Weyl's Law

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Minicourse,
São Carlos, September 2024

2020 Mathematics Subject Classification: Primary: 58-01, 35P20. Secondary: 35-01.

Preface

The Weyl law is a formula that counts asymptotically the number of eigenstates of a quantum system corresponding to energy levels below a given value of the energy. The problem originated at the dawn of quantum mechanics, with papers by Lorentz[1] [**19**] and Sommerfeld[2] [**30**], the first asking the question and suggesting its answer, the second proposing the problem to mathematicians. The problem was quickly taken up by H. Weyl [**36**] who analyzed the problem mathematically and gave a complete answer to the problem modeling the case of electromagnetic waves in the interior of a closed bounded chamber with perfectly reflection boundary in the form of an asymptotic formula: For the problem

$$\begin{cases} \Delta\mathfrak{U} + \nu^2\mathfrak{U} = 0, \quad \operatorname{div}\mathfrak{U} = 0 \text{ in } \Omega, \\ \mathfrak{U} \text{ normal on } \partial\Omega. \end{cases} \tag{$\dagger$}$$

on an open set $\Omega \subset \mathbb{R}^3$ with C^3 boundary, one has [**36**, p.470, p. 479]

$$\#\{\nu : \nu^2 \text{ is an eigenvalue of } (\dagger) \text{ and } \nu < \mu\} \sim \frac{\operatorname{Vol}(\Omega)}{3\pi^2}\mu^3.$$

Since then there have been many variants, extension, refinements, and generalizations by numerous authors. Extensions and variants include more general elliptic operators, changing the domain Ω to a closed manifold, adding a potential, working out more terms (or the impossibility of such) in the asymptotic expansion, for instance [**12, 16, 20, 21, 29**]. For more on the classical version of the problem as described above, and references, see [**15**].

Our goal here is to discuss known proofs of Weyl's formula using approaches by Seeley [**29**] and Hörmander [**12**]. The underlying differential-geometric set-up will be a closed manifold $\mathcal{M}$ (compact without boundary) with a given smooth positive density (a smooth measure) $\mathfrak{m}$ and a Hermitian vector bundle $E \to \mathcal{M}$ with which one has L^2-spaces of sections. The operator $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ is differential (pseudodifferental and scalar in the case of Hörmander), elliptic, and non-negative, that is,

$$(Pu, u) \geq 0 \quad \text{for all } u \in C^\infty(\mathcal{M}; E).$$

Ellipticity of P and compactness of $\mathcal{M}$ ensures that the spectrum of P is discrete of finite multiplicity, and positivity, that the spectrum consists of non-negative numbers.

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[1]Page 1248: "Here is the problem: to prove that the number of higher overtones between n and $n + dn$ is independent of shape of the cavity and is proportional only to its volume."

[2]Page 1062: "The general and rigorous proof of this asymptotic behavior of eigenvalues seems to me to be an important mathematical problem."

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CHAPTER I

The Weyl law

1. Set up and the formula

Throughout these notes we will denote by $\mathcal{M}$ a closed n-manifold, by $\mathfrak{m}$ a smooth density on $\mathcal{M}$, and let $E \to \mathcal{M}$ be a Hermitian vector bundle of rank r. Write $(v, w)_x$ for the inner product of $v, w \in E_x$ and $|v|_x$ for the norm. Let $\pi : T^*\mathcal{M} \to \mathcal{M}$ denote the canonical projection.

The space $C^\infty(\mathcal{M}; E)$ of smooth sections of E is a pre-Hilbert space with the inner product

$$(u, v)_0 = \int_{\mathcal{M}} (u(x), v(x))_x \, d\mathfrak{m}(x) \quad u, v \in C^\infty(\mathcal{M}; E).$$

Define $L^2(\mathcal{M}; E)$ as the completion of $C^\infty(\mathcal{M}; E)$ with respect to the induced norm.

Suppose $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ is an elliptic differential operator of order m with the property that there is $c \in \mathbb{R}$ such that

$$(Pu, u)_0 \geq c\|u\|_0^2. \tag{1.1}$$

Let

$$\mathcal{D} = \{u \in L^2(\mathcal{M}; E) : Pu \in L^2(\mathcal{M}; E)\}$$

Then

$$P : \mathcal{D} \subset L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$$

is an unbounded closed operator. The ellipticity of P implies that $\mathcal{D}$ is $H^m(\mathcal{M}; E)$, the Sobolev space of order m, and that $\mathrm{spec}(P)$, the spectrum of P, is is a discrete subset of $\mathbb{C}$ without points of accumulation consists of eigenvalues of finite multiplicity.

The semiboundedness property (1.1) implies $\mathrm{spec}(P) \subset [c, \infty) \subset \mathbb{R}$. Let $\{\lambda_\nu\}_{\nu=0}^\infty$ be an enumeration of $\mathrm{spec}(P)$, let d_ν be the multiplicity of λ_ν. The eigenvalue counting function of P is

$$N_P(\lambda) = \sum_{\{\nu : \lambda_\nu < \lambda\}} d_\nu$$

Let $\boldsymbol{\sigma}(P)$ denote the principal symbol of P. This is for each $\boldsymbol{\xi} \in \mathbb{T}^*\mathcal{M}\backslash 0$ a linear function $\boldsymbol{\sigma}(P)(\boldsymbol{\xi}) : E_x \to E_x$, where $x = \pi(x)$. The map

$$T_x^*\mathcal{M} \ni \boldsymbol{\xi} \mapsto \boldsymbol{\sigma}(P)(\xi) \in \mathrm{End}(E_x)$$

is a homogeneous polynomial of degree m. Ellipticity means that $\boldsymbol{\sigma}(P)(\xi)$ is invertible. The property (1.1) implies that $\boldsymbol{\sigma}(P)(\xi)$ is symmetric with non-negative (hence strictly positive) eigenvalues. Therefore $\boldsymbol{\xi} \mapsto \mathrm{tr}(\boldsymbol{\sigma}(P)(\boldsymbol{\xi}))$ is a smooth function of $\boldsymbol{\xi}$, strictly positive on $T^*\mathcal{M}\backslash 0$.

Finally, let $d\mathcal{L}$ denote the Liouville density of $T^*\mathcal{M}$.

THEOREM 1.2 (Weyl's Law). *The function N_P satisfies*

$$N_P(\lambda) = \frac{\mathrm{Vol}(\mathcal{B})}{(2\pi)^n}\lambda^{n/m} + \mathcal{O}(\lambda^{n/m-1/m}) \tag{1.3}$$

Here $\mathcal{B} = \{\boldsymbol{\xi} \in T^*\mathcal{M} : \mathrm{tr}(\boldsymbol{\varpi}(P)(\boldsymbol{\xi})) < 1\}$ and $\mathrm{Vol}(\mathcal{B})$ is its volume relative to the Liouville density.

2. Brief history

Weyl's Law has a long history that we will not attempt to follow in detail. The earlier versions were concerned with the Laplacian on bounded open sets in $\mathbb{R}^n$ with Dirichlet or Neumann boundary condition (we will deal only with closed manifolds, no boundary, which is simpler). When dealing with the Laplacian we will always use the positive Laplacian,

$$\Delta = \sum_j D_{x_j}^2, \qquad D_{x^j} = \frac{1}{i}\frac{\partial}{\partial x^j}$$

on $\mathbb{R}^n$ likewise with the Laplace-Beltrami operator on a Riemannian manifold.

The first reference to Weyl's work gives the formula

$$\lim_{k\to\infty} \frac{k}{\lambda_k} = \frac{\mathrm{Vol}(\Omega)}{4\pi}$$

in the case of a planar domain Ω [**35**, Theorem X, p. 114]. (In the paragraph before the statement of the theorem it is asserted that the same methods work in higher dimension.) From that formula one derives

$$N_\Delta(\lambda) = \frac{\mathrm{Vol}(\Omega)}{4\pi}\lambda + o(\lambda). \tag{2.1}$$

To see that this matches the general formula note that the principal symbol of the (positive) Laplacian in $\mathbb{R}^2$ is $|\xi|^2$, so the region $\mathcal{B} \subset T^*\Omega \approx \Omega \times \mathbb{R}^2$ is

$$\mathcal{B} = \{(x,\xi) \in \Omega \times \mathbb{R}^2 : |\xi|^2 < 1\}.$$

The Liouville density is $|d\mathcal{L}| = dx\,dy$ (Lebesgue measure on $\mathbb{R}^2 \times \mathbb{R}^2$), and

$$\frac{\mathrm{Vol}(\mathcal{B})}{(2\pi)^2} = \frac{\pi\,\mathrm{Vol}(\Omega)}{4\pi^2}$$

which is the coefficient of λ in (2.1); the exponent of λ there is $1 = 2/2$.

Weyl's proof in detail appeared in [**36**]. Carleman's paper [**3**] in 1935 is interesting in that he is the first to apply the Wiener-Ikehara Tauberian theorem [**38**, p. 44] (quoted here as Theorem 3.5 on page 5). Carleman's method, different from Weyl's, has been applied in many cases as it became one of the preferred methods of the theory. He considered the case of the Dirichlet problem for the Laplacian on a bounded planar domain Ω whose boundary is reasonably good. Letting $\{\lambda_k\}$ be the sequence of eigenvalues, each repeated according to its multiplicity, and $\{\varphi_k\}$ a corresponding orthonormal basis of real-valued eigenfunctions he lets

$$F(x,\lambda) = 2\pi\sum_k \frac{\varphi_k(x)^2}{\lambda_k(\lambda_k - \lambda)},$$

the restriction to the diagonal of $\Omega \times \Omega$ of the Schwartz kernel of $P^{-1} \circ (P - \lambda)^{-1}$. The λ_k^2 in the denominator ensures convergence of the series, and estimates of the Green'f function of $\Delta + \lambda I$ ensures the convergence of the line integral

$$\sum_k \varphi_k(x)^2 \lambda_k^{-s} = -\frac{i}{(2\pi)^2} \int_{a-i\infty}^{a+i\infty} F(x,\lambda)\lambda^{-s+1}\, d\lambda$$

when $\Re s$ is sufficiently large. Here a is positive and less than the first eigenvalue. Integration yields

$$\sum_k \varphi_k(x)^2 \lambda_k^{-s}$$

and integration over Ω gives the zeta function of the operator,

$$\zeta_\Delta(s) = \sum_k \frac{1}{\lambda_k^s}$$

(0 is not an eigenvalue) This function turns out to be holomorphic on $\Re s > 1$ with a simple pole at 1, and $(s-1)\zeta_\Delta$ is continuous up to $\Re s = 1$. Wiener-Ikehara says that the residue at 1 is the coefficient in the right hand side of the Weyl estimate. The residue is the coefficient in the Weyl formula. Carleman dealt with a more general second order elliptic operator in a domain in $\mathbb{R}^3$ again with Dirichlet condition [**4**]. Later we say much more in connection with the zeta function.

Extending to compact Riemannian manifolds, Minakshisundaram and Pleijel [**21**] used Carleman's approach via the zeta function for the Laplace-Beltrami operator—Dirichlet or Neumann condition if there is a boundary—in using the Wiener-Ikehara Tauberian theorem [**38**]. They showed

$$N_\Delta(\lambda) = \frac{\mathrm{Vol}(\mathcal{B})}{(2\pi)^n}\lambda^{n/2} + o(\lambda^{n/2}).$$

Avakumović [**1**] improved the estimate of the error, still for the Laplace-Beltrami operator with Dirichlet or Neumann condition to $\mathcal{O}(\lambda^{n/2-1/2})$; his paper actually deals with $n = 3$ but asserts the validity of the statement in any dimension. He also shows that in the case of the sphere the estimate cannot be improved to a term $c\lambda^{n/2-1/2} + o(\lambda^{n/2-1/2})$, so in general there are no more terms in the asymptotics of the eigenvalue counting function. The references in [**1**] and [**21**] are a good lead to earlier work on the estimates. The paper of Ivrii [**15**], one of the movers of the field, contains more recent references. Strichartz [**31**] has an exact formula for $N_\Delta(\lambda)$ on spheres.

3. Some details

The above mentioned works actually establish estimates on the Schwartz kernel of the resolvent of the operator. The most complete results in this respect are in Hörmander's paper [**12**], which deals with arbitrary semibounded pseudodifferential operators rather than just differential, but requires scalar operators (no vector bundle). His method extracts spectral information from the singularity at 0 of the trace of the solution operator for the Cauchy problem

$$\frac{1}{i}\frac{\partial u}{\partial t} - Pu = 0 \text{ on } \mathcal{M} \times \mathbb{R}, \quad u|_{t=0} = u_0 \tag{3.1}$$

when P is of first order. This approach works very well because $-i\partial_t - P$ is of real principal type, which fits the theory of Fourier integral operators perfectly. This

would not be the case for a general P of higher order because of possible presence of characteristics of varying multiplicity.

Briefly, assuming P is elliptic of first order, scalar, bounded below (necessarily pseudodifferential), and $\mathcal{M}$ closed, then the solution operator for (3.1) is

$$u = e^{itP}u_0, \quad e^{itP} = \sum_\nu e^{i\lambda_\nu t}\Pi_{\lambda_\nu}$$

where $\Pi_{\lambda_\nu} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is the orthogonal projection onto the eigenspace $\mathcal{E}_{\lambda_\nu}$ corresponding to the eigenvalue λ_ν. Since $\operatorname{Tr}\Pi_{\lambda_\nu} = \dim \mathcal{E}_{\lambda_\nu} = d_\nu$, the trace of e^{itP} is

$$\operatorname{Tr} e^{itP} = \sum_\nu d_\nu e^{-i\lambda_\nu t} \tag{3.2}$$

On the other hand, the construction of e^{itP} as a Fourier integral operator shows that $\operatorname{Tr} e^{itP}$ is a distribution with $t = 0$ as an isolated singular point (it may have other singularities, but not near 0). This information gives the behavior as a distribution of the sum (3.2) from which the estimate follows. Duistermaat's book [**6**] is an excellent introduction to the theory of Fourier integral operators.

A different approach uses the initial value problem for the heat equation. This is the path used by Minakshisundaram and Pleijel motivated by Carleman's technique in [**3, 4**] passing through the Wiener-Ikehara Tauberian theorem [**38**]. The assumption is that $c \geq 0$ in (1.1); the operator P can be differential or pseudodifferential, of order $m > 0$, and need not be a scalar operator (i.e. act on sections of a vector bundle). The technique is thus very robust. We keep the assumption that P is a differential operator.

On the functional analysis side, the solution at time $t > 0$ of the initial value problem

$$\frac{\partial u}{\partial t} + Pu = 0 \text{ on } \mathcal{M} \times (0, \infty), \quad u|_{t=0} = u_0 \tag{3.3}$$

is

$$u = e^{-tP}u_0, \quad e^{-tP} = \sum_\nu e^{-\lambda_\nu t}\Pi_{\lambda_\nu}.$$

As above, $\Pi_{\lambda_\nu} : L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$ is the orthogonal projection onto the eigenspace $\mathcal{E}_{\lambda_\nu}$ corresponding to λ_ν and the trace of $H(t)$ is

$$\operatorname{Tr} e^{-tP} = \sum_\nu d_\nu e^{-\lambda_\nu t}$$

The Mellin transform of $\operatorname{Tr} e^{-tP}$ (removing the term corresponding to Π_0 if 0 is an eigenvalue) is

$$\mathcal{M}(\operatorname{Tr} e^{-tP})(s) = \int_0^\infty \operatorname{Tr} e^{-tP} t^s \, \frac{dt}{t}$$

But

$$\int_0^\infty e^{-\lambda_\nu t} t^s \, \frac{dt}{t} = \frac{1}{\lambda_\nu^s} \int_0^\infty e^{-t} t^s \, \frac{dt}{t} = \Gamma(s)\frac{1}{\lambda_\nu^s}$$

Thus omitting the eigenvalue 0 if necessary,

$$\mathcal{M}(\operatorname{Tr} e^{-tP})(s) = \Gamma(s)\zeta_P(s) \tag{3.4}$$

with the notation

$$\zeta_P(s) = \sum_{\{\nu:\lambda_\nu>0\}} \frac{d_\nu}{\lambda_\nu^s}.$$

This function, the zeta function of P, is shown to be holomorphic in $\Re s > n/m$ with a simple pole at n/m. By the Wiener-Ikehara Tauberian theorem, the residue at that pole is the coefficient of $\lambda^{n/m}$ in (1.3), since

$$\zeta_P(s) = \int_{0^+}^{\infty} \lambda^{-s}\, dN_P(\lambda).$$

THEOREM 3.5 (Ikehara, in Wiener [**38**, p. 44]). *Let $\alpha(x)$ be a monotone increasing function, and let*

$$f(u) = \int_1^{\infty} x^{-u}\, d\alpha(x) = f(u)$$

converge for $\Re(u) > 1$. Let

$$f(u) - \frac{A}{u-1} = g(u)$$

converge uniformly to a finite limit as $\Re u \to 1$ over any finite interval of the line $\Re(u) = 1$. Then

$$A = \lim_{N\to\infty} \frac{\alpha(N)}{N}.$$

To move beyond a merely abstract assertion one needs to do some work, for instance construct a parametrix for the problem (3.3) and find the asymptotics at 0 of $Tr(e^{-tP})$. In the case of the Laplace-Beltrami operator Minakshisundaram and Pleijel construct a parametrix for the problem whose Schwartz kernel has the form

$$H_\nu(t,x,x') = \frac{1}{(4\pi t)^{n/2}} e^{-r(x,x')^2/4t} U(t,x,x')$$

when x and x' are close enough, where $U(t,x,x') = \sum_{k=1}^{\nu} a_k(x,x')t^k$ is a polynomial in t of arbitrarily large order ν with smooth coefficients. The coefficients are chosen so that

$$\big(\frac{\partial}{\partial t} + \Delta\big) H_\nu(t,x,x') = \frac{1}{(4\pi t)^{n/2}} e^{-r(x,x')^2/4t} V_\nu(x,x') t^\nu$$

with $U(0,x,x) = 1$ and some V_ν. Thus $\operatorname{op}(H_\nu)(t)$ is a good approximation to e^{-tP} for small t. Restricting H_ν to $x = x'$ and integrating in x produces

$$\operatorname{Tr}\operatorname{op}(H_\nu)(t) = \sum_{k=0}^{\nu} \frac{1}{(4\pi t)^{n/2}} p_k t^k, \quad p_0 = \int_{\mathcal{M}} U(0,x,x)\, d\mathfrak{m}(x) = \operatorname{Vol}(\mathcal{M}).$$

These are the first few terms of the asymptotic expansion of $\operatorname{Tr}(e^{-t\Delta})$ at $t = 0$. Let $\chi \in C^\infty(\mathbb{R})$ be supported in a neighborhood of 0 also equal to 1 near 0. The convergence of the Mellin transform of $\chi \operatorname{Tr}(\operatorname{op}(H_\nu))$,

$$\mathcal{M}(\chi \operatorname{Tr}\operatorname{op}(H_\nu))(s) = \int_0^{\infty} \sum_{k=0}^{\nu} \frac{\chi(t)}{(4\pi)^{n/2}} p_k t^{k+s-n/2}\, \frac{dt}{t}$$

is controlled by the most singular term, $k = 0$; it only requires $\Re s > n/2$ the remaining terms are holomorphic on $\Re s > n/2 - 1$ or better. Multiplying by $(s - n/2)$

get

$$\begin{aligned}\frac{(s-n/2)}{(4\pi)^{n/2}}\int_0^\infty p_0\chi(t)t^{s-n/2}\,\frac{dt}{t} &= \frac{1}{(4\pi)^{n/2}}\int_0^\infty p_0\chi(t)\frac{d}{dt}t^{s-n/2}\,dt\\ &= \frac{-1}{(4\pi)^{n/2}}\int_0^\infty p_0\chi'(t)t^{s-n/2+1}\,\frac{dt}{t}\end{aligned}$$

Since $\chi'(t)=0$ near 0, this is an entire function of s. So $\mathcal{M}(\chi\operatorname{Tr}e^{-tP})(s)$ is meromorphic in $\Re s > n/2-1$ with single simple pole at $n/2$ with residue

$$\operatorname{res}_{n/2}(\mathcal{M}(\chi\operatorname{Tr}e^{-tP})) = \frac{\operatorname{Vol}(\mathcal{M})}{(4\pi)^{n/2}}$$

Using (3.4) we thus have

The zeta function of the Laplacian on a closed n-manifold $\mathcal{M}$,

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)}\,\mathcal{M}(\operatorname{Tr}e^{-\Delta})(s)$$

is meromorphic on $\Re s > n/2-1$, has a single simple pole at $n/2$ with residue

$$\frac{\operatorname{Vol}(\mathcal{M})}{(4\pi)^{n/2}\Gamma(n/2)}.$$

This is the coefficient in the asymptotics for the eigenvalue counting function of the Laplace-Beltrami operator.

CHAPTER II

Examples

1. The standard torus

Consider the standard Laplacian on the n-torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$,

$$\Delta = \sum_j D_{x^j}^2, \quad D_{x^j} = \frac{1}{i}\frac{\partial}{\partial x^j}$$

where the x^j is the angular variable in the j-th factor.[1] A complete system of orthogonal functions is given by

$$f_\nu(x) = e^{i\sum_\ell \nu_\ell x^\ell}, \quad \nu \in \mathbb{Z}^n$$

with corresponding eigenvalue $\|\nu\|^2 = \sum |\nu_\ell|^2$. Thus

$$N_\Delta(\lambda) = \#\{\nu \in \mathbb{Z}^d : \|\nu\|^2 < \lambda\} = \#\{\nu \in \mathbb{Z}^d : \|\lambda^{-1/2}\nu\|^2 < 1\}.$$

The set $\{\lambda^{-1/2}\nu : \nu \in \mathbb{Z}^n\} \subset \mathbb{R}^n$ is an orthogonal lattice of evenly spaced points, $N_\Delta(\lambda)$ counts how many of these points there are within the the unit ball $B^n \subset \mathbb{R}^n$. Since they are the centers of a tiling of $\mathbb{R}^n$ by cubes of side $1/\lambda^{1/2}$, the sum of the volumes of those with center within B^n, of which there are $N_\Delta(\lambda)$, is an approximation to the volume of the ball:

$$N_\Delta(\lambda)\frac{1}{\lambda^{n/2}} \approx \mathrm{Vol}(B^n).$$

Thus

$$N_\Delta(\lambda) \approx \frac{\mathrm{Vol}(\mathbb{T}^n)\,\mathrm{Vol}(B^n)}{(2\pi)^n}\lambda^{n/2}.$$

2. The general flat torus

Let $\omega_1, \ldots, \omega_n \in \mathbb{R}^n$ be independent vectors. They generate a lattice

$$\mathcal{L} = \{\sum_j m^j\omega_j : m^j \in \mathbb{Z}\}.$$

Let $\mathcal{M} = \mathbb{R}^n/\mathcal{L}$ with the measure $\mathfrak{m}$ defined by the Lebesgue measure of $\mathbb{R}^n$. Since the Laplace operator $\Delta = -\sum_j \partial_{x^j}^2$ on $\mathbb{R}^n$ is invariant under translations, it induces a second order operator P on $\mathcal{M}$. This operator is elliptic and non-negative. We will find its eigenfunctions and eigenvalues and use them to find explicitly the coefficient of Weyl's law for P.

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[1] Thus $P = -\sum \partial_{x^j}^2$. Observe that

$$\int_{\mathbb{R}^n} \Delta u(x)\,\overline{v(x)}\,dx = \int \nabla u(x)\cdot\overline{\nabla v(x)}\,dx = \int_{\mathbb{T}^n} u(x)\,\overline{\Delta v(x)}\,dx, \quad u, v \in C^\infty(\mathbb{T}^n).$$

The first task is to find a set of functions that are candidates to being eigenfunctions of P. View vectors in $\mathbb{R}^n$ as column vectors with respect to the canonical basis $\mathrm{e}_1, \dots, \mathrm{e}_n$ of $\mathbb{R}^n$. The map

$$\beta : \mathbb{R}^n \to \mathbb{R}^n, \quad \beta(y) = \frac{1}{2\pi} by, \quad b = \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{bmatrix}$$

maps the cube $\mathcal{F}' = [0, 2\pi]^n$ onto the convex hull $\mathcal{F}$ of $\{\sum_j m^j \omega_j : m^j = 0, 1\}$, which is a fundamental domain of $\mathbb{R}^n \to \mathcal{M}$, since $\beta(2\pi \mathrm{e}_j) = \omega_j$. Being linear, β sends the lattice $\mathcal{L}'$ generated by the vectors $2\pi \mathrm{e}_j$ onto $\mathcal{L}$, therefore it induces a diffeomorphism $\boldsymbol{\beta} : \mathbb{T}^n \to \mathcal{M}$ (with $\mathbb{T}^n = \mathbb{R}^n / \mathcal{L}'$) through the requirement that

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\beta} & \mathbb{R}^n \\ {\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \mathbb{T}^n & \xrightarrow{\boldsymbol{\beta}} & \mathcal{M} \end{array}$$

commutes.

Functions $\mathcal{M} \to \mathbb{C}$ lift to functions on $\mathbb{R}^n$ which are periodic with respect to $\mathcal{L}$. So if $u : \mathcal{M} \to \mathbb{C}$, then $\tilde{u} = u \circ q \circ \beta$ is periodic with respect to $\mathcal{L}'$ and if u is smooth, then we have the standard Fourier series expansion

$$\tilde{u}(y) = \sum_{\nu \in \mathbb{Z}^n} u_\nu e^{i(\sum_\ell \nu_\ell y^\ell)}$$

with suitable coefficients $u_\nu \in \mathbb{C}$. Letting a be the inverse of b and defining $\alpha(x) = 2\pi a x$ (thus if a^ℓ is the ℓ-th row of a then $y^\ell = 2\pi a^\ell x = \alpha^\ell(x)$ if $x = \beta(y)$) we have

$$u(\pi(x)) = \sum_{m \in \mathbb{Z}} \tilde{u}_\nu e^{i(\sum_\ell \nu_\ell \alpha^\ell(x))}.$$

The functions

$$f_\nu(x) = e^{i(\sum_\ell \nu_\ell \alpha^\ell(x))}, \quad \nu \in \mathbb{Z}^n,$$

satisfy $f_\nu(x + \omega_j) = f_\nu(x)$, $m \in \mathbb{Z}^n$. Indeed, since $\alpha^\ell(\omega_j) = 2\pi \delta_j^\ell$ (because $\alpha(\omega_j) = 2\pi \mathrm{e}_j$),

$$\sum_\ell \nu_\ell \alpha^\ell(x + \omega_j) = \sum_\ell \nu_\ell \alpha^\ell(x) + 2\pi \nu_j.$$

We claim that the functions f_ν form a complete orthogonal system in $L^2(\mathcal{F})$. Indeed, if $u : \mathcal{F} \to \mathbb{C}$ is continuous (or just in L^1), then

$$\int_{\mathcal{F}} u(\pi(x))\, dx = \frac{|\det b|}{(2\pi)^n} \int_{\mathcal{F}'} u(\pi(\beta(y)))\, dy$$

using the change of variables $x = \beta(y)$. Since $\{f_\nu \circ \beta\}_{\nu \in \mathbb{Z}^n} = \{e^{i(\sum \nu_j y^j)}\}_{\nu \in \mathbb{Z}^n}$ is a complete orthogonal system on $\mathcal{F}'$, so is $\{f_\nu\}_{\nu \in \mathbb{Z}^n}$ on $\mathcal{F}$. It also follows that

$$\mathrm{Vol}(\mathcal{F}) = |\det b|.$$

Thus if $\mathfrak{m}$ is the measure on $\mathcal{M}$ determined by the Lebesgue measure on $\mathcal{F}$, then $\mathrm{Vol}(\mathcal{M}) = |\det b|$ and the functions $\phi_\nu : \mathcal{M} \to \mathbb{C}$ defined by

$$(\phi_\nu \circ \pi)(x) = \frac{1}{\mathrm{Vol}(\mathcal{M})^{1/2}} e^{i(\sum_\ell \nu_\ell \alpha^\ell(x))}, \quad \nu \in \mathbb{Z}^n, \tag{2.1}$$

form a complete orthonormal system of $L^2(\mathcal{M}, \mathfrak{m})$. Each of these functions is an eigenfunction of P, with corresponding eigenvalue

$$\lambda_\nu = (2\pi)^2 \sum_{i=1}^{n} (\sum_{\ell} \nu_\ell a_i^\ell)^2. \tag{2.2}$$

Indeed,

$$\Delta f_\nu = \sum_{i=1}^{n} (2\pi \sum_{\ell} \nu_\ell a_i^\ell)^2 \, f_\nu,$$

using $\alpha^\ell(x) = 2\pi a x$.

Now that we have an explicit formula for the eigenvalues of P we can estimate

$$N_P(\lambda) = \#\{\nu \in \mathbb{Z}^n : \lambda_\nu < \lambda\}.$$

The number λ_ν is the square of the Euclidean norm of

$$2\pi\nu a = 2\pi \begin{bmatrix} \nu_1 & \cdots & \nu_n \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}$$

Replacing a by its polar decomposition $\sigma\rho$, σ orthogonal, ρ symmetric positive definite, we rewrite the counting function in the form

$$N_\Delta(\lambda) = \#\{\nu \in \mathbb{Z}^n : (2\pi)^2 \|\nu\sigma\rho\|^2 < \lambda\}$$

where we observe that because σ is unitary, the elements $2\pi\nu\sigma$ with $\nu \in \mathbb{Z}^n$ form an orthogonal lattice $\mathcal{L}_1''$. Consequently,

$$N_\Delta(\lambda) = \#\{w \in \mathcal{L}_1'' : \|w\rho\|^2 < \lambda\}$$

If λ is positive, the condition on $w \in \mathcal{L}_1''$ to be counted is that $\|\lambda^{-1/2} w\rho\| < 1$. Let then

$$\mathcal{L}_\lambda'' = \{\lambda^{-1/2} w : w \in \mathcal{L}_1''\}$$

The elements of $\mathcal{L}_\lambda''$ are the centers of cubes of side length $2\pi/\sqrt{\lambda}$. The union of those cubes with $\|w\rho\| < 1$ is an approximation to

$$\mathcal{E} = \{w \in \mathbb{R}^n : \|w\rho\| < 1\}.$$

So

$$\mathrm{Vol}(\mathcal{E}) \approx N_\Delta(\lambda) \frac{(2\pi)^n}{\lambda^{n/2}}$$

Consequently

$$N_\Delta(\lambda) \approx \frac{\mathrm{Vol}(\mathcal{E})}{(2\pi)^n} \lambda^{n/2}$$

The approximation to $\mathrm{Vol}(\mathcal{E})$ by the volumes of the cubes improves as $\lambda \to \infty$ so we have an asymptotic formula for the counting function of the eigenvalues of P. It only remains to determine $\mathrm{Vol}(\mathcal{E})$.

Since ρ is symmetric positive definite, there is an orthonormal basis $\mathrm{e}_1', \ldots, \mathrm{e}_n'$ of $\mathbb{R}^n$ consisting of its eigenvectors: $\rho \mathrm{e}_j' = r_j \mathrm{e}_j'$. The eigenvalues are positive because ρ is positive. Thus

$$\mathcal{E} = \{\sum z_j \mathrm{e}_j' : \sum_j z_j^2 r_j^2 < 1\}$$

which shows that $\mathcal{E}$ is an ellipsoid with semi-radii $1/r_1, \dots, 1/r_n$. Hence

$$\operatorname{Vol}(\mathcal{E}) = \frac{\operatorname{Vol}(B^n)}{r_1 r_2 \cdots r_n} = \frac{\operatorname{Vol}(B^n)}{\det(\rho)} = \operatorname{Vol}(B^n)\operatorname{Vol}(\mathcal{M})$$

where B^n is the unit ball in $\mathbb{R}^n$ and we used that $\det \rho = |\det a| = 1/|\det b|$. Thus

$$N_P(\lambda) \approx \frac{\operatorname{Vol}(B^n)\operatorname{Vol}(\mathcal{M})}{(2\pi)^n}\lambda^{n/2} \quad \text{as } \lambda \to \infty.$$

3. Spheres

The Laplacian of $\mathbb{R}^{n+1}$ in spherical coordinates, $x = ry$, $(r, y) \in [0, \infty) \times S^n$, is

$$\Delta = -\frac{1}{r^2}\big[\big(r\frac{\partial}{\partial r}\big)^2 + (n-1)r\frac{\partial}{\partial r} + \Delta_{S^n}\big]$$

If f is a harmonic function on $\mathbb{R}^{n+1}$ homogeneous of degree k, then $f(rx) = r^k f(x)$ gives

$$0 = r^2\Delta f = -[k^2 + (n-1)k + \Delta_{S^n}]f$$

so $f|_{S^n}$ is an eigenfunction of Δ_{S^n} with eigenvalue $\lambda_k = k(k+n-1)$:

$$\Delta_{S^n} f = \lambda_k f.$$

This idea, applied to the space of harmonic polynomials on $\mathbb{R}^n$ of degree k gives all eigenfunctions of Δ_{S^n} with the above eigenvalue. For a very nice exposition see the chapter on spherical harmonics in [**22**, Neri].

Let $\mathcal{E}_{\lambda_k}$ be the eigenspace. The dimension of $\mathcal{E}_{\lambda_k}$, the multiplicity of λ_k, is

$$\dim \mathcal{E}_{\lambda_k} = \binom{k+n}{n} - \binom{k+n-2}{n}, \quad k \geq 2.$$

$\mathcal{E}_{\lambda_0}$ consists of the constant functions and $\mathcal{E}_{\lambda_1}$ of all linear functions (dimensions 1 and $n+1$, respectively). Thus

$$N_{\Delta_{S^n}}(\lambda) = \sum_{k:\lambda_k < \lambda} \dim \mathcal{E}_{\lambda_k}.$$

Simplifying we get the estimate

$$\dim \mathcal{E}_{\lambda_k}(\lambda) = \frac{(2k+n-1)(k+n-2)!}{k!\,(n-1)!} = \frac{2k^{n-1}}{(n-1)!} + \mathcal{O}(k^{n-2})$$

if k is large.

The condition that $\lambda_k < \lambda$ is

$$0 \leq k < \frac{1}{2}\big(-n+1+\sqrt{n^2+4\lambda-2n+1}\big)$$

which for large λ essentially says $0 \leq k \leq \lambda^{1/2}$. Thus

$$N_{\Delta_{S^n}}(\lambda) \sim \frac{2}{(n-1)!}\sum_{k<\lambda^{1/2}} k^{n-1} = \frac{2}{n!}\lambda^{n/2} + \mathcal{O}(\lambda^{(n-1)/2}).$$

We now verify that the coefficient is $\operatorname{Vol}(S^n)\operatorname{Vol}(B^n)$. Consulting for instance Trèves [**33**, p. 74] we know

$$\operatorname{Vol}(S^n) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$$

from which one gets

$$\mathrm{Vol}(B^n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$$

by integrating $\mathrm{Vol}(S^{n-1})r^{n-1}$ with respect to r from 0 to 1. The product of these numbers is

$$\mathrm{Vol}(S^n)\,\mathrm{Vol}(B^n) = 2\frac{(2\pi)^n}{n!}.$$

Hence

$$N_{\Delta_{S^n}}(\lambda) \sim \frac{\mathrm{Vol}(S^n)\,\mathrm{Vol}(B^n)}{(2\pi)^n}\lambda^{n/2} + \mathcal{O}(\lambda^{(n-1)/2}) \quad \text{as } \lambda \to \infty.$$

Avakumović [**1**] showed that in a Riemannian manifold with Δ the Laplace-Beltrami operator, the same formula holds. He also answered in the negative the question whether in general the bound $\mathcal{O}(\lambda^{n/2-1/2})$ for the error can be improved to $o(\lambda^{(n-1)/2})$ by using the 3-sphere as a counterexample. Hórmander [**12**] gives the same kind of estimate for the error for a general positive pseudodifferential operator.

4. The general flat torus, again

Is the same as the first paragraph of the previous section, but we will use Hörmander's approach in [**12**] which in this case works fairly easily. Later we will consider the general case.

Recall that $\mathfrak{m}$ is the measure on $\mathcal{M} = \mathbb{R}^n/\mathcal{L}$ induced by the Lebesgue measure on $\mathbb{R}^n$. The operator $P : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ has an extension

$$P : H^2(\mathcal{M}) \subset L^2(\mathcal{M}) \to L^2(M)$$

where $H^2(\mathcal{M})$ is the L^2-based Sobolev space of order 2 on $\mathcal{M}$. As such, P has a non-negative square root defined abstractly via functional analysis, given explicitly by

$$P^{1/2}(u) = \sum_{\nu\in\mathbb{Z}^n} \lambda_\nu^{1/2}(u,\phi_\nu)\phi_\nu \tag{4.1}$$

in terms of the orthonormal basis of $L^2(\mathcal{M})$ consisting of eigenvectors $\{\phi_\nu\}$ of P, see (2.1), with corresponding eigenvalues $\{\lambda_\nu\}$, see (2.2), or by (a regularization of) the formula

$$P^{1/2} = \frac{i}{2\pi}\int_\Gamma \lambda^{1/2}(P-\lambda I)^{-1}\,d\lambda \tag{4.2}$$

where Γ is the path obtained by traversing the negative real axis from $-\infty$ to $-\varepsilon$ ($\varepsilon > 0$ suitably small), followed once by the clockwise oriented path $|\lambda| = \varepsilon$, then back to $-\infty$ along the negative semi-axis (see the figure on page 36). The number ε is small enough that it does not enclose any element of $\mathrm{spec}(P)$ except possibly 0. The formula as written does not converge in norm, but replacing the power $1/2$ by $s \in \mathbb{C}$ with sufficiently negative real part, it does. Doing that gives a holomorphic function of s which has a meromorphic extension which is regular at $1/2$. This is part of Seeley's work [**29**] who showed that in particular (4.2) defined via a regularization is equal to (4.1) and a pseudodifferential operator with principal symbol

$$\boldsymbol{\sigma}(P^{1/2}) = \boldsymbol{\sigma}(P)^{1/2}.$$

We will discuss Seeley's paper later more fully. At the moment we only need to know that $P^{1/2}$ is a pseudodifferential operator with principal symbol given by the above formula.

Hörmander's approach applied to the situation at hand consists of analyzing the singularity at $t = 0$ of the trace of the solution operator U of the Cauchy problem

$$\begin{cases} D_t u - P^{1/2} u = 0 \quad \text{in } \mathbb{R} \times \mathcal{M}, \\ u = v \text{ on } \{0\} \times \mathcal{M}. \end{cases} \tag{4.3}$$

The fact that $P^{1/2}$ is a first order symmetric elliptic operator makes $D_t - P^{1/2}$ be a scalar operator of real principal type (more on this later). It is for such operators that the theory of Fourier integral operators applies very well and in the end why the approach is successful.

The aim is to estimate the eigenvalue counting function of $P^{1/2}$,

$$N_{P^{1/2}}(\sigma) = \{\nu \in \mathbb{Z}^n : \sqrt{\lambda_\nu} < \sigma\}$$

which can also be written as

$$N_{P^{1/2}}(\sigma) = \sum_{\nu \in \mathbb{Z}^n} H(\sigma - \sqrt{\lambda_\nu}). \tag{4.4}$$

using the Heaviside function, $H(\sigma) = 0$ if $\sigma \leq 0$ and $= 1$ otherwise. This series converges in $L^1_{\text{loc}}(\mathbb{R})$. In fact it is polynomially bounded, as the previous section shows and as we will also show independently later.

