^0_{r,s} = D_{r,s}/D_{r-1,s+1}$. Equip $E^0_{r,s}$ with the differential $d^0: E^0_{r,s} \to E^0_{r-1,s}$ induced by $\partial$ on $C_{r+s}(\Delta(B, b_0); F)$. Then the following diagrams commute

$$
\begin{array}{ccc}
E^0_{r,s} & \xrightarrow{\phi_0} & C_r(B; C_s(F; F)) \\
\downarrow d^0 & & \downarrow \partial_F \otimes 1 \\
E^0_{r-1,s} & \xrightarrow{\phi_1} & C_{r-1}(B; H_s(F; F))
\end{array}
$$

where $\partial_t$ is the differential associated with the local system of coefficients on $C^*_s(B)$. Together, this gives an isomorphism $E^2_{r,s} \cong H_r(B; H_s(F; F))$.

The above description uses the simplicial structure. There is a cellular approach [32] that can be seen to depend on the fibration structure more directly. Suppose $B$ is a CW-complex with a single vertex $b_0$ and $F = p^{-1}(b_0)$. The $k$-cells have characteristic functions

$$
x: (e^k, S^{k-1} \to (B^{(k)}, B^{(k-1)}),
$$

and we can pull back the fibration pair $(J^{(k)}, J^{(k-1)})$ over $x$. Since $e^k$ is contractible, it is a standard result in homotopy theory that $p^{-1}(e^k) \cong e^k \times F$. In our context, the equivalence is obtained by filling $e^k$ with paths from a base point on the boundary to another point on the boundary. Such a path may be sent via $x$ to $(B^{(k)}, B^{(k-1)}) \to (B, *)$ giving a loop at $b_0$. Then we get a mapping $e^k \times F \to p^{-1}(e^k)$ by passing through $h: \Omega B \times F \to F$. This analysis of the equivalence leads to a local coefficient system.

Main example: If $G$ is a group acting on a space $X$, then the Borel construction gives a fibration, $X \hookrightarrow X \times_G EG \twoheadrightarrow BG$, where $EG$ is a contractible free $G$-space, and $BG = EG/G$ is the classifying space for $G$. When $G$ is a discrete group, $BG = K(G, 1)$ and so the fibration action $\Omega BG \times X \simeq G \times X \to X$ is the group action. Hence, the group action data are present in the initial pages of the associated spectral sequence. We will consider Borel constructions in what follows.

7. Combinatorial applications

A number of important results in combinatorics, discrete geometry, and theoretical computer science have been proved by surprising applications of algebraic topology.

Jiří Matoušek [31]

Let us consider the following simple problems:

I. Suppose $\gamma: S^1 \to \mathbb{R}^2$ is a Jordan curve. Are there 3 points on it that form an equilateral triangle?
II. The classical Ham Sandwich Theorem. Given two slabs of bread and a slice of ham is there a straight knife cut that divides the three bits into two parts with equal halves of bread and of ham?

The Ham Sandwich theorem can be more technically presented:

Given \( n \) finite Borel measures on \( \mathbb{R}^n \). Is there a hyperplane in \( \mathbb{R}^n \) for which \( \mu_i(\text{half-space}) = (1/2)\mu_i(\mathbb{R}^n) \) for \( i = 1, \ldots, n \)?

There are many generalizations of the Ham Sandwich theorem that are approachable using the tools of algebraic topology. The manner by which we can find a use for topology is a paradigm introduced by \( \check{Z} \)ivaljević [43]:

**The Configuration Space/Test Map Paradigm:** Let \( X \) denote a space of configurations of a combinatorial problem. Let \( V \) denote a test space (often \( \mathbb{R}^d \)) and \( Z \subset V \) a subspace of distinguished values (a discriminant). A mapping \( f: X \to V \) is defined by a combinatorial problem for which we ask

**Does \( f(X) \) meet the discriminant \( Z \)?**

Suppose further that a group \( G \) acts on \( X \) and on \( V \) and on \( V \setminus Z \).

