A characterization of quadric constant mean curvature hypersurfaces of spheres

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Two families of geometric functions on hypersurfaces

- Let $\phi : M \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be an immersion of a complete n-dimensional oriented manifold.
- Given a fixed vector $v \in \mathbb{R}^{n+2}$, let us define the functions $\ell_v : M \to \mathbb{R}$ and $f_v : M \to \mathbb{R}$ by $\ell_v(x) = \langle \phi(x), v \rangle$ and $f_v(x) = \langle \nu(x), v \rangle$, where $\nu : M \to \mathbb{S}^{n+1}$ is a Gauss map.
- When we consider all possible $v \in \mathbb{R}^{n+2}$ we obtain the families
$$V_1 = \{\ell_v : v \in \mathbb{R}^{n+2}\} \quad \text{and} \quad V_2 = \{f_v : v \in \mathbb{R}^{n+2}\}.$$
- These two families are very useful in the study of the spectrum of important elliptic operators defined on M like the Laplacian and the stability operator.
- For example, Solomon (1990) computed the whole spectrum for the Laplace operator of every minimal isoparametric hypersurface of degree 3 in spheres using these two families of functions.
- For the totally umbilical spheres $\mathbb{S}^n(v,c) \subset \mathbb{S}^{n+1}$ we have that if $c = 0$, then $\dim(V_1) = n+1$ and $\dim(V_2) = 1$. Indeed, it is not difficult to prove that if for some compact hypersurface M^n in $\mathbb{S}^{n+1}$, we have that either $\dim(V_1) < n+2$ or $\dim(V_2) < n+2$, then $M = \mathbb{S}^n(v,0)$ for some unit vector $v \in \mathbb{R}^{n+2}$ (Perdomo, 2001).

Examples

Example 1: Totally umbilical spheres

Let $v \in \mathbb{R}^{n+2}$ be a fixed unit vector and c a real number with $|c| < 1$. Let us define

$$\mathbb{S}^n(v,c) = \{x \in \mathbb{S}^{n+1} : \langle x, v\rangle = c\}.$$

As is well known, $\mathbb{S}^n(v,c)$ are the only totally umbilical complete hypersurfaces of $\mathbb{S}^{n+1}$ The map $\nu : \mathbb{S}^n(v,c) \to \mathbb{S}^{n+1}$ given by

$$\nu(x) = \frac{1}{\sqrt{1-c^2}}\,(v - cx)$$

is a Gauss map along $\mathbb{S}^n(v,c)$. In this case

$$H = \frac{c}{\sqrt{1-c^2}} \quad \text{and} \quad \|A\|^2 = \frac{nc^2}{1-c^2}$$

are both constant on $\mathbb{S}^n(v,c)$.
If we take $c \neq 0$, we can observe that if $w \in \mathbb{R}^{n+2}$ is a vector perpendicular to the vector v, then

$$f_w = -\frac{c}{\sqrt{1-c^2}}\ell_w.$$

Example 2: Clifford hypersurfaces

Given an integer $k \in \{1, \ldots, n-1\}$ and a real number $r \in (0,1)$, define $M_k(r) = \{(x,y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k+1} : \|x\|^2 = r^2 \text{ and } \|y\|^2 = 1 - r^2\} \subset \mathbb{S}^{n+1}$.

It is not difficult to see that the map $\nu : M_k(r) \to \mathbb{S}^{n+1}$ given by

$$\nu(x,y) = \Big(\frac{\sqrt{1-r^2}}{r}x, -\frac{r}{\sqrt{1-r^2}}y\Big)$$

defines a Gauss map on $M_k(r)$, and its principal curvatures are $\kappa_1 = \cdots = \kappa_k = -\frac{\sqrt{1-r^2}}{r}$ and $\kappa_{k+1} = \cdots = \kappa_n = \frac{r}{\sqrt{1-r^2}}$. Thus, we also have that $H = \frac{nr^2-k}{nr\sqrt{1-r^2}}$ and $\|A\|^2 = \frac{k}{r^2} + \frac{n-k}{1-r^2} - n$ are both constant.
$M_k(r)$ are the only complete isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures.
If we take $w = (w_1, \ldots, w_{k+1}, 0, \ldots, 0) \in \mathbb{R}^{n+2}$ then we have that

