Rigidity of surfaces in the Lorentzian Heisenberg group Nil_3

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1. Introduction

Homogeneous simply connected Riemannian 3-manifolds are classified by a celebrated result of W. Thurston (see [7]). This classification shows the existence of merely 8 such geometries: $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2\times\mathbb{R}$, $\mathbb{H}^2\times\mathbb{R}$, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Nil_3 and Sol_3.

A complete classification of compact locally homogeneous *Lorentz* 3-manifolds, with non-compact (local) isotropy group has been obtained recently by S. Dumitrescu and A. Zeghib (see [3]). In this case, there are only 4 geometries: two with constant sectional curvature, and two with left-invariant metrics on the Heisenberg group Nil_3 and the Sol_3 group.

A natural geometrical question is to study how a given Riemannian (Lorentzian) surface is isometrically embedded inside these geometries (Riemannian or Lorentzian). Existence results of isometric immersions into homogeneous Riemannian 3-manifolds whose isometry group has dimension 4 have been established by B. Daniel (see [1], [2]). In [4], S. Lodovici has proved an analogous result for isometric immersions into the solvable group Sol_3 endowed with a left-invariant homogeneous Riemannian metric.

Using the theory of G-structures developed in [6], we have recently proved in [5] one result of isometric immersions of semi-Riemannian surfaces inside the Lorentzian Heisenberg group Nil_3, one of the two Lorentzian geometries which remain to be understood (Theorem 1).

Another important problem in Submanifolds Theory is the question of (local) isometric rigidity. We say that an isometric immersion $f:M^m\to N^n$ is *rigid* if, given any other isometric immersion $g:M^m\to N^n$, there is an isometry $\rho:N\to N$ such that $g=\rho\circ f$. In [5], we discuss the notion of G-rigidity for G-structure preserving isometric immersions, and we prove a rigidity result for immersions into the Lorentzian Heisenberg group. Here we present such result using the usual language of Submanifolds Theory (Theorem 2).

2. The Heisenberg group Nil_3

The 3-dimensional Heisenberg group Nil_3 is the group of upper diagonal 3×3 unipotent matrices

$$\mathrm{Nil}_3=\left\{\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}:x,y,z\in\mathbb{R}\right\}. \tag{1}$$

Identifying the matrix (1) with the triple $(x,y,z)\in\mathbb{R}^3$, the multiplication rule in Nil_3 is given by:

$$(x,y,z)\cdot(x',y',z')=(x+x',y+y',z+z'+xy').$$

Remark 1. The manifold Nil_3 is a fibration over the Minkowski 2-space. The fibers are geodesics. We will denote by ξ a unit vector field on Nil_3 tangent to the fibers; it will be called the *vertical* vector field. It is a Killing field (corresponding to translations along the fibers).

We consider the following left-invariant Lorentz metric on Nil_3:

$$\overline{g}=-\mathrm{d}x^2+\mathrm{d}y^2+(\mathrm{d}z-x\mathrm{d}y)^2.$$

The metric $\overline{g}$ is geodesically complete and $\dim\big(\mathrm{Iso}(\mathrm{Nil}_3)\big)=4$. The manifold Nil_3 has a left-invariant orthonormal frame (E_1,E_2,E_3), with $E_1=\xi$, and satisfying:

$$[E_1,E_2]=E_3\quad\text{and}\quad[E_1,E_3]=[E_2,E_3]=0;$$

it suffices to take $E_1=\frac{\partial}{\partial z}$, $E_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}$ and $E_3=\frac{\partial}{\partial x}$. For all vector field X we have

$$\overline{\nabla}_X\xi=\frac{1}{2}X\times\xi,$$

where $\times$ denotes the vector product in Nil_3. Denoting by $\overline{\nabla}$ the Levi-Civita connection of $\overline{g}$, we have the following:

Lemma 1. *The curvature tensor $\overline{R}$ of $\overline{\nabla}$ is given by:*

$$\overline{R}(X,Y)=\frac{3}{4}R_{\mathrm{o}}(X,Y)+R_1(X,Y),$$

where $R_1(X,Y)=\langle Y,\xi\rangle X\wedge\xi-\langle X,\xi\rangle Y\wedge\xi$ *and* $R_{\mathrm{o}}(X,Y)=X\wedge Y$.

3. The compatibility equations for surfaces in Nil_3

Let M be an oriented surface in Nil_3, ∇ the Levi-Civita connection of M, N the unit normal to M, A the shape operator of M and J the canonical complex structure of M.