Also assume that the mapping \( f \) is \( G \)-equivariant. We can ask for the difficult topological condition:

**Is there a \( G \)-equivariant mapping \( F: X \to V \setminus Z \)?**

If not, then \( f(X) \) meets \( Z \).

To give the basic example of the paradigm in action, recall the well known result:

**The Borsuk-Ulam Theorem.**

I. If \( f: S^n \to \mathbb{R}^n \) is antipodal, that is, \( f \) is continuous and \( f(-x) = -f(x) \) for all \( x \in S^n \), then there is a point \( y \in S^n \) with \( f(y) = 0 \).

II. If \( f: S^n \to \mathbb{R}^n \) is continuous, then there is a point \( x \in S^n \) with \( f(x) = f(-x) \).

III. If \( m < n \), then there is no antipodal mapping \( f: S^n \to S^m \).

The antipodal mappings of the theorem are \( \mathbb{Z}/2 \)-equivariant mappings \( S^n \to \mathbb{R}^n \), or \( S^n \to S^m \). To fit the Borsuk-Ulam theorem into the CS/TM paradigm, let’s consider formulation I. We suppose that there is an antipodal mapping \( f: S^n \to \mathbb{R}^n \), that is, \( f(-v) = -f(v) \) for all \( v \in S^n \). The discriminant is simply the point \( Z = \{0\} \). Thus \( S^n \) plays the role of configuration space, and \( \mathbb{R}^n \) the test space. We assume that there is an antipodal map for which \( f(v) \neq 0 \) for all \( v \in S^n \). Then we really have a \( \mathbb{Z}/2 \)-equivariant map \( f: S^n \to \mathbb{R}^n \setminus \{0\} \). Now we can ask if there is such an equivariant mapping. In both cases, \( S^n \) and \( \mathbb{R}^n \setminus \{0\} \), the \( \mathbb{Z}/2 \)-action is free with finite dimensional projective spaces as quotients.

Consider the Leray-Serre spectral sequences for the Borel constructions in each case. Because the actions are free, there is a differential. On cohomology, the mapping \( f \) induces a mapping of spectral sequences that is the identity on \( E^{*,0}_{2} = H^*(BG; H^{0}(S^{n-1}; F)) \to H^*(BG; H^{0}(S^n; F)) = E^{*,0}_{2} \). The differentials commute with \( f^* \), and here is the rub: the non-zero elements in \( E^{*,+}_{2} \) are concentrated in row \( n-1 \) for \( \mathbb{R}^n \setminus \{0\} \) and in row \( n \) for \( S^n \). However, \( 0 \neq d_{n-1}(1 \otimes [S^{n-1}]) = l \otimes 1 = f^*(l) \otimes 1 = 0 \),
a contradiction. Since we made this argument based on the existence of such a mapping $f$, no such mapping exists.

Formulation II follows from I by taking any continuous mapping $f: S^n \to \mathbb{R}^n$ and turning attention to $g(x) = f(x) - f(-x)$, an equivariant mapping. Then formulation I proves the existence of a zero of $g(x)$, proving II. Finally, for $m < n$, the equivariant mapping $f: S^n \to S^m$ can be extended to $f: S^n \to \mathbb{R}^n$ by composing with the inclusion of $S^m$ into $\mathbb{R}^{m+1} \subset \mathbb{R}^n$.

The Ham Sandwich theorem provides a model example for mass partition problems. The idea is to construct the right configuration space. Let $\mu_1, \mu_2$, and $\mu_3$ be finite Borel measures (two pieces of bread and the meat) in $\mathbb{R}^3$. Consider $v \in S^3$, $v = (v_0, v_1, v_2, v_3)$. Then $v$ defines half-spaces in $\mathbb{R}^3$:

$$h^+(v) = \{(x, y, z) \mid v_1 x + v_2 y + v_3 z \leq v_0\}, \text{ and } h^-(v) = \{(x, y, z) \mid v_1 x + v_2 y + v_3 z \geq v_0\}. $$

Notice that the $\mathbb{Z}/2$-action on $S^3$ is $v \mapsto -v$, and $h^+(v) = h^-(v)$.