The strategy is to write $\operatorname{Tr} U(t)$ in two ways, each complementing the information the other naturally provides. The first formula for $U(t)$ is

$$U(t)v = \sum_{\nu \in \mathbb{Z}^n} e^{it\sqrt{\lambda_\nu}} (v, \phi_\nu) \phi_\nu.$$

The sum clearly converges in norm and shows that $U(t)$ is unitary. The Schwartz kernel of $U(t)$ is

$$K(t, x, x') = \sum_{\nu \in \mathbb{Z}^n} e^{it\sqrt{\lambda_\nu}} \phi_\nu(x) \overline{\phi}_\nu(x').$$

and the trace is

$$\operatorname{Tr} U(t) = \int_{\mathcal{M}} K(t, x, x)\, d\mathfrak{m}(x) = \sum_{\nu \in \mathbb{Z}^n} e^{it\sqrt{\lambda_\nu}}. \tag{4.5}$$

where the sum converges in the sense of distributions. That this sum converges as a distribution requires proof, but it does. Further, as shown by Chazarain [**5**], and Duistermaat and Guillemin [**7**], the singular support of $\operatorname{Tr} U$ is contained in

$$\{t : t = 0 \text{ or there is a closed geodesic of length } t\},$$

a discrete set. So for small $t \neq 0$, $\operatorname{Tr} U$ is smooth. This will matter later.

The link between $\operatorname{Tr} U(t)$ and (4.4) is provided by the Fourier transform: from (4.5) we get

$$\widehat{\operatorname{Tr} U}(\sigma) = 2\pi \sum_{\nu \in \mathbb{Z}^n} \delta(\sigma - \sqrt{\lambda_\nu}) = \frac{d}{d\sigma} N_{P^{1/2}}(\sigma).$$

Accepting for now that (4.4) converges as a Schwartz distribution we at least know that $\operatorname{Tr} U$ is well defined as a distribution.

For the second formula for $\operatorname{Tr} U(t)$ Hörmander begins by writing U as a Fourier integral operator; below we follow and adapt the exposition in [**11**]. In our case

here we lift all to $\mathbb{R}^n$ and view the diffeomorphism $\alpha : \mathcal{F} \to \mathcal{F}'$ as providing local coordinates on $\mathcal{M}$ (or $\mathcal{F}$) and rewrite Δ. Using $(y_1, \dots, y^n) = \alpha(x)$ we have

$$\frac{\partial}{\partial x^j} = \sum_\ell \frac{\partial y^\ell}{\partial x^j} \frac{\partial}{\partial y^\ell} = 2\pi \sum_\ell a_j^\ell \frac{\partial}{\partial y^\ell},$$

hence

$$\Delta = (2\pi)^2 \sum_{k,\ell} g^{k\ell} D_{y^k} D_{y^\ell}$$

with the definition

$$g^{k\ell} = \sum_j a_j^k a_j^\ell.$$

The principal symbol of Δ (in fact its total symbol) in the y-coordinates is

$$\boldsymbol{\sigma}(\Delta)(\boldsymbol{\eta}) = (2\pi)^2 \sum_{k,\ell} g^{k\ell} \eta_k \eta_\ell$$

when $\boldsymbol{\eta} = \sum_k \eta_k \, dy^k$, that of its square root is then

$$\boldsymbol{\sigma}(\sqrt{\Delta})(\boldsymbol{\eta}) = (\sum_{k,\ell} (2\pi)^2 g^{k\ell} \eta_k \eta_\ell)^{1/2},$$

and thus

$$\Delta^{1/2} u(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(y-y')\cdot\eta} (\sum_{k,\ell} (2\pi)^2 g^{k\ell} \eta_k \eta_\ell)^{1/2} \, u(y') \, dy' \, d\eta$$

as an operator on $L^2(\mathbb{R}^n)$ (which by the way exhibits $\Delta^{1/2}$ as a pseudodifferential operator). This formula works because $p(y, \eta) = \boldsymbol{\sigma}(\Delta)(\boldsymbol{\eta})$ is actually independent of y.

The solution to (4.3) in the coordinates y when $\mathcal{M}$ is replaced by $\mathbb{R}^n$ is

$$u(t, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i[(y-y')\cdot\eta + t\sqrt{p(y,\eta)}]} v(y') \, d\eta \, dy'$$

if v is compactly supported (or just in $L^2(\mathbb{R}^n)$). While this gives the exact solution on $\mathbb{R}^n$, it does not yield a solution to (4.3), i.e., on the the torus, since it is not periodic in y. Indeed, even if v is supported in the interior of a fundamental domain then as soon as $t \neq 0$ the solution need no longer be supported on $\mathcal{F}'$ and is generally not periodic.

However, the most singular part of the trace of the Schwartz kernel of the actual solution operator $U(t)$ at $t = 0$ comes from

$$K(t, y, y') = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i[(y-y')\cdot\eta + t\sqrt{p(y,\eta)}]} \, d\eta.$$

We will take this as our starting point, set $y = y'$ and integrate over $\mathcal{F}'$:

$$\operatorname{Tr} U(t) = \frac{1}{(2\pi)^n} \int_{\mathcal{F}'} \int_{\mathbb{R}^n} e^{it\sqrt{p(y,\eta)}} \, d\eta \, dy + \epsilon(t) \tag{4.6}$$

where ϵ is a distribution defined in an interval around $t = 0$ which is smooth for $t \neq 0$. The singularity of $\epsilon(t)$ is weaker than that contributed by K; this is expressed by the condition that

> if $\rho \in C^\infty(\mathbb{R})$ has support in a sufficiently small neighborhood of 0, then there is c such that $\widehat{\rho\varepsilon}(\sigma) = \mathcal{O}(\sigma^{n-2})$

To compute the integral, introduce spherical (ellipsoidal) coordinates:

$$\eta = s\tilde{\eta}, \quad s \in [0,\infty),\ \tilde{\eta} \in \mathcal{S}_g = \{\eta \in \mathbb{R}^n : p(y,\eta) = 1\}, \quad d\varsigma = \text{ surface measure}$$

(since $p(y,\eta)$ does not depend on y, neither does $\mathcal{S}_g$). Get

$$\begin{aligned}\operatorname{Tr} U(t) &= \frac{1}{(2\pi)^n}\int_{\mathcal{F}'}\int_{\mathcal{S}_g}\int_0^\infty s^{n-1}e^{its}\,ds\,d\varsigma\,dy + \epsilon(t)\\ &= \operatorname{Vol}(\mathcal{S}_g)\int_0^\infty s^{n-1}e^{its}\,ds + \epsilon(t) \qquad (4.7)\end{aligned}$$

using that the integrand is independent of y and $\tilde{\eta} \in \mathcal{S}_g$.

Combining (4.5) and (4.7) gives

$$\sum_{\nu\in\mathbb{Z}^n} e^{it\sqrt{\lambda_\nu}} = \operatorname{Vol}(\mathcal{S}_g)\int_0^\infty s^{n-1}e^{its}\,ds + \epsilon(t); \qquad (4.8)$$

this equality is valid only for $|t| < \delta$ for some positive δ. To take this into account, but more importantly, to isolate the singularity of $\operatorname{Tr} U(t)$ at 0 from the rest of its singularities one multiplies both sides by a function $\rho \in C_c^\infty(\mathbb{R})$ supported in $(-\delta,\delta)$. The analysis then proceeds by taking the Fourier transform.

Following [**12**], multiply both sides of (4.8) by ρ and take the Fourier transform. Get

$$\sum_{\nu\in\mathbb{Z}^n} \hat{\rho}(\sigma - \sqrt{\lambda_\nu}) = \frac{\operatorname{Vol}(\mathcal{S}_g)}{2\pi}(s_+^{n-1} * \hat{\rho})(\sigma) + \widehat{\rho\epsilon}(\sigma) \qquad (4.9)$$

where $s_+^{n-1} = s^{n-1}H(s)$. The details for the following arguments will come later. Assuming only that $\rho \in C_c^\infty(\mathbb{R})$ is supported in $(-\delta,\delta)$ and that $\rho(0) = 1$ one shows that

$$(s_+^{n-1} * \hat{\rho})(\sigma) = 2\pi\sigma_+^{n-1} + \mathcal{O}(\sigma^{n-2}).$$

Hence

$$\sum_{\nu\in\mathbb{Z}^n} \hat{\rho}(\sigma - \sqrt{\lambda_\nu}) = \operatorname{Vol}(\mathcal{S}_g)\sigma^{n-1} + \mathcal{O}(\sigma^{n-1}) \text{ as } \sigma \to \infty. \qquad (4.9')$$

The left hand side of this last formula is the derivative of $\hat{\rho} * N_{P^{1/2}}$ (in the distributional sense, so we need not bother about discontinuities). Therefore

$$(\hat{\rho} * N_{P^{1/2}})(\sigma) = \int_0^\sigma \sum_{\nu\in\mathbb{Z}^n} \hat{\rho}(\sigma - \sqrt{\lambda_\nu})\,d\sigma$$

since $N_{P^{1/2}}(\sigma) = 0$ if $\sigma \le 0$. Replacing the integrand by the right hand side of (4.9′) and integrating gives

$$(\hat{\rho} * N_{P^{1/2}})(\sigma) = \frac{\operatorname{Vol}(\mathcal{S}_g)}{n}\sigma^n + \mathcal{O}(\sigma^{n-1}) \text{ as } \sigma \to \infty$$

With a more careful choice of ρ (in addition to the above require $\hat{\rho} \ge 0$) one can upgrade the left hand side to

$$(\hat{\rho} * N_{P^{1/2}})(\sigma) = N_{P^{1/2}}(\sigma) + \mathcal{O}(\sigma^{n-2}),$$

which leads to

$$N_{P^{1/2}}(\sigma) = \frac{\operatorname{Vol}(\mathcal{S}_g)}{n}\sigma^n + \mathcal{O}(\sigma^{n-1}).$$

It only remains to find $\mathrm{Vol}(\mathcal{S}_g)$. This coefficient appears for the first time in the passage from

$$\operatorname{Tr} U(t) = \frac{1}{(2\pi)^n} \int_{\mathcal{F}'} \int_{\mathbb{R}^n} e^{it\sqrt{p(y,\eta)}}\, d\eta\, dy + \epsilon(t) \tag{4.6$'$}$$

to

$$\begin{aligned} \operatorname{Tr} U(t) &= \frac{1}{(2\pi)^n} \int_{\mathcal{F}'} \int_{\mathcal{S}_g} \int_0^\infty s^{n-1} e^{its}\, ds\, d\sigma\, dy + \epsilon(t) \qquad (4.7') \\ &= \mathrm{Vol}(\mathcal{S}_g) \int_0^\infty s^{n-1} e^{its}\, ds + \epsilon(t) \end{aligned}$$

through the change of variables. With the n in the denominator it is

$$\frac{1}{(2\pi)^n} \int_{\mathcal{F}'} \int_{\{\eta : p(y,\eta) \le 1\}} d\eta\, dy,$$

the volume of the region

$$\{(y,\eta) : y \in \mathcal{F}',\ p(y,\eta) \le 1\} \tag{4.10}$$

with respect to the measure $dy\, d\eta$. This is the Liouville density, obtained from $\theta = \sum dy^j \wedge d\eta_j$ as

$$d\mathcal{L} = \frac{1}{n!} \underbrace{|\theta \wedge \cdots \wedge \theta|}_{n \text{ times}} = |dy^1 \wedge d\eta_1 \wedge \cdots \wedge \cdots \wedge dy^n \wedge d\eta_n|.$$

To compute (4.10) recall that

$$p(y,\eta) = \sum_{k.\ell} g^{k\ell} \eta_k \eta_\ell \quad g^{k\ell} = (2\pi)^2 \sum_j a_j^k a_\ell^k$$

It is natural to change variables setting

$$x^j = \frac{1}{2\pi} \sum_k b_k^j y^k, \quad \xi_j = 2\pi \sum_\ell a_j^\ell \eta_\ell$$

(recall that we interpreted $y = \alpha(x)$ as coordinates on $\mathcal{F}$). With this change of coordinates

$$\begin{aligned} \sum_j dx^j \wedge d\xi_j &= \sum_j \Big(\sum_k b_k^j\, dy^k\Big) \wedge \Big(\sum_\ell a_j^\ell\, d\eta_\ell\Big) \\ &= \sum_{k,\ell} \sum_j a_j^\ell b_k^j\, dy^k \wedge d\eta_\ell = \sum_{k,\ell} \delta_k^\ell\, dy^k \wedge d\eta_\ell = \theta \end{aligned}$$

so

$$\begin{aligned} \frac{1}{(2\pi)^n} \int_{\mathcal{F}'} \int_{\{\eta : p(y,\eta) \le 1\}} d\eta\, dy &= \frac{1}{(2\pi)^n} \int_{\mathcal{F}} \int_{\{\eta : p(x,\xi) \le 1\}} d\xi\, dx \\ &= \frac{\mathrm{Vol}(B^n)\, \mathrm{Vol}(\mathcal{M})}{(2\pi)^n} \end{aligned}$$

since $p(x,\xi) = |\xi|^2$. This is the coefficient in the last formula of the previous section.

What we have is for $P^{1/2}$:

$$N_{P^{1/2}}(\sigma) = \frac{\mathrm{Vol}(B^n)\, \mathrm{Vol}(\mathcal{M})}{(2\pi)^n} \sigma^n + \mathcal{O}(\sigma^{n-1}).$$

Using now $N_{P^{1/2}}(\sigma) = N_P(\sigma^2)$ we obtain

$$N_P(\lambda) = \frac{\mathrm{Vol}(B^n)\,\mathrm{Vol}(\mathcal{M})}{(2\pi)^n}\lambda^{n/2} + \mathcal{O}(\sigma^{(n-1)/2})$$

CHAPTER III

Eigenvalues

1. Semibounded elliptic operators

Assume $\mathcal{M}$ is closed, $\mathfrak{m}$ is a smooth density, $E \to \mathcal{M}$ is a Hermitian vector bundle, and $L^2(\mathcal{M}; E)$ is the closure of $C^\infty(\mathcal{M}; E)$ under the norm defined by the inner product

$$(u, v)_0 = \int_{\mathcal{M}} (u(x), v(x))_x \, dm(x), \quad u, v \in C^\infty(\mathcal{M}; E).$$

DEFINITION 1.1. An operator $P : C^\infty(\mathcal{M}; E) \subset L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$ is said to be

(i) symmetric if $(Pu, v) = (u, Pv)$ for all $u, v \in C^\infty(\mathcal{M}; E)$;

(ii) bounded below if

there is c such that $(Pu, u)_0 \geq c\|u\|_0$ for all $u, v \in C^\infty(\mathcal{M}; E)$; (1.2)

(iii) non-negative if it is bounded below with $c \geq 0$.

LEMMA 1.3. *The operator P is symmetric iff $(Au, u)_0$ is real for all $u \in C^\infty(\mathcal{M}; E)$.*

PROOF. If P is symmetric, then

$$2\Im(Pu, u)_0 = (Pu, u)_0 - \overline{(Pu, u)}_0 = (Pu, u)_0 - (u, Pu)_0 = 0 \text{ for all } u \in C^\infty(\mathcal{M}; E).$$

Thus $(Pu, u)_0$ is real valued for each smooth u.

Conversely, suppose (Pu, u) is real valued for each smooth u. For any $u, v \in C^\infty(\mathcal{M}; E)$ we have

$$(Pu, v)_0 + (Pv, u)_0 = (P(u + v), u + v)_0 - (Pu, u)_0 - (Pv, v)_0 \in \mathbb{R}$$

therefore $\Im(Pu, v) = -\Im(Pv, u) = \Im(u, Pv)$ using $(Pv, u) = \overline{(u, Pv)}$. Replacing v by $-iv$ in the same formula we have that since $i(Pu, v) - i(Pv, u)$ is real, $\Re(Pu, v) = \Re(Pv, u) = \Re(u, Pv)$. It follows that $(Pu, v) = (v, Pu)$. □

LEMMA 1.4. *If P is symmetric, then*

$$\|(P - \lambda I)u\|_0 \geq |\Im\lambda| \|u\|_0, \quad u \in C^\infty(\mathcal{M}; E).$$

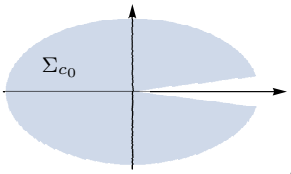
If in addition P is non-negative, then

$$\|(P - \lambda I)u\|_0 \geq |\lambda| \|u\|_0 \quad \text{if } \Re\lambda < 0.$$

Before giving the proof we draw the conclusion we will use later. Let $c_0 > 0$ and let

$$\Sigma_{c_0} = \mathbb{C} \backslash \{\lambda : \Re\lambda \geq 0 \text{ and } |\Im\lambda| \leq c_0 \Re\lambda\} \tag{1.5}$$

This is the complement of a closed sector $\Sigma^c_{c_0}$ containing the positive real axis in its interior. Combining both estimates (P non-negative) gives: for each $c_0 > 0$ there is c_1 such that



$$\|(P - \lambda I)u\|_0 \geq c_1|\lambda|\|u\|_0 \quad \text{if } \lambda \in \Sigma_{c_0}. \tag{1.6}$$

Indeed, if $\Re\lambda \geq 0$ and $|\Im\lambda| > c_0\Re\lambda$ then

$$|\lambda|^2 < (1 + (1/c_0^2))(\Im\lambda)^2$$

so the first assertion of the lemma gives that (1.6) holds for $\lambda \in \Sigma_{c_0}$ when $\Re\lambda \geq 0$, with c_1 positive $\leq 1/\sqrt{1 + 1/c_0^2}$. Together with the second assertion of the lemma we get that (1.6) holds in Σ_{c_0} with the constant $c_1 = \min\{1, 1/\sqrt{1 + 1/c_0^2}\}$.

Proof. Suppose P is symmetric. Starting with the Cauchy-Schwarz inequality we have for arbitrary $u \in C^\infty(\mathcal{M}; E)$,

$$\|(P - \lambda I)u\|_0 \, \|u\|_0 \geq |\big((P - \lambda I)u, u\big)| = |(Pu, u) - \lambda(u, u)| \geq |\Im\lambda|\|u\|_0^2$$

where in the last inequality we used $|z| \geq |\Im z|$ for any complex number z. Simplifying we get the first assertion.

Suppose now that (Pu, u) is non-negative. Then

$$\|(P - \lambda I)u\|_0^2 = \|Pu\|_0^2 + |\lambda|^2\|u\|_0^2 - 2\Re\lambda(Pu, u).$$

If $\Re\lambda < 0$, then $-2\Re\lambda(Pu, u) \geq 0$, hence the right hand side is bounded from below by $|\lambda|^2\|u\|_0^2$. This proves the second assertion. □

2. Maximal domain

To discuss the spectrum of $P \in \mathrm{Diff}^m(\mathcal{M}; E)$ one has to specify a domain. The initial definition of P is

$$P : C^\infty(\mathcal{M}; E) \subset L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E). \tag{2.1}$$

The maximal domain of this operator is the domain, momentarily denoted $\mathcal{D}$, of the Hilbert space adjoint of

$$P^\star : C^\infty(\mathcal{M}; E) \subset L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E),$$

the formal adjoint of P (see Section VII.3). This means:

> $u \in \mathcal{D}$ if and only if the functional $C^\infty(\mathcal{M}; E) \ni v \mapsto (u, P^\star v)_0 \in \mathbb{C}$ is continuous in the L^2-norm.

With $\mathcal{D}$ as domain, P is closed (the graph of P is closed) and densely defined ($C^\infty(\mathcal{M}; E)$ is a subspace of $\mathcal{D}$ which is dense in $L^2(\mathcal{M}; E)$). If $\mathcal{D}' \subset L^2(\mathcal{M}; E)$ is a subspace containing $C^\infty(\mathcal{M}; E)$ with which P is a closed operator, then $\mathcal{D}' \subset \mathcal{D}$. Indeed, Pu is always well defined as a distribution. The definition of $\mathcal{D}$ requires u to be such that Pu, as a distribution, is an element of $L^2(\mathcal{M}; E)$.

If P is elliptic, then $\mathcal{D} = H^m(\mathcal{M}; E)$ by part (*ii*) of Proposition 1.3. The Hilbert space adjoint of $P^\star$ is thus

$$P : H^m(\mathcal{M}; E) \subset L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E), \tag{2.2}$$

an extension of the operator (2.1). Since $C^\infty(\mathcal{M}; E)$ is dense in $H^m(\mathcal{M}; E)$, (2.2) is the closure of (2.1). (This is Corollary IX.1.6.)

PROPOSITION 2.3. *Suppose $P \in \mathrm{Diff}^m(\mathcal{M}; E)$ is elliptic. The Hilbert space adjoint of* (2.2) *is*

$$P^\star : H^m(\mathcal{M}; E) \subset L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E).$$

PROOF. The domain of the Hilbert space adjoint of (2.2) is

$$\mathfrak{D} = \{v \in L^2(\mathcal{M}; E) : \mathcal{D} \ni u \mapsto (Pu, v)_0 \text{ is continuous in the } L^2\text{-norm}\}.$$

In particular, since $C^\infty(\mathcal{M}; E) \subset \mathcal{D}$, if $v \in \mathfrak{D}$ then v belongs to the maximal domain of $P^\star$. So $\mathfrak{D} \subset H^m(\mathcal{M}; E)$ (since $P^\star$ is elliptic). We show that if $v \in H^m(\mathcal{M}; E)$, then $v \in \mathfrak{D}$. For arbitrary $u \in H^m(\mathcal{M}; E)$ we have

$$(Pu, v)_0 = (u, P^\star v)_0$$

by Corollary IX.1.10. But then

$$|(Pu, v)_0| \leq \|u\|_0 \, \|P^\star v\|_0$$

gives that $H^m(\mathcal{M}; E) \ni u \mapsto (Pu, v) \in \mathbb{C}$ is L^2-continuous. So $v \in \mathfrak{D}$. □

3. Selfadjointness, spectrum

PROPOSITION 3.1. *If $P \in \mathrm{Diff}^m(\mathcal{M}; E)$ is elliptic and non-negative, see Part* (*ii*) *of Definition 1.1, then*

- (*i*) *the operator* (2.2) *is selfadjoint;*
- (*ii*) *With this domain,* $\mathrm{spec}(P)$ *is a discrete subset of* $[0, \infty)$ *without points of accumulation;*
- (*iii*) *each element of* $\mathrm{spec}(P)$ *is an eigenvalue of finite multiplicity.*

PROOF. The property of being bounded from below includes the requirement that (Pu, u) is real for any $u \in C^\infty(\mathcal{M}; E)$. Hence P with domain $C^\infty(\mathcal{M}; E)$ is symmetric by Lemma 1.3, i.e. $P^\star = P$ on smooth sections. Consequently, the Hilbert space adjoint of (2.2) is itself (they have the same domain). This proves (*i*).

Lemma 1.4 gives

$$\|(P - \lambda I)u\| \geq c_1 |\lambda| \|u\|^2 \text{ for any } u \in C^\infty(\mathcal{M}; E) \text{ if } \lambda \in \Sigma_{c_0}$$

via (1.6) with c_1 depending on $c_0 > 0$ (Σ_{c_0} is defined in (1.5)). By continuity, the estimate holds also when $u \in H^m(\mathcal{M}; E)$ and shows that if $\lambda \in \Sigma_{c_0}$ then

$$(P - \lambda I) : H^m(\mathcal{M}; E) \subset L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E) \tag{3.2}$$

is injective. The Hilbert space adjoint of this operator is

$$(P - \overline{\lambda} I) : H^m(\mathcal{M}; E) \subset L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$$

since (2.2) is selfadjoint. Since Σ_{c_0} is closed under conjugation, also the above operator is injective, therefore the closure of the range of (3.2) is $L^2(\mathcal{M}; E)$. But (3.2) has closed range by Corollary IX.2.7, so it is onto. It follows that $\mathrm{spec}(P)$ is contained in the complement of Σ_{c_0} for any $c_0 > 0$, hence $\mathrm{spec}(P) \subset [c, \infty)$.

Let $(P - \lambda I)^{-1} : L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$ be the inverse of (3.2) for those λ for which it exists, certainly if $\lambda \notin [0, \infty)$. Since P is elliptic, $(P - \lambda)^{-1}$ is a pseudodifferential operator of order $-m$ (Proposition VIII.3.3) and is therefore a compact operator (Corollary IX.2.8). When λ is real, less than c, then $(P - \lambda)^{-1}$ is also selfadjoint.

Before continuing the proof we state a lemma which we prove later.

LEMMA 3.3. *Let H be a Hilbert space, $T : H \to H$ be compact, selfadjoint, and positive. The spectrum of T is a subset of $[0, \|\mathbb{T}\|]$ with only 0 as point of accumulation. Each nonzero $\kappa \in \operatorname{spec}(T)$ is an eigenvalue of T of finite multiplicity, and H is the closure of*

$$\ker T \oplus \bigoplus_{\substack{\kappa \in \operatorname{spec}(T) \\ \kappa > 0}} \mathcal{E}_\kappa, \quad \mathcal{E}_\kappa = \ker(T - \kappa I).$$

If $H = L^2(\mathcal{M}; E)$ and T is the inverse $L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$ of a selfadjoint elliptic differential operator, then $\ker T = \{0\}$.

Fix some $\lambda_* < 0$. Then $T = (P - \lambda_* I)^{-1}$ satisfies the conditions of the lemma. Thus

$$L^2(\mathcal{M}; E) = c\ell \bigoplus_{\substack{\kappa \in \operatorname{spec}(T) \\ \kappa > 0}} \mathcal{E}_\kappa.$$

If $u \in \mathcal{E}_\kappa$, then $Tu = \kappa u \implies u = \kappa(P - \lambda_8)u$ so

$$Pu = (\lambda_* + \frac{1}{\kappa})u$$

and conversely. Therefore the eigenspaces of $P - \lambda_0$ are those of T, eigenspaces of finite multiplicity with eigenvalues $\lambda_\nu = (\lambda_0 + 1/\kappa_\nu^{-1})$ tending to ∞ as $\nu \to \infty$. □

The following proof uses parts of Kato [**17**, p. 185], Trèves [**34**, Theorem 48.1], and Schatten [**28**, pp. 15–16].

PROOF OF THE LEMMA. Since T is positive, $\operatorname{spec}(T) \subset [0, \infty)$ and since T is selfadjoint (normal), is spectral radius is equal to its operator norm. The number 0 belongs to the spectrum because, if the range is closed, it is finite-dimensional, or else, T is not surjective. Thus $\operatorname{spec}(T) \subset [0, \|T\|]$. Furthermore (see Rudin [**27**, Theorem 12.25]), the norm of T is

$$\kappa_0 = \sup_{\|u\|=1} (Tu, u).$$

Note that $\kappa_0 \neq 0$, since T is not the zero operator. Pick a sequence $\{u_\nu\} \subset H$ be such that $\|u_\nu\| = 1$ and $(Tu_\nu, u_\nu) \to \kappa_0$. Passing to a subsequence, assume $\{Tu_\nu\}$ converges. We have

$$\|Tu_\nu - \kappa u_\nu\|^2 = \|Tu_\nu\|^2 - 2\kappa_0(Tu_\nu, u_\nu) + \kappa_0^2 \leq 2\kappa^2 - 2\kappa_0(Tu_\nu, u_\nu)$$

which tends to 0 as $\nu \to \infty$. Since $\{Tu_\nu\}$ converges, $\{u_\nu\}$ converges to some u and we have $Tu = \kappa_0 u$ with $\|u\| = 1$.

Thus $\mathcal{E}_{\kappa_0} = \ker(T - \kappa_0 I) \neq \{0\}$. If $\mathcal{E}_{\kappa_0}$ is infinite-dimensional, it has an orthonormal basis $\{v_\nu\}$. Compactness of T implies $\{Tv_\nu\}$ has a convergent subsequence, that is $\{\kappa_0 v_\nu\}$ has a convergent subsequence, which is not possible since $\kappa_0 \neq 0$. Therefore $\mathcal{E}_{\kappa_0}$ is finite-dimensional.

Let H_1 be the subspace of $H_0 = H$ orthogonal to $\mathcal{E}_{\kappa_0}$. If $u \in H_1$, then $Tu \in H_1$. Indeed, if $(u, v) = 0$ for all $v \in \mathcal{E}_{\kappa_0}$, then $(Au, v) = (u, Av) = \kappa_0(u, v) = 0$. The restriction $T_1 = T|_{H_1} : H_1 \to H_1$ satisfies the same hypotheses as T. If $\|T\| = \|T_1\|$ then κ_0 is an eigenvalue of T_1, which is not possible. So $\kappa_1 = \|T_1\| < \|T\|$ is an eigenvalue of T and $\mathcal{E}_{\kappa_1} = \ker(T_1 - \kappa_1 I) \subset H_1$ is orthogonal to $\mathcal{E}_{\kappa_0}$, and there are no eigenvalues of T between κ_1 and κ_0.

Suppose we have obtained, for $\mu = 1, \dots, \nu$, T-invariant spaces $H_\mu \subset H_{\mu-1}$ such that $H_{\mu-1} = H_\mu \oplus \mathcal{E}_{\kappa_{\mu-1}}$ as an orthogonal, where $\mathcal{E}_{\kappa_\mu} = \ker(T - \kappa_\mu I)$ with $\kappa_\mu = \|T_\mu\| > 0$, $T_\mu = T|_{H_\mu} : H_\mu \to H_\mu$ if $\mu < \nu$.

If $T_\nu \neq 0$, let $\kappa_\nu = \|T_\nu\|$, let $\mathcal{E}_{\kappa_\nu} = \ker(T_\nu - \kappa_\nu)$, and let $H_{\nu+1}$ be the subspace of H_ν orthogonal to $\mathcal{E}_{\kappa_\nu}$. This process continues while $T_\nu \neq 0$.

If no $T_\nu = 0$ we have a strictly decreasing sequence $\{\kappa_\nu\}_{\nu=0}^\infty \subset [0, \|T\|]$ which converges to some $\kappa \geq 0$. Pick, for each ν, an element $u_\nu \in \mathcal{E}_{\kappa_\nu}$, $u_\nu \neq 0$. There is a subsequence $\{u_{\nu_\mu}\}$ such that Tu_{ν_μ} converges. Since $(u_{\nu_\mu}, u_{\nu_{\mu'}}) = 0$ if $\mu \neq \mu'$,

$$\|\kappa_{\nu_\mu} u_{\nu_\mu} - \kappa_{\nu_{\mu+1}} u_{\nu_{\mu+1}}\|^2 = \kappa_\nu^2 + \kappa_{\nu+1}^2 \to 2\kappa^2 \text{ as } \mu \to \infty,$$

for such μ, μ', while

$$\|Tu_{\mu_\nu} - Tu_{\mu_{\nu+1}}\|^2 \to 0 \text{ as } \mu \to \infty,$$

therefore $\kappa = 0$.

Or else, $T_\nu = 0$ and then $H_\nu = \ker T$. In this case T has finite rank equal to the direct sum of the finitely many spaces $\mathcal{E}_{\kappa_\nu}$.

When T is the inverse $L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$ of a selfadjoint elliptic differential operator, say of order m, then $T : L^2(\mathcal{M}; E) \to H^m(\mathcal{M}; E)$ is an isomorphism, so $\ker T = \{0\}$. □

Note that the spectrum of P and the eigenspaces are actually properties of P itself, independent of the Hilbert space structure of $L^2(\mathcal{M}; E)$ or $H^m(\mathcal{M}; E)$.

PROPOSITION 3.4. *Let $P \in C^\infty(\mathcal{M}; E)$ be elliptic and non-negative (Definition 1.1, Part (iii)). Then*

$$(P - \lambda I) : L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$$

exists for $\lambda \notin \operatorname{spec}(P)$ and as such, for any c_0 There is c_1' such that

$$\|(P - \lambda I)^{-1}\| \geq c_1' |\lambda|^{-1} \quad \text{for } \lambda \in \Sigma_{c_0}. \tag{3.5}$$

This is a consequence of (1.6) and the fact that $C^\infty(\mathcal{M}; E)$ (which is where that inequality holds) is dense in $H^m(\mathcal{M}; E)$ (Corollary IX.1.5). We can take $c_1' = 1/c_1$ where c_1 is the constant in (1.6).

We end the section with an example that illustrates that being elliptic does not imply discreteness of the spectrum and a theorem with a sufficient condition for the spectrum of an elliptic to be discrete.

EXAMPLE 3.6. Let $\mathcal{M} = S^1$, let $P = e^{-ix} D_x^2$, where x is the standard angular variable. Then P is elliptic and $\operatorname{spec}(P) = \mathbb{C}$. For arbitrary $\lambda \in \mathbb{C}$,

$$u_\lambda(x) = \sum_{k>0} \frac{\lambda^k}{(k!)^2} e^{ikx}$$

is an eigenfunction of P with eigenvalue λ.

PROPOSITION 3.7. *Let $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ be an elliptic operator of order m, extend it to its closure (i.e., give it its maximal domain):*

$$P : H^m(\mathcal{M}; E) \subset L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$$

Suppose $\operatorname{spec}(P) \neq \mathbb{C}$. *Then* $\operatorname{spec}(P)$ *is a set without points of accumulation.*

PROOF. Let λ_0 be a point of the boundary of $\operatorname{spec}(P)$. Let $\mathcal{K} = \ker(P - \lambda_0)$,

$$\iota_{\mathcal{K}} : \mathcal{K} \to L^2(\mathcal{M}; E), \quad \iota_{\hat{\mathcal{K}}^\perp} : \hat{\mathcal{K}}^\perp \to L^2(\mathcal{M}; E)$$

be the inclusion maps. By $\hat{\mathcal{K}}^\perp$ we mean $H^m(\mathcal{M}; E) \cap \mathcal{K}^\perp$, with $\mathcal{K}^\perp$ the subspace of $L^2(\mathcal{M}; E)$ orthogonal to $\mathcal{K}$; since $\mathcal{K} \subset C^\infty(\mathcal{M}; E)$, the orthogonal projection $\pi_{\mathcal{K}^\perp} : L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$ onto $\mathcal{K}^\perp$ is the identity on $\hat{\mathcal{K}}^\perp$, so

$$H^m(\mathcal{M}; E) = \mathcal{K} \oplus \hat{\mathcal{K}}^\perp.$$

Let $\mathcal{R} = \operatorname{rg}(P - \lambda_0 I)$. Let

$$\pi_{\mathcal{R}} : L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E), \quad \pi_{\mathcal{R}^\perp} : L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$$

be the orthogonal projections on the subspaces indicated by the subindex. Using these inclusions and projections we rewrite $(P - \lambda I)$ as

$$\begin{bmatrix} \mathcal{P}_{11}(\lambda) & \mathcal{P}_{12}(\lambda) \\ \mathcal{P}_{21}(\lambda) & \mathcal{P}_{22}(\lambda) \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \hat{\mathcal{K}}^\perp \end{matrix} \to \begin{matrix} \mathcal{R}^\perp \\ \oplus \\ \mathcal{R} \end{matrix}.$$

For instance $\mathcal{P}_{22}(\lambda) = \pi_{\mathcal{R}}(P - \lambda I)\iota_{\hat{\mathcal{K}}^\perp}$. Since $P_{22}(\lambda_0)$ is invertible, also $P_{22}(\lambda)$ is invertible for λ close to λ_0. Indeed,

$$\mathcal{P}_{22}(\lambda) = P_{22}(\lambda_0) - (\lambda - \lambda_0)\pi_{\mathcal{R}} \circ \iota_{\hat{\mathcal{K}}^\perp}$$

is invertible when $\lambda - \lambda_0$ is small. The system

$$\begin{cases} \mathcal{P}_{11}(\lambda)u + \mathcal{P}_{12}(\lambda)v = f \\ \mathcal{P}_{21}(\lambda)u + \mathcal{P}_{22}(\lambda)v = g \end{cases} \tag{3.8}$$

is solved by the pair (u, v) if and only if

$$v = \mathcal{P}_{22}(\lambda)^{-1}(g - \mathcal{P}_{11}(\lambda))u$$

and u solves

$$\big(\mathcal{P}_{11}(\lambda) - P_{12}(\lambda)P_{22}(\lambda)^{-1}\mathcal{P}_{21}(\lambda)\big)u = f - \mathcal{P}_{12}\mathcal{P}_{22}(\lambda)^{-1}g$$

in which case λ lies in the resolvent set of P.

The operator on the left, call it $\mathcal{Q}(\lambda)$, depends analytically on λ, maps the finite-dimensional space $\mathcal{K}$ into the (finite-dimensional) space $\mathcal{K}^\perp$. Since λ_0 lies on the boundary of $\operatorname{spec}(P)$, any disc centered at λ_0 will contain points where $P_-\lambda_I$ is invertible. At such points where $\mathcal{Q}(\lambda)$ is invertible. It follows that $\dim \mathcal{K} = \dim \mathcal{R}^\perp$.

The linear map $Q(\lambda)$ is zero when $\lambda = \lambda_0$, but is not identically 0: as pointed out already any disc centered at λ_0 will contain points where $\mathcal{Q}(\lambda)$ is invertible. Using bases on $\mathcal{K}$ and $\mathcal{R}^\perp$ to write a determinant for $\mathcal{Q}(\lambda)$ we see that $\det \mathcal{Q}(\lambda)$ vanishes at λ_0 but not at nearby points. Since $\det \mathcal{Q}(\lambda)$ is holomorphic, only λ_0 can be a zero in a sufficiently small disk centered at λ_0. Thus (3.8) can be solved in a punctured disc about λ_0, which means that as an element of $\operatorname{spec}(P)$, λ_0 is isolated. This proves the proposition. □

4. Polynomial bounds

Let $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ be an elliptic differential operator of order m. Assume

$$(Pu, u) \geq \|u\|^2 \quad \text{for all } u \in C^\infty(\mathcal{M}; E).$$

The spectrum of P is a discrete subset of $[0, \infty)$ consisting of eigenvalues of finite multiplicity. Let $\{\lambda_k\}_{k=0}^\infty$ be the sequence of eigenvalues, each eigenvalue repeated according to its multiplicity, in non-decreasing. Let

$$N_P(\lambda) = \ \#\{k : \lambda_k < \lambda\}$$

PROPOSITION 4.1. *There are C and $s_0 > n/m$ such that $N_P(\lambda) \leq c\lambda^{s_0}$.*

The following proof is copied from Lesch [**18**, Prop. 1.4.7, available form the arXiv] who in turn adapted it from Gilkey [**10**, Lemma 1.6.3], who credits the idea to B. Allard.