$$f_w = \frac{\sqrt{1-r^2}}{r}\ell_w.$$

Also, if we take $w = (0, \ldots, 0, w_{k+2}, \ldots, w_{n+2}) \in \mathbb{R}^{n+2}$, we have

$$f_w = -\frac{r}{\sqrt{1-r^2}}\ell_w.$$

A characterization result for quadratic hypersurfaces

We will prove that these two examples are the only hypersurfaces with constant mean curvature in $\mathbb{S}^{n+1}$ where the relation $f_w = \lambda \ell_w$, for some non-zero vector $w \in \mathbb{R}^{n+2}$, is possible.

Theorem 1 (Alías, Brasil and Perdomo, 2008)

Let $\phi : M \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be an immersion with constant mean curvature of a complete n-dimensional oriented manifold. If for some non-zero vector $v \neq \mathbf{0}$ and some real number λ, we have that $\ell_v = \lambda f_v$, then, $\phi(M)$ is either a totally umbilical sphere or a Clifford hypersurface.

The proof of Theorem 1 is based on a geometric argument divided in the following steps

- Step 1: The integral curves of $v^\top$ in M are Euclidean circles.
- Step 2: The intersection $N = M \cap \mathbb{S}^n(v, 0)$ is non-empty.
- Step 3: The study of the intersection $N = M \cap \mathbb{S}^n(v, 0)$ as a hypersurface of M and as a hypersurface of $\mathbb{S}^n(v, 0)$.
- Step 4: Computation of the principal curvatures of M along the integral curves of $v^\top$.
- Step 5: M is isoparametric with at most two distinct principal curvatures.

Stability index of CMC hypersurfaces

As an application of our Theorem 1, we will prove the following result.

Theorem 2 (Alías, Brasil and Perdomo, 2008)

Let M^n be a compact orientable hypersurface immersed into the Euclidean sphere $\mathbb{S}^{n+1}$ with constant mean curvature. If M has constant scalar curvature and M is neither a Clifford nor an umbilical hypersurface, then the weak stability index of M is greater than or equal to $2n+4$.

Sketch of the proof:

- When $H=0$, let $V_1=\{\ell_v : v\in\mathbb{R}^{n+2}\}$ and $V_2=\{f_v : v\in\mathbb{R}^{n+2}\}$.
- Then, it is not difficult to see that the functions $\ell_v\in V_1$ and $f_v\in V_2$ are eigenfunctions of the stability operator J with negative eigenvalues $-\|A\|^2$ and $-n$, respectively.
- Therefore,

$$\mathrm{Ind}_T(M)\geq \dim(V_1\oplus V_2)=\dim V_1+\dim V_2=n+2+n+2=2n+4$$

- When $H \neq 0$, we will work with test functions of the form $\ell_v - \alpha_\pm f_v$, where

$$\alpha_\pm = \frac{\|A\|^2 - n \pm \sqrt{D}}{2nH} \quad \text{with} \quad D = (\|A\|^2 - n)^2 + 4n^2H^2 > 0.$$

- Let

$$U_+ = \{\ell_v - \alpha_+ f_v : v \in \mathbb{R}^{n+2}\} \quad \text{and} \quad U_- = \{\ell_v - \alpha_- f_v : v \in \mathbb{R}^{n+2}\}.$$

- Then, it can be seen that $Ju + \lambda_\pm u = 0$ for every $u \in U_\pm$, with

$$\lambda_- = \frac{-(n + \|A\|^2) - \sqrt{D}}{2} < \lambda_+ = \frac{-(n + \|A\|^2) + \sqrt{D}}{2} < 0,$$

  and therefore

$$\mathrm{Ind}_T(M) \geq \dim(U_+ \oplus U_-) = \dim U_+ + \dim U_-.$$

- Finally, since M is neither a totally umbilical sphere nor a Clifford hypersurface, our Theorem 1 implies that $\dim U_+ = \dim U_- = n + 2$, and then we conclude that $\mathrm{Ind}_T(M) \geq 2n + 4$.

References

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