Consider the mapping $f: S^3 \to \mathbb{R}^3$ given by $f(v) = (\mu_1(h^+(v)), \mu_2(h^+(v)), \mu_3(h^+(v)))$. The assumption of finite Borel measures implies that $f$ is continuous. Formulation II of the Borsuk-Ulam theorem determines a direction $v \in S^3$ with $f(v) = f(-v)$, that is, $\mu_i(h^+(v)) = \mu_i(h^-(v))$ for $i = 1, 2, 3$. The hyperplane $h^+(v) \cap h^-(v)$ is the knife that cuts the sandwich. The reader will want to extend this proof to higher dimensions.

In order to extend this sort of argument in a general manner, Blagojević and I [5] introduced the following notion for a $G$-equivariant mapping. A $G$-map, $f: X \to Y$, induces a mapping of associated fibrations for the Borel constructions,

$$
\begin{align*}
X & \xrightarrow{f} Y \\
EG \times_G X & \xrightarrow{1 \times_G f} EG \times_G Y \\
BG & \xrightarrow{id} BG
\end{align*}
$$

The map of fibrations induces a mapping of the Leray-Serre spectral sequences associated to the fibrations:

$$E_2^{*,*}(f): E_2^{*,*}(Y \times_G EG) \to E_2^{*,*}(X \times_G EG).$$

**Definition.** A spectral sequence witness of a pair of $G$-spaces $X$ and $Y$, with coefficients in $F$, is any nonzero element

$$l \in H^{n+1}(BG; F) = E_2^{n+1,0}(EG \times_G X) = E_2^{n+1,0}(EG \times_G Y),$$

for some fixed integer $n \geq 2$, satisfying

(A) In the spectral sequence for $Y$, for $2 \leq i < n$, $l$ is not an coboundary, and $l$ is in the image of the transgression, that is,

$$l = d_n(z), \text{ for some } z \in E_n^{0,n}(EG \times_G Y).$$
(B) In the spectral sequence for $X$, $l$ survives to $E_\infty$, that is,

$$l \notin \text{im}(d_i: E_i^{n-i,i}(EG \times_G X) \to E_i^{n+1,0}(EG \times_G X)),$$

for all $2 \leq i \leq n$.

The set of all spectral sequence witnesses is denoted by $W(X,Y;R)$.

Notice that the witnesses are defined independently of a choice of $G$-map.

**Theorem 7.1.** Let $X$ and $Y$ be connected $G$-spaces. If $W(X,Y;F) \neq \emptyset$, then there is no $G$-equivariant map $X \to Y$.

Proof. Let $l \in W(X,Y;F)$. The condition (A) of the definition of a witness implies that

$$l \notin \text{image} \left( d_i: E_i^{n-i,i}(EG \times_G X) \to E_i^{n+1,0}(EG \times_G X) \right)$$

for some fixed $n > 2$ and every $2 \leq i < n$. The morphism $E_i^{n,0}(f)$ induced by the $G$-equivariant mapping $f$ is the identity on $l \in W(X,Y;F)$ for $2 \leq i \leq n$. By condition (B), we can make the identification

$$E_n^{n+1,0}(EG \times_G Y) \ni l \xrightarrow{E_n^{0,0}(f)} l \in E_n^{n+1,0}(EG \times_G X).$$

Since $l \neq 0$, by condition (A), we can write $l = d_n(z)$ for some nonzero $z \in E_n^{0,n}$. Applying the morphism of spectral sequences we have

$$l = E_n(f)(l) = E_n(f)(d_n(z)) = d_n(E_n(f)(z)) = d_n(f^*(z)).$$

But, $l$ is a permanent cycle, so we get a contradiction, and $f$ cannot exist. \qed

For those familiar with the work of Fadell and Husseini [19], there is a connection between their *ideal-valued index* and the set of witnesses. Certainly all witnesses lie in their index. The notion of witness may be a bit more general, and it is clearly relevant via the connection to spectral sequences.

Witnesses give a proof of a theorem of Dold [14]:

![Diagram](image-url)
Theorem 7.2. Suppose $X$ and $Y$ are finite CW free $G$-spaces for $G$ a nontrivial finite group. If $f: X \to Y$ is a $G$-equivariant mapping, then $\dim(Y) \geq 1 + \text{connectivity}(X)$.