PROOF. Let $\{\phi_k\}_{k=0}^\infty$ be an orthonormal basis of $L^2(\mathcal{M}; E)$ with $P\phi_k = \lambda_k\phi_k$. We need to estimate

$$N_P(\lambda) = \sum_{\{k:\lambda_k<\lambda\}} \|\phi_k\|_0^2.$$

We will write $(v, w)_x$ for the inner product of $v, w \in E_x$ and $|v|_x$ for the norm. The immediate task is to show the pointwise estimate

there are $\ell \in \mathbb{N}$ and $C > 0$ such that

$$\sum_{\{k:\lambda_k<\lambda\}} |\phi_k(x)|_x^2 \leq rC(1 + \lambda^{2\ell})^{1/2}\big(\sum_{\{k:\lambda_k<\lambda\}} |\phi_k(x)|_x^2\big)^{1/2} \quad \text{for all } \lambda. \tag{4.2}$$

Here $r = \operatorname{rk} E$. Simplifying and squaring yields

there are $\ell \in \mathbb{N}$ and $C > 0$ such that

$$\sum_{\{k:\lambda_k<\lambda\}} |\phi_k(x)|_x^2 \leq r^2C^2(1 + \lambda^{2\ell}) \quad \text{for all } \lambda$$

which gives

$$N_P(\lambda) \leq r^2C^2 \operatorname{Vol}(\mathcal{M})(1 + \lambda^{2\ell}) \tag{4.3}$$

after integration over $\mathcal{M}$.

We now prove (4.2). Fix some $x_0 \in \mathcal{M}$. Let $\eta_1, \dots, \eta_r$ be an orthonormal basis of E_{x_0}. If $v \in E_{x_0}$, then

$$|v|_{x_0}^2 = \sum_{j=1}^r |(\eta_j, v)|_{x_0}^2 = \sum_{j=1}^r (\eta_j, v)_{x_0}\overline{(\eta_j, v)}_{x_0} = \sum_{j=1}^r \big(\eta_j, (\eta_j, v)_{x_0}v\big)_{x_0}$$

where the first equality is by the Pythagoras theorem. Thus

$$\begin{aligned}
\sum_{\{k:\lambda_k<\lambda\}} \big|\phi_k(x_0)\big|_{x_0}^2 &= \sum_{\{k:\lambda_k<\lambda\}} \sum_{j=1}^r \big(\eta_j, \big(\eta_j, \phi_k(x_0)\big)_{x_0}\phi_k(x_0)\big)_{x_0} \\
&= \sum_{j=1}^r \big(\eta_j, \sum_{\{k:\lambda_k<\lambda\}} \big(\eta_j, \phi_k(x_0)\big)_{x_0}\phi_k(x_0)\big)_{x_0} \\
&\leq \sum_{j=1}^r \big| \sum_{\{k:\lambda_k<\lambda\}} \big(\eta_j, \phi_k(x_0)\big)_{x_0}\phi_k(x_0)\big|_{x_0}
\end{aligned}$$

using Cauchy's inequality in E_{x_0}. With our fixed x_0 and arbitrary $x \in \mathcal{M}$, let

$$u_j(x) = \sum_{\{k:\lambda_k<\lambda\}} \big(\eta_j, \phi_k(x_0)\big)_{x_0} \phi_k(x);$$

These are smooth section of E with which the above is

$$\sum_{\{k:\lambda_k<\lambda\}} \big|\phi_k(x_0)\big|^2_{x_0} \le \sum_{j=1}^{r} |u_j(x_0)|_{x_0}.$$

If $\ell \in \mathbb{N}$ is such that $m\ell > n/2$ then, combining the estimate in the Sobolev embedding theorem (IX.3.2) and the a priori estimate (IX.1.4) yields

$$|u_j(x_0)|^2_{x_0} \le C^2(\|P^\ell u_j\|_0^2 + \|u_j\|_0^2) \tag{4.4}$$

with some $C > 0$. Computing carefully, using that the ϕ_k are orthonormal, we have

$$\begin{aligned}
\|P^\ell u_j\|_0^2 + \|u_j\|_0^2 &= \|P^\ell \sum_{\{k:\lambda_k<\lambda\}} \big(\eta_j, \phi_k(x_0)\big)_0 \phi_k\|_0^2 + \|\sum_{\{k:\lambda_k<\lambda\}} \big(\eta_j, \phi_k(x_0)\big)_0 \phi_k\|_0^2 \\
&= \|\sum_{\{k:\lambda_k<\lambda\}} \lambda_k^\ell \big(\eta_j, \phi_k(x_0)\big)_0 \phi_k\|_0^2 + \|\sum_{\{k:\lambda_k<\lambda\}} \big(\eta_j, \phi_k(x_0)\big)_0 \phi_k\|_0^2 \\
&= \sum_{\{k:\lambda_k<\lambda\}} \lambda_k^{2\ell} |\big(\eta_j, \phi_k(x_0)\big)_0|^2 + \sum_{\{k:\lambda_k<\lambda\}} |\big(\eta_j, \phi_k(x_0)\big)_0|^2 \\
&= \sum_{\{k:\lambda_k<\lambda\}} (1+\lambda_k^{2\ell}) |\big(\eta_j, \phi_k(x_0)\big)_0|^2 \\
&\le \sum_{\{k:\lambda_k<\lambda\}} (1+\lambda^{2\ell}) |\big(\eta_j, \phi_k(x_0)\big)_0|^2 \\
&= (1+\lambda^{2\ell}) \|u_j\|_0^2.
\end{aligned}$$

But

$$\|u_j\|_0^2 = \sum_{\{k:\lambda_k<\lambda\}} |\big(\eta_j, \phi_k(x_0)\big)_{x_0}|^2 \le \sum_{\{k:\lambda_k<\lambda\}} \sum_{j'} |\big(\eta_{j'}, \phi_k(x_0)\big)_{x_0}|^2 = \sum_{\{k:\lambda_k<\lambda\}} |\phi_k(x_0)|^2_{x_0}$$

using the Pythagoras theorem in E_{x_0} again in the last equality. So

$$(1+\lambda^{2\ell}) \|u_j\|_0^2 \le (1+\lambda^{2\ell}) \sum_{\{k:\lambda_k<\lambda\}} |\phi_k(x_0)|^2_{x_0},$$

thus each term in (4.4) is bounded by

$$|u_j(x_0)|_{x_0} \le C(1+\lambda^{2\ell})^{1/2} \big(\|\sum_{\{k:\lambda_k<\lambda\}} |\phi_k(x_0)|^2_{x_0}\big)^{1/2},$$

Hence

$$\sum_{\{k:\lambda_k<\lambda\}} \big|\phi_k(x_0)\big|^2_{x_0} \le \sum_{j=1}^{r} |u_j(x_0)|_{x_0} \le rC(1+\lambda^{2\ell})^{1/2} \big(\sum_{\{k:\lambda_k<\lambda\}} |\phi_k(x_0)|^2_{x_0}\big)^{1/2}$$

This completes the proof of (4.2) and of the proposition. □

In the proof, ℓ is chosen so that $m\ell > n/2$. If we could use arbitrary powers ℓ and arbitrary real order Sobolev spaces, not just integer powers and integer Sobolev orders as we are constrained to do here, then (4.3) would give

$$N_P(\lambda) \le r^2 C^2 (1+\lambda^{n/m+\varepsilon}) \tag{4.5}$$

with arbitrary $\varepsilon > 0$, C depending on the choice of ε. But as it is, the smallest integer ℓ we can take is $\ell = (\lfloor n/2 \rfloor + 1)/m$, so the best bound of $N_P(\lambda)$ we have so far is $C\lambda^{2(\lfloor n/2 \rfloor + 1)/m}$. The Weyl estimate is of course better and more precise. We will get this later.

COROLLARY 4.6. *The series*

$$\zeta(s) = \sum_k \frac{1}{\lambda_k^s}.$$

converges absolutely and uniformly on compact subsets of

$$\{s \in \mathbb{C} : \Re s > 2(\lfloor n/2 \rfloor + 1)/m\}.$$

Thus $\zeta(s)$ is holomorphic in that set.

PROOF. Suppose $N(\lambda) \leq c\lambda^\mu$. In

$$\#\{k : \lambda_k < \lambda\} \leq c\lambda^\mu$$

replace $\lambda = \lambda_j$. The set consists of all k such that $\lambda_k < \lambda_j$, which, since λ_k is monotone, has j elements. Thus

$$j \leq c\lambda_j^\mu$$

which gives $\lambda_j \geq (j/c)^{1/\mu}$, so

$$\frac{1}{|\lambda_j^s|} \leq \frac{c^{\Re s/m}}{j^{\Re s/\mu}}$$

Thus the series converges if $s/\mu > 1$.

□

CHAPTER IV

Hörmander's estimate

1. The Cauchy problem

Let $P : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ be a classical first order elliptic pseudodifferential operator with real-valued principal symbol. We will discuss the solution operator of the problem

$$\begin{cases} \frac{1}{i}\frac{\partial u}{\partial t} - Pu = 0 & \text{in } \mathbb{R}\times\mathcal{M} \\ u|_{t=0} = u_0 & \text{on } \{0\}\times\mathcal{M} \end{cases}$$

for small t. The ansatz is that in a neighborhood $(-\varepsilon,\varepsilon)\times\mathcal{U}$ of $(0,x_0)$, in coordinates (t,x), the problem is solved by

$$Qu_0(t,x) = \frac{1}{(2\pi)^n}\int e^{i(\phi(t,x,\theta)-x'\cdot\theta)}q(t,x,x',\theta)u_0(x')\,dx'\,d\theta. \tag{1.1}$$

where q is a classical symbol and the phase $\phi : \mathcal{U}\times\mathbb{R}^n \to \mathbb{R}$ is, in principle, also a first order symbol (homogeneous of degree 1 in θ for large $|\theta|$). Setting $t = 0$ should produce the identity operator, thus

$$\phi(0,x,\theta - x'\cdot\theta) = (x-x')\cdot\theta \text{ and } q(0,x,x',\theta) = 1;$$

that is,

$$\phi(0,x,\theta) = x\cdot\theta$$

and we add to the ansatz that q is of order 0 but $q(0,x,x',\theta)$ is equal to 1 only in some compact subset $\mathcal{K}\times\mathcal{K}$ and otherwise 0 in $U\times\mathcal{U}\times\mathbb{R}^n$. This would reflect the use of partitions of unity in the construction of the parametrix for the Cauchy problem. This justifies some integration by parts we will carry out later.

Letting $P \in \Psi^1_{c\ell}(\mathcal{M})$ be the operator

$$Pv(x) = \frac{1}{(2\pi)^n}\int e^{i(x-x')\cdot\xi}p(x,\xi)v(x')\,dx'\,d\xi$$

one gets

$$(PQv)(t,x) = \frac{1}{(2\pi)^n}\int e^{i(\phi(t,x,\theta)-x''\cdot\theta)}(p\circ_\phi q)(t,x,x'',\theta)u_0(x'')\,dx''\,d\theta$$

where

$$(p\circ_\phi q)(t,x,x'',\theta) = \frac{1}{(2\pi)^n}\int e^{i(x-x')\cdot\xi - i[\phi(t,x,\theta)-\phi(t,x',\theta)]}p(x,\xi)q(t,x',x''\theta)\,dx'\,d\xi.$$

That $(p \circ_\phi q)$ is a symbol of order 1 is essentially proved by the following argument. The expression in brackets in the phase can be written as

$$\phi(t,x,\theta) - \phi(t,x',\theta) = \sum_j \phi_j(t,x,x',\theta)(x^j - x'^j)$$

by way of the first order Taylor expansion at $x = x'$; the coefficients satisfy

$$\phi_j(t,x',x',\theta) = \frac{\partial\phi}{\partial x^j}(x',\theta)$$

Defining $\Phi = (\phi_1,\dots,\phi_n)$, the phase is

$$(x-x')\cdot\xi - (x-x')\cdot\Phi(t,x,x',\theta)$$

Introducing the change of variables

$$x' = x+y, \quad \xi = \eta + \Phi(t,x,x-y,\theta)$$

we get

$$(p\circ_\phi q)(t,x,x'',\theta) = \frac{1}{(2\pi)^n}\int e^{-iy\cdot\eta} p(x,\eta+\Phi) q(x+y,x'',\theta)\,dy\,d\eta.$$

Replacing the Taylor polynomial with remainder (expansion in η at $\eta = 0$),

$$p(x,\eta+\Phi) = \sum_{|\alpha|\le N} \frac{1}{\alpha!}\frac{\partial^{|\alpha|}p}{\partial\xi^\alpha}(x,\Phi(t,x,x+y,\theta))\eta^\alpha + \sum_{|\alpha|=N+1}\tilde p_{N,\alpha}\eta^\alpha$$

in the integral and integrating by parts yields a sum of terms

$$\begin{aligned}
&\frac{1}{(2\pi)^n}\frac{1}{\alpha!}\int e^{-iy\cdot\eta}\frac{\partial^{|\alpha|}p}{\partial\xi^\alpha}(x,\Phi(t,x,x+y,\theta))\eta^\alpha q(x+y,x'',\theta)\,dy\,d\eta\\
&\quad= \frac{1}{(2\pi)^n}\frac{1}{\alpha!}\int (-D_y)^\alpha e^{-iy\cdot\eta}\frac{\partial^{|\alpha|}p}{\partial\xi^\alpha}(x,\Phi(t,x,x+y,\theta))q(x+y,x'',\theta)\,dy\,d\eta\\
&\quad= \frac{1}{(2\pi)^n}\frac{1}{\alpha!}\int e^{-iy\cdot\eta} D_y^\alpha\big[\frac{\partial^{|\alpha|}p}{\partial\xi^\alpha}(x,\Phi(t,x,x+y,\theta))q(x+y,x'',\theta)\big]\,dy\,d\eta\\
&\quad= \frac{1}{\alpha!}\int D_y^\alpha\big[\frac{\partial^{|\alpha|}p}{\partial\xi^\alpha}(x,\Phi(t,x,x+y,\theta))q(x+y,x'',\theta)\delta(y)\big]\,dy\\
&\quad= \frac{1}{\alpha!}D_y^\alpha\big|_{y=0}\big[\frac{\partial^{|\alpha|}p}{\partial\xi^\alpha}(x,\Phi(t,x,x+y,\theta))q(x+y,x'',\theta)\big]
\end{aligned}$$

plus the remainder. The last expression is a classical symbol of order $1-|\alpha|$. A similar expression results also from the remainder term. The highest order term (order 1) has $\alpha = 0$ and comes from the principal parts of p and q:

$$p_1(x,\Phi(t,x,x,\theta))q_0(x,x'',\theta) = p_1\big(x,\nabla_x\phi(t,x,\theta)\big)q_0(x,x'',\theta).$$

(since p_1 is homogeneous of degree 1, we must ensure $\nabla_x\phi(x,\xi)\neq 0$).

Thus

$$(\frac{1}{i}\frac{\partial}{\partial t} - P)Qu_0 = \int e^{i(\phi(x,\theta)-x'\cdot\theta)}(\phi_t q + D_t q - p\circ_\phi q)u_0(x')\,dx'\,d\theta$$

where $(\phi_t q - D_t q - p\circ_\phi q)$ is a classical symbol of order 1 which is required to vanish. The term of order 1 is

$$\big(\phi_t(t,x,x'',\theta) - p_1(x,\nabla_x\phi(t,x,x'',\theta))\big)q_0$$

Thus the phase ϕ should satisfy

$$\begin{cases}\phi_t(t,x,\theta) - p_1\big(x, \nabla_x\phi(t,x,\theta)\big) = 0\\ \phi(0,x,\theta) = x\cdot\theta.\end{cases}$$

Since $\nabla_x\phi(0,x) = \theta$, the condition $\nabla_x\phi(x,\xi)\neq 0$ is guaranteed as long as t is small ($\theta = 0$ is not relevant). This first order equation has a unique solution for that exists for small t. Thus

$$\phi(t,x,\theta) = x\cdot\theta + tp(x,\theta) + t^2\phi_2(t,x,\theta)$$

with some $h(t,x,\theta)$. The term of order 0 in is a first order differential equation $-i\mathcal{L}q_0 + cq_0 = 0$ in which q_0 appears linearly. The coefficients of $\mathcal{L}$, a vector field with real coefficients, depend p_1 and its first order derivatives, and c depends on p_0. This equation, called the first transport equation comes with the initial condition $q_0(0,x,x'',\theta) = 1$. The rest of the terms p_{-j} are solutions of $-i\mathcal{L}q_j + cq_j = v_j$ with $q_j = 0$ at $t = 0$ where v_j depends on $p_0,\dots,p_j$ and $q_0,\dots,q_{j-1}$.

2. The trace

In setting $x' = x$ in (1.1) the phase becomes

$$\phi(t,x,\theta) - x\cdot\theta = tp(x,\theta) + t^2\phi_2(t,x,\theta)$$

so the trace of Q for small t (set $x = x'$) is (with $\mathfrak{m} = \mathfrak{m}_0 dx$ in coordinates)

$$\operatorname{Tr}(Q)(t) = \frac{1}{(2\pi)^n}\int_{\mathcal{U}\times\mathbb{R}^n} e^{i(tp(x,\theta)+t^2\phi_2(t,x,\theta))}q(t,x,x,\theta)\,m_0(x)dx\,d\theta,$$

a distribution.

Lemma 2.1. *The distribution* $\operatorname{Tr}(Q)$ *is smooth in some punctured neighborhood of* 0

Proof. The function

$$\psi(t,x,\theta) = p(x,\theta) + t\phi_2(t,x,\theta)$$

is homogeneous of degree 1 in θ, smooth (when $\theta\neq 0$) and nonzero for small t since $p(x,\theta)\neq 0$. With such t as a parameter let

$$S_{\psi,t} = \partial\mathcal{B}_{\psi,t} = \{(x,\theta)\in U\times\mathbb{R}^n : \psi(t,x,\theta) = 1\};$$

this is a cylinder with axis $\mathcal{U}$ and section diffeomorphic to the $n-1$ sphere in $\mathbb{R}^n$. Define

$$\wp_t : \mathbb{R}^+\times S_{\psi,t}\to U\times\mathbb{R}^n,\quad \wp_t(s,x,\hat\theta) = (x, s\hat\theta).$$

Then

$$\wp_t^*(\mathfrak{m}_0 dx^1\wedge\cdots\wedge dx^n\wedge d\theta_1\wedge\cdots\wedge d\theta_n) = s^{n-1}ds\wedge d\Theta$$

where $d\Theta$ is surface measure on $\partial\mathcal{B}_{\psi,t}$. Then

$$\operatorname{Tr}(Q)(t) = \frac{1}{(2\pi)^n}\int_0^\infty e^{ist}s^{n-1}\int_{\partial\mathcal{B}_{\psi,t}} q(t,x,s\hat\theta)\,d\Theta\,ds$$

The inner integral is a smooth function $a(t,s)$, a symbol of order 0 with t as space variables and s as phase variable.

Choose some $\chi\in C_c^\infty(\mathbb{R})$ with $\chi(s) = 1$ then s is close to 0. Then

$$\int_0^\infty e^{ist}\chi(s)a(t,s)s^{n-1}\,ds$$

is smooth. And if $t \neq 0$, then for arbitrary $N \in \mathbb{N}$ one has

$$\begin{aligned}\int_0^\infty e^{ist}(1-\chi(s))a(t,s)s^{n-1}\,ds &= \frac{1}{t^N}\int_0^\infty t^N e^{ist}(1-\chi(s))a(t,s)s^{n-1}\,ds \\ &= \frac{1}{t^N}\int_0^\infty e^{ist}(-D_s)^N(1-\chi(s))a(t,s)s^{n-1}\,ds\end{aligned}$$

exhibits $\operatorname{Tr}(Q)(t)$ as a C^k function of t once $N > n+k+1$ since the amplitude is bounded by $c(1-\chi(s))s^{-k-2}$ then. □

We now write a global formula still valid only for small t. The subbundle of $T^*(\mathcal{M}\times\mathcal{M}) \approx \bigsqcup_{x,x'} T^*_x\mathcal{M}\oplus T^*_{x'}\mathcal{M}$ conormal to $\operatorname{diag}(\mathcal{M}\times\mathcal{M})$, denoted Λ_0 for convenience, is

$$\{(\boldsymbol{\xi}\oplus\boldsymbol{\eta}) \in T^*_{(x,x)}(\mathcal{M}\times\mathcal{M}) : x\in\mathcal{M},\ \iota^*(\boldsymbol{\xi}+\boldsymbol{\eta})=0\}$$

where $\iota : \mathcal{M}\to\mathcal{M}\times\mathcal{M}$ is the diagonal map. Thus

$$\Lambda_0 = \{(\boldsymbol{\xi}\oplus(-\boldsymbol{\xi})) \in T^*_{(x,x)}(\mathcal{M}\times\mathcal{M}) : x\in\mathcal{M},\ \boldsymbol{\xi}\in T^*_x\mathcal{M}\}$$

Pick an auxiliary Riemannian metric g on $T^*\mathcal{M}$. The restriction to Λ_0 of the exponential map of $T^*(\mathcal{M}\times\mathcal{M})$ is

$$T^*_x\mathcal{M} \ni \boldsymbol{\xi} \mapsto (\exp_x\boldsymbol{\xi}, \exp_x(-\boldsymbol{\xi})) \in \mathcal{M}\times\mathcal{M}$$

using $\boldsymbol{\xi}$ in place of $(\boldsymbol{\xi}\oplus(-\boldsymbol{\xi}))$ to identify Λ_0 with $T^*\mathcal{M}$. This is a diffeomorphism of a neighborhood $\mathcal{B}$ of 0 in $T^*\mathcal{M}$ to a neighborhood $\mathcal{W}$ of the diagonal; for the sake of concreteness pick $\mathcal{B}$ as the bundle of balls radius equal to half the injectivity radius. Let $\delta(x,x') : \mathcal{W}\to\mathcal{B}$ be the inverse of $\boldsymbol{\eta}\mapsto(\exp_y\boldsymbol{\eta}/2, \exp_y(-\boldsymbol{\eta}/2))$. This map plays the role of $x-x'$ in the coordinate version: if the metric is Euclidean, the first map is

$$(y,\eta)\mapsto(x,x') = (y+\eta/2, y-\eta/2)$$

and the inverse is $x-x'=\eta$ (omitting $y=(x+x')/2$). Define

$$\delta^*T^*\mathcal{M} = \{(x,x',\xi)\in\mathcal{W}\times T^*\mathcal{M} : \xi\in T^*_y\mathcal{M},\ y=\pi\delta(x,x')\}$$

where $\pi : T^*\mathcal{M}\to\mathcal{M}$ is the canonical projection. Define $\phi_0 : \delta^*T^*\mathcal{M}\to\mathbb{R}$ by

$$\phi_0(x,x',\boldsymbol{\xi}) = g(\delta(x,x'),\boldsymbol{\xi}). \tag{2.2}$$

In this formula, $\delta(x,x')\in T^*_y\mathcal{M}$ and thus $\boldsymbol{\xi}\in T^*_y\mathcal{M}$, so the pairing is well defined. This function plays the role of $(x-x')\cdot\xi$. (For more on global representations of pseudodifferential operators see Widom [**37**], Pflaum [**25**].) The phase ϕ, the solution of the problem

$$\begin{cases} d_t\phi - \boldsymbol{\sigma}(P)(d_x\phi) = 0 \\ \phi|_{t=0} = \phi_0 \end{cases}$$

has the form

$$\phi_0(x,x',\boldsymbol{\xi}) + t\,\boldsymbol{\sigma}(P)(d\phi_0(x,x',\boldsymbol{\xi})) + t^2\phi_2(t,x,x',\boldsymbol{\xi}).$$

Along the diagonal, $\phi_0=0$ and $\boldsymbol{\sigma}(P)(d\phi_0(x,x',\boldsymbol{\xi})) = \boldsymbol{\sigma}(P)(\boldsymbol{\xi})$, which gives

$$\phi(t,x,x,\boldsymbol{\xi}) = tp(\boldsymbol{\xi}) + t^2\phi_2(x,x,\boldsymbol{\xi}),\ x=\pi(\boldsymbol{\xi}).$$

Let

$$\psi(t,\boldsymbol{\xi}) = \boldsymbol{\sigma}(P)(\boldsymbol{\xi}) + t\phi_2(x,x\boldsymbol{\xi}).$$

The global statement is:

$$\operatorname{Tr} Q(t) = \frac{1}{(2\pi)^n} \int_{T^*\mathcal{M}} e^{t\psi(t,\boldsymbol{\xi})} q(\boldsymbol{\xi})\, d\mathcal{L}(\xi)$$

where $q = 1 + q_{-1}$ where q_1 is a symbol of order -1. Taking advantage of this last fact,

$$\operatorname{Tr} Q(t) = \frac{1}{(2\pi)^n} \int_{T^*\mathcal{M}} e^{t\psi(t,\boldsymbol{\xi})}\, d\mathcal{L}(\xi) + \epsilon(t)$$

where $\epsilon(t)$ is smooth when $t \neq$ and a weaker singularity at 0 than the main term.

3. The asymptotic formula

Let $\Sigma = \{\sigma_k\}_{k=0}^{\infty}$ be a nondecreasing sequence of positive numbers such that

$$N_\Sigma(\sigma) = \{\nu \in \mathbb{N}_0 : \sigma_\nu < \sigma\}$$

is polynomially bounded. Equivalently, using the Heaviside function,

$$N_\Sigma(\sigma) = \sum_{\nu=0}^{\infty} H(\sigma - \sigma_\nu)$$

is polynomially bounded, which implies that N_Σ is a Schwartz distribution, and so is its derivative,

$$\frac{dN_\Sigma}{d\sigma}(\sigma) = \sum_{\nu=0}^{\infty} \delta(\sigma - \sigma_\nu).$$

Consequently

$$\sum_{\nu=0}^{\infty} e^{it\sigma_\nu}.$$

converges as a Schwartz distribution.

Let $\psi : (-\delta, \delta) \times \dot{T}^*\mathcal{M} \to \mathbb{R}$ be smooth, strictly positive, homogeneous of degree 1 in $\boldsymbol{\xi}$ ($\psi(t, \tau\boldsymbol{\xi}) = \tau\psi(t, \boldsymbol{\xi})$ if $\tau > 0$), let $d\mathcal{L}$ be the Liouville density.

THEOREM 3.1 (Hörmander, [**12**]). *Suppose*

$$\sum_{\nu=0}^{\infty} e^{it\sigma_\nu} = \frac{1}{(2\pi)^n} \int_{T^*\mathcal{M}} e^{it\psi(t,\boldsymbol{\xi})}\, d\mathcal{L}(\boldsymbol{\xi}) + \epsilon(t) \qquad \text{if } |t| < \delta \tag{3.2}$$

where ϵ is a distribution defined in $(-\delta, \delta)$ which is smooth for $t \neq 0$ and satisfies

$$\widehat{\rho\epsilon}(\sigma) = \mathcal{O}(\sigma^{n-2})$$

for any $\rho \in C^\infty(\mathbb{R})$ with support contained in $(-\delta, \delta)$. Then

$$N_\Sigma(\sigma) = \frac{\operatorname{Vol}(\mathcal{B}_{\psi,0})}{(2\pi)^n} \sigma^n + \mathcal{O}(\sigma)^{n-1}$$

where

$$\mathcal{B}_{\psi,t} = \{\boldsymbol{\xi} \in T^*\mathcal{M} : \psi(t, \boldsymbol{\xi}) \leq 1\},$$

PROOF, TAKEN FROM [**11**]. We begin by introducing a change of coordinates in the integral in (3.2) similar to spherical (or rather cylindrical) coordinates. Let $\partial\mathcal{B}_{\psi,t}$ be the boundary of $\mathcal{B}_{\psi,t}$,

$$\partial\mathcal{B}_{\psi,t} = \{\boldsymbol{\xi} \in T^*\mathcal{M} : \psi(t, \boldsymbol{\xi}) = 1\}$$

Define $\wp_t : \mathbb{R}^+ \times \partial\mathcal{B}_{\psi,t} \to T^*\mathcal{M}$ by $\wp_t(s,\tilde{\boldsymbol{\xi}}) = s\tilde{\boldsymbol{\xi}}$. The Liouville density $d\mathcal{L}$ transforms to

$$\wp_t^*(d\mathcal{L}) = s^{n-1}|ds \wedge d\Theta|$$

where $d\Theta$ is surface measure on $\partial\mathcal{B}_{\psi,t}$. To clarify this, choose local coordinates $x^1, \dots, x^n$, let $\xi_1, \dots, \xi_n$ be the dual coordinates, so $\boldsymbol{\xi} = \sum_j \xi_j dx^j$. Then

$$\partial\mathcal{B}_{\psi,t} = \{(x,\xi) : \psi(t,x,\xi) = 1\}.$$

Write ϑ_j for the restriction of ξ_j to $\partial\mathcal{B}_{\psi,t}$. These functions together with the x^j provide local coordinates on $\partial\mathcal{B}_{\psi,t}$. The map $\wp_t$ in these coordinates (and s) is

$$\wp_t(s,x,\vartheta) = (x^1, \dots, x^n, s\vartheta_1, \dots, s\vartheta_n)$$

and thus

$$\begin{aligned}\wp^*(d\mathcal{L}) &= |dx^1 \wedge \cdots \wedge dx^n \wedge (\vartheta_1 ds + s d\vartheta_1) \wedge \cdots \wedge (\vartheta_n ds + s d\vartheta_n)| \\ &= |dx^1 \wedge \cdots \wedge dx^n \wedge \big(s^{n-1} ds \wedge \sum (-1)^{j+1} \vartheta_j d\vartheta_1 \wedge \cdots \wedge \widehat{d\vartheta_j} \wedge \cdots \wedge d\vartheta_n\big)|\end{aligned}$$

where as usual the caret means to omit that term. This formula says

$$d\Theta = |dx^1 \wedge \cdots \wedge dx^n \wedge \big(\sum (-1)^{j+1} \vartheta_j d\vartheta_1 \wedge \cdots \wedge \widehat{d\vartheta_j} \wedge \cdots \wedge d\vartheta_n\big)|$$

in coordinates. (When $\psi(t,\boldsymbol{\xi}) = (\sum \xi_i^2)^{1/2}$, the part in parenthesis gives the density, the element of surface area, of the sphere.)

With this change of variables, the integral is

$$\int_0^\infty \int_{\partial\mathcal{B}_{\psi,t}} s^{n-1} e^{ist}\, d\Theta\, ds$$

in which the inner integral produces a smooth function of t,

$$\int_{\partial\mathcal{B}_{\psi,t}} d\Theta = \int_{\partial\mathcal{B}_{\psi,0}} d\Theta + t\upsilon(t) = \mathrm{Vol}(\partial\mathcal{B}_{\psi,0}) + t\upsilon(t).$$

Hence

$$\int_0^\infty \int_{\partial\mathcal{B}_{\psi,t}} s^{n-1} e^{ist}\, d\Theta\, ds = \mathrm{Vol}(\partial\mathcal{B}_{\psi,0}) \int_0^\infty s^{n-1} e^{ist}\, ds + t\upsilon(t) \int_0^\infty s^{n-1} e^{ist}\, ds.$$

Now (3.2) reads

$$\sum_{\nu=0}^\infty e^{it\sigma_\nu} = \frac{\mathrm{Vol}(\partial\mathcal{B}_{\psi,0})}{(2\pi)^n} \int_0^\infty s^{n-1} e^{ist}\, ds + \epsilon(t) \quad \text{if } |t| < \delta \tag{3.2$'$}$$

with a different function ϵ however with the same properties of smoothness and regularity because of the factor t.

Pick some $\rho \in C_c^\infty(\mathbb{R})$ supported in $(-\delta,\delta)$ with $\int \hat\rho(t)\, dt = 1$. Keep in mind that $\hat\rho$ is a Schwartz function. Multiply both sides of (3.2$'$) by ρ and take the Fourier transform. Get

$$\sum_{\nu=0}^\infty \hat\rho(\sigma - \sigma_\nu) = \frac{\mathrm{Vol}(\partial\mathcal{B}_{\psi,0})}{(2\pi)^n} (s_+^{n-1} * \hat\rho)(\sigma) + \widehat{\rho\epsilon}(\sigma) \tag{3.3}$$

where $s_+^{n-1} = s^{n-1} H(s)$.

Since s_+^{n-1} vanishes when $s<0$, $s_+^{n-1}*\hat\rho$ is rapidly decreasing as $\sigma\to-\infty$ (we do not need this). To estimate the convolution when $\sigma>0$, we proceed as follows.

$$\begin{aligned}(s_+^{n-1}*\hat\rho)(\sigma)&=\int_{-\infty}^{\sigma}(\sigma-s)^{n-1}\hat\rho(s)\,ds\\&=\sigma^{n-1}\int_{-\infty}^{\sigma}\hat\rho(s)+\sum_{k=0}^{n-2}c_{n-1,k}\,\sigma^k\int_{-\infty}^{\sigma}s^{n-k-1}\hat\rho(s)\,ds\end{aligned}$$

with some constants $c_{n-1,k}$. We have

$$\sigma^{n-1}\int_{-\infty}^{\sigma}\hat\rho(s)\,ds=\sigma^{n-1}\int_{-\infty}^{\infty}\hat\rho(s)\,ds-\sigma^{n-1}\int_{\sigma}^{\infty}\hat\rho(s)\,ds=\sigma^{n-1}+\psi(\sigma)$$

with $\psi(\sigma)$ rapidly decreasing as $\sigma\to\infty$. Similarly we get

$$\big|\sigma^k\int_{-\infty}^{\sigma}s^{n-k-1}\hat\rho(s)\,ds\big|\le C\sigma^k$$

for $k=0,\dots,n-2$ and some C if $\sigma\ge0$. By hypothesis

$$|\widehat{\rho\epsilon}(\sigma)|\le C\sigma^{n-2}.$$

In conclusion, (3.3) gives (see [**11**, Ch 6, formula 8.11])

$$\sum_{\nu=0}^{\infty}\hat\rho(\sigma-\sigma_\nu)=\frac{\mathrm{Vol}(\partial\mathcal{B}_{\psi,0})}{(2\pi)^n}\sigma^{n-1}+\mathcal{O}(\sigma^{n-1})\text{ as }\sigma\to\infty.\tag{3.3$'$}$$

As was pointed out earlier, the left hand side is the derivative of $\hat\rho*N_\Sigma$, so

$$(\hat\rho*N_\Sigma)(\sigma)=\frac{\mathrm{Vol}(\partial\mathcal{B}_q)}{n(2\pi)^n}\sigma^n+\mathcal{O}(\sigma^{n-1})\text{ as }\sigma\to\infty\tag{3.4}$$

We now want to rewrite the left hand side as $N_\Sigma(\sigma)$ plus a "small" error: assuming $\int\hat\rho\,ds=1$ we have

$$N_\Sigma(\sigma)=(\hat\rho*N_\Sigma)(\sigma)+\int_{\mathbb{R}}\hat\rho(s)\big(N_\Sigma(\sigma)-N_\Sigma(\sigma-s)\big)\,ds.\tag{3.5}$$

For the first term on the right we already have a formula. To estimate the integral we will need an estimate in the difference in parenthesis. Here a more careful choice of ρ comes in.

To construct a convenient function ρ (see [**11**, p. 377]), choose $\rho_0\in C_c^\infty(\mathbb{R})$ non-negative, even, with $\rho_0^2(0)=1/2\pi$ and supported in $(-\delta/2,\delta/2)$, so $\int\rho_0\ne0$ and $\hat\rho_0$ is real-valued. Let $\rho=\rho_0*\rho_0$. Then

$$\mathrm{supp}(\rho)\subset(-\delta,\delta),\quad\int\hat\rho(s)\,ds=1,\quad\hat\rho=\hat\rho_0^2\ge0,\quad\hat\rho(0)=(\int\rho_0)^2>0.$$

The key estimate comes now:

Lemma 3.6. *There is c such that*

$$N_\Sigma(\sigma'')-N_\Sigma(\sigma')\le c(\sigma''+\sigma')^{n-1}(\sigma''-\sigma')$$

whenever $\sigma''\ge\sigma'\ge0$.

PROOF. Pick $\kappa > 0$ such that

$$|\hat\rho(\sigma)| > \hat\rho(0)/2 \quad \text{if } |\sigma| < \kappa/2. \tag{3.7}$$

For arbitrary σ, the first inequality in

$$\sum_{\nu=0}^{\infty} \hat\rho(\sigma-\sigma_\nu) \ge \sum_{\substack{\nu\in\mathbb{N}_0\\ |\sigma-\sigma_\nu|\le\kappa/2}} \hat\rho(\sigma-\sigma_\nu) \ge \frac{\hat\rho(0)}{2}\#\{\nu\in\mathbb{Z}^n : |\sigma-\sigma_\nu|\le\kappa/2\}$$

follows from the non-negativity of $\hat\rho$, the second from (3.7). In conjunction with (3.3′) this gives

$$\#\{\nu\in\mathbb{N}_0 : |\sigma-\sigma_\nu|\le\kappa/2\} \le c\sigma^{n-1}. \tag{3.8}$$

Let $m = \lceil(\sigma''-\sigma')/\kappa\rceil$ be the smallest integer greater than or equal to $(\sigma''-\sigma')/\kappa$, let $\kappa' = (\sigma''-\sigma')/m$. The numbers $\sigma_\ell = \sigma' + \ell\kappa'$ partition $[\sigma',\sigma'']$, $\ell = 0,\dots,m$ into m subintervals of length κ', with respective midpoints

$$\sigma' + (\ell+1/2)\kappa', \quad \ell = 0,\dots,m-1.$$

The estimate (3.8) gives the first inequality in

$$\begin{aligned}\#\{\nu\in\mathbb{N}_0 : \sigma_\nu\in[\sigma_\ell,\sigma_{\ell+1}]\} &\le c(\sigma'+(\ell+1/2)\kappa')^{n-1}\\ &\le c(\sigma'+\sigma'')^{n-1},\end{aligned}$$

the second just because $\sigma_\ell < \sigma''$ if $\ell < m$. Consequently

$$\begin{aligned}\#\{\nu\in\mathbb{N}_0 : \sigma_\nu\in[\sigma_\ell,\sigma_{\ell+1}]\} &\le cm(\sigma'+\sigma'')^{n-1}\\ &= c(\sigma'+\sigma'')^{n-1}(\sigma''-\sigma')\end{aligned}$$

which completes the proof of the lemma. □

If $\sigma' \le 0 \le \sigma$ then

$$N_\Sigma(\sigma) - N_\Sigma(\sigma') \le c\sigma^{n-1}$$

because $N_\Sigma(\sigma') = 0 = N_\Sigma(0)$.