Proof. If $\text{connectivity}(Y) + 1 > \dim(X)$, then any nonzero differentials for $X$ occur for $d_k$, $2 \leq k \leq n$. However, there is a patch of trivial vector spaces to receive any candidate under $f^*$ in the spectral sequence for $Y$, so we see that $W(X, Y; \mathbb{F}) \neq \emptyset$.

We next consider some particular test spaces and discriminants. If algebraic topology is going to be useful, then the algebraic topology of the test space minus the discriminant must be computable. A useful class of examples is provided by arrangements.

Definition. An arrangement in $\mathbb{R}^d$ is a collection $A = \{L_1, \ldots, L_m\}$ of affine subspaces of $\mathbb{R}^d$. An arrangement determines a poset

$$P(A) = \{\bigcap_i L_{k_i} \mid L_{k_i} \in A\}.$$ 

The poset is ordered by reverse inclusion.

Discriminants often arise as arrangements. An affine hyperplane is defined by a single linear equation: $l(x) = a_1x_1 + a_2x_2 + \cdots + a_dx_d = b$. A family of linear equations determines an arrangement. The Borsuk-Ulam theorem requires only the trivial subspace $\{0\}$. Mass partition problems can be posed for which a winning partition (configuration) forces the associated masses to satisfy a linear condition, that is, to land in an arrangement.

Here is a geometric example: Suppose $\gamma: S^1 \to \mathbb{R}^2$ is a Jordan curve. Let

$$C_3(\gamma) = \{(\gamma(z_1), \gamma(z_2), \gamma(z_3)) \mid z_i \in S^1, \text{for all } i, \text{ and } z_i \neq z_j \text{ for } i \neq j\},$$

the set of ordered triples of distinct points in $\gamma(S^1)$. A test map can be constructed

$$f: C_3(\gamma) \to \mathbb{R}^3, \quad f(\gamma(z_1), \gamma(z_2), \gamma(z_3)) = (\|\gamma(z_2) - \gamma(z_1)\|, \|\gamma(z_3) - \gamma(z_2)\|, \|\gamma(z_1) - \gamma(z_3)\|).$$

Let $L$ denote the diagonal subspace in $\mathbb{R}^3$, that is, $\{(r, r, r) \mid r \in \mathbb{R}\}$. Then there is a set of three points on $\gamma$ that form an equilateral triangle if and only if $f(C_3(\gamma)) \cap L \neq \emptyset$.

Notice that there is an $\mathbb{Z}/3$-action on $C_3(\gamma)$ and on $\mathbb{R}^3$ given by $[1] \cdot (a, b, c) = (c, a, b)$. Furthermore, $f$ as defined is a $\mathbb{Z}/3$-equivariant mapping.

It turns out that this formulation does not enjoy sufficient algebraic topology for the CS/TM paradigm, but it is possible to fashion a suitable configuration space and test map to serve [3]. This provides a warm-up for an investigation of the square peg problem: Is there a set of four points on $\gamma$ that form a square?
An arrangement determines a poset \( P(\mathcal{A}) \). A poset \((P, <)\) determines a simplicial set, \( \Delta_*(P) \), the nerve of the poset. A \( k \)-simplex of \( \Delta_*(P) \) is a \( k+1 \)-tuple of elements of \( P \) \((a_0, a_1, \ldots, a_k)\) satisfying \( a_0 < a_1 < \cdots < a_k \). Goresky and MacPherson [25] computed \( H_*(\mathbb{R}^d \setminus \bigcup_i L_i ; \mathbb{R}) \), the homology of the complement of the arrangement and expressed the answer in terms of the nerve of sub-posets of \( P(\mathcal{A}) \). If \( a < b \) are in \( P \), then the interval between \( a \) and \( b \) is the sub-poset of \( P \) given by \((a, b) = \{ x \in P \mid a < x < b \} \). Let \( \Delta_*((a, b)) \) denote the nerve of \((a, b)\). We can also define the sub-poset \([a, b]\) as expected.