We take advantage of the lemma by splitting the integral in (3.5) as

$$\int_{-\infty}^{\sigma} \hat\rho(s)\big(N_\Sigma(\sigma)-N_\Sigma(\sigma-s)\big)\,ds + \int_{\sigma}^{\infty} \hat\rho(s)\big(N_\Sigma(\sigma)-N_\Sigma(\sigma-s)\big)\,ds$$

The integrand in the first integral is bounded by

$$c\hat\rho(s)(2\sigma-s)^{n-1}|s| \le c_N\sigma^{n-1}(1+|s|)^{-N}, \quad s\le\sigma$$

while that in the second is bounded by

$$c\hat\rho(s)\sigma^{n-1} \le c_N\sigma^{n-1}(1+|s|)^{-N}, \quad s\ge\sigma$$

which gives the bound $c'\sigma^{n-1}$ for the sum of the two integrals. Using this in (3.5) gives

$$N_\Sigma(\sigma) = (\hat\rho * N_{P^{1/2}})(\sigma) + \mathcal{O}(\sigma^{n-1})$$

which replaced in (3.4) produces

$$N_\Sigma(\sigma) = \frac{\mathrm{Vol}(\partial\mathcal{B}_q)}{n(2\pi)^n}\sigma^n + \mathcal{O}(\sigma^{n-1})$$

But $\mathrm{Vol}(\partial\mathcal{B}_q)/n = \mathrm{Vol}(\mathcal{B}_q)$. □

CHAPTER V

Complex powers

1. Motivation

All this chapter is based on Seeley's paper [**29**]. The principal aim is to get the residue of the first pole of the zeta function of P. The Wiener-Ikehara theorem (see Theorem I.3.5 or [**38**, p. 44]) then gives the Weyl asymptotics of the operator. Seeley's paper is more general in that it handles pseudodifferential operators with positivity replaced by existence of a ray of minimal growth. By dealing only with non-negative elliptic differential operators we avoid some technical complications that would only obscure the technique.

As before $\mathcal{M}$ is a closed manifold with smooth density $\mathfrak{m}$ and $E \to \mathcal{M}$ is a Hermitian vector bundle. We let $P \in \mathrm{Diff}(\mathcal{M}; E)$ be a non-negative elliptic operator. From Proposition III.3.1 we know that

$$P : H^m(\mathcal{M}; E) \subset L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$$

is selfadjoint with discrete spectrum contained in $[0, \infty)$ consisting only of eigenvalues of finite multiplicity. Let $\{\lambda_k\}_{k=0}^{\infty}$ be the sequence of eigenvalues as a non-decreasing sequence with each eigenvalue repeated according to its multiplicity.

Lemma 1.1. *The series*

$$\zeta_P(s) = \sum_{\lambda_k > 0} \frac{1}{\lambda_k^s} \tag{1.2}$$

converges in $\{s \in \mathbb{C} : \Re s > n/m + \varepsilon\}$ *(for some* $\varepsilon > 0$*) to a holomorphic function.*

Proof. From Proposition III.4.1 we know that the eigenvalue counting function of P,

$$N_P(\lambda) = \#\{k : \lambda_k < \lambda\}$$

is bounded by $C\lambda^{n/m+\varepsilon}$ for $\lambda > 0$ (see (III.4.3)) for some C depending on ε. Evidently

$$\#\{k : \lambda_k < \lambda_{k_0}\} = k_0,$$

so for any k we have $k \leq C\lambda_k^{n/m+\varepsilon}$. This gives

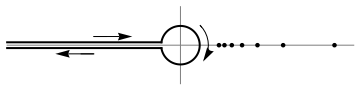
$$\frac{1}{\lambda_k} \leq \left(\frac{C}{k}\right)^{1/(n/m+\varepsilon)}$$

and so (1.2) converges absolutely and uniformly on compact subsets of $\mathbb{C}$ with $\Re s > n/m + \varepsilon$ for any $\varepsilon > 0$. □

Following Seeley [**29**], we extract more information about ζ_P using the formula

$$\zeta_P(s) = \operatorname{Tr} \frac{i}{2\pi} \int_\Gamma \lambda^{-s} (P - \lambda I)^{-1} \, d\lambda, \tag{1.3}$$

which is valid, as we shall see, if $m\Re s > n$, where Γ is the path obtained by traversing the negative real axis from $-\infty$ to $-\varepsilon$ ($\varepsilon > 0$ suitably small), followed once by the clockwise oriented path $|\lambda| = \varepsilon$, then back to $-\infty$ along the negative semi-axis. The number ε is small enough that it does not enclose any element of $\operatorname{spec}(P)$ except possibly 0. The function $\lambda \mapsto \lambda^s$ is defined in the complement of the real negative semiaxis using the principal branch of the logarithm. Along the first part of the path ($\lambda < 0$), $\lambda^{-s} = |\lambda|^{-s}e^{-is\pi}$ and along the third, $\lambda^{-s} = |\lambda|^{-s}e^{is\pi}$. The motivation for the formula comes from



$$\eta^{-s} = \frac{i}{2\pi}\int_\Gamma \lambda^{-s}(\eta-\lambda)^{-1}\,d\lambda$$

if $\eta \in (\varepsilon, \infty]$, $\Re s > 0$. Used with

$$(P-\lambda I)^{-1} = \sum_{\eta\in\operatorname{spec}(P)} \frac{1}{\eta-\lambda}\Pi_\eta,$$

where $\Pi_\eta : L^2(\mathcal{M};E) \to L^2(\mathcal{M};E)$ is the orthogonal projection on $\mathcal{E}_\eta$ and proceeding formally we get

$$\operatorname{Tr}(P-\lambda I)^{-1} = \sum_{\eta\in\operatorname{spec}(P)} \frac{1}{\eta-\lambda}\operatorname{Tr}\Pi_\eta,$$

from which one gets the relation between $\zeta_P(s)$ an the trace of

$$P^{-s} = \frac{i}{2\pi}\int_\Gamma \lambda^{-s}(P-\lambda I)^{-1}\,d\lambda. \tag{1.4}$$

Because of the estimate in Proposition III.3.4, this integral converges in norm if $\Re s > 0$. We will see later that P^{-s} is a pseudodifferential operator of order $-m\Re s$. Given this, it is trace class when its Schwartz kernel is, which by Lemma VIII.4.1 will happen if its order is less than $-n$. Thus P^{-s} is trace class for sure if $\Re s > n/m$.

Incidentaly, the factor i in the numerator in (1.4) takes into account that we wrote $(P-\lambda I)^{-1}$ instead of $(\lambda I - P)^{-1}$.

2. Complex powers of P

Since the aim is to count eigenvalues, we may assume that P is strictly positive by replacing it with $P+I$, for instance, if 0 is an eigenvalue.

The simplest way to define the integral (1.4) is by appealing to the theory of line integrals for holomorphic functions. Recall form Proposition III.3.4 that

$$\|(P-\lambda I)^{-1}u\|_0 \le \frac{c_1'}{|\lambda|}\|u\|_0, \quad u\in L^2(\mathcal{M};E), \quad \lambda\in\Sigma_{c_0}$$

for any c_0 and some $c_1' = 1/c_1$. The region Σ_{c_0} was defined in (III.1.5). If $u, v \in L^2(\mathcal{M};E)$, then

$$\lambda \mapsto \lambda^{-s}\big((P-\lambda I)^{-1}u, v\big)$$

is a holomorphic function in $\mathbb{C}\backslash(\operatorname{spec}(P)\cup\overline{\mathbb{R}}_-)$, bounded by

$$|\lambda^{-s}\big((P-\lambda I)^{-1}u, v\big) \le c_1'|\lambda|^{-\Re s-1}\|u\|_0\|v\|_0.$$

in Σ_{c_0}. Therefore: if $\Re s > 0$, then the line integral

$$\frac{i}{2\pi}\int_\Gamma \lambda^{-s}\big((P-\lambda I)^{-1}u, v\big)\,d\lambda$$

with Γ as in the previous section converges absolutely and is bounded by

$$C_s\|u\|_0\|v\|_0$$

with

$$C_s = c\int_\Gamma |\lambda|^{-\Re s-1}\,|d\lambda|.$$

In particular, the functional

$$L^2(\mathcal{M};E)\ni v\mapsto \frac{i}{2\pi}\int_\Gamma \lambda^{-s}\big((P-\lambda I)^{-1}u,v\big)\,d\lambda\in\mathbb{C}$$

is linear continuous. Via the Riesz representation theorem it defines an element of $L^2(\mathcal{M};E)$, naturally denoted $P^{-s}u$. This element depends linearly on u and satisfies

$$\|P^{-s}u\|_0\le C_s\|u\|_0.$$

Thus we do have a continuous operator $P^{-s}: L^2(\mathcal{M};E)\to L^2(\mathcal{M};E)$ whose norm is $\le C_s$. Alternatively one can use that

$$\lambda\mapsto \lambda^{-s}(P-\lambda I)^{-1}u\in L^2(\mathcal{M};E)$$

is continuous and the extension of the theory of Riemann integrals (in this case, improper) to continuous functions with values in a Hilbert (or Banach) space.

In any case, the definition of P^{-s} as an integral works only when $\Re s>0$. It gives a family of operators depending holomorphically on s (has a Taylor expansion around any s_0 with $\Re s_0>0$ with continuous operator coefficients that converges in operator norm if $|s-s_0|$ is small enough).

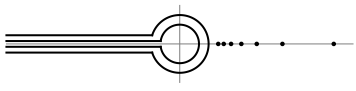
The following proposition justifies the notation, but more importantly, it will allow for the extension of the definition of P^{-s} to arbitrary $s\in\mathbb{C}$. Recall that we are assuming that P is invertible.

PROPOSITION 2.1. *If $s,s'\in\mathbb{C}$ have positive real part, then*

$$P^{-s}\circ P^{-s'}=P^{-s-s'}.$$

If $\Re s>1$, then $P\circ P^{-s}=P^{-s+1}$. If s is a positive integer, then P^{-s} is the s-th power of the inverse of P.

PROOF. Because of the holomorphicity, the precise path used to define P^{-s} is immaterial, only its general behavior matters. Keep Γ as the original path and let and let Γ' be similarly described as follows. Choose $\varepsilon'>\varepsilon$ but $<\min\operatorname{spec}(P)$, pick δ positive $<\varepsilon'$. The path Γ' consist of the segment parallel to the real axis in the second quadrant with imaginary part δ until meeting the circle of radius ε', then the arc of this circle traversed clockwise until it meets the line segment $\Im z=-\delta$ in the third quadrant, and from there to $-\infty+i\delta$ parallel to the real axis.



To get a formula for $(P^{-s}\circ P^{-s'})(u)$, $u\in L^2(\mathcal{M};E)$ use the definition of then integral

$$\frac{i}{2\pi}\int_{\Gamma'}\mu^{-s}(P-\mu I)^{-1}(u)\,d\mu$$

in the previous section to deduce

$$(P-\lambda I)^{-1}\Big(\int_{\Gamma'}\mu^{-s'}(P-\mu I)^{-1}(u)\,d\mu\Big) \\ = \int_{\Gamma'}\mu^{-s'}(P-\lambda I)^{-1}\circ(P-\mu I)^{-1}(u)\,d\mu.$$

This gives

$$(P^{-s}\circ P^{-s'}) = \Big(\frac{i}{2\pi}\Big)^2\int_\Gamma\int_{\Gamma'}\lambda^{-s}\mu^{-s'}(P-\lambda I)^{-1}\circ(P-\mu I)^{-1}\,d\mu\,d\lambda.$$

Once we reach this conclusion the order of integration becomes immaterial. Now use

$$\begin{aligned}(P-\lambda I)^{-1}-(P-\mu I)^{-1} &= (P-\lambda I)^{-1}\circ(P-\mu I-(P-\lambda I))\circ(P-\mu I)^{-1}\\ &= (\lambda-\mu)(P-\lambda I)^{-1}\circ(P-\mu I)^{-1}\end{aligned}$$

to obtain

$$(P^{-s}\circ P^{-s'}) = \Big(\frac{i}{2\pi}\Big)^2\int_\Gamma\int_{\Gamma'}\frac{\lambda^{-s}\mu^{-s'}}{\lambda-\mu}(P-\lambda I)^{-1}\,d\mu\,d\lambda \\ -\Big(\frac{i}{2\pi}\Big)^2\int_{\Gamma'}\int_\Gamma\frac{\lambda^{-s}\mu^{-s'}}{\lambda-\mu}(P-\mu I)^{-1}\,d\lambda\,d\mu. \quad (2.2)$$

Rewrite the first term on the right as

$$\Big(\frac{i}{2\pi}\Big)^2\int_\Gamma\Big[\int_{\Gamma'}\frac{\mu^{-s'}}{\lambda-\mu}\,d\mu\Big]\lambda^{-s}(P-\lambda I)^{-1}\,d\lambda$$

and observe that in the integral in brackets we can change the path of integration, Γ', to a new one, Γ_r, defined in the same way as γ' but with r in place of ε', still $r>\varepsilon$, because λ is constrained to lie in the path Γ; the spectrum of P does not obstruct this deformation. Using the estimate

$$\Big|\frac{\mu^{-s'}}{\lambda-\mu}\Big| \le C|\mu|^{-\Re s'-1}$$

with fixed λ and letting $r\to\infty$ we conclude that this integral vanishes, since $\Re s'>0$ and $|\mu|>r$ on Γ_r.

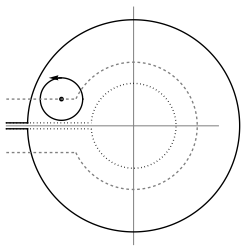
Rewrite the second term on the right hand side of (2.2), including the sign, as

$$-\Big(\frac{i}{2\pi}\Big)^2\int_{\Gamma'}\Big[\int_\Gamma\frac{\lambda^{-s}}{\lambda-\mu}\,d\lambda\Big]\mu^{-s'}(P-\mu I)^{-1}\,d\mu$$

Here the integral in brackets is equal to

$$\oint_{|\lambda-\mu|=\rho}\frac{\lambda^{-s}}{\lambda-\mu}\,d\lambda+\int_{\Gamma''}\frac{\lambda^{-s}}{\lambda-\mu}\,d\lambda$$

where Γ'' is defined as Γ was, but with parameter $\varepsilon''>2|\mu|$ so that μ and the circular (counterclockwise oriented) path $|\lambda-\mu|=\rho$ with $0<\rho<\mu$ lies in the interior of the circular part of Γ''. The second integral is then 0 by the same argument as above, while the



first is

$$\oint_{|\lambda-\mu|=\rho} \frac{\lambda^{-s}}{\lambda-\mu}\,d\lambda = 2\pi i \mu^{-s}$$

because the circle is contained in the complement of the negative real axis. Thus

$$\begin{aligned}
-\Big(\frac{i}{2\pi}\Big)^2 \int_{\Gamma'} \Big[\int_{\Gamma} \frac{\lambda^{-s}}{\lambda-\mu}\,d\lambda\Big]\mu^{-s'}(P-\mu I)^{-1}\,d\mu \\
&= \frac{i}{2\pi}\int_{\Gamma'} \Big[\frac{1}{2\pi i}\oint_{|\lambda-\mu|=\rho} \frac{\lambda^{-s}}{\lambda-\mu}\,d\lambda\Big]\mu^{-s'}(P-\mu I)^{-1}\,d\mu \\
&= \frac{i}{2\pi}\int_{\Gamma'} \big[\mu^{-s}\big]\mu^{-s'}(P-\mu I)^{-1}\,d\mu = \frac{i}{2\pi}\int_{\Gamma'} \mu^{-s-s'}(P-\mu I)^{-1}\,d\mu.
\end{aligned}$$

In conclusion, $P^{-s}\circ P^{-s'} = P^{-s-s'}$.

To prove the second assertion of the proposition we use

$$\begin{aligned}
\lambda^{-s}P\circ(P-\lambda I)^{-1} &= \lambda^{-s}(P-\lambda I)\circ(P-\lambda I)^{-1} + \lambda^{-s+1}(P-\lambda I)^{-1} \\
&= \lambda^{-s} I + \lambda^{-s+1}(P-\lambda I)^{-1}
\end{aligned}$$

and argue that because $|\lambda^{-s}| < |\lambda|^{-\Re s}$ and $\Re s > 1$,

$$\int_{\Gamma} \lambda^{-s}\,d\lambda = 0$$

by the same argument as in the first part of the proof.

If s is a positive integer, the limits

$$\lim_{\substack{\lambda\to\lambda_0\\ \Im\lambda>0}} \lambda^{-s}, \quad \lim_{\substack{\lambda\to\lambda_0\\ \Im\lambda<0}} \lambda^{-s}$$

with $\lambda_0 < 0$ are equal. This gives a cancellation in the integral defining P^{-s} to

$$P^{-s} = \frac{i}{2\pi}\oint_{|\lambda|=\varepsilon} \lambda^{-s}(P-\lambda I)^{-1}\,d\lambda.$$

If $s = 1$, this evaluates $(P-\lambda)^{-1}$ at $\lambda = 0$. (Again, keep in mind that P is invertible). If $s > 1$ (integer), we use the second statement of the proposition. □

We use the second assertion of the proposition to extend the defintion of P^s to the whole complex plane.

COROLLARY 2.3. *The family $s \mapsto P^{-s}$ has an extension as en entire map.*

PROOF. Let k be a positive integer and let $\Omega_k = \{s : \Re s > -k+1\}$. The map $s \mapsto P^{-s-k}$ is defined and holomorphic for $s \in \Omega_k$. Then also the new map $s \mapsto P^{-s}_{(k)} = P^k P^{-s-k}$ is holomorphic in Ω_k. By the proposition, if $\Re(s-k) > 1$, then $P^k P^{-s-k} = P^{-s}$. That is, $P^{-s}_{(k)}$ is a holomorphic extension of P^{-s}. We extend P^{-s} to Ω_k defining $P^{-s} = P^{-s}_{(k)}$. □

3. The resolvent as a pseudodifferential operator

If $P \in \mathrm{Diff}^m(\mathcal{M}; E)$ is elliptic then any of its parametrices is a pseudodifferential operator (in $\Psi^{-m}(\mathcal{M}; E)$). In particular, the resolvent of $(P - \lambda I)$ is a pseudodifferential operator. To construct a good parametrix we need to investigate $(P - \lambda I)$ in more detail.

We fix once and for all an auxiliary Riemannian metric on $T^*\mathcal{M}$ and write $|\boldsymbol{\xi}|_x$, or just $|\xi|$, for the norm of $\boldsymbol{\xi} \in T^*_x\mathcal{M}$.

First, $\boldsymbol{\sigma}(P)(\boldsymbol{\xi}_0) : E_{x_0} \to E_{x_0}$, $x_0 = \pi(\boldsymbol{\xi}_0)$, is selfadjoint non-negative for each $\boldsymbol{\xi}_0 \in \dot{T}^*\mathcal{M}$. We prove this using (VII.1.9). Pick $f \in C^\infty(\mathcal{M})$ real-valued, $df(x_0) = \boldsymbol{\xi}_0$. Given $u \in C^\infty(\mathcal{M}; E)$ with support on the set where df does not vanish, the positivity of P implies

$$(Pe^{i\tau f}u, e^{i\tau f}u) \geq 0.$$

But

$$\begin{aligned}\frac{1}{\tau^m}(Pe^{i\tau f}u, e^{i\tau f}u) &= (\frac{1}{\tau^m}e^{-i\tau f}Pe^{i\tau f}u, u)\\ &\to (\boldsymbol{\sigma}(P)(df)(u), u) \text{ cuando } \tau \to \infty,\end{aligned}$$

so

$$(\boldsymbol{\sigma}(P)(df)(u), u) \geq 0.$$

Now choose local coordinates $x^1, \dots, x^n$ centered at x_0, say, on $\mathcal{U}$ and $\chi \in C^\infty_c(\mathbb{R})$ with $\chi(0) = 1$. In place of u tale $k^{n/2}e^{-k^2|x|^2}u$ with $\nu \in \mathbb{N}$. Writing the density as $d\mathfrak{m} = \mathfrak{m}_0(x)\,dx$ we get

$$\begin{aligned}&\big(\boldsymbol{\sigma}(P)(df)(k^{n/2}e^{-k^2|x|^2}u), k^{n/2}e^{-k^2|x|^2}u\big)\\ &\quad= \int_{\mathcal{M}} \big(\boldsymbol{\sigma}(P)(df(x))(k^{n/2}e^{-k^2|x|^2}u(x)), k^{n/2}e^{-k^2|x|^2}u(x)\big)_x \mathfrak{m}_0(x)\,dx\\ &\quad= \int_{\mathcal{M}} k^n e^{-2k^2|x|^2}\big(\boldsymbol{\sigma}(P)(df(x))(u(x)), u(x)\big)_x \mathfrak{m}_0(x)\,dx\\ &\quad= \int_{\mathbb{R}^n} \big(e^{-2|x|^2}\,\boldsymbol{\sigma}(P)(df(x/k))(u(x/k)), u(x/k)\big)_{x/k} \mathfrak{m}_0(x/k)\,dx\end{aligned}$$

after a change of variables. The limit as $k \to \infty$ is

$$\big(\boldsymbol{\sigma}(P)(df(x_0))(u(x_0)), u(x_0)\big)_{x_0} \mathfrak{m}(x_0) \int_{\mathbb{R}} e^{-|x|^2}\,dx$$

Thus

$$\big(\boldsymbol{\sigma}(P)(\boldsymbol{\xi}_0)(w), w\big)_{x_0} \geq 0.$$

where $w = u(x_0) \in E_{x_0}$ is arbitrary.

This immediately implies that $(\boldsymbol{\sigma}(P)(\xi) - \lambda I) \in \mathrm{Hom}(E_x)$ is invertible when $\lambda \in \Sigma_{c_0}$. We can in fact get better information.

LEMMA 3.1. *Let Σ_{c_0} be defined as in* (1.5)*, pick some Riemannian metric on $T^*\mathcal{M}$, write $|\boldsymbol{\xi}|$ for the norm of $\boldsymbol{\xi}$. There is c_2 such that*

$$\begin{aligned}&\text{for all } \boldsymbol{\xi} \in \dot{T}^*\mathcal{M} \text{ and } \lambda \in \Sigma_{c_0} :\ |(\boldsymbol{\sigma}(P)(\boldsymbol{\xi}) - \lambda I)(w)|^2_x \geq c_2(|\boldsymbol{\xi}|^{2m} + |\lambda|^2)|w|^2_x\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } w \in E_x \to E_x,\ x = \pi(\boldsymbol{\xi}).\end{aligned}$$

PROOF. Fix $\boldsymbol{\xi}$, write $\boldsymbol{\sigma}(P)(\boldsymbol{\xi}) = p$. As we just saw, $p : E_x \to E_x$ is selfadjoint and non-negative: there is an orthonormal basis $e_1, \dots, e_r$ of E_x and positive numbers $\varrho_1, \dots, \varrho_r$ such that $p(e_j) = \varrho_j e_j$. Let π_j be the orthogonal projection on e_j. Let $\lambda = \alpha + i\beta$ with α and β real. With arbitrary $w \in E_x$ we have

$$|p(w) - \lambda w|_x^2 = \sum_j \big((\varrho_j - \alpha)^2 + \beta^2\big)|\pi_j(w)|_x^2$$

To get a bound from below we show that for a general $\varrho > 0$ there is $\kappa > 0$ depending only on c_0 such that

$$(\varrho - \alpha)^2 + \beta^2 \geq \kappa(\varrho^2 + \alpha^2 + \beta^2) \quad \text{for all } \alpha + i\beta \in \Sigma_{c_0}.$$

To see this let

$$F(\alpha, \beta) = \frac{(\varrho - \alpha)^2 + \beta^2}{\varrho^2 + \alpha^2 + \beta^2} \tag{3.2}$$

This is a smooth function of $(\alpha, \beta) \in \mathbb{R}^2$ (since $\varrho > 0$). It has critical points only at $(\varrho, 0)$ and $(-\varrho, 0)$; the former is a minimum (in the complement of Σ_{c_0}), the latter is a maximum. This implies that the minimum of F on the closure of Σ_{c_0} is attained on the boundary of Σ_{c_0} (or the infimum, 1, at ∞). Replacing $\beta = \pm c_0\alpha$ in F and differentiating with respect to α we get an expression whose numerator is

$$2\varrho((1 + c_0^2)\alpha^2 - \varrho^2)$$

This vanishes when $\alpha_0 = \varrho/\sqrt{1 + c_0^2}$ (the negative solution does not correspond to a point on $\partial\Sigma_{c_0}$). That this is a minimum is verified using the second derivative test. Replacing (α_0, β_0), $\beta_0 = \pm c_0\alpha_0$ in (3.2) yields

$$F(\alpha_0, \beta_0) = \frac{1 + c_0^2 - \sqrt{1 + c_0^2}}{1 + c_0^2} = \kappa$$

(Since $\kappa < 1$, this is the absolute minimum value) Consequently

$$|p(w) - \lambda w|_x^2 \geq \kappa \sum_j (\varrho_j^2 + |\lambda|^2)|\pi_j w|_x^2.$$

For any $\boldsymbol{\xi}$ let $\varrho_{\min}(\boldsymbol{\xi})$ be the smallest eigenvalue of $\boldsymbol{\sigma}(P)(\boldsymbol{\xi})$. This is a continuous function of $\boldsymbol{\xi}$, homogeneous of degree m, strictly positive for $\boldsymbol{\xi} \neq 0$. With the chosen Riemanniann metric on $T^*\mathcal{M}$, pick c between 0 and 1 such that $c|\boldsymbol{\xi}|^m \leq \varrho_{\min}(\boldsymbol{\xi})$. Then

$$|\,\boldsymbol{\sigma}(P)(\boldsymbol{\xi})(w) - \lambda w|_x^2 \geq c\kappa(|\boldsymbol{\xi}|^{2m} + |\lambda|^2)|w|_x.$$

This completes the proof of the lemma. □

The mapping

$$T^*\mathcal{M} \times \Sigma_{c_0} \ni (\boldsymbol{\xi}, \lambda) \mapsto (\boldsymbol{\sigma}(P)(\boldsymbol{\xi}) - \lambda I)$$

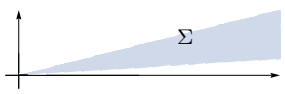
is smooth, anisotropically homogeneous of degree m: If $\tau > 0$, then

$$(\boldsymbol{\sigma}(P)(\tau\boldsymbol{\xi}) - \tau^m\lambda I) = \tau^m(\boldsymbol{\sigma}(P)(\boldsymbol{\xi}) - \lambda I)$$

because $(\boldsymbol{\xi}, \lambda) \mapsto (\boldsymbol{\sigma}(P)(\boldsymbol{\xi}) - \lambda I)$ is of degree m. Rather than working with this anisotropy, we will replace λ by υ^m. The function

$$\boldsymbol{\sigma}(P)(\boldsymbol{\xi}) - \upsilon^m I \quad \text{on } T^*\mathcal{M} \times \mathbb{C}$$

is then smooth (m is a positive integer), homogeneous of degree m for the action $\tau\cdot(\boldsymbol{\xi}, \upsilon) = (\tau\boldsymbol{\xi}, \tau\upsilon)$. We will restrict υ to belong to the sector



$$\Sigma_{c_0}^{1/m} = \{\upsilon \in \mathbb{C} : \upsilon \neq 0,\ \theta_0/m < \arg(\upsilon) < (2\pi - \theta_0)/m\} \tag{3.3}$$

where θ_0 is smallest positive number with $re^{i\theta_0} = (1 + ic_0)$ to ensure $\upsilon^m \in \Sigma_{c_0}$ if $\upsilon \in \Sigma_{c_0}^{1/m}$. We will write Σ in place of $\Sigma_{c_0}^{1/m}$, the reference to c_0 and m will be omitted. The lemma then gives

$$c(|\boldsymbol{\xi}|^2 + \upsilon^2)^{m/2} \leq \big|\, \boldsymbol{\sigma}(P)(\boldsymbol{\xi}) - \upsilon^m I\big|_x,\ (\boldsymbol{\xi}, \upsilon) \in T^*\mathcal{M} \times \Sigma,\ x = \pi(\boldsymbol{\xi}).$$

where $|\cdot|_x$ denotes the operator norm in $\mathrm{End}(E_x)$, with some other constant c. The immediate consequence of the lemma is:

Corollary 3.4. *Define* $b_{-m} : T^*\mathcal{M} \times \Sigma \to \mathrm{End}(E)$ *by*

$$b_{-m}(\boldsymbol{\xi}, \upsilon) = \big(\, \boldsymbol{\sigma}(P)(\boldsymbol{\xi}) - \upsilon^m I\big)^{-1}.$$

This function is smooth on its domain and homogeneous of degree $-m$*. With the operator norm on* $\mathrm{Hom}(E)$*, there is* $c_3 > 0$ *depending only on* c_0 *such that*

$$\big|b_{-m}(\boldsymbol{\xi}, \upsilon)\big|_x \leq c_3(|\boldsymbol{\xi}|^2 + |\upsilon|^2)^{-m/2}, \quad x = \pi(\boldsymbol{\xi}),\ \upsilon \in \Sigma.$$

(The constant c_3 *is* $1/c$*.)*

The corollary motivates the following local definition:

Definition 3.5. Let $\mathcal{U} \subset \mathbb{R}^n$ be open and $\mu \in \mathbb{R}$. Then $S^\mu(\mathcal{U}, \mathbb{R}^n \times \Sigma)$ is the space of smooth functions $a : \mathcal{U} \times \mathbb{R}^n \times \Sigma$ such that for all $\mathcal{K} \Subset \mathcal{U}$, for all k and indices $\alpha, \beta, \gamma, \gamma'$, there is C such that

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\upsilon^\gamma \partial_{\overline{\upsilon}}^{\gamma'} a(x, \xi, \upsilon)| \leq C(1 + |\xi| + |\upsilon|)^{\mu - \beta} \ \text{ for all } x \in \mathcal{K},\ (\xi, \upsilon) \in \mathbb{R}^n \times \Sigma.$$

The space $S^\mu_{c\ell}(\mathcal{U}, \mathbb{R}^n \times \Sigma)$ consists of those a such that there are

$$a_{\mu - j} \in C^\infty(\mathcal{U}, \mathbb{R}^n \times \Sigma),\ j = 0, 1, \ldots$$

positively homogeneous of degree $\mu - j$ in (ξ, υ) such that if $\chi \in C_c^\infty(\mathbb{R})$ is equal to 1 in a neighborhood of 0, then

$$a - \sum_{j=0}^{N} (1 - \chi(\xi)) a_{\mu - j} \in S^{\mu - N - 1}(\mathcal{U}, \mathbb{R}^n \times \Sigma)$$

In the definition any open sector Σ would work, by of course for us the one defined above is the relevant one. The point $(0, 0)$ is not in $\mathbb{R}^n \times \Sigma$ since $0 \in \mathbb{C}$ is not in the sector, so smoothness of homogeneous functions with the given domain sis not an issue.

If $a \in S^\mu(\mathcal{U}, \mathbb{R}^n \times \Sigma)$, then

$$A(\upsilon) : C_c^\infty(\mathcal{U}) \to C^\infty(U)$$

is the operator

$$A(\upsilon)(u) = \mathrm{op}(a)(\upsilon)(u) = \frac{1}{(2\pi)^n} \int a(x, \xi, \upsilon) \hat{u}(\xi)\, d\xi.$$

This operator depends on $\upsilon \in \Sigma$ as a parameter. If $\chi_0, \chi_1 \in C_c^\infty(U)$, and $\mu \leq 0$, then $\mathrm{op}(a_{\chi_0, \chi_1})(\upsilon)(u) = \chi_1 \,\mathrm{op}(a)(\chi_0 u)$ defines a continuous operator $L^2(\mathcal{U}) \to L^2(\mathcal{U})$ and there is $C > 0$ such that

$$\|\,\mathrm{op}(a_{\chi_0, \chi_1})\| \leq C(1 + |\upsilon|)^\mu$$

for υ in any subsector $\Sigma' \Subset \Sigma^{1/m}_{c_0}$,

$$\Sigma' = \{\upsilon \in \mathbb{C} : \upsilon \neq 0, \theta_0/m + \varepsilon < \arg \upsilon < (2\pi - \theta_0/m) - \varepsilon\}$$

defined with small enough $\varepsilon > 0$; C depends on ε.

On $\mathcal{M}$ and the vector bundle $E \to \mathcal{M}$ one constructs operators

$$A(\upsilon) \in \Psi^{\mu}_{c\ell}(\mathcal{M}, \Sigma; E)$$

using local coordinates and trivializations. The principal symbol is now obtained as

$$\boldsymbol{\sigma}_{\mu}(A)(df(x_0), \upsilon)(u) = \frac{1}{\tau^{\mu}}(e^{-i\tau f(x_0)} A(\tau^{\mu}\upsilon))(e^{i\tau f}u)(x_0)$$

when $df(x_0) \neq 0$ and u has support in the set where $df \neq 0$. This symbol is homogeneous of degree μ:

$$\boldsymbol{\sigma}_{\mu}(A)(\tau\boldsymbol{\xi}, \tau\upsilon) = \tau^{\mu}\, \boldsymbol{\sigma}(A)(\boldsymbol{\xi}, \upsilon).$$

The space of such symbols is

$$S^{\mu}_{h}(T^*\mathcal{M} \times \Sigma; E).$$

An element a of this space is a smooth section $a : T^*\mathcal{M} \times \Sigma \to \operatorname{Hom}(\pi^*_{\Sigma}E)$ where $\pi_{\Sigma} : T^*\mathcal{M} \times \Sigma \to \mathcal{M}$ is the canonical projection. It is "elliptic with parameter" if the pointwise inverse exists. If it does, it is an element of $S^{-\mu}_{h}(T^*\mathcal{M} \times \Sigma; E)$.

For instance if $A = P - \mu^m I$ with P differential of order m, elliptic, positive, then $A \in \Psi^{m}_{c\ell}(\mathcal{M}, \Sigma; E)$ and

$$\boldsymbol{\sigma}(A)(\boldsymbol{\xi}, \upsilon) = \boldsymbol{\sigma}(P)(\boldsymbol{\xi}) - \upsilon^m I.$$

As we saw above, this is invertible.

The space of residual operators is

$$\Psi^{-\infty}_{c\ell}(\mathcal{M}, \Sigma; E) = \bigcap_{\mu} \Psi^{\mu}_{c\ell}(\mathcal{M}, \Sigma; E).$$

If $R(\mu)$ is residual, then $\|R(\mu)\|$, its norm as an operator on $L^2(\mathcal{M}; E)$, decreases rapidly as $|\upsilon| \to \infty$ uniformly on subsectors $\Sigma' \Subset \Sigma$.

THEOREM 3.6. *There is an exact sequence*

$$0 \to \Psi^{\mu-1}_{c\ell}(\mathcal{M}, \Sigma; E) \xrightarrow{\iota} \Psi^{\mu}_{c\ell}(\mathcal{M}, \Sigma; E) \xrightarrow{\boldsymbol{\sigma}_{\mu}} S^{\mu}_{h}(T^*\mathcal{M} \times \Sigma; E) \to 0. \tag{3.7}$$

If $A \in \Psi^{\mu}(\mathcal{M}, \Sigma; E)$ is elliptic with parameter, that is, there exists

$$b \in S^{-\mu}_{h}(T^*\mathcal{M} \times \Sigma; E)$$

such that $ab = I$, then there is $B \in \Psi^{-\mu}(\mathcal{M}, \Sigma; E)$ such that $A \circ B = I - R$, where R is residual.

The proof runs along the same lines as those of Theorems VIII.2.3 and VIII.3.1.

We apply the theorem in the case of $P - \lambda I$, $P \in \operatorname{Diff}^m(\mathcal{M}; E)$ elliptic and strictly positive. We obtain an operator $B(\mu)$ with principal symbol

$$\boldsymbol{\sigma}(B)(\boldsymbol{\xi}, \upsilon) = \big(\boldsymbol{\sigma}(P)(\boldsymbol{\xi}) - \upsilon^m I\big)^{-1}$$

such that $(P - \upsilon^m I)B(\mu) = I - R_{\mu}$ where R_{μ} is smoothing with norm tending to 0 rapidly as $\mu \to \infty$ uniformly in a subsector $\Sigma' \Subset \Sigma$. The principal symbol of $B(\mu)$ is

We now replace $\upsilon = \lambda^{1/m}$ using the inverse of the bijection $\Sigma \to \Sigma_{c_0}$ given by $\upsilon \mapsto \upsilon^m$ and write $B(\lambda)$ in place of $B(\lambda^{1/m})$ and $R(\lambda)$ in place of $R(\lambda^{1/m})$.

Since $\|R_\lambda\|$ decreases as $|\lambda| \to \infty$ uniformly in any subsector $\Sigma' \Subset \Sigma_{c_0}$, once $\|R(\mu)\| < 1$ we can use the Neumann series

$$\big(I - R(\lambda)\big)^{-1} = \sum_{k=0}^{\infty} R(\lambda)^k \quad \text{for large enough } |\lambda|$$

to invert $I - R(\lambda)$. Then $B(\lambda)\big(I - R(\lambda)\big)^{-1}$ inverts $P - \lambda I$ for large $|\lambda|$.

4. Local calculations

To show that the operators P^{-s} are pseudodifferential operators of order $-sm$ we resort to local constructions. After fixing local coordinates $x^1, \dots, x^n$ with sufficiently small domain over which E is trivial transfer all to an open set $\mathcal{U}$ in $\mathbb{R}^n$ and deal with matrix valued operators and symbols. The density $\mathfrak{m}$ is $\mathfrak{m}_0(x)dx$. We use an orthonormal frame of E so the Hermitian product looks like the canonical one. Thus P is now an $r \times r$ matrix of differential operators of order m, strictly positive (with respect to $\mathfrak{m} = \mathfrak{m}_0\, dx$), the principal symbol of P is a positive $r \times r$ matrix $p_m(x, \xi)$ ($\boldsymbol{\xi} = \sum \xi_j\, dx^j \in T^*_x\mathcal{U}$) and

$$p_m(x, \xi) - \lambda I_r$$

(I_r for the identity $r \times r$ matrix) is invertible for λ not an eigenvalue of the matrix $p(x, \xi)$ (of course).

Eventually we will restrict λ to a sector Σ_{c_0} (union a small neighborhood of 0) because the need to have bounds from below. But for the moment we can proceed formally.

As at the end of the previous section, instead of using υ as parameter, we will use $\lambda = \upsilon^m$. Homogeneity will be with respect to $\tau \cdot (\xi, \lambda) = (\tau\xi, \tau^m\lambda)$.

We have

$$P = \sum_{|\alpha| \le m} p_\alpha(x) D_x^\alpha$$

with smooth matrix valued functions $p_\alpha(x)$. Let

$$p(x, \xi, \lambda) = \sum_{j=0}^{m} p_{m-j}(x, \xi, \lambda),$$

where

$$p_m(x, \xi, \lambda) = p_m(x, \xi) - \lambda I_r, \quad p_{m-j}(x, \xi, \lambda) = \sum_{|\alpha| = m-j} a_\alpha(x)\xi^\alpha \text{ for } j > 0.$$

Suppose $b(x, \xi, \lambda) \in S^{-m}_{c\ell}(U, \Sigma, \mathbb{R}^n)$ is an $r \times r$ matrix of symbols,

$$b(x, \xi, \lambda) \sim \sum_{k=0}^{\infty} b_{-m-k}(x, \xi, \lambda),$$

$b_{-m-k}(x, \xi, \lambda)$ homogeneous of degree $-m - k$:

$$b_{-m-k}(x, \tau\xi, \tau^m\lambda) = \tau^{-m-k} b_{-m-k}(x, \xi, \lambda).$$

If $u : \mathcal{U} \to \mathbb{C}^r$ is compactly supported and smooth and

$$B(\lambda)u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} b(x,\xi,\lambda) u(x')\, \mathfrak{m}_0(x') dx\, d\xi,$$

then

$$P \circ B(\lambda)(u)(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha b(x,\xi,\lambda) u(x)\, \mathfrak{m}_0(x) dx\, d\xi$$

as in (VIII.2.5). So $(P - \lambda I) \circ B(\lambda)$ is the operator

$$(P - \lambda I) \circ B(\lambda)(u) = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} c(x,\xi,\lambda) u(x)\, \mathfrak{m}_0(x) dx\, d\xi$$

with

$$\begin{aligned} c(x,\xi,\lambda) &= \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha b(x,\xi,\lambda) \\ &\sim \sum_{\ell=0}^\infty \sum_{j+k+|\alpha|=\ell}^\infty \frac{1}{\alpha!} \partial_\xi^\alpha p_{m-j}(x,\xi,\lambda) D_x^\alpha b_{-m-k}(x,\xi,\lambda) \\ &= \sum_{\ell=0}^\infty c_{-\ell}(x,\xi,\lambda) \end{aligned}$$

The condition that $(P - \lambda I) \circ B(\lambda) = I$ suggests setting

$$\begin{aligned} c_0(x,\xi,\lambda) &= I_r \\ c_{-\ell}(x,\xi,\lambda) &= 0 \text{ if } \ell > 0 \end{aligned}$$

(this procedure is the same as when finding a pseudodifferential parametrix of an elliptic operator). These become explicit conditions on the b_{-m-k}:

$$\begin{aligned} \big(p_m(x,\xi) - \lambda I_r\big) b_{-m}(x,\xi,\lambda) &= I_r, \\ \sum_{j+k+|\alpha|=\ell} \frac{1}{\alpha!} \partial_\xi^\alpha p_{m-j}(x,\xi) D_x^\alpha b_{-m-k}(x,\xi,\lambda) &= 0, \quad \ell > 0. \end{aligned} \tag{4.1}$$

The index j runs from 0 to m, since P is a differential operator but the formulas look simpler if we set $p_{m-j}(x,\xi) = 0$ for $j > m$. (If P is pseudodifferential then p_{m-j} need not vanish when $j > m$ but the formulas are still true). Subject to these conditions, the b_{-m-k} are found iteratively, starting with

$$b_{-m}(x,\xi,\lambda) = \big(p_m(x,\xi) - \lambda I_r\big)^{-1} \tag{4.2}$$

then observing that for $\ell > 0$

$$\begin{aligned} c_\ell(x,\xi,\lambda) &= \sum_{k=0}^{\ell} \sum_{j+|\alpha|=\ell-k} \frac{1}{\alpha!} \partial_\xi^\alpha p_{m-j}(x,\xi,\lambda) D_x^\alpha b_{-m-k}(x,\xi,\lambda) \\ &= \big(p_m(x,\xi) - \lambda I_r\big) b_{-m-k}(x,\xi,\lambda) \\ &\qquad + \sum_{k=0}^{\ell-1} \sum_{j+|\alpha|=\ell-k} \frac{1}{\alpha!} \partial_\xi^\alpha p_{m-j}(x,\xi,\lambda) D_x^\alpha b_{-m-k}(x,\xi,\lambda) \end{aligned}$$

so the vanishing of c_ℓ gives $b_{-m-\ell}$ in terms of b_{-m-k} with $k<\ell$:

$$b_{-m-\ell}(x,\xi,\lambda) = \\ -b_{-m}(x,\xi,\lambda)\sum_{k=0}^{\ell-1}\sum_{j+|\alpha|=\ell-k}\frac{1}{\alpha!}\partial_\xi^\alpha p_{m-j}(x,\xi,\lambda)D_x^\alpha b_{-m-k}(x,\xi,\lambda) \quad (4.3)$$

The set of eigenvalues of $p_m(x,\xi)$, $\xi\neq 0$, is a finite set $\mathrm{sp}(x,\xi)\subset(0,\infty)$ with the property that $\mathrm{sp}(x,\tau\xi)=\tau^m\mathrm{sp}(x,\xi)$. Thus if ξ is close to 0 the singularities (in λ) of $b_m(x,\xi,\lambda)$ are also close to 0. Unless we do something this will be a problem later.

We modify $p_m(x,\xi)$ to $\tilde{p}_m(x,\xi,\lambda)$ so that the spectrum of $\tilde{p}(x,\xi)$ remains above an arbitrary threshold when ξ is small. We will do this in a global fashion. Using the notation leading to IV.2.2, let

$$\phi_0 : \delta^*T^*\mathcal{M}\to\mathbb{R}$$

be the phase function defined there. We used this in Chapter IV to represent scalar pseudodifferential operators (rather, their Schwartz kernels) in the neighborhood $\mathcal{W}$ of the diagonal of $\mathcal{M}\times\mathcal{M}$. Her we will use it to define an operator on $C^\infty(\mathcal{M};E)$.

Pick a smooth non-negative function $\rho_0\in C_c^\infty(\mathbb{R})$ supported in $|t|<r/3$, equal to 1 if $|t|<r/4$ (r is the injectivity radius of the metric used in the definition of ϕ_0). Then $\rho_0(|\delta(x,x')|)$ is supported in $\mathcal{W}$, equal to 1 near the diagonal, and is symmetric:

$$\rho_0(\delta(x,x')) = \rho_0(\delta(x',x)).$$

Next, pick a Hermitian connection on E. View the identity homomorphism $E\to E$ as a section of $E\otimes E^*$. This section can be lifted to a smooth section $\mathcal{I}$ of $E\boxtimes E^*$ along the diagonal in $M\times\mathcal{M}$. Using the connection on E and the induced connection on E^*, extend $\mathcal{I}$ to a section of $E\boxtimes E^*$ over $\mathcal{W}$ by using parallel transport along the curves

$$t\mapsto\big(\exp_y(t\boldsymbol{\eta}/2),\exp_y(-t\boldsymbol{\eta}/2)\big),\ t\in[0,1],|\boldsymbol{\eta}|<r/2$$

Continue to denote this extension by $\mathcal{I}$. It defines a homomorphism $E_{x'}\to E_x$ for each $(x,x')\in\mathcal{W}$ which again we continue to denote by $\mathcal{I}$.

Let $\chi\in C_c^\infty(\mathbb{R})$ be another smooth function, strictly positive when $|\tau|<1$ equal to 1 in $|\tau|<1/2$, 0 when $|\tau|>1$. Recall form the proof of Lemma 3.1 that $\varrho_{\min}(\boldsymbol{\xi})$ is the smallest eigenvalue of $\boldsymbol{\sigma}(P)(\boldsymbol{\xi})$. This function is continuous, strictly positive for $\boldsymbol{\xi}\neq 0$, and positively homogeneous of degree m. Let

$$c=\min_{\boldsymbol{\xi}\in\dot{T}^*\mathcal{M}}\varrho_{\min}(\boldsymbol{\xi})/|\boldsymbol{\xi}|^m.$$

Finally, let

$$K_\varepsilon(x,x')=\frac{\varepsilon c}{(2\pi)^n}\int_{T^*_{\delta(x,x')}\mathcal{M}}e^{i\phi_0(x,x',\boldsymbol{\xi})}\chi(|\boldsymbol{\xi}|)\rho_0(\delta(x,x'))\mathcal{I}\,d\boldsymbol{\xi}$$

where $d\boldsymbol{\xi}$ refers to integration along the fiber (use an orthonormal basis of the fiber to identify with $\mathbb{R}^n$, then use the Lebesgue density). This is the Schwartz kernel of an operator $C^\infty(\mathcal{M};E)\to C^\infty(\mathcal{M};E)$, in fact a smoothing operator, symmetric with small norm if ε is small. Thus, with small enough $\varepsilon>0$, $P+\mathrm{op}(K_\varepsilon)$ is again strictly positive, essentially P again.

We view

$$\tilde{p}_\varepsilon(\boldsymbol{\xi}) = \big(\boldsymbol{\sigma}(P)(\boldsymbol{\xi}) + \varepsilon c\chi(|\boldsymbol{\xi}|)I\big) : E_x \to E_x, \quad x = \pi(\boldsymbol{\xi})$$

as the principal symbol of $P + \operatorname{op}(K_\varepsilon)$ (this is just a device); ρ_0 is absent because $\delta(x,x) = 0$. The eigenvalues of $\tilde{p}_\varepsilon(\boldsymbol{\xi})$ are all no less than

$$c_{\min} = \min\{\varrho_{\min}(\boldsymbol{\xi}) + \varepsilon c\chi(|\boldsymbol{\xi}|) : \boldsymbol{\xi} \in T^*\mathcal{M}\} \geq c \min_{\tau \geq 0}(\tau^m + \varepsilon\chi(\tau));$$

The effect of this is that we can make the curve Γ in the functional-analytic definition of A^{-s} have the circular part of radius $< c_{\min}$, thus ensuring there is no singularity of b_{-m} on the closed disc with that radius, eventually allowing us to make a change in the order of a certain integral.

We now replace $b_{-m}(\boldsymbol{\xi}, \lambda)$ by

$$b_{-m}(\boldsymbol{\xi}, \lambda) = \big(\boldsymbol{\sigma}(P)(\boldsymbol{\xi}) + (\varepsilon c\chi(\boldsymbol{\xi}) - \lambda)I\big)^{-1},$$

which satisfies the same estimates as the original symbol, and redefine all successive local $b_{-m-k}(x, \xi, \lambda)$ using the redefined b_{-m}. We will not indicate in the notation that the symbols gave been redefined. For the record, we point out that

$$\tilde{p}_\varepsilon(\boldsymbol{\xi}, \lambda) = \boldsymbol{\sigma}(P)(\boldsymbol{\xi}, \lambda) \text{ if } |\boldsymbol{\xi}| \geq 1 \tag{4.4}$$

In coordinates this reads

$$\tilde{p}_\varepsilon(x, \xi) = p_m(x, \xi) \text{ if } \sum_{jk} g^{jk}(x)\xi_j\xi_k \geq 1. \tag{4.4$'$}$$

We will write

$${}^\varepsilon P = P + \operatorname{op}(K_\varepsilon) \tag{4.5}$$

and regrad

$${}^\varepsilon p(\boldsymbol{\xi}) = \boldsymbol{\sigma}(P)(\boldsymbol{\xi}) + \varepsilon c\chi(\boldsymbol{\xi})I \tag{4.6}$$

as its principal symbol.

5. Pseudodifferential resolvent

We now assemble the pseudodifferential parametrix of $(P - \lambda I)$. Pick an open cover $\mathcal{M}$ by domains of local charts over which E is trivial and a partition of unity subordinate to this cover. Since $\mathcal{M}$ is compact, we may take the cover to be finite, and the functions in the partition to be indexed by the same index as the cover: on $\mathcal{U}_i$, the function $\varphi_{0,i} \in C_c^\infty(\mathcal{U}_i)$. Next, take for each i, a function $\varphi_{1,i} \in C_c^\infty(\mathcal{U}_i)$ such that $\varphi_{1,i} = 1$ in a neighborhood of $\operatorname{supp}\varphi_{0,i}$.

On $\mathcal{U}_i$, let $b_{i,-m-\ell}(x, \xi, \lambda)$ be the symbols obtained iteratively as in (4.1) on $\mathcal{U}_i$ but for ${}^\varepsilon P$ (see (4.5)), let $b_i \in S^{-m}(\mathcal{U}_u, \Sigma_{c_0}, \mathbb{R}^n)$ be such that

$$b_i \sim \sum_\ell b_{i,-m-\ell} \tag{5.1}$$

and finally, let

$$B_i(\lambda)u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-x')\cdot\xi} b_i(x, \xi, \lambda)u(x')\, dx\, d\xi,$$

thus

$$({}^\varepsilon P - \lambda I) \circ B_i = I - R_i(\lambda)$$

where $R_\lambda \in \Psi^{-\infty}(\mathcal{U}_i, \mathbb{R}^n, \Sigma_{c_0})$. We will leave implicit the dependence on ε of these symbols. Define

$$B(\lambda)(u) = \sum_i \varphi_{1,i} B_i(\lambda)(\varphi_{0,i} u).$$

We now show that $B(\lambda)$ is a parametrix for $A - \lambda I$.

Pick $x_0 \in \mathcal{M}$. Let $\mathfrak{I}_{x_0}$ be the set of indices i such that $x_0 \in \operatorname{supp} \varphi_{0,i}$. There is a neighborhood $\mathcal{V} \subset \bigcap_{i \in \mathfrak{I}_{x_0}} \mathcal{U}_i$ of x_0 such that for each $i \in \mathfrak{I}_{x_0}$: $\varphi_{1,i} = 1$ on $\mathcal{V}$. If $x \in \mathcal{V}$, then on $\mathcal{V}$,

$$\begin{aligned}[({}^\varepsilon P - \lambda I) \circ \varphi_{1,i} B_i(\lambda)(\varphi_{0,i} u)](x) &= [({}^\varepsilon P - \lambda I) \circ B_i(\lambda)(\varphi_{0,i} u)](x) \\ &= [(I - R_i)(\lambda)(\varphi_{0,i} u)](x) \\ &= (\varphi_{0,i} u)(x) - [\varphi_{1,i} R_i(\lambda)(\varphi_{0,i})](u)(x)\end{aligned}$$

because $\varphi_{1,i} = 1$ on $\mathcal{V}$ Thus

$$({}^\varepsilon P - \lambda I) \circ \Big(\sum_{i \in \mathfrak{I}_{x_0}} \varphi_{1,i} B_i(\lambda) \varphi_{0,i} \Big) = I - \Big(\sum_{i \in \mathfrak{I}_{x_0}} \varphi_{1,i} R_i(\lambda) \varphi_{0,i} \Big)$$

in a neighborhood of x_0.

Continuing with x_0, the open set $\bigcap_{i \notin \mathfrak{I}_{x_0}} (\mathcal{M} \setminus \operatorname{supp} \varphi_i)$ contains an open neighborhood of x_0. For $i \notin \mathfrak{I}_{x_0}$, B_i is smoothing near x_0. Thus

$$({}^\varepsilon P + \operatorname{op} K_\varepsilon - \lambda I) \varphi_{1,i} B(\lambda) \varphi_{0,i} = -R_i(\lambda)$$

is also smoothing with $R(\lambda)$ smoothing. Thus, in a neighborhood of x_0 we have

$$({}^\varepsilon P - \lambda I) \circ \Big(\sum_i \varphi_{1,i} B(\lambda) \varphi_{0,i} \Big) = I - R(\lambda)$$

with $R(\lambda)$ smoothing. We conclude that

$$({}^\varepsilon P - \lambda I) \circ B(\lambda) = I - R(\lambda)$$

globally with $R(\lambda) \in \Psi_{c\ell}^{-N}(\mathcal{M}; E)$ for any N.

6. Estimates

We continue with the notation of the previous section. The symbols $b_{i,-m-\ell}$ are smooth on $\mathcal{U}_i \times \mathbb{R}^n \times \Sigma_{c_0}$ since ${}^\varepsilon p_m(x,\xi) - \lambda$ (defined in (4.6)) is non singular when $\lambda \notin [0, \infty)$. So in the asymptotic sum (5.1) we may take arbitrarily (finitely) many terms as the actual $b_{i,-m-\ell}$. Since

$$b_{i,-m-\ell} \in S_{c\ell}^{-m-\ell}(\mathcal{U}_i, \mathbb{R}^n \times \Sigma_{c_0})$$

satisfies estimates such as

$$\partial_x^\alpha \partial_\xi^\beta b_{i,m-\ell}(x,\xi,\lambda) \le C_{\mathcal{K},\alpha,i} (1 + |\xi| + |\lambda|^{1/m})^{-m-\ell-|\beta|}$$

when $x \in \mathcal{K} \Subset \mathcal{U}_i$ and $\xi \in \mathbb{R}^n$ and $\lambda \in \Sigma_{c_0}$, see Definition 3.5, we also have

$$\partial_x^\alpha \partial_\xi^\beta b_{i,m-\ell}(x,\xi,\lambda) \le C_{\mathcal{K},\alpha,i} (1 + |\lambda|)^{-1-\ell/m} (1 + |\xi|)^{-|\beta|}$$

which exhibits $b_{i,-m-\ell}$ as an element of $S^0_{c\ell}(\mathcal{U}_i \times \mathbb{R}^n)$ with a bound that depends on $|\lambda|$. By Theorem VIII.6.1 and according to (VIII.6.2), we have the bound

$$\|\varphi_{i,1} \operatorname{op}(b_{i,-m-\ell})(\lambda) \varphi_{0,i}\| \le C(1 + |\lambda|)^{-1-\ell/m}, \quad \lambda \in \Sigma_{c_0}$$

for the norm of

$$\varphi_{i,1}\operatorname{op}(b_{i,-m-\ell})(\lambda)\varphi_{0,i} : L^2(\mathcal{M};E)\to L^2(\mathcal{M};E).$$

The same observation holds for

$$\varphi_{i,1}\operatorname{op}(b_i)(\lambda)\varphi_{0,i} : L^2(\mathcal{M};E)\to L^2(\mathcal{M};E)$$

(the asymptotic sum of the $b_{i,-m-\ell}$) and for

$$\varphi_{i,1}\operatorname{op}(b_i\varphi_{0,i})(\lambda)-\sum_{\ell=0}^{N}\varphi_{i,1}\operatorname{op}(b_{i,-m-\ell})(\lambda)\varphi_{0,i} : L^2(\mathcal{M};E)\to L^2(\mathcal{M};E)$$

for the same reason. Thus

$$\|B(\lambda)\|\le C(1+|\lambda|)^{-1},\quad \lambda\in\Sigma_{c_0}, \tag{6.1}$$

and

$$\|B(\lambda)-\sum_{\ell=0}^{N}B_{-m-\ell}(\lambda)\|\le C(1+|\lambda|)^{-1-(N+1)/m},\quad \lambda\in\Sigma_{c_0}, \tag{6.2}$$

where

$$B_{-m-\ell}(\lambda)=\sum_i\varphi_{1,i}\operatorname{op}(b_{i,-m-\ell})(\lambda)\varphi_{0,i}. \tag{6.3}$$

The norm of this operator is estimated by

$$\|B_{-m-\ell}(\lambda)\|\le C(1+|\lambda|)^{1-\ell/m},\quad \lambda\in\Sigma_{c_0}. \tag{6.4}$$

One consequence of this and (6.2) is:

Proposition 6.5. *The remainder*

$$R(\lambda) : L^2(\mathcal{M};E)\to L^2(\mathcal{M};E)$$

is a smoothing operator with norm rapidly decreasing in $\lambda\in\Sigma_{c_0}$: for each N there are $C, M>0$ such that

$$\|R(\lambda)\|\le C_N(1+|\lambda|^{1/m})^{-N},\quad \lambda\in\Sigma_{c_0},\ |\lambda|>M.$$

7. The analytic group of operators

This section's is the same title as the corresponding section [**29**, p. 299] of Seeley's paper.

Theorem 7.1. *Let Γ be the path defined in Section 1. The family of operators*

$$s\mapsto {}^{\varepsilon}P_{-s}=\frac{i}{2\pi}\int_\Gamma\lambda^{-s}B(\lambda)\,d\lambda$$

is well defined and holomorphic on $\Re s>0$. Further, the family

$$s\mapsto {}^{\varepsilon}P_{-s}-\sum_{\ell=0}^{N}\frac{i}{2\pi}\int_\Gamma\lambda^{-s}B_{-m-\ell}(\lambda)$$

is holomorphic in $\Re s>-(N+1)/m$. If $\Re s>1$ then

$${}^{\varepsilon}P\circ{}^{\varepsilon}P_{-s}-{}^{\varepsilon}P_{-s+1}$$

is an entire family of smoothing operators. If $\Re s$ and $\Re s'$ are positive, then

$${}^{\varepsilon}P_{-s}{}^{\varepsilon}P_{-s'}-{}^{\varepsilon}P_{-s-s'}$$

is also an entire family of smoothing operators.

PROOF. The first two statements are simply consequence of analyticity and the estimates (6.1) and (6.2).

The formula

$$ {}^{\varepsilon}P \circ B(\lambda) = ({}^{\varepsilon}P - \lambda I) \circ B(\lambda) + \lambda B(\lambda) = I - R(\lambda) + \lambda B(\lambda), $$

gives

$$ {}^{\varepsilon}P \circ {}^{\varepsilon}P_{-s} - {}^{\varepsilon}P_{-s+1} = \big(\frac{i}{2\pi}\int_{\Gamma} \lambda^{-s}\, d\lambda\big) I - \frac{i}{2\pi}\int_{\Gamma} \lambda^{-s} R(\lambda)\, d\lambda. $$

The first operator on the right vanishes: there are no poles and $\Re s > 1$. For the second term, by Proposition 6.5, the operator involving the remainder is an entire (in s) smoothing family.

For the last statement we use

$$ B(\lambda) \circ ({}^{\varepsilon}P - \lambda I) + R(\lambda) = I $$

and the corresponding formula with μ instead of λ to get

$$ \begin{aligned} B(\lambda) - B(\mu) &= B(\lambda) \circ [({}^{\varepsilon}P - \mu I) \circ B(\mu) + R(\mu)] - [B(\lambda) \circ ({}^{\varepsilon}P - \lambda I) + R(\lambda)] \circ B(\mu) \\ &= (\lambda - \mu) B(\lambda) \circ B(\mu) + B(\lambda) \circ R(\mu) - R(\lambda) B(\mu) \end{aligned} $$

then proceed as in the case of ${}^{\varepsilon}P^{-s}$ in the proof of Proposition 2.1 □

COROLLARY 7.2. *The holomophic family ${}^{\varepsilon}P_{-s}$, initially defined for $\Re s > 0$, extends to an entire family.*

The proof is the same as that of the corresponding statement in Proposition 2.1, starting with ${}^{\varepsilon}P \circ {}^{\varepsilon}P_{-s} = {}^{\varepsilon}P_{-s+1}+$ entire smoothing. In some detail: If $k > 0$ is a positive integer, then the left hand side of

$$ {}^{\varepsilon}P_{-s} - {}^{\varepsilon}P^{k} \circ {}^{\varepsilon}P_{-s-k} = \mathcal{R}_{s,k} $$

is holomorphic when $\Re s > 0$. But $\mathcal{R}_{s,k}$ is entire and smoothing, so since ${}^{\varepsilon}P_{-s-k}$ is holomorphic in $\Re s > -k$, ${}^{\varepsilon}P_{-s}$, initially defined and holomorphic in $\Re s > 0$, has a holomorphic extension to $\Re s > -k$, namely

$$ {}^{\varepsilon}P_{s} = {}^{\varepsilon}P^{k} \circ {}^{\varepsilon}P_{-s-k} + \mathcal{R}_{s,k}. $$

We next extract information from these families by once again writing them as families of pseudodifferential operators.

Let $\mathcal{U}$ be one of the sets $\mathcal{U}_i$, let $b \in S^{-m-\ell}_{c\ell}(\mathcal{U}, \mathbb{R}^n \times \Sigma_{c_0})$ be one of the symbols associated with the index i.

LEMMA 7.3. *Suppose $\Re s > -(m + \ell)$. The operator*

$$ {}^{\varepsilon}P_{-ms-\ell}(u) = \frac{i}{2\pi}\int_{\Gamma} \lambda^{-s} B_{-m-\ell}(\lambda)(u)\, d\lambda \tag{7.4} $$

is a pseudodifferential of order $-m\Re s - \ell$ depending holomorphically on s in $\Re s > -\ell/m$.

PROOF. That ${}^{\varepsilon}P_{-ms-\ell}$ is holomorphic in $\Re s > -\ell/m$ follows from the estimate (6.4).

In view of (6.3), to prove the statement about the pseudodifferential order of ${}^{\varepsilon}P_{-ms-\ell}$ we only need to argue with

$$ u \mapsto \operatorname{op}(b_{i,m-\ell})(\lambda)(u) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} b_{i,-m-\ell}(x, \xi, \lambda) u(x')\, dx'\, d\xi. $$

Let the metric be $\sum_{jk} g^{jk}(x)\xi_j\xi_k$ in the coordinates we are using on $\mathcal{U}_i$, write $|\xi|_{g,x}$ for the norm of $\xi \cdot dx$. By (4.4′) the singularities of $b_{i,-m-\ell}(x,\xi,\lambda)$ lie in $|\xi_{g,x} > \varepsilon$, so if the circular part of the path Γ has sufficiently small radius we can exchange the order of integration:

$$\int_\Gamma \lambda^{-s} \int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} b_{i,-m-\ell}(x,\xi,\lambda) u(x')\, d\xi\, d\lambda \\ = \int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} \int_\Gamma \lambda^{-s} b_{i,-m-\ell}(x,\xi,\lambda) u(x')\, d\lambda\, d\xi$$

Let

$$ {}^{\varepsilon}p_{i,-s,-ms-\ell}(x,\xi) = \frac{i}{2\pi} \int_\Gamma \lambda^{-s} b_{i,-m-\ell}(x,\xi,\lambda)\, d\lambda \tag{7.5}$$

We show that

$$ {}^{\varepsilon}p_{i,-s,-ms-\ell}(x,\xi) = \frac{i}{2\pi} |\xi|_{g,x}^{-ms-\ell} \int_\Gamma \lambda^{-s} b_{i,-m-\ell}(x,\hat{\xi},\lambda)\, d\lambda, \quad \text{if } |\xi|_{g,x} \geq 1, \tag{7.6}$$

where $\hat{\xi} = \xi/|\xi|_{g,x}$. Because of (4.4′), $b_{i,-m-\ell}(x,\xi,\lambda)$ has the homogeneity property

$$b_{i,-m-\ell}(x,\tau\xi,\tau^m\lambda) = \tau^{-m-\ell} b_{i,-m-\ell}(x,\xi,\lambda)$$

if both $|\xi|_{g,x}$ and τ are ≥ 1. Therefore

$$b_{i,-m-\ell}(x,\xi,\lambda) = |\xi|_{g,x}^{-m-\ell} b_{i,-m-\ell}(x,\hat{\xi},\lambda) \text{ if } |\xi|_{g,x} \geq 1$$

Keeping $|\xi|_{g,x} \geq 1$, we thus have

$$\int_\Gamma \lambda^{-s} b_{i,-m-\ell}(x,\xi,\lambda)\, d\lambda = |\xi|_{g,x}^{-m-\ell} \int_\Gamma \lambda^{-s} b_{i,-m-\ell}(x,\hat{\xi},\lambda/|\xi|_{g,x}^m)\, d\lambda \\ = |\xi|_{g,x}^{-ms-\ell} \int_{|\xi|_{g,x}^{-m}\Gamma} \lambda^{-s} b_{i,-m-\ell}(x,\hat{\xi},\lambda)\, d\lambda$$

where the last integral is just a redefinition of the path of integration. When $|\xi|_{g,x} \geq 1$, the paths $|\xi|_{g,x}^{-m}\Gamma$ and Γ are homotopic within the region where $\lambda^{-s} b_{i,-m-\ell}(x,\hat{\xi},\lambda)$ is holomorphic, so the last integral is

$$|\xi|_{g,x}^{-ms-\ell} \int_\Gamma \lambda^{-s} b_{i,-m-\ell}(x,\hat{\xi},\lambda)\, d\lambda$$

This proves (7.6) and that ${}^{\varepsilon}P_{-ms-\ell}$ is a pseudodifferential operator of order $-ms-\ell$.

The estimate

$$|b_{i,-m-\ell}(x,\hat{\xi},\lambda) \leq C(1+|\lambda|)^{-1-\ell/m},\ \lambda \in \Sigma_{c_0}$$

shows that the integral defining ${}^{\varepsilon}p_{i,-s,-ms-\ell}$ converges absolutely if $-\Re s - \ell/m < 0$, so

$$ {}^{\varepsilon}p_{i,-s,-ms-\ell} \text{ is holomorphic in } \Re s > -\ell/m. \tag{7.7}$$

□

Corollary 7.8. *${}^{\varepsilon}P_{-s}$ is a classical pseudodifferential operator of order $-m\Re s$ depending holomorphically on s, $\Re s > 0$. For any N,*

$$ {}^{\varepsilon}P_{-s} - \sum_{\ell=0}^{N} {}^{\varepsilon}P_{-ms-\ell} = {}^{\varepsilon}\tilde{P}_{-s,m-N-1} \tag{7.9}$$

is of order $m\Re s - (N+1)$, holomorphic in $\Re s > -(N+1)/m$.

We will write ${}^{\varepsilon}P_{-ms}$ for the holomorphic family ($s \in \mathbb{C}$) defined by ${}^{\varepsilon}P_{-s}$. When needed, we will write P_{-ms-0} for the first operator ($\ell = 0$) in (7.9).

Parametrize Γ as

$$\gamma(t) = \begin{cases} t & \text{if } t < -\varepsilon \\ \varepsilon(\cos \pi t/\varepsilon, -\sin \pi t/\varepsilon) & \text{if } t \in [-\varepsilon, \varepsilon] \\ -t & \text{if } t > \varepsilon. \end{cases}$$

If $|t| \geq \varepsilon$, then for general $s \in \mathbb{C}$,

$$\gamma(t)^{-s} = \begin{cases} |t|^{-s} e^{-i\pi s} & \text{if } t < -\varepsilon \\ |t|^{-s} e^{+i\pi s} & \text{if } t > \varepsilon \end{cases}$$

therefore with our small ε and large enough $\Re s$,

$$\begin{aligned} &\int_\Gamma \lambda^{-s} b_{i,-m-\ell}(x,\xi,\lambda) d\lambda = \oint\!\!\!\!\!\!- \!\!\!\!\!\!- \!\!\!\!\!\!- \int_{|\lambda|=\varepsilon} \lambda^{-s} \operatorname{tr} b_{i,-m-\ell}(x,\xi,\lambda)\, d\lambda \\ &+ \int_{-\infty}^{-\varepsilon} |t|^{-s} e^{-i\pi s} \operatorname{tr} b_{i,-m-\ell}(x,\xi,t)\, dt - \int_\varepsilon^\infty |t|^{-s} e^{i\pi s} \operatorname{tr} b_{i,-m-\ell}(x,\xi,-t)\, d\gamma(t) \\ &\quad = \oint\!\!\!\!\!\!- \!\!\!\!\!\!- \!\!\!\!\!\!- \int_{|\lambda|=\varepsilon} \lambda^{-s} \operatorname{tr} b_{i,-m-\ell}(x,\xi,\lambda)\, d\lambda - 2i \sin(\pi s) \int_\varepsilon^\infty |t|^s \operatorname{tr} b_{i,-m-\ell}(x,\xi,-t)\, dt. \end{aligned}$$

Thus

$$\begin{aligned} {}^{\varepsilon}p_{i,-s,-ms-\ell}(x,\xi) = \frac{i}{2\pi} \oint\!\!\!\!\!\!- \!\!\!\!\!\!- \!\!\!\!\!\!- \int_{|\lambda|=\varepsilon} \lambda^{-s} \operatorname{tr} b_{i,-m-\ell}(x,\xi,\lambda)\, d\lambda \\ + \frac{1}{\pi} \sin(\pi s) \int_\varepsilon^\infty |t|^s \operatorname{tr} b_{i,-m-\ell}(x,\xi,-t)\, dt \end{aligned} \tag{7.10}$$

In this decomposition of ${}^{\varepsilon}p_{i-s,-ms-\ell}$, the first term, the clockwise oriented integral along the circle, is an entire function of s. Both terms are homogeneous of degree $-ms - \lambda$ when $|\xi|_{g,x} \geq 1$.

We now argue that the ${}^{\varepsilon}p_{i,-s,-ms-\ell}$ are actually entire functions of s. Define

$${}^{\varepsilon}\tilde{P}_{i,-sm} : C_c^\infty(\mathcal{U}_i, E_{U_i}) \to C^\infty(\mathcal{U}_i, E_{U_i})$$

by

$$C_c^\infty(\mathcal{U}_i; E_{\mathcal{U}_i}) \ni u \mapsto {}^{\varepsilon}P_{-ms} u\big|_{\mathcal{U}_i} \in C^\infty(\mathcal{U}; E_{\mathcal{U}_i}).$$

The open set $\mathcal{U}_i$ is the domain of a local chart as well as of an orthonormal frame of E. Decompose ${}^{\varepsilon}\tilde{P}_{i,-sm}$ as a matrix of scalar operators ${}^{\varepsilon}\tilde{P}_\nu^\mu$ according to that frame. Let $\varphi_i \in C_c^\infty(\mathcal{U}_i)$ be equal to 1 in a neighborhood of $\operatorname{supp} \varphi_{1,i}$, define

$${}^{\varepsilon}\tilde{p}_{\nu,s,-ms}^{\mu}(x,\xi) = e^{-imx\cdot\xi}\, {}^{\varepsilon}\tilde{P}_\nu^\mu(e^{i(\cdot)\cdot\xi}\varphi_i)(x)$$

and let ${}^{\varepsilon}\tilde{p}_{i,-s,-ms}$ be the matrix with these coefficients. This is, for each $s \in \mathbb{C}$, a classical symbol of order $-ms$ (really $-m\Re s$) depending holomorphically on s. It has an asymptotic expansion

$${}^{\varepsilon}\tilde{p}_{i,s,-ms}(x,\xi) \sim \sum_\ell {}^{\varepsilon}\tilde{p}_{i,s,-ms-\ell}(x,\xi),$$

${}^{\varepsilon}\tilde{p}_{i,s,-ms-\ell}(x,\xi)$ (complex-) homogeneous of degree $-ms-\ell$, and with the pseudo-differential operators operators ${}^{\varepsilon}\tilde{P}_{i,-ms}$ it defines one has that

$$ {}^{\varepsilon}P_{-ms} - \sum_i \varphi_{1,i}\, {}^{\varepsilon}P_{-ms}\varphi_{0,i} $$

is smoothing (and entire). Replacing the original ${}^{\varepsilon}p_{i,s,-ms}$ by the ${}^{\varepsilon}\tilde{p}_{i,s,-ms}$ we can then assume that the ${}^{\varepsilon}p_{i,s,-ms}$ are entire, rather than just as stated in (7.7), and redefine (7.4):

${}^{\varepsilon}p_{i,-s-ms-\ell}$ is entire, and

$$ {}^{\varepsilon}p_{i,-s-ms-\ell}(x,\tau\xi) = \tau^{-ms-\ell}\, {}^{\varepsilon}p_{i,-s-ms-\ell}(x,\xi) $$

if $|\xi|_{g,x}$ and τ are ≥ 1. Furthermore (7.11)

$$ {}^{\varepsilon}P_{-ms-\ell} = \sum_i \varphi_{1,i}\, \mathrm{op}({}^{\varepsilon}p_{-s,-ms-\ell}\varphi_{0,i}). $$

Consequently, in (7.10) both terms on the right are entire, and in (7.9), all operators are entire in s.

CHAPTER VI

The ζ function

1. Pseudodifferential definition of ζ_P

We continue with the notation and assumptions of the previous chapter, still following Seeley [**29**] very closely.

Let $K_{{}^\varepsilon P_{-s}}$ be the Schwartz kernel of ${}^\varepsilon P_{-s}$, a distribution whose singular set is contained in the diagonal of $\mathcal{M}\times\mathcal{M}$. Since ${}^\varepsilon P_{-s}\in\Psi^{-m\Re s}(\mathcal{M};E)$, $K_{{}^\varepsilon P^{-s}}$ is continuous when $m\Re s>n$. For such s, $K_{{}^\varepsilon P_{-s}}$ is trace class and we can use the pseudodifferential version of (V.1.3) to define the ζ function:

$$
{}^\varepsilon\zeta(s)=\operatorname{Tr}({}^\varepsilon P_{-s})
$$

In view of Corollary V.7.8,

$$
\operatorname{Tr}({}^\varepsilon P_{-s})=\sum_{\ell=0}^{N}\operatorname{Tr}(K_{{}^\varepsilon P_{-ms-\ell}})+\operatorname{Tr}(K_{{}^\varepsilon\tilde P_{-ms-N-1}}) \tag{1.1}
$$

with $\operatorname{Tr}({}^\varepsilon P_{-ms-\ell})$ holomorphic in $\Re s>-\ell/m$ and $\operatorname{Tr}({}^\varepsilon\tilde P_{-ms-N-1})$ holomorphic in $\Re s>-(N+1)/m$.

2. Poles and residues of the zeta function

We now analyze the individual contributions of ${}^\varepsilon p_{i,-s,-ms-\ell}$, see (V.7.11), to the trace of ${}^\varepsilon P_{-ms-\ell}$. We have

$$
K_{{}^\varepsilon P_{-s,m-\ell}}(x,x')=\frac{1}{(2\pi)^n}\sum_i\int_{\mathbb{R}^n}e^{i(x-x')\cdot\xi}\varphi_{1,i}(x)\,{}^\varepsilon p_{i,-s,-ms-\ell}(x,\xi)\varphi_{0,i}(x')\,d\xi
$$

so

$$
\operatorname{Tr}({}^\varepsilon P_{-s,m-\ell})=\sum_i\frac{1}{(2\pi)^n}\int_{\mathcal{U}_i}\int_{\mathbb{R}^n}\operatorname{tr}{}^\varepsilon p_{i,-s,-ms-\ell}(x,\xi)\varphi_{0,i}(x)\,d\xi\,\mathfrak{m}_i(x)\,dx;
$$

here $\mathfrak{m}=\mathfrak{m}_i(x)\,dx$ in the coordinates of $\mathcal{U}_i$. We used that $\varphi_{1,i}=1$ on the support of $\varphi_{1,i}$ so $\varphi_{1,i}(x)\varphi_{0,i}(x')=\varphi_{0,i}(x)$ when $x'=x$.

Fix $\chi_0\in C_c^\infty(\mathbb{R})$ with $\chi_0(\tau)=1$ neat $\tau=0$ and $\chi_0(\tau)=0$ if $|\tau|>1$. Define $\chi_1(\tau)=1-\chi_0(1/\tau)$, so

$$
{}^\varepsilon p_{i,-s,-ms-\ell}(x,\xi)=\chi_0(|\xi|_{g,\xi})\,{}^\varepsilon p_{i,-s,-ms-\ell}(x,\xi)+\chi_1(|\xi|_{g,\xi}^{-1})\,{}^\varepsilon p_{i,-s,-ms-\ell}(x,\xi).
$$

In this spliting, the first term is

$$
\chi_0(|\xi|_{g,x})\frac{i}{2\pi}\int_\Gamma\lambda^{-s}b_{i,-m-\ell}(x,\xi,\lambda)\,d\lambda,
$$

see (V.7.11), which is entire in s: the eventual contribution of this term to the trace is though a smoothing operator (so trace class already) that is an entire function

of s. To get the contribution of the second term we introduce polar coordinates, $\xi = r^{-1}\hat\xi$ (bringing ∞ to 0),

$$\hat\xi \in S^{n-1}_x = \{\xi : |\xi|_{g,x} = 1\}.$$

and recall that ${}^{\varepsilon}p_{i,-s,-ms-\ell}$ is (complex-) homogeneous (in ξ for large ξ) of order $-ms-\ell$:

$${}^{\varepsilon}p_{i,-s,-ms-\ell}(x,\xi) = \frac{i}{2\pi}|\xi|^{-ms-\ell}_{g,x}\int_\Gamma \lambda^{-s} b_{i,-m-\ell}(x,\hat\xi,\lambda)\,d\lambda$$

and entire, see (V.7.11) The contribution of this to the trace is thus

$$\frac{1}{(2\pi)^n}\int_{\mathcal{U}_i}\int_{S^{n-1}_x} \operatorname{tr}{}^{\varepsilon}p_{i,-s,-ms-\ell}(x,\hat\xi)\,d\hat\xi\,\mathfrak{m}_i(x)dx \int_0^\infty \chi_1(r)\,r^{ms+\ell-n}\frac{dr}{r},$$

with $d\hat\xi$ the surface measure of S^{n-1}_x. Keep in mind that $\chi_1(r)$ vanishes for large r.