**Theorem of Goresky and MacPherson.** The homology of the complement of an arrangement \( \mathcal{A} \), denoted \( M = \mathbb{R}^d \setminus \bigcup_i L_i \) is given by

\[
H_*(M) \cong \bigoplus_{L \in P(\mathcal{A})} H^{r(L) - i - 1}(\Delta_*([\mathbb{R}^d, L]), \Delta_*([\mathbb{R}^d, L])).
\]

Here \( r(L) \) denotes the codimension of \( L \), and we adopt the convention that \( H^{-1}(\emptyset, \emptyset) \cong \mathbb{Z} \). Thus the homology and cohomology of the complement of an arrangement is determined by the combinatorial structure of the associated poset.

If \( \text{codim}(L_i) = c \) for all \( i \) and for all \( X < Y \) in \( P(\mathcal{A}) \), \( \text{codim}_X(Y) \) is a multiple of \( c \), then the arrangement \( \mathcal{A} \) is called a \( c \)-arrangement. For example, if \( L_i \) is a complex hyperplane in \( \mathbb{C}^d \), considered as \( \mathbb{R}^{2d} \). In this special case, De Longueville and Schultz [28] have proved

**Theorem 7.3.** The integral cohomology of the complement of a \( c \)-arrangement \( M = M(\mathcal{A}) \) is generated as an algebra by \( H^{c-1}(M) \).

In [5], my coauthors and I mix the ingredients above to apply spectral sequences and witnesses to provide a tool suited to mass partition problems. *A priori*, an arrangement \( \mathcal{A} \) need not be a \( c \)-arrangement. However, this can be fixed in an interesting manner.

**Definition.** Let \( \mathcal{A} = \{L_1, \ldots, L_N\} \) denote the set of maximal elements an arrangement of affine subspaces in \( \mathbb{R}^d \). Suppose \( k_i = \text{codim}_{\mathbb{R}^d}L_i \), for \( i = 1, \ldots, N \). For each maximal element, \( L_i \), choose a linearly independent family of 1-forms \( \{\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,k_i}\} \) defining \( L_i \). The blow up of \( \mathcal{A} \) is the arrangement \( \mathcal{B}(\mathcal{A}) \) in \((\mathbb{R}^d)^{k_1 + \cdots + k_N} = (\mathbb{R}^d)^{k_1} \times \cdots \times (\mathbb{R}^d)^{k_N}\) where a subspace \( \bar{L}_i \) in \( \mathcal{B}(\mathcal{A}) = \{\bar{L}_1, \ldots, \bar{L}_n\} \) is given by the forms \( \xi_{i,j} \) on the \( j \)th subspace \( \mathbb{R}^d \) in the factor \((\mathbb{R}^d)^{k_i}\).

The main properties of \( \mathcal{B}(\mathcal{A}) \) include

(A) If the maximal elements in \( \mathcal{A} \), \( L_1, \ldots, L_N \), all have the same codimension, then \( \mathcal{B}(\mathcal{A}) \) is a \( c \)-arrangement with \( c = \text{codim}_{\mathbb{R}^d}\mathcal{A} = \min_{L \in \mathcal{A}} \text{codim}_{\mathbb{R}^d}L \). If not, \( \mathcal{B}(\mathcal{A}) \) can be arranged to give a \( c \)-arrangement.

(B) The diagonal mapping \( \mathbb{R}^d \to (\mathbb{R}^d)^{k_1 + \cdots + k_N} \) restricts to a map of complements

\[
D: M(\mathcal{A}) = \mathbb{R}^d \setminus \bigcup_{L \in \mathcal{A}} L \to M(\mathcal{B}(\mathcal{A})) = (\mathbb{R}^d)^{k_1 + \cdots + k_N} \setminus \bigcup_{L \in \mathcal{B}(\mathcal{A})} \bar{L}.
\]

If \( G \) is a finite group acting on \( \mathbb{R}^d \), then an arrangement is \( G \)-invariant if \( g \cdot L \in \mathcal{A} \) whenever \( g \in G \) and \( L \in \mathcal{A} \). Since an arrangement is a finite set, one can make an arrangement \( G \)-invariant by adding \( g \cdot L \) to \( \mathcal{A} \) for all \( L \in \mathcal{A} \) and \( g \in G \). We can take our arrangements to be \( G \)-invariant.
Theorem 7.4. Let $G$ denote a finite group and $\mathbb{F}$ a field. Let $X$ be a $G$-space satisfying $H^i(X; \mathbb{F}) = 0$ for $1 \leq i \leq n$ for some $n \geq 2$. Consider a $G$-invariant arrangement $A$ in (some subspace $V$ of) $\mathbb{R}^d$ and its $G$-invariant blow up $\mathcal{B}(A)$. Suppose further that:

(A) the codimension of all maximal elements in $A$ is $n + 1$;
(B) $G$ acts trivially on the cohomology of the complement $H^*(M(\mathcal{B}(A)); \mathbb{F})$;
(C) the map $H^*(BG; \mathbb{F}) \to H^*(EG \times_G M(\mathcal{B}(A)); \mathbb{F})$, induced by the natural projection $EG \times_G M(\mathcal{B}(A)) \to BG$, is not a monomorphism, and
(D) for all $L \in A$, $L$ contains the fixed point set $(\mathbb{R}^d)^G$, that is, $L \supseteq (\mathbb{R}^d)^G$.

Then there are no $G$-equivariant mappings $X \to M(A)$.

Sketch of proof: Everything is set up to use the De Longueville and Schultz theorem to guarantee a differential in the spectral sequence that determines witnesses for the complement of the blow up $M(\mathcal{B}(A))$. For a $G$-equivariant mapping $f: X \to M(A)$, we can form the composite $X \overset{f}{\rightarrow} M(A) \overset{D}{\rightarrow} M(\mathcal{B}(A))$, which is also $G$-equivariant. However, the assumptions are made to produce a nonzero witness.

An example of a situation that can be found is the pairing/sharing cheese problem. Suppose there are $k$ finite Borel measures on $\mathbb{R}^d$ (these are the cheeses). Around a circular table sit $m$ gourmands with preferences for how much of each cheese (measure) they want to eat. Directly across the table sits the spouse of each gourmand who also has a preference for how much of each cheese he or she will eat.

To encode their preferences, introduce a ration vector $(\alpha_1, \alpha_2, \ldots, \alpha_m; \beta_1, \beta_2, \ldots, \beta_m)$ which satisfies $\alpha_i, \beta_j \geq 0$, and $\sum_i (\alpha_i + \beta_i) = 1$. For convenience, let’s assume that $\alpha_i = \beta_i$ for all $i$ (the couple want to eat the same portions).

To describe the configurations of the slices of cheese we change the geometry of the problem a bit and modify the measures. There is a central point and a sphere of sufficient radius that contains the support of all the measures. Centrally project from this point to induce measures on $S^{d - 1}$ that correspond to the $\mu_i$. Now we can introduce the knives.

To cut the cheeses, choose a 2-frame in $\mathbb{R}^d$, $(u, v_1)$. Let $L = \text{Span}(u)$. Denote by $H_1$ the hyperplane $v_1$. Since $v_1$ is in $L^\perp$, rotate $v_1$ in $L^\perp$ to $v_2$ so that $(1), (u, v_2) \in V_2(\mathbb{R}^d)$, and $(2)$ the hyperplane $H_2 = v_2$ determines a (generalized) octant $O_1 \subset \mathbb{R}^d$ with $mu_1(O_1) = 1/2n$. The choice of octant is determined by the sides of the hyperplanes where $v_1$ goes to $v_2$. Continue this process all of the way around the sphere until all of the desired portions are allotted to everyone.