LEMMA 2.1. *The radial integral converges absolutely if $\Re s > (n-\ell)/m$ but extends as a meromorphic function on all of $\mathbb{C}$ with a single simple pole at $s = n/m - \ell/m$ with residue $1/m$ (independently of χ_1).*

PROOF. We have

$$\begin{aligned}(s-(n-\ell)/m)\int_0^\infty \chi_1(r)\,r^{ms+\ell-n}\frac{dr}{r} &= \frac{1}{m}\int_0^\infty \chi_1(r)\,\frac{d}{dr}r^{ms+\ell-n}\,dr\\ &= -\frac{1}{m}\int_0^\infty \chi_1'(r)\,\frac{d}{dr}r^{ms+\ell-n}\,dr\end{aligned}$$

using integration by parts. Since $\chi_1(r)$ is contant when r is small or when r is large, the integral converges for any s and thus defines am entire function of s. Its value at $s = (n-\ell)/m$ is the residue. This is

$$-\frac{1}{m}\int_0^\infty \chi'(r)\,dr = -\frac{1}{m}\chi(r)\Big|_{r=0}^{r=\infty} = \frac{1}{m}.$$

□

The value $s = (n-\ell)/m$ lies in the set $\{s\in\mathbb{C} : \Re s > -\ell/m\}$ where

$$\frac{1}{(2\pi)^n}\int_{\mathcal{U}_i}\int_{S^{n-1}_x} \operatorname{tr}{}^{\varepsilon}p_{i,-s,-ms-\ell}(x,\hat\xi)\,d\hat\xi\,\mathfrak{m}_i(x)dx$$

is holomorphic, so $\operatorname{Tr}({}^{\varepsilon}P_{-ms-\ell})$ is meromorphic on $\Re s > -\ell/m$ with a pole at $n/m-\ell/m$ and residue

$$c_{(n-\ell)/m} = \frac{i}{(2\pi)^{n+1}m}\sum_i\int_{U_i}\int_{S^{n-1}_x}\int_\Gamma \lambda^{-(n-\ell)/m}\operatorname{tr} b_{i,-m-\ell}(x,\hat\xi,\lambda)\,d\lambda\,d\hat\xi\,\mathfrak{m}_i(x)dx$$

at that point.

PROPOSITION 2.2. *If $(n-\ell)/m$ is an integer ≤ 0, then $c_{(n-\ell)/m} = 0$.*

PROOF. In (7.10), if $s = (n-\ell/s)$ is an integer, the term involving the integral over $[\varepsilon,\infty)$ vanishes simply because $\sin(\pi s) = 0$ because $e^{-i\pi s} = e^{i\pi s}$ in that case. If in addition the integer s is non-positive, then the closed integral vanishes because there is no pole at 0. □

In view of (1.1) we have:

THEOREM 2.3. *The function ${}^{\varepsilon}\zeta(s)$ is meromorphic on $\mathbb{C}$ with simple poles on the set*

$$\{s : s = (n-\ell)/m,\ \ell \in \mathbb{N}_0\}$$

The residue at $(n-\ell)/m$ is

$$\operatorname{Res}({}^{\varepsilon}\zeta, (n-\ell)/m) = c_{(n-\ell)/m}$$

which vanishes if $(n-\ell)/m$ is an integer ≤ 0.

By the Ikehara Tauberian theorem (Wiener [**38**, p. 44]),

COROLLARY 2.4. *Let $\{\lambda_k\}_{k=0}^{\infty}$ be the sequence of eigenvalues of P, as a monotone sequence with eigenvalues repeated accodring to their multiplicity. Let*

$$N_P(\lambda) = \#\{k : \lambda_k < \lambda\}.$$

Then

$$N_P(\lambda) \sim c_{n/m}\lambda^{n/m} \quad \text{as } \lambda \to \infty.$$

CHAPTER VII

Differential operators

1. Definition, characterization

Let $E \to \mathcal{M}$ be a vector bundle. Viewing E as a smooth manifold on its own, it makes sense to speak about the set of smooth maps $\mathcal{M} \to E$ and its topology. In particular there is a topology on $C^\infty(\mathcal{M}; E)$.

Let $E, F \to \mathcal{M}$ be complex vector bundles. Differential operators are defined as follows, inductively on the order:

DEFINITION 1.1. Let $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; F)$ be a continuous linear operator.

(*i*) P is a differential operator of order -1 iff $P = 0$.

(*ii*) If $m \geq 0$, then P is of order m iff for every $f \in C^\infty(\mathcal{M})$, the operator

$$C^\infty(\mathcal{M}; E) \ni u \mapsto P(fu) - fP(u) \in C^\infty(\mathcal{M}; F) \tag{1.2}$$

is a differential operator of order $m - 1$.

For the sake of simplifying the notation we will assume $F = E$ and write $\mathrm{Diff}^m(\mathcal{M}; E)$ for the space of differential operators $C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ of order at most m. Also, write $[P, f]$ for the operator (1.2). (If $F \neq E$ then this is not a commutator but the expression still makes sense.)

The advantage of this definition, later the characterization of differential operators in Theorem 1.10, and the definition of pseudodifferential operators, is that it is independent of coordinates. So any conclusion that can be drawn in some coordinate system will also hold when done in a different coordinate system. We mostly follow Hörmander [**14**] and Nirenberg [**23**]

If $P \in \mathrm{Diff}^0(\mathcal{M}; E)$, then the condition $[P, f] = 0$ implies P is given by a smooth map $p : E \to E$ such that $p(x)$ maps $E_x \to E_x$ linearly. In terms of this homomorphism, P is given by

$$\text{If } u \in C^\infty(\mathcal{M}; E) \text{ then } Pu \in C^\infty(\mathcal{M}; E) \text{ is the section } x \mapsto p(x)(u(x)).$$

This implies that $P(u)$ vanishes on any open set on which u vanishes. By induction this will hold for $P \in \mathrm{Diff}^m(\mathcal{M}; E)$: for such P suppose u vanishes on the open set $\mathcal{U}$. Let $f \in C^\infty(\mathcal{M})$ be such that $f = 0$ in an open subset $\mathcal{V} \subset \mathcal{U}$ but $f = 1$ in the complement of $\mathcal{U}$. Then $u = fu$, hence

$$P(u) = P(fu) = fP(u) + [P, f](u).$$

Since $[P, f]$ is of order $m - 1$, $[P, f](u)$ vanishes in $\mathcal{U}$. And $fP(u)$ vanishes in the set where f vanishes, which includes $\mathcal{V}$. So $P(u)$ vanishes in any subset of $\mathcal{U}$, hence in $\mathcal{U}$.

PROPOSITION 1.3. *Differential operators are local:* $\operatorname{supp} Pu \subset \operatorname{supp} u$.

Since P is local, it can be analyzed on arbitrary open sets. From this and some more work one gets that P is a differential operator in the familiar sense, a matrix of differential operators.

PROPOSITION 1.4. *If $P \in \mathrm{Diff}^m(\mathcal{M}; E)$ then:*

(i) for any smooth $f : \mathcal{M} \to \mathbb{R}$ and section u of E,

$$p(f, u, \tau) = e^{-i\tau f} P(e^{i\tau f} u)$$

is a polynomial in τ of degree $\leq m$ with values in $C^\infty(\mathcal{M}; E)$. Let

$$p(f, u, \tau) = \sum_{\ell=0}^{m} p_\ell(f, u)\tau^\ell.$$

(ii) The leading coefficient of $p(f, u, \tau)$,

$$p_m(f, u) = \lim_{\tau\to\infty} \frac{1}{\tau^m} e^{-i\tau f} P(e^{i\tau f} u),$$

is, as a function of u, a differential operator of order 0, thus defines a homomorphism $p_m(f) : E \to E$. In general, $u \mapsto p_\ell(f, u)$ is a differential operator of order $m - \ell$.

(iii) For any $f : \mathcal{M} \to \mathbb{R}$ and $x_0 \in \mathcal{M}$, this homomorphism depends only on $df(x_0) \in T^*_{x_0}\mathcal{M}$.

(iv) If $p_m(f) = 0$, then P is a differential operator of order $m - 1$.

The point of view in this proposition is that of Hörmander [**14**], already mentioned, who uses it to motivate the definition of pseudodifferental operators later. We will also use his approach when defining the latter. (One can take f to be complex valued, but that will not be convenient.)

Observe that if P is of order m and h is a smooth function, then $u \mapsto P(hu)$ is again a differential operator of order m (also $u \mapsto hP(u)$, trivially): One has, for any f and u:

$$P(hfu) - fP(hu) = (P(hfu) - hfP(u)) - (fP(hu) - fhP(u)).$$

But $u \mapsto P(hfu) - hfP(u)$ and $u \mapsto P(hu) - hP(u)$ are of order $m - 1$.

PROOF. We prove Part (i) by induction. Suppose $P \in \mathrm{Diff}^0(\mathcal{M}; E)$, let f and u be arbitrary as stated. Then

$$\begin{aligned} \frac{\partial}{\partial\tau} p(f, u, \tau) &= ie^{-i\tau f}\big(P(fe^{i\tau f}u) - fP(e^{i\tau f}u)\big) \\ &= e^{-i\tau f}[P, f](e^{i\tau f}u) \end{aligned} \tag{1.5}$$

which vanishes because $u \mapsto P(e^{i\tau f}u)$ is a differential operator of order 0. Suppose now that the statement is true for differential operators of order $m - 1$ for some $m > 0$. The expression in the last line of (1.5) is a polynomial of degree $m - 1$, so $p(f, u, \tau)$ is a polynomial of degree m. This completes the proof of (i).

For part (ii) note first that

$$p_m(f, u) = \frac{1}{m!}\frac{d^m}{d\tau^m} p(f, u, \tau) = \sum_{k=0}^{m} \frac{1}{m!}\binom{m}{k}(-if)^{m-k} P((if)^k u). \tag{1.6}$$

If $h : \mathcal{M} \to \mathbb{C}$ is arbitrary, then

$$\begin{aligned} p_m(f, hu) - hp_m(f, u) &= \sum_{k=0}^{m} \frac{1}{m!}\binom{m}{k}(-if)^{m-k}P((if)^k hu) \\ &\qquad - h\sum_{k=0}^{m} \frac{1}{m!}\binom{m}{k}(-if)^{m-k}P((if)^k u) \\ &= \frac{1}{m!}\frac{d^m}{d\tau^m}e^{-i\tau f}[P, h](e^{i\tau f}u) \end{aligned}$$

which vanishes because $[P, h]$ is a polynomial of degree $m - 1$. Thus $u \mapsto p_m(f, u)$ is of order 0. A similar argument proves the second assertion of Part (*ii*).

For Part (*iii*) we show first:

$$\begin{aligned} &\text{if } P \in \mathrm{Diff}^k(\mathcal{M}; E), \text{ if } f_1, \dots, f_{k+1} \text{ vanish at } x_0, \text{ then} \\ &P(f_1, \dots, f_{k+1}u)(x_0) = 0. \end{aligned} \tag{1.7_k}$$

This holds when $k = 0$. Suppose this holds for $k = m - 1$, let $f_1, \dots, f_m$ be smooth and vanish at x_0 and let $u \in C^\infty(\mathcal{M}; E)$. The right hand side of

$$P(f_1 \cdots f_{m+1}u) = f_{m+1}P(f_1 \cdots f_m u) + [P, f_{m+1}](f_1 \cdots f_m u)$$

evaluated at x_0 vanishes because f_{m+1} does and $[P, f_{m+1}]$ is of order $m - 1$. Thus $P(f_1 \cdots f_{m+1}u)(x_0) = 0$.

Observe now that $p(f, u, \tau)(x_0) = p(f - f(x_0), u, \tau)(x_0)$, so we may assume $f(x_0) = 0$. Keeping this assumption throughout, the assertion in Part (*iii*) is that if f and $\tilde{f}$ agree to first order at x_0, then $p_m(f)(x_0) = p_m(\tilde{f})(x_0)$. Suppose then that $f = f_1 + f_2$ where f_1 vanishes at x_0 and f_2 vanishes to first order. Using (1.6) we have

$$\begin{aligned} p_m(f, u)(x_0) &= \frac{1}{m!}P((i(f_1 + f_2))^m u)(x_0) \\ &= \frac{1}{m!}P((if_1)^m u + f_1^{m-1}hu)(x_0) \\ &= \frac{1}{m!}P((if_1)^m u) + \frac{1}{m!}P(f_1^{m-1}hu)(x_0) \end{aligned}$$

where h vanishes to order 2 at x_0. The first term in the last line is $p_m(f_1, u)(x_0)$. The second vanishes, because $f_1^{m-1}h$ is a linear combination of products of $m + 1$ functions vanishing at x_0. Thus $p_m(f, u)(x_0) = p_m(f_1, u)(x_0)$.

We leave the proof of Part (*iv*) to the reader. □

The principal symbol of a differential operator is defined only on $\dot{T}\mathcal{M} = T^*\mathcal{M}\backslash 0$ (on a zero covector it would be 0 anyway if the order of the operator is ≥ 1). Let $\dot{\pi} : \dot{T}^*\mathcal{M} \to \mathcal{M}$ be the restriction of the projection.

Definition 1.8. Let $P \in \mathrm{Diff}(\mathcal{M}; E)$, let $\boldsymbol{\xi}_0 \in T^*\mathcal{M}$, $\boldsymbol{\xi} \neq 0$, let f be smooth, real valued, defined near $x_0 = \pi(\boldsymbol{\xi}_0)$ with $df(x_0) = \boldsymbol{\xi}_0$. The principal symbol of P is the homomorphism

$$\boldsymbol{\sigma}(P) \in C^\infty(\dot{T}^*\mathcal{M}; \dot{\pi}^*E),\ \boldsymbol{\sigma}(P)(df(x_0))(u) = p_m(f, u),\ df(x_0) \neq 0,\ u \in E_{x_0}.$$

The operator P is elliptic if $\boldsymbol{\sigma}(P)(\boldsymbol{\xi})$ is invertible for all $\boldsymbol{\xi} \in T^*\mathcal{M}\backslash 0$.

Part (*ii*) of Proposition 1.4 gives a practical formula for the primcipal symbol of P: If $\boldsymbol{\xi}_0 = df(x_0)$ and $u_0 = u(x_0)$, then

$$\boldsymbol{\sigma}(P)(\boldsymbol{\xi}_0)(u_0) = \lim_{\tau\to\infty} \frac{1}{\tau^m} e^{-i\tau f(x_0)} P(e^{i\tau f}u)(x_0). \tag{1.9}$$

The property (1.7_k) is equivalent to the statement that that $P(fu)(x_0)$ vanishes if f vanishes to order k. With $k = m$ it implies that P is a differential operator of order m in the usual sense. To simplify the notation assume that P is a scalar operator. Pick $x_0 \in \mathcal{M}$ let $x^1, \dots, x^n$ be coordinates in a neighborhood $\mathcal{U}$ of x_0. Let $\chi \in C_c^\infty(\mathcal{M})$ be supported in $\mathcal{U}$, equal to 1 in an open set $\mathcal{V}$ where $\chi(x) = 1$, x_0. Let u be an arbitrary smooth function. If $x' \in \mathcal{V}$, then

$$u(x) = \sum_{|\alpha|\le m} \frac{1}{\alpha!}\frac{\partial^{|\alpha|}u}{\partial x^\alpha}(x')\chi(x)(x-x')^\alpha \\ + \chi(x)R_{m+1}(u,x,x') + (1-\chi(x))u(x).$$

As function of x, the functions on the second line vanish to order $m+1$ or higher when $x = x'$. Thus

$$P(u)(x') = \sum_{|\alpha|\le m} \frac{1}{\alpha!} P(\chi(\cdot)(\cdot - x')^\alpha)\frac{\partial^{|\alpha|}f}{\partial x^\alpha}(x').$$

The following characterization of differential operators is more relevant to us.

THEOREM 1.10. *If $P : C^\infty(\mathcal{M};E) \to C^\infty(\mathcal{M};E)$ is linear, continuous and local, and satisfies the condition that for any smooth $f : \mathcal{M} \to \mathbb{R}$ and smooth section u of E there is $C > 0$ such that*

$$|p(f,u,\tau)| \le C(1+|\tau|)^m, \tag{1.11}$$

then P is a differential operator of order m. In the estimate we use some Hermitian inner product on E.

The proof below is a second proof that P is a differential operator in the usual sense. Given that P is local, if $\mathcal{M}$ is a compact manifold the estimate is not necessary to conclude that P is a differential operator (of some order). If $\mathcal{M}$ is not compact, then the conclusion without the estimate is still true, but the order of the operator is finite only locally. Peetre's theorem [**24**] is stronger: If $P : C_c^\infty(\mathcal{M}) \to C^{-\infty}(\mathcal{M})$ is linear and local, then P is a differential operator (locally of finite order); in particular, continuity is not needed.

PROOF. Let $\mathcal{U} \subset \mathcal{M}$ be the domain of a system of local coordinates $x^1, \dots, x^n$, so small that E is trivial over $\mathcal{U}$. Fix local non-vanishing sections η of E and η^* of E^* over $\mathcal{U}$ (part of local frames). Let $\chi_1, \chi_0 \in C_c^\infty(\mathcal{U})$ be such that $\chi_1 = 1$ on $\operatorname{supp}\chi_0$. The operator

$$C^\infty(\mathcal{M}) \ni \varphi \mapsto P_{\chi_1,\chi_0,\eta,\eta^*}(\varphi) = \langle \theta^*, \chi_1 P(\chi_0\varphi\eta)\rangle \in C^\infty(\mathcal{M})$$

where the pairing is meant to be fiberwise so it produces a function, is still continuous, linear, and local, and satisfies the same estimate as P: for any smooth f and φ there is $C > 0$ such that

$$|e^{-i\tau f} P_{\chi_1,\chi_0,\eta,\eta^*}(e^{i\tau f}\varphi)(x)| \le C(1+|\tau|^m).$$

We will omit the reference to η and η^* in the next paragraphs. P_{χ_1,χ_0} acts on functions and produces functions, and is local. Let

$$p_{\chi_1,\chi_0}(x,\xi) = e^{-ix\cdot\xi} P_{\chi_1,\chi_0}(e^{i(\cdot)\cdot\xi}),$$

a smooth function bounded by $C(1+|\xi|)^m$, as is seen by writing $\xi = |\xi|\breve{\xi}$, $|\breve{\xi}| = 1$.

2. Local representation

We continue with the notation of the previous section.

THEOREM 2.1. *If $\varphi \in C_c^\infty(\mathcal{U})$, then*

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} p_{\chi_1,\chi_0}(x,\xi)\hat{\varphi}(\xi)\,d\xi = P_{\chi_1,\chi_0}(\varphi)(x). \tag{2.2}$$

The integral converges absolutely because p_{χ_1,χ_0} is tempered in ξ and $\hat{\varphi}$ is rapidly decreasing.

PROOF. Indeed,

$$\begin{aligned}
&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} p_{\chi_1,\chi_0}(x,\xi)\hat{\varphi}(\xi)\,d\xi \\
&\qquad = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}[e^{-ix\cdot\xi} P_{\chi_1,\chi_0}(e^{i(\cdot)\cdot\xi})(x)]\hat{\varphi}(\xi)\,d\xi \\
&\qquad = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} P_{\chi_1,\chi_0}(e^{i(\cdot)\cdot\xi}\hat{\varphi}(\xi))(x)\,d\xi \\
&\qquad = P_{\chi_1,\chi_0}\big(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\cdot)\cdot\xi}\hat{\varphi}(\xi)\big)(x)\,d\xi \\
&\qquad = P_{\chi_1,\chi_0}(\varphi)(x)
\end{aligned}$$

using linearity in the second equality, continuity and linearity in the third, and the Fourier inversion formula in the fourth. (Since P is local, χ_1 is not really relevant here; later it will be useful). □

The operator $\varphi \mapsto P_{\chi_1,\chi_0}(\varphi)$ has Schwartz kernel

$$K_{\chi_1,\chi_0}(x,x') = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} p_{\chi_1,\chi_0}(x,\xi)\,d\xi,$$

a distribution on $\mathcal{U}\times\mathcal{U}$. (Since $p_{\chi_1,\chi_0}(x,\xi)$ is tempered in ξ, the partial inverse Fourier transform

$$\check{p}_{\chi_1,\chi_0}(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{iy\cdot\xi} p_{\chi_1,\chi_0}(x,\xi)\,d\xi$$

is well defined in the distributional sense; replace $y = x - x'$.) The kernel K_{χ_1,χ_0} has te property that

$$\langle P_{\chi_1,\chi_0}(\varphi),\psi\rangle = \langle K_{\chi_1,\chi_0}, \psi\boxtimes\varphi\rangle, \quad (v\boxtimes u)(x,x') = v(x)u(x')$$

for any $u, v \in C_c^\infty(\mathcal{U})$. Consequently the distribution K_{χ_1,χ_0} vanishes on finite linear combinations of functions of the form $v\boxtimes u$ with disjoint supports. In general the space of finite linear combinations of functions of the form $v\boxtimes u$ is dense in $C^\infty(\mathcal{U}\times\mathcal{U})$ (see Ehrenpreis [**8**, Lemma 1]). The conclusion is that K_{χ_1,χ_0} vanishes on functions on $\mathcal{U}\times\mathcal{U}$ whose support is disjoint from the diagonal in $\mathcal{U}\times\mathcal{U}$. That is, $\operatorname{supp} K_{\chi_1,\chi_0} \subset \operatorname{diag}(\mathcal{U}\times\mathcal{U})$.

The fact that $\operatorname{supp} K_{\chi_1,\chi_0} \subset \operatorname{diag}(\mathcal{U}\times\mathcal{U}) \cap \operatorname{supp}(\chi_1 \boxtimes \chi_0)$ is compact implies there are C and M such that

$$|\langle K_{\chi_1,\chi_0}, w\rangle| \le \sup_{(x,x')\in S, |\alpha|+|\beta|\le M} |(\partial_x - \partial_{x'})^\alpha (\partial_x + \partial_{x'})^\beta w(x,x')| \tag{2.3}$$

We show (with a proof adapted from Hörmander [**13**, p. 12]):

> if $(\partial_x - \partial_{x'})^\alpha w(x,x') = 0$ for $|\alpha| \le M$ on $\operatorname{diag}(\mathcal{U}\times\mathcal{U})$, then $\langle K_{\chi_1,\chi_0}, w\rangle = 0$.

For such w one has

$$w(x.x') = \sum_{|\alpha|=M+1} w_\alpha(x,x')(x-x')^\alpha$$

Let $\chi \in C_c^\infty$ be equal to 1 in a neighborhood of 0. With the function $\chi_\varepsilon(x,x') = \chi(\varepsilon^{-1}(x-x'))$ we have

$$|(\partial_x - \partial_{x'})^\alpha (\partial_x + \partial_{x'})^\beta \chi_\varepsilon w(x,x')| \le C\varepsilon^{M+1-|\alpha|}.$$

Since K_{χ_1,χ_0} is supported on the diagonal and χ_ε is equal to 1 in a neighborhood of $\operatorname{supp} K_{\chi_1,\chi_0}$, we have $K_{\chi_1,\chi_0} = \chi_\varepsilon K_{\chi_1,\chi_0}$. It follows that $\langle K_{\chi_1,\chi_0}, w\rangle = \langle \chi_\varepsilon K_{\chi_1,\chi_0}, w\rangle = \langle K_{\chi_1,\chi_0}, \chi_\varepsilon w\rangle$ and from this and (2.3) that

$$|\langle K_{\chi_1,\chi_0}, w\rangle| \le C\varepsilon.$$

Consequently, $(x-x')^\alpha K = 0$ if $|\alpha| \ge M+1$. Indeed, with such α

$$\langle (x-x')^\alpha K_{\chi_1,\chi_0}, w\rangle = \langle K_{\chi_1,\chi_0}, (x-x')^\alpha w\rangle = 0$$

for any compactly supported smooth function w. But

$$\begin{aligned}(x-x')^\alpha K_{\chi_1,\chi_0} &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}} (x-x')^\alpha e^{i(x-x')\cdot\xi}\, p_{\chi_1,\chi_0}(x,\xi)\, d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}} D_\xi^\alpha e^{i(x-x')\cdot\xi}\, p_{\chi_1,\chi_0}(x,\xi)\, d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}} e^{i(x-x')\cdot\xi}\, (-D_\xi)^\alpha p_{\chi_1,\chi_0}(x,\xi)\, d\xi\end{aligned}$$

Replacing $(x-x') = y$, get

$$0 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}} e^{iy\cdot\xi}\, (-D_\xi)^\alpha p_{\chi_1,\chi_0}(x,\xi)\, d\xi$$

The Fourier transform in y yields $(-D_\xi)^\alpha p_{\chi_1,\chi_0}(x,\xi) = 0$ whenever $|\alpha| \ge M+1$. So $\xi \mapsto p_{\chi_1,\chi_0}(x,\xi)$ is a polynomial of degree $\le M$. Because of the estimate (1.11), $p_{\chi_1,\chi_0}(x,\xi)$ is a polynomial in ξ of degree (at most) m. Therefore, by (2.2), $\varphi \mapsto P_{\chi_1,\chi_0}(\varphi)$ is a differential operator of order m. Since this is true for any χ_0 and χ_1 and local non-vanishing sections η of E and θ^* of E^*, so P itself is a differential operator of order m. □

The following proposition follows from the proof

Proposition 2.4. *With the hypotheses of Proposition 1.10, suppose $\mathcal{U}$ is the domain of a system of local coordinates $x^1,\dots,x^n$, so small that E is trivial over $\mathcal{U}$. Let $\eta_1\dots,\eta_r$ be a frame of E over $\mathcal{U}$. Then there are (scalar) differential operators*

$$P_j^k(x,D_x)\varphi = \sum_{|\alpha|\le m} p_{j\alpha}^k(x) D_x^\alpha \varphi$$

such that

$$P(\sum_j u^j \eta_j) = \sum_{k,j} P_j^k(u^j)\eta_k.$$

The fact that if P is a differential operator, then $\operatorname{supp} Pu \subset \operatorname{supp} u$ implies: if $u = v$ near p_0, then $Pu = Pv$ near x_0.

Let $\mathcal{K} \subset \mathcal{M}$ be compact, let $N \in \mathbb{N}_0$. If $P \in \operatorname{Diff}^m(\mathcal{M}; E)$, then

$$\wp_{P,\mathcal{K}}(u) = \sup_{x\in\mathcal{K}} |Pu(x)|, \quad u \in C^\infty(\mathcal{M}; E),$$

is a seminorm. The topology of $C^\infty(\mathcal{M}; E)$ described at the beginning of this section is the largest topology generated by the collection of seminorms of the form $\wp_{P,\mathcal{K}}$ with $P \in \operatorname{Diff}^N(\mathcal{M}; E)$ for any N and $\mathcal{K} \Subset \mathcal{M}$.

We finish with an auxiliary result.

LEMMA 2.5. *Let $P \in \operatorname{Diff}(\mathcal{M}; E)$. Then, for any $u \in C^\infty(\mathcal{M}; E)$, the operator $C^\infty \ni f \mapsto P(fu) \in C^\infty(\mathcal{M}; E)$ is a differential operator of order m.*

PROOF. By induction. This holds if P is of order 0. Suppose it holds for any operator of order $m-1$ and any u. Suppose P is of order m. Since

$$P(fu) = fP(u) + [P, f]u,$$

we see that we need to show that $f \mapsto [P, f]u$ is of order m. We do this by showing that if $h \in C^\infty(\mathcal{M})$, then

$$C^\infty \ni f \mapsto [P, hf]u - h[P, f]u \in C^\infty(\mathcal{M}; E) \tag{2.6}$$

is of order $m-1$ for any $h \in C^\infty(\mathcal{M})$. Expanding the brackets we get

$$[P, hf]u - h[P, f]u = [P, h](fu).$$

Since $[P, h] \in \operatorname{Diff}^{m-1}(\mathcal{M}; E)$, the induction hypothesis gives that (2.6) is of order $m-1$. □

Locally, in coordinates $x_1, \ldots, x^n$ and a local frame η_j of E on U,

$$P(\sum_j u^j \eta_j) = \sum_k (\sum_{\alpha,j} p_{j,\alpha}^k D_x^\alpha u^j)\eta_k$$

and

$$P(\sum_j u^j \eta_j)(fu) = \sum_k \sum_{\alpha,j} \sum_{\alpha'+\alpha''=\alpha} \frac{\alpha!}{\alpha'!\alpha''!} p_{j,\alpha}^k (D_x^{\alpha'} f)(D_x^{\alpha''} u^j)\eta_k$$

3. Adjoints

Suppose now that $E \to \mathcal{M}$ is Hermitian and $\mathfrak{m}$ is a positive density on $\mathcal{M}$. Define

$$(u, v)_0 = \int_{\mathcal{M}} (u(x), v(x))_x \, d\mathfrak{m}(x), \quad u, v \in C^\infty(\mathcal{M}; E)$$

and let $L^2(\mathcal{M}; E)$ be the closure of $C_c^\infty(\mathcal{M}; E)$ by the norm defined with this inner product. The formal adjoint of $P \in \operatorname{Diff}^m(\mathcal{M}; E)$ is the operator $P^\star : C_c^\infty(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$ such that

$$(P^\star u, v) = (u, Pv)_0 \quad \text{for all } u, v \in C_c^\infty(\mathcal{M}; E).$$

It exists because of the Riesz representation theorem; uniqueness holds because $C_c^\infty(\mathcal{M}; E)$ is dense in $L^2(\mathcal{M}; E)$.

THEOREM 3.1. *If $P \in \mathrm{Diff}^m(\mathcal{M};E)$, then $P^\star \in \mathrm{Diff}^m(\mathcal{M};E)$.*

The shortest argument is by using the local representation of Proposition 2.4 and integration by parts.

LEMMA 3.2. *For any $P \in \mathrm{Diff}^m(\mathcal{M};E)$, $\boldsymbol{\sigma}(P^*)(\boldsymbol{\xi}) = \boldsymbol{\sigma}(P)(\boldsymbol{\xi})^*$.*

PROOF. Let $f : \mathcal{M} \to \mathbb{R}$ be real valued, $u, v \in C_c^\infty(\mathcal{M})$. Then

$$(e^{-i\tau f} P(e^{i\tau f}u, v))_0 = (u, e^{-i\tau f} P^\star(e^{i\tau f}v))_0$$

Assuming u and v with support in the set $\mathcal{U}$ where $df \neq 0$, this gives

$$(\boldsymbol{\sigma}(P)(df)(u), v)_0 = (u, \boldsymbol{\sigma}(P^\star)(df)(v))_0$$

after dividing m and letting $\tau \to \infty$. Hence

$$(u, \boldsymbol{\sigma}(P)(df)^*(v))_0 = (u, \boldsymbol{\sigma}(P^\star)(df)(v))_0$$

for any $u \in C_c^\infty(\mathcal{U})$. □

EXAMPLE 3.3. A connection on E is a continuous operator

$$\nabla : C^\infty(\mathcal{M};E) \to C^\infty(\mathcal{M}; E \otimes T^*\mathcal{M})$$

such that

$$\nabla(fu) = f\nabla u + u \otimes df, \quad f \in C^\infty(\mathcal{M}),\ u \in C^\infty(\mathcal{M};E).$$

Connections are first order operators: $u \mapsto \nabla(fu) - f\nabla u = u \otimes df$ is evidently of order 0. The principal symbol of ∇ is computed using (1.9): using

$$e^{-i\tau f}\nabla(e^{i\tau f}u) = \nabla u + u \otimes i\tau df$$

get

$$\boldsymbol{\sigma}(\nabla)(\boldsymbol{\xi})(\cdot) = \lim_{\tau\to\infty} \frac{1}{\tau} e^{-i\tau f}\nabla(e^{i\tau f}u) = i(\cdot) \otimes \boldsymbol{\xi}.$$

For $\boldsymbol{\xi}_0 \in \dot{T}^*_{x_0}$, the mapping $E_x \ni u_0 \mapsto iu_0 \otimes \boldsymbol{\xi}_0$ is injective.

Now pick a metric g on $T^*\mathcal{M}$, use it together with the Hermitian form of E to get an inner product on the fibers of $E \otimes T^*\mathcal{M}$ and with thia, the formal adjoint of ∇. For $\boldsymbol{\eta}, \boldsymbol{\xi} \in T^*\mathcal{M}$ define $\boldsymbol{\eta} \rfloor \boldsymbol{\xi} = g(\boldsymbol{\eta}, \boldsymbol{\xi})$. Then

$$\boldsymbol{\sigma}(\nabla^\star)(\boldsymbol{\xi})(u \otimes \boldsymbol{\xi}) = \boldsymbol{\sigma}(\nabla)(\boldsymbol{\xi})^*(u \otimes \boldsymbol{\xi}) = (\boldsymbol{\xi} \rfloor \boldsymbol{\xi})u = |\boldsymbol{\xi}|^2 u.$$

This is, the principal symbol of $\nabla^\star\nabla$ is $|\boldsymbol{\xi}|^2 I$.

CHAPTER VIII

Pseudodifferential operators

1. Symbols

Let $\mathcal{M}$ be a smooth manifold, not necessarily closed, $E \to \mathcal{M}$ a vector bundle, $\pi : \mathcal{M} \times \mathbb{R}^+ \to \mathcal{M}$ the projection.

DEFINITION 1.1. Let $m \in \mathbb{R}$. The space $S^m(\mathcal{M}; E)$ of symbols on $\mathcal{M}$ of order m with values in E consists of all $a \in C^\infty(\mathcal{M} \times \mathbb{R}^+; \pi^* E)$ such that for any seminorm $\wp_E$ of $C^\infty(\mathcal{M}; E)$ and $k \in \mathbb{N}_0$ there is C such that

$$\wp_E(\frac{d^k}{d\tau^k} a(\cdot, \tau)) \leq C(1+\tau)^{m-k}$$

The functional $\wp_{E,k} : S^m(\mathcal{M}; E) \to \mathbb{R}$ given by

$$\wp_{E,k}(a) = \sup_{\tau \in \mathbb{R}^+} \frac{\wp_E(d^k a/d\tau^k(\cdot, \tau))}{(1+\tau)^{m-k}}$$

is a seminorm on $S^m(\mathcal{M}; E)$. The topology of $S^m(\mathcal{M}; E)$ is the largest topology generated by these seminorms.

A section $a \in C^\infty(\mathcal{M} \times \mathbb{R}^+; \pi^* E)$ with the property that $\tau^{-m} a_m(\cdot, \tau)$ is independent of τ for large τ uniformly on compact subsets of $\mathcal{M}$ is said to be eventually positively homogeneous of degree m. Let $S^m_h(\mathcal{M}; E)$ denote the space of such elements.

DEFINITION 1.2. The space of classical symbols on $\mathcal{M}$ of order m with values in E, denoted $S^m_{c\ell}(\mathcal{M}; E)$, consists of those $a \in S^m(\mathcal{M}; E)$ for which there is $a_m \in S^m_h(\mathcal{M}; E)$ such that

(*i*) $a - a_m \in S^{m-1}(\mathcal{M}; E)$

(*ii*) $a - a_m$ is classical of order $m-1$.

The section

$$\tau^m \lim_{\tau \to \infty} \tau^{-m} a_m(\cdot, \tau)$$

is the principal part of a; often we write a_m also for this element.

If $a \in S^m_{c\ell}(\mathcal{M}; E)$ and a_m is its principal part, then $a - a_m \in S^{m-1}_{c\ell}(\mathcal{M}; E)$ has a principal part a_{m-1} and $a - a_m - a_{m-1} \in S^{m-2}(\mathcal{M}; E)$. More generally:

LEMMA 1.3. *Let $a \in S^m_{c\ell}(\mathcal{M}; E)$. There are*

$$a_{m-j} \in S^{m-j}_h(\mathcal{M}; E),\ j = 0, 1, \ldots$$

such that

$$\text{for any } N \in \mathbb{N}_0,\ a - \sum_{j=0}^{N} a_{m-j} \in S^{m-N-1}(\mathcal{M}; E). \tag{1.4}$$

The a_{m-j} are essentially unique: if $\tilde{a}_{m-j} \in S_h^{m-j}(\mathcal{M};E)$ also satisfy (1.4)*, then $a_{m-j} - \tilde{a}_{m-j} = 0$ for large τ uniformly over compact sets.*

The uniqueness is proved by induction beginning with

$$\tau^{-m} a_m(\cdot,\tau) = \lim_{s\to\infty} s^{-m} a(\cdot,s)$$

for large τ

If $a \in S^m_{c\ell}(\mathcal{M};E)$ and $a_{m-j} \in S^{m-j}_{c\ell}(\mathcal{M};E)$ are such that (1.4) holds for all N, then we say that a is asymptotic to $\sum a_{m-j}$, written

$$a \sim \sum_{j=0}^{\infty} a_{m-j}$$

2. Pseudodifferential operators

If $A : C_c^\infty(\mathcal{M};E) \to C^\infty(\mathcal{M};E)$ is linear and continuous, if $f : \mathcal{M} \to \mathbb{R}$ is a smooth function and u a smooth section of E, then

$$a(f,u,\tau) = e^{-i\tau f} A(e^{i\tau f} u)$$

is a smooth section of $\pi^*(E)$. In the next definition we write $C^\infty_{\neg 0}(\mathcal{U})$ for the subset of $C^\infty(\mathcal{U},\mathbb{R})$ whose elements are real-valued and satisfy $df(x) \neq 0$ for all $x \in \mathcal{U}$. We give $C^\infty_{\neg 0}(\mathcal{U})$ the subspace topology. Also, $E_{\mathcal{U}}$ means the part of E over $\mathcal{U}$. The map $(f,u) \mapsto a(f,u,\tau)$ is continuous, but we need the asymptotic behavior as $\tau \to \infty$. This is why we will require df nowhere 0 on $\operatorname{supp}(u)$.

Definition 2.1. Let $A : C_c^\infty(\mathcal{M};E) \to C^\infty(\mathcal{M};E)$ be a continuous linear operator.

(*i*) A is a pseudodifferential operator of order m if for every open $\mathcal{U} \subset \mathcal{M}$,

$$a(f,u,\tau) \in \mathcal{S}^m(\mathcal{U},E_{\mathcal{U}}) \quad \text{for all } f \in C^\infty_{\neg 0}(\mathcal{U}) \text{ and } u \in C_c^\infty(\mathcal{U},E_{\mathcal{U}}),$$

and additionally, for all $f_0 \in C^\infty_{\neg 0}(\mathcal{U})$, for all $\varepsilon > 0$, for all seminorms $\wp_E$ of $C^\infty(\mathcal{U};E_{\mathcal{U}})$ there are seminorms $\wp_{\neg 0}$ of $C^\infty(\mathcal{M})$ and $\wp'_E$ of $C^\infty(\mathcal{U};E_{\mathcal{U}})$ such that

$$\wp_E\Big(\frac{d^k}{d\tau^k}(a(f,u,\tau) - a(f_0,u,\tau))\Big) \leq \varepsilon(1+\tau)^{m-k}\wp'_E(u)$$

if $\wp_{\neg 0}(f - f_0) < 1$ and $u \in C_c^\infty(\mathcal{U},E_{\mathcal{U}})$. The space of pseudodifferential operators of order m is denoted $\Psi^m(\mathcal{M};E)$.

In particular, $a(f,u,\tau) \in S^m(\mathcal{U},E)$. (If just $\wp_{\neg 0}(f-f_0) < r$, $r \neq 0$, for a given seminorm, then $r^{-1}\wp_{\neg 0}(f-f_0) < 1$, so we can adjust the seminorm according to our convenience.) The estimates express the continuity of $(f,u) \mapsto p(f,u,\tau)$ written out explicitly.

(*ii*) An operator $A \in \Psi^m(\mathcal{M};E)$ is a classical pseudodifferential operator, denoted $A \in \Psi^m_{c\ell}(M;E)$, if for every open $\mathcal{U} \subset \mathcal{M}$, any $f \in C^\infty_{\neg 0}(\mathcal{U})$ and $u \in C_c^\infty(\mathcal{U};E_{\mathcal{U}})$ there are $a_{m-j}(f,u,\tau) \in S_h^{m-j}(\mathcal{U};E_{\mathcal{U}})$ continuous as a map $(f,u) \mapsto a_{m-j}(f,u,\tau)$ such that

$$u \mapsto \tau^{-(m-j)} a_{m-j}(f,u,\tau)$$

is a differential operator of order j independent of τ for large τ uniformly on compact sets of $\mathcal{U}$, such that

$$a(f,u,\tau) - \sum_{j=0}^{N} a_{m-j}(f,u,\tau) \in S^{m-N-1}(\mathcal{U}, E_{\mathcal{U}})$$

for any $N \in \mathbb{N}_0$.

For an excellent short exposition on pseudodifferential operators, see Nirenberg [**23**]. Much of this section is based on that paper.

We now define the principal symbol of $A \in \Psi^m_{c\ell}(\mathcal{M}; E)$. Let $x_0 \in \mathcal{M}$, and $f, \tilde{f} \in C^\infty(\mathcal{M})$ be real-valued with $df(x_0) = d\tilde{f}(x_0) \neq 0$. Let $\mathcal{U}$ be a neighborhood of x_0 where df and $d\tilde{f}$ do not vanish. In general, as with differential operators, $a(f,u,\tau)(x_0) = a(f - f(x_0), u, \tau)(x_0)$ for any u, so we may assume $f(x_0) = \tilde{f}(x_0) = 0$. The task is to show that $\tau^{-m} a_m(f,u,\tau)(x_0)$ depends only of $df(x_0)$ and $u(x_0)$ when τ is large.

First, if $\tilde{f} = f$ in some neighborhood $\mathcal{V} \subset \mathcal{U}$ of x_0, then $a(f,u,\tau) = a(\tilde{f},u,\tau)$ if $\operatorname{supp} u \subset \mathcal{V}$, so also $a_m(f,u,\tau) = a_m(\tilde{f},u,\tau)$ for such u. Since $u \mapsto \tau^{-m} a_m(f,u,\tau)$ is of order 0 and independent of τ for large τ,

$$\tau^{-m} a_m(f,\cdot,\tau)(x_0) = \tau^m a_m(\tilde{f},\cdot,\tau)(x_0) \text{ for large } \tau.$$

Next, assume only that $\tilde{f} - f$ vanishes to first order at x_0. Let $\chi \in C^\infty_c(\mathcal{U})$ be equal to 1 near x_0 and supported in a neighborhood $\mathcal{V}$ of x_0, so small that

$$d\big(f + s(\tilde{f} - f)\chi\big) \neq 0 \text{ on } \mathcal{V} \text{ when } s \in [0,1].$$

Let $h = \chi(\tilde{f} - f)$. We will show $a_m(f,u,\tau)(x_0) = a_m(f+h,u,\tau)(x_0)$ for large τ. From the previous paragraph we have

$$a_m(f+h,u,\tau)(x_0) = a_m(f + (\tilde{f} - f),u,\tau)(x_0) \text{ for large } \tau$$

hence

$$\tau^{-m} a_m(f,\cdot,\tau)(x_0) = \tau^m a_m(\tilde{f},\cdot,\tau)(c_0) \text{ for large } \tau \text{ if } df(x_0) = d\tilde{f}(x_0).$$

Inserting

$$e^{i\tau(f+h)} = e^{i\tau f}(1 + i\tau h \int_0^1 e^{i\tau s h}\, ds)$$

in the definition of $a(f+h,u,\tau)$, then using linearity and continuity and reorganizing, yields

$$\begin{aligned}
&e^{-i\tau(f+h)} A(e^{i\tau f}(1 + i\tau h \int_0^1 e^{i\tau s h}\, ds)\, u) \\
&\qquad = e^{-i\tau(f+h)} A(e^{i\tau f} u) + i\tau e^{-i\tau(f+h)} A(e^{i\tau f} h \int_0^1 e^{i\tau s h}\, ds\, u) \\
&\qquad = e^{-i\tau h} a(f,u,\tau) + i\tau e^{-i\tau(f+h)} \int_0^1 A(e^{i\tau f} h e^{i\tau s h} u)\, ds \\
&\qquad = e^{-i\tau h} a(f,u,\tau) + i\tau \int_0^1 e^{-i\tau(1-s)h} a(f + sh, hu, \tau)\, ds
\end{aligned}$$

Thus

$$a_m(f+h,u,\tau)(x_0) = \\ e^{-i\tau h(x_0)}a_m(f,u,\tau)(x_0) + i\tau\int_0^1 e^{-i\tau(1-s)h(x_0)}a_m(f+sh,hu,\tau)(x_0)\,ds$$

Since $u \mapsto a_m(f+sh,hu,\tau)$ is of order 0,

$$a_m(f+sh,hu,\tau)(x_0) = a_m(f+sh,h(x_0)u(x_0),\tau) = 0.$$

Thus

$$a_m(f+h,u,\tau)(x_0) = p_m(f,u,\tau)(x_0).$$

DEFINITION 2.2. Given the covector $\boldsymbol{\xi} \in \dot{T}^*\mathcal{M}$ at x_0, let $f : \mathcal{M} \to \mathbb{R}$ be smooth with $df(x_0) = \boldsymbol{\xi}$. For $A \in \Psi^m_{c\ell}(\mathcal{M};E)$ define

$$\boldsymbol{\sigma}_m(A)(\boldsymbol{\xi}) : E_{x_0} \to E_{x_0}, \quad \boldsymbol{\sigma}_m(A)(\boldsymbol{\xi})(u_0) = \lim_{\tau\to\infty}\frac{1}{\tau^m}e^{-i\tau f}A(e^{i\tau f}u)(x_0)$$

where $u \in C^\infty_c(\mathcal{M};E)$ is any extension of u_0 with support in $\{x : df(x) \neq 0\}$. Thus

$$\boldsymbol{\sigma}_m(A)(\boldsymbol{\xi})(u_0) = \tau^{-m}a(f,u,\tau)(x_0), \quad \tau \text{ large}.$$

The symbol of $A \in \Psi^m_{c\ell}(\mathcal{M};E)$ is a smooth section of $\operatorname{Hom}(\dot\pi^*E,\dot\pi^*E)$, positively homogeneous of degree m:

$$\boldsymbol{\sigma}_m(A)(s\boldsymbol{\xi}) = s^m\,\boldsymbol{\sigma}_m(A)(\boldsymbol{\xi}), \quad s > 0$$

thus an element of

$$S^m_h(\dot T^*\mathcal{M};\pi^*E) = \{a \in C^\infty(\dot T^*\mathcal{M};\operatorname{End}(\dot\pi^*E)) : \\ a \text{ is positively homogeneous of degree } m\}$$

Observe also that $\Psi^{m-1}_{c\ell}(\mathcal{M};E) \subset \Psi^m_{c\ell}(\mathcal{M};E)$ and that by the definition, $\boldsymbol{\sigma}_m(A)$, vanishes if $A \in \Psi^{m-1}_{c\ell}(\mathcal{M};E)$. In fact:

THEOREM 2.3 (Symbol sequence). *The sequence*

$$0 \to \Psi^{m-1}(\mathcal{M};E) \xrightarrow{\iota} \Psi^m(\mathcal{M};E) \xrightarrow{\boldsymbol{\sigma}_m} S^m_h(\dot T^*\mathcal{M};\pi^*E) \to 0$$

is exact.

THEOREM 2.4. *Let $P \in \operatorname{Diff}^m(\mathcal{M};E)$, let $A \in \Psi^{m'}_{c\ell}(\mathcal{M};E)$. Then*

(i) $P\circ A$ and $A\circ P$ belong to $\Psi^{m+m'}_{c\ell}(\mathcal{M};E)$;

(ii) $\boldsymbol{\sigma}_{m+m'}(P\circ A)(\boldsymbol{\xi}) = \boldsymbol{\sigma}_m(P)(\boldsymbol{\xi})\circ\boldsymbol{\sigma}_{m'}(A)(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \dot T^*\mathcal{M}$, likewise for $A\circ P$.

This also holds when P is a pseudodifferential operator under some mild conditions, but the proof is more delicate. We will restate it after the proof for the sake reference in the form we will use it, just adding the hypothesis that $\mathcal{M}$ is closed. The issue in general is to have some control of supports.

PROOF. Let $\mathcal{U} \subset \mathcal{M}$ be open, $f \in C^\infty_{-0}(\mathcal{U})$, $u \in C^\infty_c(u)$. Then

$$\begin{aligned} e^{-i\tau f}(P\circ A)(e^{i\tau f}u) &= e^{-i\tau f}P(A(e^{i\tau f}u)) \\ &= e^{-i\tau f}P(e^{i\tau f}e^{-i\tau f}A(e^{i\tau f}u)) \\ &= e^{-i\tau f}P(e^{i\tau f}a(f,u,\tau)) \end{aligned}$$

But $e^{-i\tau f}P(e^{i\tau f}v) = \sum_\ell \tau^{m-\ell} p_{m-\ell}(f,v)$ where $v \mapsto p_{m-\ell}(f,v)$ is a differential operator of order ℓ; here the support of v or whether df may vanish in some sets is not an issue. Using this,

$$e^{-i\tau f}(P\circ A)(e^{i\tau f}u) = \sum_{\ell=0}^{m} \tau^{m-\ell} p_{m-\ell}(f, a(f,u,\tau)).$$

But $a_{m-j}(f,a(f,u,\tau))$ is an element of $S^{m-j+m'}(\mathcal{M};E)$ depending continuously on (f,u), so $P\circ A \in \Psi^{m+m'}(\mathcal{M};E)$.

To show that $P\circ A$ is classical, observe that

$$b_{m+m'-k}(f,u,\tau) = \sum_{\substack{j+\ell=k\\ \ell\le m}} \tau^{m-\ell} p_{m-\ell}(f, a_{m'-j}(f,u,\tau))$$

belongs to $S_h^{m+m'-k}(\mathcal{U},E_{\mathcal{U}})$, that $u \mapsto \tau^{-m-m'+k} b_{m+m'-k}(f,u,\tau)$ is a differential operator of order k, is independent of τ when τ is large, and that

$$\sum_{\ell=0}^{m} \tau^{m-\ell} p_{m-\ell}(f,a(f,u,\tau)) - \sum_{k=0}^{N} b_{m+m'-k}(f,u,\tau) \in S^{m+m'-N-1}(\mathcal{U};E_{\mathcal{U}}).$$

The statement about the principal symbols follows from

$$b_{m+m'} = p_m(f, a_{m'}(f,u,\tau),\tau).$$

This completes the proof of the theorem. □

Locally, the proof of Part (i) goes as follows. Suppose that A and A are scalar operators. In the domain U of the local chart $x^1,\dots,x^n$, the operators are

$$Pu(x) = \sum_{|\beta|\le m} p_\beta(x) D_x^\beta u(x), \quad Av(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi} a(x,\xi)\hat v(\xi)\,d\xi.$$

when $u \in C^\infty(U)$ and $v \in C_c^\infty(U)$. By the Leibniz rule,

$$\begin{aligned} D_x^\beta\big(e^{ix\cdot\xi}a(x,\xi)\big) &= e^{ix\cdot\xi}\sum_{\alpha\le\beta}\frac{\beta!}{\alpha!(\beta-\alpha)!}\xi^{\beta-\alpha}D_x^\alpha a(x,\xi)\\ &= \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha\xi^\beta\, D_x^\alpha a(x,\xi)\end{aligned}$$

So with $p(x,\xi) = \sum_{|\beta|\le m} a_\beta(x),\xi^\beta$ we have

$$(P\circ A)(v) = \frac{1}{(2\pi)^n}\int \sum_\alpha \frac{1}{\alpha!}\partial_x^\alpha p(x,\xi)\, D_x^\alpha a(x,\xi)\hat v(\xi)\,d\xi. \tag{2.5}$$

This is a finite sum because $p(x,\xi)$ is polynomial in ξ. Note that

$$Pu(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}p(x,\xi)\hat u(\xi)\,d\xi.$$

when u is supported in U. The formula for the symbol representing the composition of two pseudodifferential operators generalizes this. If $b(x,\xi)$ gives B on U, then (with some care about supports), the symbol c corresponding to $B\circ A$ is

$$c(x,\xi) \sim \sum_\alpha \partial_\xi^\alpha b(x,\xi)\, D_x^\alpha a(x,\xi).$$

THEOREM 2.6. *Let $\mathcal{M}$ be closed, let $B \in \Psi^m_{c\ell}(\mathcal{M};E)$ and $A \in \Psi^{m'}_{c\ell}(\mathcal{M};E)$. Then*

(i) $B\circ A$ and $A\circ B$ belong to $\Psi^{m+m'}_{c\ell}(\mathcal{M};E)$;

(ii) $\boldsymbol{\sigma}_{m+m'}(B\circ A)(\boldsymbol{\xi}) = \boldsymbol{\sigma}_m(B)(\boldsymbol{\xi})\circ\boldsymbol{\sigma}(A)(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}\in \dot{T}^*\mathcal{M}$, similarly for $A\circ B$.

PROPOSITION 2.7. *If $A \in \bigcap_{N\in\mathbb{Z}}\Psi^N(\mathcal{M};E)$, then the Schwartz kernel of A is a smooth section of $E^*\boxtimes E$.*

This will be proved later as part of Proposition 4.2. Let $\Psi^{-\infty}(\mathcal{M};E) = \bigcap_{N\in\mathbb{Z}}\Psi^N(\mathcal{M};E)$.

THEOREM 2.8 (Asymptotic summability). *Given $A_{m-j}\in\Psi^{m-j}(\mathcal{M};E)$, $j = 0,1,\dots$, there is $A\in\Psi^m(\mathcal{M};E)$ such that for each N,*

$$A - \sum_{j=0}^{N} A_{m-j} \in \Psi^{m-N-1}(\mathcal{M};E). \tag{2.9}$$

If A' satisfies the same property, then $A-A'\in\Psi^{-\infty}(\mathcal{M};E)$.

One says that A is the asymptotic sum of the operators A_{m-j}. We will show later (Lemma 4.3) that the operators in $\Psi^{-\infty}(\mathcal{M};E)$ are precisely the smoothing operators, those whose Schwartz kernel is smooth. So operators with the same asymptotic sum differ by a smoothing operator.

3. Parametrices of elliptic differential operators

THEOREM 3.1. *If $P\in\mathrm{Diff}^m(\mathcal{M};E)$ is elliptic then there is $A\in\mathrm{Diff}^{-m}(\mathcal{M};E)$ such that*

$$P\circ A = I - R', \quad A\circ P = I - R''$$

where $R'', R'\in\Psi^{-\infty}(\mathcal{M};E)$ are pseudodifferential operators of arbitrary negative order.

PROOF. Ellipticity means that $\boldsymbol{\sigma}_m(P)(\boldsymbol{\xi})$ is invertible for all $\boldsymbol{\xi}\in\dot{T}^*\mathcal{M}$. Let $a_{-m}(\boldsymbol{\xi}) = \boldsymbol{\sigma}_m(P)(\boldsymbol{\xi})^{-m}$; this is positively homogeneous of degree $-m$, so there is $A_{-m}\in\Psi^{-m}_{c\ell}(\mathcal{M};E)$ such that $\boldsymbol{\sigma}(A_{-m}) = \boldsymbol{\sigma}_m(P)^{-1}$. Then $P\circ A_m\in\Psi^0_{c\ell}(\mathcal{M};E)$ and furthermore, $\boldsymbol{\sigma}_0(P\circ A_m) = \mathrm{id}\in C^\infty(\dot{T}^*\mathcal{M};\mathrm{End}(\dot{\pi}^*E))$. Thus

$$\boldsymbol{\sigma}_0(P\circ A_{-m} - I) = 0$$

because $\mathrm{id} = \boldsymbol{\sigma}_0(I)$, hence $P\circ A_{-m} - I\in\Psi^{-1}_{c\ell}(\mathcal{M};E)$. Let $R_{-1} = I - P\circ A_{-m}$, let $A_{-m-1} = A_{-m}\circ R_{-1}$. Then $A_{-m-1}\in\Psi^{-m-1}_{c\ell}(\mathcal{M};E)$ and

$$P\circ(A_{-m}+A_{-m-1}) = (I-R_1) + (I-R_{-1})R_{-1} = I - R_{-1}^2.$$

In general, let $A_{-m-j} = A_{-m}R_{-1}^j$. Then $A_{-m-j}\in\Psi^{-m-j}_{c\ell}(\mathcal{M},E)$ and

$$P\circ\sum_{j=0}^{N} A_{-m-j} = (I-R_1)\sum_{j=0}^{N} R_1^j = I - R_{-1}^{N+1}$$

Let $A\in\Psi^{-m}_{c\ell}(\mathcal{M};E)$ be such that (2.9) holds, let $R' = -I - P\circ A$. Since

$$P\circ A - I = P\circ(A - \sum_{j=0}^{N} A_{-m-j}) + P\circ\sum_{j=0}^{N} A_{-m-j} \in S^{-N-1}_{c\ell}(\mathcal{M};E)$$

for any N, $R \in S^{-\infty}(\mathcal{M}; E)$. Thus there is $A \in \Psi^{-m}(\mathcal{M}; E)$ such that

$$P \circ A = I - R'.$$

Likewise there is A_{L} such that $A_{\mathrm{L}} \circ P = I - R$ with $R \in \Psi^{-\infty}(\mathcal{M}; E)$. But

$$\begin{aligned} A_{\mathrm{L}} &= A_{\mathrm{L}} \circ (I - R') + A \circ R' \\ &= A_{\mathrm{L}} \circ P \circ A + A \circ R' \\ &= (I - R) \circ A + A \circ R' \\ &= A - (R \circ A - A \circ R') \end{aligned}$$

Since $(R \circ A - A \circ R') \in \Psi^{-\infty}(\mathcal{M}; E)$, also $A \circ P = I - R''$ with a suitable R''. □

Example 3.2. (Let $\mathcal{M} = S^1$, let $P = e^{ix} D_x^2$, where x is the standard angular variable. Then P is elliptic. Indeed, $\boldsymbol{\sigma}(P)(\xi\, dx) = e^{ix}\xi^2$, which is invertible ($\xi \neq 0$). We can construct a parametrix using Fourier series: the equation

$$e^{ix} D_x^2 u = f,\ f \in L^2(S^1),\ f = \sum_k f_k e^{ikx},\ f_1 = 0$$

is solved by

$$u(x) = \sum_{k \neq 0} \frac{1}{k^2} f_{k+1} e^{ikx}.$$

In other words, if Π_1 is the orthogonal projection on $\operatorname{span}(e^{ix})$ (a smoothing operator, its Schwartz kernel is $e^{ix}e^{-ix'}/2\pi$), then

$$(Af)(x) = \frac{1}{2\pi} \int_{S^1} \sum_{k \neq 0} \frac{1}{k^2} e^{ikx - i(k+1)x'} (f - \Pi_1 f)(x')\, dx'$$

solves

$$PAf = f - \Pi_1 f.$$

Thus A is a parametrix of P. Note that P has non-trivial kernel.

A left (resp., right) parametrix for P is an operator A such that $A \circ P - I$ (resp., $P \circ A = I$) is a smoothing operator.

Proposition 3.3. *Left or right parametrices of an elliptic differential operator are pseudodifferential.*

Proof. Suppose Q is a left parametrix of P, $Q \circ P = I - R$ with R smoothing. Let A be a pseudodifferential right parametrix for P, so $P \circ A = I - R'$. Then, as in the proof above,

$$Q = A - (R \circ A - A \circ R')$$

which says that Q is equal a pseudodifferential operator plus a smoothing operator. But smoothing operators are pseudodifferental operators (of any order). Hence Q is pseudodifferential. □

Suppose now that $\mathcal{U}$ is the domain of a local chart $x_1, \ldots, x_n$ of $\mathcal{M}$ over which E is trivial. As in the case of a differential operator, using a local frame for E over $\mathcal{U}$ we reduce the next few paragraphs to the case of a fixed scalar pseudodifferential operator. Let then $A \in \Psi^m_{c\ell}(\mathcal{M})$, let $\chi_0, \chi_1 \in C_c^\infty(\mathcal{M})$ be supported in $\mathcal{U}$, define, as before,

$$A_{\chi_1,\chi_0}\varphi = \chi_1(x) A(\chi_0 \varphi), \quad a_{\chi_1,\chi_0}(x,\xi) = e^{-ix\cdot\xi} A_{\chi_1\chi_0}(e^{i(\cdot)\cdot\xi}).$$

Then a_{χ_1,χ_0} is a smooth function. Writing $\xi = |\xi|\breve{\xi}$ for $\xi \neq 0$, the definition of A being pseudodifferential gives:

$$\begin{aligned}&\text{For all } \mathcal{K} \Subset \mathcal{U}, \text{ for all multiindices } \alpha, \beta \text{ there exists } C > 0 \text{ such}\\ &\text{that}\\ &\qquad |D_x^\alpha \partial_\xi^\beta a_{\chi_1,\chi_0}(x,\xi)| \le (1+|\xi|)^{m-|\beta|} \quad \text{if } (x,\xi) \in \mathcal{K} \times \mathbb{R}^n.\end{aligned} \tag{3.4}$$

This property of $a_{\chi_1,\chi_0}(x,\xi)$ expresses the fact that $a_{\chi_1,\chi_0} \in S^m(\mathcal{U}, \mathbb{R}^n)$. Further, the assumption that A is classical implies that

$$a_{\chi_1,\chi_0}(x,\xi) \sim \sum_{j=0}^{\infty} a_{\chi_1,\chi_0,m-j}(x,\xi)$$

where $a_{\chi_1,\chi_0,m-j}(x,\xi)$ is smooth and

$$|\xi|^{-(m-j)} a_{\chi_1,\chi_0,m-j}(x,\xi)$$

is independent of $|\xi|$ for large $|\xi|$.

THEOREM 3.5. *If $\varphi \in C^\infty(\mathcal{M})$, then*

$$A_{\chi_1,\chi_0}(\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}} e^{ix\cdot\xi} a_{\chi_1,\chi_0}(x,\xi) \hat{\varphi}(x')\, dx'\, d\xi, \quad \varphi \in C_c^\infty(\mathcal{U}).$$

This is Theorem 2.1, now in the context of pseudodifferential operators; the proof is the same.

4. Schwartz kernel

Being continuous operators, pseudodifferential operators have a Schwartz kernel, in general a distribution on $\mathcal{M} \times \mathcal{M}$ with singular support on the diagonal.

LEMMA 4.1. *The Schwartz kernel of A_{χ_1,χ_0} is*

$$K_{\chi_1,\chi_0}(x,x') = \int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} a_{\chi_1,\chi_0}(x,\xi)\, d\xi$$

If $m < -n-k$, then K_{χ_1,χ_0} is C^k function.

The lemma is proved using integrability in ξ and estimates of derivatives based on the fact that a_{χ_0,χ_1} is a symbol.

PROPOSITION 4.2. *If $\operatorname{supp}\chi_0 \cap \operatorname{supp}\chi_1 = \emptyset$, then there is a smooth function K_{χ_1,χ_0} such that*

$$\chi_1(x) A(\chi_0\varphi)(x) = \int_{\mathcal{U}} K_{\chi_1,\chi_0}(x,x')\varphi(x')\, dx'.$$

Let K be the Schwartz kernel of A . Then sing-supp$(K) \subset \operatorname{diag}(\mathcal{M}\times\mathcal{M})$. *If $m < -n-k$ then K is of class C^k. In particular, if $A \in \Psi^{-\infty}(\mathcal{M}; E)$, then K is smooth.*

PROOF. The hypothesis about the supports of χ_0 and χ_1 implies that for each multiindex α there is $c_\alpha > 0$ such that

$$c_\alpha < |(x-x')^\alpha| \quad \text{when } x \in \operatorname{supp}\chi_1,\ x' \in \operatorname{supp}\chi_0$$

hence

$$\frac{1}{(x-x')^\alpha} a_{\chi_1,\chi_0}(x,\xi)$$

is smooth and an element of $S^m(\mathcal{U}; R^n)$ for any α. Using this and

$$\begin{aligned} K_{\chi_1,\chi_0}(x,x') &= \int_{\mathbb{R}^n} (x-x')^\alpha e^{i(x-x')\cdot\xi} \frac{a_{\chi_1,\chi_0}(x,\xi)}{(x-x')^\alpha}\, d\xi \\ &= \int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} \frac{(-D_\xi)^\alpha a_{\chi_1,\chi_0}(x,\xi)}{(x-x')^\alpha}\, d\xi \end{aligned}$$

shows that K_{χ_1,χ_0} is C^k for any k since we can take α such that $|\alpha| > m+n+k$.

Suppose now $(x_1, x_0) \notin \operatorname{diag}(\mathcal{M}\times\mathcal{M})$, that is, $x_1 \neq x_0$. We may assume $x_0, x_1 \in \mathcal{U}$ (the domain of our local chart of $\mathcal{M}$). Let $\chi_0, \chi_1 \in C_c^\infty(\mathcal{M})$ be supported in $\mathcal{U}$, have disjoint supports but with $\chi_0(x_0)$ and $\chi_1(x_1)$ both different from 0. Let K be the Schwartz kernel of A. Then $\chi_1(x)K(x,x')\chi_0(x) = K_{\phi_0,\phi_1}(x,x')$. By the first part, $\chi_1(x)K(x,x')\chi_0(x)$ is smooth. So K is smooth in the open set $\{(x,x') : x_0(x') \neq 0,\ \chi_1(x) \neq 0\}$.

Suppose now $m < -n-k$, let x_0 be arbitrary. Again assuming $x_0 \in \mathcal{U}$, let $\chi_1 = \chi_0$, with $\chi(x_0) \neq 0$. Then $\chi_1(x)K(x,x')\chi(x_0)$ is of class C^k. and thus so is K in a neighborhood of x_0. □

LEMMA 4.3. *Let $A : C_c^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ be linear and continuous with smooth Schwartz kernel. Then $A \in \Psi^{-\infty}(\mathcal{M})$*

PROOF. Let K be the Schwartz kernel of A. Suppose that $\mathcal{U}$ is the domain of a local chart as above, let $f \in C_{-0}^\infty(\mathcal{U})$ and $\varphi \in C_c^\infty(\mathcal{U})$ be arbitrary. We will show that $a(f,\varphi,\tau)(x)$ is an element of $S^{-\infty}(\mathcal{U}, \mathbb{R}^n)$. The vector field

$$L = \frac{1}{|\nabla'_x f(x')|^2} \nabla'_x f(x') \cdot D'_x$$

has the property that $\tau^{-1} L e^{i\tau f(x')} = e^{i\tau f(x')}$. Here ∇ means the gradient in coordinates. Thus (writing $d\mathfrak{m} = \mu\, dx$ for the density $d\mathfrak{m}$ in coordinates),

$$\begin{aligned} a(f,\varphi,\tau)(x) &= \int_{\mathcal{U}} e^{i\tau(f(x')-f(x))} K(x,x')\varphi(x')\,\mu(x')dx \\ &= \tau^{-N} \int_{\mathcal{U}} L^N(e^{i\tau(f(x')-f(x))}) K(x,x')\varphi(x')\,\mu(x')dx \\ &= \tau^{-N} \int_{\mathcal{U}} e^{i\tau(f(x')-f(x))} (L')^n (K(x,x')\varphi(x'))\,\mu(x')dx \end{aligned}$$

The integrand is smooth, compactly supported, and bounded independently of τ. So

$$|a(f,\varphi,\tau)(x)| \leq C(1+\tau|)^{-N} \wp(\varphi)$$

□

The trick of using the property of L and integration by parts is standard in the theory.

COROLLARY 4.4. *Let $A \in \Psi_{c\ell}^m(\mathcal{M}; E)$, let $\mathcal{W}$ be a neighborhood of the diagonal in $\mathcal{M}\times\mathcal{M}$. There is $A' \in \Psi_{c\ell}^m(\mathcal{M}; E)$ whose Schwartz kernel is supported in $\mathcal{W}$ such that $A - A' \in \Psi^{-\infty}(\mathcal{M}; E)$.*

PROOF. Let $\chi \in C^\infty(\mathcal{M}\times\mathcal{M})$ be supported in $\mathcal{W}$ and equal to 1 in a neighborhood of the diagonal. Split the Schwartz kernel K of A as

$$K = \chi K + (1-\chi)K$$

and let A', A'' have, respectively, Schwartz kernel χK, and $(1-\chi)K$. Since $(1-\chi)K$ is smooth, $A'' \in \Psi^{-\infty}(\mathcal{M}, E)$, hence $A' = A - A''$ is a pseudodifferential operator of order m. □

In particular, sing-supp $K \subset \operatorname{diag}(\mathcal{M}\times\mathcal{M})$.

A convenient decomposition of A is as follows. Pick an open cover $\mathfrak{U} = \{\mathcal{U}_\kappa\}$ of $\mathcal{M}$ by domains of local charts, perhaps chosen so E is trivial over the open sets of the cover. Let $\{\chi_{0,\nu}\}$ be a partition of unity subordinate to $\mathfrak{U}$, say with $\operatorname{supp}\{\chi_{0,\nu}\} \subset U_{\rho(\nu)}$. Pick, for each ν, $\tilde\chi_{1,\nu} \in C_c^\infty(\mathcal{U}_{\rho(\nu)})$ be strictly positive on $\operatorname{supp}\chi_{\nu,0}$. Let

$$\chi_{\nu,1} = \frac{1}{\sum_{\nu'}\tilde\chi_{\nu',1}}\tilde\chi_{\nu,1}$$

These functions give another partition of unity subordinate to $\mathfrak{U}$ with the advantage that $\chi_{\nu,0}$ and $\chi_{\nu,1}$ have support on the domain of the same local chart. So we can define $a_\nu(x,\xi) = a_{\chi_{0,\mu},\chi_{1,\mu}}(x,\xi)$ (with the local chart depending on ν). Each operator A_ν thus defined has Schwartz kernel K_ν with $\operatorname{supp}K_\nu \subset \operatorname{supp}\chi_{\nu,1}\boxtimes\chi_{\nu,0}$. The sum of these kernels is a distribution supported near the diagonal and $K - \sum K_\nu$ is smoothing.

5. Extension to $C_c^{-\infty}$

PROPOSITION 5.1. *Pseudodifferential operators extend to continuous operators* $C_c^{-\infty}(\mathcal{M};E) \to C^{-\infty}(\mathcal{M};E)$.

PROOF. That A is continuous means that for any $\mathcal{K} \Subset \mathcal{M}$, its restriction to $C_c^\infty(\mathcal{K};E) = \{u \in C_c^\infty(\mathcal{M};E) : \operatorname{supp}u \subset \mathcal{K}\}$ is continuous. To establish continuity, one only needs to verify continuity on all $C_c^\infty(\mathcal{K};E)$ where the $\mathcal{K}$ are arbitrary compact subsets of the elements of an open cover of $\mathcal{M}$. We conveniently take the open cover to be a collection of domains of local charts. Also, as above, we may assume A is a scalar operator.

Let $\mathcal{U}$ be the domain of a local chart $x^1,\dots,x^n$, let $\mathcal{K} \Subset \mathcal{U}$, let $\chi_0 \in C_c^\infty(\mathcal{U})$ be such that $\chi_0(x) = 1$ in a neighborhood of $\mathcal{K}$, and let $\chi_1 \in C_c^\infty(\mathcal{U})$ be equal to 1 in a neighborhood of $\operatorname{supp}\chi_0$. If $\varphi \in C^\infty(K)$, then

$$A(\varphi) = \chi_1 A(\chi_0\varphi) + (1-\chi_1)A(\chi_0\varphi)$$

because $\varphi = \chi_0\varphi$. The Schwartz kernel of $(1-\chi_1)A\chi_0$ is $(1-\chi_1)\boxtimes\chi_0 K$, which is smooth because $\operatorname{supp}((1-\chi_1)\boxtimes\chi_0) \cap \operatorname{diag}\mathcal{M} = \emptyset$. Consequently,

$$C_c^\infty(\mathcal{M}) \ni \varphi \mapsto (1-\chi_1)A\chi_0 \in C^\infty(\mathcal{M})$$

extends to a continuous operator $C_c^{-\infty}(\mathcal{M}) \to C^{-\infty}(\mathcal{M})$

With $a_{\chi_0,\chi_1}(x,\xi)$ and A_{χ_0,χ_1} as defined above we have

$$A_{\chi_0,\chi_1}(\varphi) = \int_{\mathcal{U}\times\mathbb{R}^n} e^{i(x-x')\cdot\xi} a_{\chi_0,\chi_1}(x,\xi)\varphi(x')\,dx'\,d\xi$$

for $\varphi \in C_c^\infty(\mathcal{M})$ where the integration in x' is to be done first. The proof exploits that if φ is a distribution with support in $\mathcal{U}$, then $|\hat\varphi(\xi)| \le c(1+|\chi|)^M$ for some c and M, so if A is of order $< -n - M$ then the formula above makes sense.

Let

$$Q = \frac{1}{1+|\xi|^2}(1 + \sum_j \xi_j D_{x^j}),$$

note that $Qe^{i(x-x')\cdot\xi} = e^{i(x-x')\cdot\xi}$. Using

$$\begin{aligned} &Q(e^{i(x-x')\cdot\xi})a_{\chi_0,\chi_1} \\ &\quad = e^{i(x-x')\cdot\xi}\frac{a_{\chi_0,\chi_1} - \sum_j \xi_j D_{x^j} a_{\chi_0,\chi_1}}{1+|\xi|^2} + \sum_j D_{x^j}\big(e^{i(x-x')\cdot\xi}\frac{a_{\chi_0,\chi_1}\xi_j}{1+|\xi|^2}\big) \\ &\quad = e^{i(x-x')\cdot\xi}a'_{\chi_0,\chi_1} + \sum_j D_{x^j}\big(e^{i(x-x')\cdot\xi}a''_{\chi_0,\chi_1,j}\big) \end{aligned}$$

we can rewrite the above formula for A_{χ_0,χ_1} as

$$\operatorname{op}(a_{\chi_0,\chi_1}) = \operatorname{op}(a'_{\chi_0,\chi_1}) + \sum_j D_{x^j}\operatorname{op}(a''_{\chi_0,\chi_1,j})$$

with symbols a'_{χ_0,χ_1}, $a''_{\chi_0,\chi_1,j}$ in $S^{m-1}(\mathcal{U})$. Using this repeatedly we get that for any N there are symbols $a_{\chi_0,\chi_1,\alpha}$ in $S^{m-N}(\mathcal{U})$, $|\alpha| \leq N$, such that

$$A_{\chi_0,\chi_1} = \sum_{|\alpha|\leq N} D_{x^\alpha}\operatorname{op}(a_{\chi_0,\chi_1,\alpha})$$

If $\varphi \in C_c^{-\infty}(\mathcal{M})$ is of finite order M, take $N > n+M+m$ let $\varphi_\varepsilon \in C_c^\infty(\mathcal{U})$ converge to φ in $C_c^{-\infty}(\mathcal{U})$ as $\varepsilon \to 0$. Then $\operatorname{op}(a_{\chi_0,\chi_1,\alpha})(\varphi_\varepsilon)$ converges in $C_c(\mathcal{M})$ and thus $A_{\chi_0,\chi_1}(\varphi_\varepsilon)$ converges in $C_c^{-\infty}(\mathcal{M})$. □

THEOREM 5.2. *Pseudodifferential operators are pseudolocal: that is, for any $A \in \Psi^m_{c\ell}(\mathcal{M};E)$ and $u \in C_c^{-\infty}(\mathcal{M};E)$,* sing-supp$(Au) \subset$ sing-supp(u).

PROOF. Suppose $x_0 \notin$ sing-supp(u), Suppose u is smooth in some open set $\mathcal{U}$, let $x_0 \in U$, and let $\chi_1 \in C_c^\infty(U;E)$, $\chi_1(x) = 1$ for x near x_0. Let $\chi_- \in C_c^\infty(\mathcal{U})$, $\chi_0(x) = 1$ for $x \in \operatorname{supp}(\chi_1)$. Since $\chi_1 u$ is smooth, so is $A(\chi_0 u)$, then also $\chi_1 A(\chi_0 u)$. The Schwartz kernel of $\chi_1 A(1-\chi_0)$ is smooth, so $\chi_1 A(1-\chi_0)u$ is smooth. Hence Au is smooth on the set where $\chi_1 \neq 0$. □

Using locally finite partitions of unity one can reconstruct A using the A_{χ_1,χ_0}. The latter can be expressed using the symbols p_{χ_0,χ_1}. and allow linking the principal symbol if A with the local representation via:

$$\boldsymbol{\sigma}(A_{\chi_0,\chi_1})(df(x)) = a_{\chi_0,\chi_1}(x, \nabla_x f(x)).$$

This is the main tool in the proofs of Theorems 2.3, 2.4, and 2.8.

6. Continuity of pseudodifferential operators

Assume $\mathcal{M}$ closed with smooth density $\mathfrak{m}$, let $E \to \mathcal{M}$ be a Hermitian vector bundle, define $L^2(\mathcal{M};E)$.

THEOREM 6.1. *Let $A : C^\infty(\mathcal{M};E) \to C^\infty(\mathcal{M};E)$ be a (classical) pseudodifferential operator of order* 0. *Then A extends to a continuous operator $L^2(\mathcal{M};E) \to L^2(\mathcal{M};E)$.*

We already know that A has an extension as a continuous operator

$$C^{-\infty}(\mathcal{M};E)\to C^{-\infty}(\mathcal{M};E),$$

so part of the statement is that if $u\in L^2(\mathcal{M};E)\subset C^{-\infty}(\mathcal{M};E)$, then also $Au\in L^2(\mathcal{M};E)$. The other part of the statement is that the resulting operator is continuous.

The proof below is from Taylor's book [**32**, p. 49] (based on [**14**, p. 510]); he deals with a more general class than the one we are discussing here. The most general theorem on the continuity of pseudodifferential operators is due to Calderón and Vaillancourt [**2**].