This process produces a point $(L; v_1, v_2, \ldots, v_{2n})$, or $(L; O_1, O_2, \ldots, O_{2n})$ in a space $X_{\mu_1,2n}$. However, this space is simply $V_2(\mathbb{R}^d)$. The advantage of $X_{\mu_1,2n}$ is that the group $D_{2n}$ acts on it. The dihedral group $D_{4n}$ is presented as $D_{4n} = \langle \epsilon, \sigma | \epsilon^{2n} = \sigma^2 = 1, \epsilon^{n-1} \sigma = \sigma \epsilon \rangle$. The action $X_{\mu_1,2n}$ is given by

$\epsilon(L; v_1, v_2, \ldots, v_{2n}) = (L; v_{2n}, v_1, \ldots, v_{2n-2}, v_{2n-1}), \text{ and } \sigma(L; v_1, v_2, \ldots, v_{2n}) = (L; v_{2n}, v_{2n-1}, \ldots, v_1)$. 

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We can define the same action on coordinates on $\mathbb{R}^{2n}$. From this we can define a $D_{4n}$-equivariant mapping

$$F(L; O_1, \ldots, O_{2n}) = ((\mu_i(O_1) - \frac{1}{n}, \ldots, \mu_i(O_{2n}) - \frac{1}{n}))_j = 2^m \in (W_{2n})^\oplus(m-1).$$

Here $W_{2n} \subset \mathbb{R}^{2n}$ is given by $W_{2n} = \{(x_1, \ldots, x_{2n}) \mid \sum x_i = 0\}$. Notice that $W_{2n}$ is $D_{4n}$-stable with the described action.

The mapping $F$ is defined in such a way that it is $D_{4n}$-equivariant. Finally, let’s consider the discriminant $Z \subset (W_{2n})^\oplus(m-1)$. Denote a point $x \in (W_{2n})^\oplus(m-1)$ by

$$x = (x_{1,2}, x_{2,2}, \ldots, x_{2n,2}; x_{1,3}, x_{2,3}, \ldots, x_{2n,3}; \cdots; x_{1,m}, x_{2,m}, \ldots, x_{2n,m}).$$

The choice of $n$ as least common denominator means that we can take consecutive octants with unions of measure $b_i/n = \alpha_i = \beta_i$. For example, if $\mu_1(O_1) = \frac{1}{n}$ for all $i$, then

$$\mu_1(O_1 \cup O_2 \cup \cdots \cup O_{b_1}) = \frac{b_1}{n} = \alpha_1.$$

Continuing in this manner, we can obtain the desired partition for all gourmands and spouses.

In the coordinates of $W_{2n}$ we have linear systems: For each $i = 2, \ldots, m$:

$$x_{1,i} + \cdots + x_{b_1,i} = x_{b_2+1,i} + \cdots + x_{b_k+b_k+1,i},$$

$$x_{b_1+1,i} + \cdots + x_{b_1+b_2,i} = x_{b_k+b_k+1+1,i} + \cdots + x_{b_k+b_k+1+b_k+2,i},$$

$$\vdots$$

$$x_{b_1+\cdots+b_{m-1}+1,i} + \cdots + x_{m,i} = x_{b_k+\cdots+b_{2k-1}+1,i} + \cdots + x_{2m,i}$$

This family of linear equations determine an arrangement $\mathcal{A}$ in $(W_{2n})^\oplus(m-1)$. The $D_{4n}$-action on $X_{\mu_1,2n}$ determines a $D_{4n}$-action on $V_2(\mathbb{R}^d)$, whose algebraic topology we know.

**Theorem 7.5.** If there are no $D_{4n}$-equivariant mappings

$$f : V_2(\mathbb{R}^d) \to M(\mathcal{A}) = (W_{2n})^\oplus(m-1) \setminus \bigcup_{L \in \mathcal{A}} L,$$

then for any ration $(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_m)$ (with $\alpha_i = \beta_i$ for all $i$) and for any collection of $m$ finite Borel measures on $S^{d-1}$, there is a $2m$-fan realizing the ration.

In this formulation we applied the general method to prove that condition $(m-1)k < d-1$ implies no $D_{4n}$-mappings exist [5].

In general, the tools from algebraic topology are standard and the arguments blunt. The insight lies in the formulation of the problem. I look forward to many other applications of the CS/TM paradigm in geometric combinatorics, and in other areas of mathematics as well.
References


27. Jones, J.D.S.; McCleary, J., String homology, and closed geodesics on manifolds which are elliptic spaces, preprint 2014.