Proof. We take advantage of the decomposition given at the end of Section 4. We assume the sets of the cover so small that E is trivial over any one of them to reduce the problem to scalar operators. Let then $\mathcal{U}$ be the domain of a local chart $x^1,\dots,x^n$, let $A\in\Psi^0(\mathcal{U})$ have Schwartz kernel with support contained in $\mathcal{K}\boxtimes\mathcal{K}\subset\mathcal{U}\times\mathcal{U}$ for some $\mathcal{K}\Subset\mathcal{U}$. As before, define

$$a(x,\xi)=e^{-ix\cdot\xi}A(e^{i(\cdot)\cdot\xi}).$$

Then a is smooth with support contained in $\mathcal{K}\times\mathbb{R}^n$ and satisfies (3.4). By Theorem 3.5, if $u\in C_c^\infty(\mathcal{U})$, then

$$Au(x)=\frac{1}{(2\pi)^n}\int a(x,\xi)e^{ix\cdot\xi}\hat u(\xi)\,d\xi$$

The key is to interpret this in a convenient form as a family of pseudodifferential operators with constant coefficients. Let

$$\hat a(\eta,\xi)=\frac{1}{(2\pi)^n}\int_{\mathcal{U}}e^{-ix\cdot\eta}a(x,\xi)\,dx.$$

Then with fixed η we have $|\hat a(\eta,\xi)|\le c(\eta)$ so

$$\int\Big|\int\hat a(\eta,\xi)e^{ix\cdot\xi}\hat u(\xi)\,d\xi\Big|^2dx\le c(\eta)^2\|u\|_0^2$$

using Plancherel's theorem twice. We will prove in a moment that there is an integrable bound $c(\eta)$. If we admit this, then the above estimate and Minkowski's inequality give

$$\begin{aligned}\|Au\|_0&=\frac{1}{(2\pi)^n}\Big[\int\Big|\int e^{ix\cdot\eta}\int\hat a(\eta,\xi)e^{ix\cdot\xi}\hat u(\xi)\,d\xi\,d\eta\,\Big|^2dx\Big]^{1/2}\\&\le\frac{1}{(2\pi)^n}\int\Big[\int\Big|\int\hat a(\eta,\xi)e^{ix\cdot\xi}\hat u(\xi)\,d\xi\,\Big|^2dx\Big]^{1/2}d\eta\\&\le\int c(\eta)\|u\|_0\,d\eta=C\|u\|_0.\end{aligned}$$

This shows continuity, since $C_c^\infty(\mathcal{U})$ is dense in $L^2(\mathcal{U})$,

To find a good estimate for $\hat a(\eta,\xi)$ we use

$$\begin{aligned}(1+|\eta|^2)\hat a(\eta,\xi)&=\frac{1}{(2\pi)^n}\int_{\mathcal{U}}[(1+\sum D_{x^j}^2)^Ne^{-ix\cdot\eta}]a(x,\xi)\,dx\\&=\frac{1}{(2\pi)^n}\int_{\mathcal{U}}e^{-ix\cdot\eta}(1+\sum D_{x^j}^2)^Na(x,\xi)\,dx\end{aligned}$$

using integration by parts, taking advantage of the fact that $a(x,\xi)$ vanishes if $x \notin \mathcal{K}$. It follows that

$$|\hat{a}(\eta,\xi)| \leq \frac{1}{(2\pi)^n} \frac{\sup_{x,\xi} \left|(1+\sum D_{x^j}^2)^N a(x,\xi)\right|}{(1+|\eta|^2)^N} = \frac{1}{(2\pi)^n} \frac{\wp_N(a)}{(1+|\eta|^2)^N}.$$

The numerator is a seminorm of $a \in S^0_{c\ell}(U,\mathbb{R}^n)$. Let $c(\eta) = \wp_N(a)/(1+|\eta|^2)^N$. Then $c(\eta)$ is integrable if $N > n/2$ and we conclude

$$\|Au\|_0 \leq \frac{C}{(2\pi)^n} \wp_N(a) \|u\|_0 \tag{6.2}$$

□

LEMMA 6.3. *Let $\mathcal{U}$ be the domain of a local chart of $\mathcal{M}$, let $\mathcal{K} \Subset \mathcal{U}$. Suppose $A \in \Psi^0(\mathcal{U})$ and $a \in S^0(\mathcal{U},\mathbb{R}^n)$ are as in the proof above. Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 1$ if $|\xi| < 1$ and let $\chi_k(\xi) = k^{-1}\xi)$. If $u \in L^2(\mathcal{U})$, then*

$$w_k = \int e^{ix\cdot\xi} a(x,\xi)(1-\chi_k(\xi))\hat{u}(\xi)\, d\xi$$

tends to 0 in $L^2(\mathcal{U})$ as $k \to \infty$.

PROOF. If $u \in L^2(\mathcal{U})$ (view $\mathcal{U}$ as as subset of $\mathbb{R}^n$), then the functions $\hat{v}_k(\xi) = (1-\chi_k(\xi))\hat{u}(\xi)$ form a sequence tending to 0 in $L^2(\mathbb{R}^n)$, therefore the same is true of the $v_k \in L^2(\mathbb{R}^n)$ (again using Plancherel). Because of the control on the support of the Schwartz kernel of A it makes sense to view w_k as equal to Av_k. But then by the previous proof,

$$\|w_k\|_0 \leq C\|v_k\|_0$$

for some C and thus $\|w_k\|_0 \to 0$ as $k \to \infty$. □

CHAPTER IX

Sobolev spaces

1. Definitions

Suppose $\mathcal{M}$ is a closed n-manifold with smooth measure $\mathfrak{m}$ and $E \to \mathcal{M}$ a Hermitian vector bundle (a complex vector bundle with Hermitian metric). Write $(v, w)_x$ for the inner product of $v, w \in E_x$ and $|v|_x$ for the norm.

The space $C^\infty(\mathcal{M}; E)$ of smooth sections of E is a pre-Hilbert space with the inner product

$$(u, v)_0 = \int_{\mathcal{M}} (u(x), v(x))_x \, d\mathfrak{m}(x) \quad u, v \in C^\infty(\mathcal{M}; E).$$

Define $L^2(\mathcal{M}; E)$ as the completion of $C^\infty(\mathcal{M}; E)$ with respect to the induced norm.

We will actually need L^2-based Sobolev spaces of any real order s. This can be done using local coordinates and partitions of unity. On the other hand a definition that "only" depends on differential operators is cleaner but restricts us to Sobolev spaces of integer order. The statements below with s or s' are correct even when these are not in $\mathbb{N}_0$.

With the Hermitian metric of E and some Riemannian metric on $T^*\mathcal{M}$ get a Hermitian metric on $E \otimes T^*\mathcal{M}$, make $C^\infty(\mathcal{M}; E \otimes T^*\mathcal{M})$ into a pre-Hilbert space following the above recipe and pass to its completion. Pick a connection

$$\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes T^*\mathcal{M}).$$

Connections have injective principal symbol (see Example VII.3.3), therefore $\Delta = \nabla^\star \nabla$ is elliptic.

Define

$$H^0(\mathcal{M}; E) = L^2(\mathcal{M}; E).$$

Proceeding inductively (thus m is a non-negative integer), once $H^{2m}(\mathcal{M}; E)$ with inner product $(\cdot, \cdot)_{2m}$ has been defined for any vector bundle over $\mathcal{M}$, define

$$H^{2m+1}(\mathcal{M}; E) = \{u \in H^{2m}(\mathcal{M}; E) : \nabla u \in H^{2m}(\mathcal{M}; E \otimes T^*\mathcal{M})\}$$
$$(u, v)_{2m+1} = (u, v)_{2m} + (\nabla u, \nabla v)_{2m}$$

and

$$H^{2m+2}(\mathcal{M}; E) = \{u \in H^{2m}(\mathcal{M}; E) : \Delta u \in H^{2m}(\mathcal{M}; E)\}$$
$$(u, v)_{2m+2} = (u, v)_{2m} + (\Delta u, \Delta v)_{2m}.$$

The condition $\nabla u \in H^{2m}(\mathcal{M}; E \otimes T^*\mathcal{M})$ means

$$\begin{aligned} &\text{there is } f \in H^{2m}(\mathcal{M}; E \otimes T^*\mathcal{M}) \text{ such that} \\ &\qquad (u, \nabla^\star v)_{2m} = (f, v)_{2m} \text{ for all } v \in C^\infty(\mathcal{M}; E \otimes T^*\mathcal{M}). \end{aligned} \tag{1.1}$$

where

$$\nabla^\star : C^\infty(\mathcal{M}; E \otimes T^*\mathcal{M}) \to C^\infty(\mathcal{M}; E)$$

is the formal adjoint of ∇, similarly for the meaning of $\Delta u \in H^{2m}(\mathcal{M}; E)$.

When $m = 0$, (1.1), expresses the fact that $H^1(\mathcal{M}; E)$ is the domain of the Hilbert space adjoint of

$$\nabla^\star : C_c^\infty(\mathcal{M}; E \otimes T^*\mathcal{M}) \subset L^2(\mathcal{M}; E \otimes T^*\mathcal{M}) \to L^2(\mathcal{M}; E).$$

Since the Hilbert space adjoint of an operator is closed, i.e. the graph is closed, $H^1(\mathcal{M}; E)$ with the given inner product is complete. The same argument applies to the higher order spaces. They all contain $C^\infty(\mathcal{M}; E)$ as a dense subspace.

The inner products can also be written as

$$(u, v)_{2m} = \sum_{j=0}^{m} (\Delta^j u, \Delta^j v)_0,$$

$$(u, v)_{2m+1} = (\nabla\Delta^m u, \nabla\Delta^m v)_0 + \sum_{j=0}^{m} (\Delta^j u, \Delta^j v)_0;$$

By definition, $H^m(\mathcal{M}; E) \subset H^k(\mathcal{M}; E)$ for $k = 0, \dots, m$ with continuous inclusions. (The inclusions are in fact compact except when $k = m$, of course.)

As we will see in a moment, this works because ∇ is a first order differential operator with injective symbol, which makes Δ an elliptic operator:

$$\boldsymbol{\sigma}(\Delta)(\boldsymbol{\xi}) = \boldsymbol{\sigma}(\nabla)(\boldsymbol{\xi})^\star \circ \boldsymbol{\sigma}(\nabla^\star)(\boldsymbol{\xi})$$

(see Lemma VII.3.2).

Let $A_2 \in \Psi_{c\ell}^{-2}(\mathcal{M}; E)$ be a parametrix for Δ:

$$A_{-2} \circ \Delta = I - R_2, \quad R_2 \text{ smoothing.}$$

Then $A_{-1} = A_2 \circ \nabla^\star \in \Psi^{-1}(\mathcal{M}; E \otimes T^*\mathcal{M}, E)$ is a parametrix for ∇:

$$A_{-1}\nabla = A_2\Delta = I - R_2.$$

In general, let $A_{-2j} = A_{-2}^j$ and $A_{-2j-1} = A_{-2j-2}\nabla^\star$. Denote

$$Q_{2m} = \Delta^m, \; Q_{2m+1} = \nabla\Delta^m$$

Then $A_{-\ell}$ is a parametrix for Q_ℓ. For example

$$A_{-2j}Q_{2j}^j = A_2^j\Delta^j = A_{-2}^{j-1}(A_{-2}\Delta)\Delta^{j-1} = A^{j-1}\Delta^{j-1} - A^{j-1}R_1\Delta^{j-1}$$

and by induction

$$A_{-2j}Q_{2j} = I - R_{2j}$$

with R_{2j} smoothing.

We will only consider Sobolev spaces of non-negative integer order; s, s', t will denote such integers. However, the statements are all true for general real orders. We will write E for the vector bundle even when occasionally (because of the self-imposed restriction to integers) the correct bundle is $E \otimes T^*\mathcal{M}$. In the case where $\mathcal{M}$ is not compact, the statements typically require hypotheses concerning compact support or local behavior. We do not need this generality.

PROPOSITION 1.2. *If $u \in H^s(\mathcal{M}; E)$ and $B \in \Psi^{-s'}$ then $Bu \in H^{s+s'}(\mathcal{M}; E)$.*

PROOF. From $u = (I - R_s)u + R_s = A_{-s}Q_s u + R_s u$ get

$$Bu = BA_{-s}Q_s u + BR_s u.$$

Since $BA_{-s} \in \Psi^{-s-s'}(\mathcal{M}; E)$, $Q_t BA_{-s}$ is pseudodifferential of order ≤ 0 if $0 \leq t \leq s + s'$. Since $Q_s u + R_s u \in L^2(\mathcal{M}; E)$,

$$Q_t(BA_{-s}Q_s u + BR_s u) \in L^2(\mathcal{M}; E)$$

for such t. Thus $Bu \in H^{s+s'}(\mathcal{M}; E)$. □

Similarly to (1.1), in the following, the statement $u \in L^2(\mathcal{M}; E)$ and $Pu \in H^s(\mathcal{M}; E)$ means

there is $f \in H^s(\mathcal{M}; E)$ such that $(f, v) = (u, P^\star v)_0$ for all $v \in L^2(\mathcal{M})$.

PROPOSITION 1.3. *Let $P : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ be a differential operator of order s'.*

(i) If $u \in H^s(\mathcal{M}; E)$ and $s' \leq s$, then $Pu \in L^2(\mathcal{M}; E)$. The map $P : H^s(\mathcal{M}; E) \to H^{s-s'}(\mathcal{M}; E)$ is continuous.

(ii) If P has injective principal symbol, then

$$u \in L^2(\mathcal{M}; E) \text{ and } Pu \in H^s(\mathcal{M}; E) \implies u \in H^{s+s'}(\mathcal{M}; E),$$

and there is C such that

$$\|u\|_{s+s'} \leq C(\|Pu\|_s + \|u\|_0). \tag{1.4}$$

PROOF. Let $u \in H^s$, let $P \in \mathrm{Diff}^{s'}$, $s' \leq s$. Using $u = A_{-s}Q_s u + R_s u$ we het

$$Pu = PA_{-s}Q_s u + PR_s u$$

Since $PA_{-s} \in \Psi^{s'-s}$ and $PR_s u$ is smooth, the right hand side belongs to $H^{s-s'}$ and then so does Pu. In addition, since $s' - s \leq 0$, $PA_{-s} : L^2(\mathcal{M}; E) \to L^2(\mathcal{M}; E)$ is continuous and PR_s is smoothing,

$$\|Pu\|_{s-s'} \leq c_1\|Q_s u\|_0 + c_2\|u\|_0 \leq c\|u\|_s.$$

Suppose now that P has injective principal symbol. Then $P^\star P$ is elliptic so has a pseudodifferential parametrix B' of order $-2s'$, and $B = B'P^\star \in \Psi^{-s'}(\mathcal{M}; E)$ is a parametrix for P: $BP = I - R$ with R smoothing. With u as in the statement, $A_{-s}Q_s Pu = Pu - R_s Pu$ gives

$$BA_{-s}Q_s Pu = BPu - BR_s Pu = u - (R + BR_s P)u.$$

In the leftmost member, $Q_s Pu \in L^2(\mathcal{M}; E)$ and BA_{-s} is pseudofifferential of order $-s - s'$, so $BA_{-s}Q_s Pu \in H^{s+s'}(\mathcal{M}; E)$ by Proposition 1.2. The term $(R + BR_s P)u$ in the rightmost member of the last displayed formula is smooth. So $u \in H^{s+s'}(\mathcal{M}; E)$. Writing that formula as

$$u = BA_{-s}Q_s Pu + (R + BR_s P)u$$

gives

$$\|u\|_{s+s'} \leq c_1\|Pu\|_s + c_2\|u\|_0.$$

This completes the proof of the proposition. □

The combination of Parts (i), (ii) gives

$$u \in H^s(\mathcal{M};E) \iff Pu \in L^2(\mathcal{M};E) \text{ for all } P \text{ of order } \le s.$$

It also gives that the spaces are independent of the connection used to define them; the inner products may depend, but the norms with different connections are equivalent. In fact, for example, any elliptic second order can be used in place of Δ: for such P, if $u \in H^2(\mathcal{M};E)$, then

$$\|u\|_2 \le C(\|Pu\|_0 + \|u_0\|_0) \le C'\|u\|_2$$

so the norm of $H^2(\mathcal{M};E)$ we defined is equivalent to the graph norm defined using P. Likewise, any first order operator with injective symbol can be used to define $H^1(\mathcal{M};E)$. And even the smooth density on $\mathcal{M}$ and the Hermitian product on E can be changed without changing the spaces; the inner products change, but the resulting norms are equivalent.

COROLLARY 1.5. *$C^\infty(\mathcal{M};E)$ is dense in $H^s(\mathcal{M};E)$.*

PROOF. Let $u \in H^s(\mathcal{M};E)$. As above, $u = A_{-s}Q_s u + R_s u$. The term $R_s u$ is smooth. We show that $A_{-s}Q_s u$ lies in the closure of $C^\infty(\mathcal{M})$. As in the proof of Theorem VIII.6.1 we use a partition of unity subordinate to an open cover by domains of local charts over each of which E is trivial.

The problem becomes: if $v \in L^2(\mathcal{M})$ (playing the role of $Q_s u$) and $A \in \Psi^{-s}(\mathcal{U})$ has with Schwartz kernel supported in $\mathcal{K} \boxtimes \mathcal{K} \Subset \mathcal{U} \times U$, then

$$Av(x) = \frac{1}{(2\pi)^n} \int e^{-ix\cdot\xi} a(x,\xi)\hat{v}(\xi)\, d\xi$$

lies in the closure of $C^\infty(\mathcal{M})$ in the norm of $H^s(\mathcal{U})$.

Let $\chi(\xi) \in C_c^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 1$ if $|\xi| < 1$, let $\chi_k(\xi) = \chi(\xi/k)$. The function

$$w_k(x) = \frac{1}{(2\pi)^n} \int e^{-ix\cdot\xi} a(x,\xi)\chi_k(\xi)\hat{v}(\xi)\, d\xi$$

which is smooth because the integrand is compactly supported. For any multiindex α with $|\alpha| \le s$, $\xi^\alpha a(x,\xi)$ is a symbol of order $-s + |\alpha| \le 0$. By Lemma 6.3,

$$D_x^\alpha\big(w(x) - w_k(x)\big) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \xi^\alpha a(x,\xi)(1 - \chi_k(\xi))\hat{v}(\xi)\, d\xi,$$

tends to 0 in $L^2(\mathcal{U})$ as $k \to \infty$. □

COROLLARY 1.6. *If $P \in \mathrm{Diff}^m(\mathcal{M};E)$ is elliptic, then*

$$P : H^m(\mathcal{M};E) \subset L^2(\mathcal{M};E) \to L^2(\mathcal{M};E). \tag{1.7}$$

is the closure of

$$P : C^\infty(\mathcal{M};E) \subset L^2(\mathcal{M};E) \to L^2(\mathcal{M};E). \tag{1.8}$$

PROOF. If $\{u_\nu\}$ is a sequence in $C^\infty(\mathcal{M};E)$ such that $u_\nu \to u$ and $Pu_\nu \to u$, then u_ν converges in $H^m(\mathcal{M};E)$ (to u), which follows immediately from (1.4). Thus the closure of

$$\mathcal{G} = \{(u, Pu) : u \in C^\infty(\mathcal{M};E)\} \subset L^2(\mathcal{M};E) \times L^2(\mathcal{M};E)$$

is contained in

$$\mathcal{G}' = \{(u, Pu) : u \in H^m(\mathcal{M};E)\}$$

Conversely, if $u \in H^m(\mathcal{M};E)$ and $u_\nu \subset L^2(\mathcal{M};E)$ converges to u in $H^m(\mathcal{M};E)$, then $\{(u_\nu, Pu_\nu)\}$ is a sequence in $\mathcal{G}$ converges to (u,Pu) in $L^2(\mathcal{M};E)\times L^2(\mathcal{M};E)$. Thus $\mathcal{G}' \subset \overline{\mathcal{G}}$, hence these spaces are equal. □

COROLLARY 1.9. *If $u \in H^s(\mathcal{M};E)$ and $f \in H^s(\mathcal{M})$, then $fu \in H^s(\mathcal{M};E)$ and*

$$\|fu\|_s \le c\|f\|_s\|u\|_s.$$

with c independent of f and u.

This holds for any real $s \ge 0$, but as usual we assume s is a non-negative integer.

PROOF. Fix $u \in C^\infty(\mathcal{M};E)$. From Lemma VII.2.5 we know that $C^\infty(\mathcal{M}) \ni f \mapsto Q_{s'}(fu) \in C^\infty(\mathcal{M};E)$ is a differential operator of order s, so Part (i) of Proposition 1.3 gives (assuming $s' \le s$)

$$\|Q_{s'}(fu)\|_0 \le C_1\|f\|_{s'} \le C_1\|f\|_s$$

for some $C_1 > 0$. Also, for fixed $f \in C^\infty(\mathcal{M})$, $u \mapsto Q_s(fu)$ is of order s, so also

$$\|Q_{s'}(fu)\|_0 \le C_2\|f\|_s$$

Any $f \in H^s(\mathcal{M})$ and $u \in H^s(\mathcal{M};E)$ are limits of smooth elements. For these the above inequalities lead to the corresponding inequalities for each $s' \le s$. This shows that $fu \in H^s(\mathcal{M};E)$ and

$$H^s(\mathcal{M}) \times H^s(\mathcal{M};E) \ni (f,u) \mapsto fu \in H^s(\mathcal{M};E)$$

is separately continuous. Since Sobolev spaces are complete, the separate continuity of the bilinear form implies the joint continuity, see Rudin [**27**, p. 52]. □

The corollary allows for localization.

COROLLARY 1.10. *If $P \in \mathrm{Diff}^m(\mathcal{M};E)$, then*

$$(Pu,v)_0 = (u,P^\star v)_0 \text{ for all } u,v \in H^m(\mathcal{M};E).$$

PROOF. If $\{u_\nu\}$ and $\{v_\nu\}$ are sequences in $C^\infty(\mathcal{M})$ with $u_\nu \to u$ and $v_\nu \to v$ in $H^m(\mathcal{M};E)$ as $\nu \to \infty$, then $Pu_\nu \to Pu$ and $P^\star v_\nu \to v$, since by Part (i) of Proposition 1.3 $P, P^\star : H^m(\mathcal{M};E) \to L^2(\mathcal{M};E)$ are continuous. So

$$(Pu,v)_0 = \lim_{\mu,\nu\to\infty}(Pu_\nu, v_\mu)_0 = \lim_{\mu,\nu\to\infty}(u_\nu, P^\star v_\mu)_0 = (u,P^\star v)_0.$$

□

2. Rellich's lemma

We continue to assume $\mathcal{M}$ closed. Rellich's lemma is the key to showing that the spectrum of an elliptic operator is discrete and unbounded; this, which was know already then, is shown by Rellich for the Laplacian with Dirichlet condition as an application of his result. The Sobolev embedding theorem gives that the eigensections of an elliptic operator are smooth, i.e, hypoellipticity.

In the original version of Rellich's lemma [**26**], the dimension is 2 and the statement concerns C^1 functions. The following statement is the modern version of Rellich's result; we will prove a more restricted with nonnegative Sobolev orders. The statements are correct for arbitrary real orders. Recall that $\mathcal{M}$ is closed.

THEOREM 2.1 (Rellich's lemma). *If $s' < s$, then the inclusion*

$$H^s(\mathcal{M}; E) \hookrightarrow H^{s'}(\mathcal{M}; E)$$

is compact.

Suppose $s' < s$ are non-negative integers. We need to show that every bounded sequence in $H^s(\mathcal{M}; E)$ has a subsequence that converges in $H^{s'}(\mathcal{M}; E)$. Using a partition of unity, trivializations, and local coordinates we only need to deal with the scalar case and with indices $s' = s - 1$; we will assume $s' = 0$. This is closer to Rellich's original statement. For the proof in the general setting (after localization and trivialization) see Folland [**9**, p. 304].

PROPOSITION 2.2. *Let $\{u_\nu\}_{\nu=1}^\infty$ be a sequence in $H^1(\mathbb{R}^n)$ with support in $(0, 2\pi)^n$. Suppose there is $C > 0$ such that*

$$\|u_\nu\|_0^2 + \sum_{j=1}^n \|\partial_{x_j} u_\nu\|^2 \le C^2$$

Then there is a subsequence $\{u_{\nu_k}\}_{k=1}^\infty$ that converges in $L^2(\mathbb{R}^n)$.

Rellich indicates that the statement can be proved using Fourier series, but uses a different method. We take his comment as a hint to produce a proof using Fourier series.

PROOF. With the notation

$$u_\nu(x) = \sum_{\alpha \in \mathbb{Z}^n} u_{\nu,\alpha} e^{i\alpha \cdot x}$$

then the hypothesis is that

$$\sum_{\alpha \in \mathbb{Z}^n} \left(|u_{\nu,\alpha}|^2 + \sum_{j=1}^n |\alpha_j u_{\nu,\alpha}|^2 \right) \le C^2$$

for all ν. Let $\hat{u}_\nu$ denote the element of $\ell^2(\mathbb{Z}^n)$ with components $u_{\nu,\alpha}$, $\alpha \in \mathbb{Z}^n$.

We construct a succession of subsequences of subsequences of $\{\hat{u}_\nu\}$. For $N = 1$ we observe that trivially $\{u_{\nu,0}\}_{\nu=1}^\infty$ (this is the component with $\alpha = 0$) is a bounded sequence, so it has a convergent subsequence: there is $\kappa_1 : \mathbb{N} \to \mathbb{N}$ strictly increasing such that $\{u_{\kappa(\nu),0}\}$ converges. Adjusting κ_1 we arrange that

$$|u_{\kappa_1(\mu),0} - u_{\kappa_1(\nu),0}|^2 < 1 \quad \text{for all } \mu, \nu.$$

This gives the subsequence $\{u_{\kappa_1(\nu)}\}_{\nu=1}^\infty$. Suppose now we have found strictly increasing functions $\kappa_1, \dots, \kappa_{N-1} : \mathbb{N} \to \mathbb{N}$ with the property that $\kappa_k(\mathbb{N}) \subset \kappa_{k-1}(\mathbb{N})$ ($\kappa_0 : \mathbb{N} \to \mathbb{N}$ is the identity function) and

$$\sum_{\substack{\alpha \in \mathbb{Z}^n \\ \max |\alpha_j| < k}} |u_{\kappa_k(\mu),\alpha} - u_{\kappa_k(\nu),\alpha}|^2 < \frac{1}{k^2} \quad \text{for all } \mu, \nu. \tag{2.3}$$

The vectors $(u_{\kappa_{N-1}(\nu),\alpha})_{\max |\alpha_j| < N}$, $\nu = 1, 2, \dots$ with components indexed by the elements of $\mathcal{I}_N = \{\alpha : \max |\alpha_j| < N\}$ form a sequence in the ball of radius C in $\mathbb{C}^{\mathcal{I}_N}$. So a subsequence of these converges and again choosing $\kappa_N : \mathbb{N} \to \kappa_{N-1}(\mathbb{N})$

strictly increasing we arrange that (2.3) holds with $k = N$. This produces the sequence $\{u_{\kappa_N(\nu)}\}_{\nu=1}^{\infty}$ as a subsequence of $\{u_{\kappa_{N-1}(\nu)}\}_{\nu=1}^{\infty}$. In particular,

$$\sum_{\substack{\alpha\in\mathbb{Z}^n\\ \max|\alpha_j|<m}} |u_{\kappa_k(\nu),\alpha} - u_{\kappa_k(\mu),\alpha}|^2 < \frac{1}{m^2} \quad \text{for all } \mu,\nu \text{ if } m \le k.$$

Now diagonalize: consider the sequence $\{\hat{u}_{\kappa_\nu(\nu)}\}_{\nu=1}^{\infty} \subset \ell^2(\mathbb{Z}^n)$. If $\nu \le \mu$, then

$$\sum_{\substack{\alpha\in\mathbb{Z}^n\\ \max|\alpha_j|<\nu}} |\hat{u}_{\kappa_\nu(\nu),\alpha} - \hat{u}_{\kappa_\mu(\nu),\alpha}|^2 < \frac{1}{\nu^2} \tag{2.4}$$

since both $\hat{u}_{\kappa_\mu(\mu)}$ and $\hat{u}_{\kappa_\mu(\nu)}$ are terms of the same sequence $\{\hat{u}_{\kappa_\nu(\rho)}\}_{\rho=1}^{\infty}$.

LEMMA 2.5. If $b = \{b_\alpha\} \in \ell^2(\mathbb{Z}^n)$ is such that

$$\sum_{\alpha\in\mathbb{Z}^n}\sum_{j=1}^{n} |\alpha_j b_\alpha|^2$$

converges, then

$$\|b\|_{\ell^2}^2 \le \sum_{\substack{\alpha\in\mathbb{Z}^n\\ \max|\alpha_j|<N}} |b_\alpha|^2 + \frac{1}{N^2}\sum_{\alpha\in\mathbb{Z}^n}\sum_{j=1}^{n} |\alpha_j b_\alpha|^2 \tag{2.6}$$

for any $N \in \mathbb{N}$.

Applying (2.6) with $b_\alpha = v_{\nu,\alpha} - v_{\mu,\alpha}$ with arbitrary $\nu < \mu$ gives

$$\begin{aligned}\|\hat{u}_{\kappa_\nu(\nu)} - \hat{u}_{\kappa_\mu(\mu)}\|_{\ell^2}^2 &\le \sum_{\substack{\alpha\in\mathbb{Z}^n\\ \max|\alpha_j|<\nu}} |\hat{u}_{\kappa_\nu(\nu),\alpha} - \hat{u}_{\kappa_\mu(\mu),\alpha}|^2 \\ &\qquad + \frac{1}{\nu^2}\sum_{\alpha\in\mathbb{Z}^n}\sum_{j=1}^{n} |\alpha_j(\hat{u}_{\kappa_\nu(\nu),\alpha} - \hat{u}_{\kappa_\mu(\mu),\alpha})|^2 \\ &\le \frac{1}{\nu^2} + \frac{2C^2}{\nu^2}.\end{aligned}$$

Thus the subsequence $\{\hat{u}_{\kappa_\nu(\nu)}\}$ converges in $\ell^2(\mathbb{Z}^n)$. □

To prove the lemma, begin with

$$\begin{aligned}\sum_{\alpha_n\in\mathbb{Z}} |b_\alpha|^2 &= \sum_{|\alpha_n|<N} |b_\alpha|^2 + \sum_{|\alpha_n|\ge N} |b_\alpha|^2 \\ &= \sum_{|\alpha_n|<N} |b_\alpha|^2 + \frac{1}{N^2}\sum_{|\alpha_n|\ge N} N^2|b_\alpha|^2 \\ &\le \sum_{|\alpha_n|<N} |b_\alpha|^2 + \frac{1}{N^2}\sum_{|\alpha_n|\ge N} |\alpha_n b_\alpha|^2 \\ &\le \sum_{|\alpha_n|<N} |b_\alpha|^2 + \frac{1}{N^2}\sum_{\alpha_n\in\mathbb{Z}} |\alpha_n b_\alpha|^2.\end{aligned}$$

Next, in

$$\sum_{\alpha_{n-1},\alpha_n\in\mathbb{Z}} |b_\alpha|^2 \le \sum_{\alpha_{n-1}\in\mathbb{Z}}\sum_{|\alpha_n|<N} |b_\alpha|^2 + \frac{1}{N^2}\sum_{\alpha_{n-1},\alpha_n\in\mathbb{Z}} |\alpha_n b_\alpha|^2$$

use

$$\sum_{\alpha_{n-1}\in\mathbb{Z}}\sum_{|\alpha_n|<N}|b_\alpha|^2 \le \sum_{|\alpha_{n-1}|,|\alpha_n|<N}|b_\alpha|^2+\frac{1}{N^2}\sum_{|\alpha_{n-1}|\ge N}\sum_{|\alpha_n|<N}N^2|b_\alpha|^2$$
$$\le \sum_{|\alpha_{n-1}|,|\alpha_n|<N}|b_\alpha|^2+\frac{1}{N^2}\sum_{\alpha_{n-1},\alpha_n\in\mathbb{Z}}|\alpha_{n-1}b_\alpha|^2,$$

to get

$$\sum_{\alpha_{n-1},\alpha_n\in\mathbb{Z}}|b_\alpha|^2 \le \sum_{|\alpha_{n-1}|,|\alpha_n|<N}|b_\alpha|^2+\frac{1}{N^2}\sum_{\alpha_{n-1},\alpha_n\in\mathbb{Z}}|\alpha_{n-1}b_\alpha|^2+|\alpha_n b_\alpha|^2.$$

Continuing by induction gives (2.6).

COROLLARY 2.7. *Let $P\in\mathrm{Diff}^m(\mathcal{M};E)$ be elliptic. Then P, as an unbounded operator*

$$P: H^m(\mathcal{M};E)\subset L^2(\mathcal{M};E)\to L^2(\mathcal{M};E)$$

has finite-dimensional kernel and closed range of finite codimension. The kernel is a subspace of $C^\infty(\mathcal{M};E)$.

PROOF. Let $\{u_\nu\}_{\nu=1}^\infty\subset L^2(\mathcal{M};E)$ be a bounded sequence in $\ker P$. Using Part (ii) of Proposition (1.3) we get $u\in H^m(\mathcal{M};E)$ and from (1.4), that

$$\|u_\nu\|_m\le C\|u_\nu\|_0.$$

Thus, since $\{u_\nu\}$ is bounded in L^2, it is bounded in $H^m(\mathcal{M};E)$, therefore has a convergent subsequence. Consequently, $\ker P$ must be finite-dimensional.

That the kernel consists of smooth sections is a consequence of Corollary 3.3, whose proof is independent of the statement we are proving. It follows that if $w\in H^m(\mathcal{M};E)$ and v is the the orthogonal projection of w on $\ker P$, then $u=w-v$ is orthogonal to $\ker P$ and still an element of $H^m(\mathcal{M};E)$.

Suppose now $\{f_\nu\}_{\nu=1}^\infty$ is a sequence in $\mathrm{rg}\,P$ that converges to f in $L^2(\mathcal{M};E)$. For each ν let $u_\nu\in H^m(\mathcal{M};E)$ be orthogonal to $\ker P$ and such that $Pu_\nu=f_\nu$.

If $\{u_\nu\}$ is not bounded in $L^2(\mathcal{M};E)$, pass to a subsequence whose norms increase to ∞. Relabeling, let $\{u_\nu\}$ be that sequence and let $\tilde{u}_\nu=\|u_\nu\|^{-1}u_\nu$. Then $P\tilde{u}_\nu=\tilde{f}_\nu=\|u_\nu\|^{-1}f_\nu\to 0$ as $\mu\to\infty$. The estimate (1.4) gives

$$\|\tilde{u}_\nu\|_m\le C(\|u_\nu\|_0^{-1}\|f_\nu\|_0+\|\tilde{u}_\nu\|_0)$$

which is bounded. By Rellich's lemma, there is (a further) subsequence of $\{u_\nu\}$ that converges in $L^2(\mathcal{M})$, to some $\tilde{u}$. Relabeling, let $\{\tilde{u}_\nu\}$ itself be that sequence. The property

$$u\tilde{u}_\nu\to\tilde{u},\quad P\tilde{u}_\nu=\tilde{f}_\nu\to\tilde{f}\text{ in }L^2(\mathcal{M};E)\text{ as }k\to\infty$$

implies that $P\tilde{u}=\tilde{f}$. But $\tilde{f}=0$, so $\tilde{u}\in\ker P$, which is impossible, since each $\tilde{u}_\nu$ is orthogonal to the kernel and of norm 1.

Consequently, the original sequence $\{u_\nu\}$ was bounded in $L^2(\mathcal{M};E)$ to begin with. Since also f_ν is bounded in $L^2(\mathcal{M};E)$ (it converges), $\{u_\nu\}$ is a bounded sequence in $H^m(\mathcal{M};E)$, so has a subsequence that converges in $L^2(\mathcal{M};E)$ to some u. Then $u\in H^m(\mathcal{M};E)$, consequently $f\in\mathrm{rg}\,P$.

To see that the orthogonal of $\mathrm{rg}\,P$ is finite-dimensional, observe that it is the kernel of the adjoint of P. □

COROLLARY 2.8. *Let $A \in \Psi^{-s}(\mathcal{M}; E)$, $s > 0$. Then A, as an operator on $L^2(\mathcal{M}; E)$, is compact.*

View $A : L^2(\mathcal{M} : E) \to L^2(\mathcal{M} : E)$ as the composition

$$L^2(\mathcal{M}; E) \xrightarrow{A} H^s(\mathcal{M}; E) \xhookrightarrow{\iota} L^2(\mathcal{M}; E)$$

of continuous operators, where ι, the inclusion, is compact.

3. Sobolev embedding theorem

THEOREM 3.1. *If $s > n/2 + k$, then $H^s(\mathcal{M}; E) \subset C^k(\mathcal{M}; E)$ and the inclusion is continuous.*

The norm in $C^0(\mathcal{M}; E)$ is

$$\|u\|_{L^\infty} = \sup_{x \in \mathcal{M}} h_x(u(x), u(x))$$

so the continuity of the embedding is that there is c such that

$$\|u\|_{L^\infty} \leq c\|u\|_s. \tag{3.2}$$

PROOF. We use the notation of the previous section. If $u \in H^s$ then $Q_s u \in L^2$ and $u = (A_{-s}Q_s u + R_s)u$: u is in the image of a pseudodifferential operator of order $-s$. Passing to local coordinates and trivializations, we need to show: Suppose $A \in \Psi^{-s}_{c\ell}(\mathcal{U})$ with Schwartz kernel supported in $\mathcal{K} \times \mathcal{K}$, $\mathcal{K} \Subset \mathcal{U}$. If $v \in L^2(\mathcal{U})$ then $u = Av \in C^k(\mathcal{U})$.

We have

$$u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} a(x, \xi)\hat{v}(\xi)\, d\xi$$

with $a \in S^{-s}(\mathcal{U}, \mathbb{R}^n)$. The functions $\xi \mapsto \xi^\alpha a(x, \xi)\hat{v}(\xi)$ is integrable on $\mathbb{R}^n$ when $|\alpha| \leq k$ so

$$w_\alpha(x, y) = \frac{1}{(2\pi)^n} \int e^{iy\cdot\xi} \xi^\alpha a(x, \xi)\hat{v}(\xi)\, d\xi.$$

is continuous. Since $u = w_0(x, x)$, u is of class C^k. (Derivatives of a with respect to the x^j don't change decay in ξ.) □

COROLLARY 3.3. *Let $P \in \mathrm{Diff}^m(\mathcal{M}; E)$ be elliptic, $u \in L^2(\mathcal{M}; E)$, $f = Pu$ (as distribution). If $f \in C^\infty(\mathcal{M}; E)$, then $u \in C^\infty(\mathcal{M}; E)$.*

PROOF. Since $f \in C^\infty(\mathcal{M}; E)$, $f \in H^s(\mathcal{M}; E)$ for any s. Then, by Part (ii) of Proposition 1.3, $u \in H^{s+m}(\mathcal{M}; E)$. Given k, take s such that $s + m > k + n/2$ to conclude $u \in C^k(\mathcal{M}; E)$. □

Thus, elliptic operators are hypoelliptic.

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