Proposition 1. *The Gauss and Codazzi equations in Nil_3 are*

$$K+\det A=\nu^2-\frac{1}{4}, \tag{2}$$

$$\nabla_XAY-\nabla_YAX-A[X,Y]=\nu(Y\wedge X)T, \tag{3}$$

where K is the Gauss curvature of M, $\nu=\langle N,\xi\rangle$ and T is the projection of ξ on TM, i.e., $T=\xi-\nu N$.

Using the fact that $\overline{\nabla}_X\xi=\frac{1}{2}X\times\xi$, we have:

Proposition 2. *For $X\in\mathbf{\Gamma}(TM)$ we have:*

$$\nabla_XT=\nu\big(AX-\frac{1}{2}JX\big), \tag{4}$$

$$\mathrm{d}\nu(X)=-\langle AX-\frac{1}{2}JX,T\rangle. \tag{5}$$

4. Isometric immersions of surfaces into Nil_3

Let M be an oriented Riemannian surface, ∇ the Levi-Civita connection of M, R its curvature tensor, K its Gauss curvature, and J the canonical complex structure of M. Let $A:TM\to TM$ be a symmetric tensor, $T\in\mathbf{\Gamma}(TM)$ such that $||T||\leq 1$ and $\nu\in\mathrm{C}^\infty(M)$ such that $\nu^2\leq 1$.

Definition 1. We say that (M,A,T,ν) satisfies the *compatibility equations* for Nil_3 if

$$||T||^2+\nu^2=1$$

and the equations (2), (3), (4) and (5) hold, for all $X,Y\in\mathbf{\Gamma}(TM)$.

The central result of [5] is the following:

Theorem 1. *(i)* ***Existence****: Let M be an oriented Riemannian surface, ∇ its Levi-Civita connection, $A:TM\to TM$ be a symmetric tensor, $T\in\mathbf{\Gamma}(TM)$ and $\nu\in\mathrm{C}^\infty(M)$ such that (M,A,T,ν) satisfies the compatibility equations for Nil_3. Then there exists an isometric immersion $f:M\to\mathrm{Nil}_3$ such that the shape operator with respect to the normal N associated to f is*

$$\mathrm{d}f\circ A\circ\mathrm{d}f^{-1}$$

and such that

$$\xi=f_*T+\nu N.$$

(ii) ***Uniqueness****: Let $f,g:M\to\mathrm{Nil}_3$ be isometric immersions of a Riemannian surface. Assume that $\phi\circ\alpha_f=\alpha_g$ for one of the vector bundle isometries $\phi:TM_f^\perp\to TM_g^\perp$. Suppose in addition that $(T_f,\nu_f)=\pm(T_g,\nu_g)$. Then there exists an isometry $\rho:\mathrm{Nil}_3\to\mathrm{Nil}_3$ such that $\rho\circ f=g$ and $\rho_*|_{TM_f^\perp}=\phi$.*

5. Local isometric rigidity of surfaces in Nil_3

Definition 2. We say that an isometric immersion $f:M\to\mathrm{Nil}_3$ is *rigid* if, given any other isometric immersion $g:M\to\mathrm{Nil}_3$, there is an isometry $\rho:\mathrm{Nil}_3\to\mathrm{Nil}_3$ such that $g=\rho\circ f$. In this case we say that f and g are *congruent*.

Let $f,g:M\to\mathrm{Nil}_3$ be isometric immersions with second fundamental forms $\alpha_f:TM\times TM\to TM_f^\perp$ and $\alpha_g:TM\times TM\to TM_g^\perp$, respectively. If f and g are congruent it is easy to check that $(T_f,\nu_f)=\pm(T_g,\nu_g)$.

On the other hand, according to the Theorem 1, in order to prove that f and g are congruent, it suffices to show:
(i) At each point $x\in M$, there exists a linear isometry $\phi:T_xM_f^\perp\to T_xM_g^\perp$ such that $\phi\circ\alpha_f(x)=\alpha_g(x)$, and these linear isometries fit together to form a smooth vector bundle isometry $\phi:TM_f^\perp\to TM_g^\perp$.
(ii) $(T_f,\nu_f)=\pm(T_g,\nu_g)$.

The next Theorem, obtained in [5] in the context of G-structures, is an answer for the particular case where f and g satisfies $(T_f,\nu_f)=(T_g,\nu_g)$.

Theorem 2. *Let $f,g:M\to\mathrm{Nil}_3$ be isometric immersions of a Riemannian surface. Assume that $(T_f,\nu_f)=(T_g,\nu_g)$ and that the set $\{x\in M:\nu_f(x)\neq 0\}$ is dense in M. Then f and g are congruent.*

Proof. This is a simple application of equation (4). As $T_f=T_g$, equation (4) implies that

$$\nu_f(A_fX-A_gX)=0, \tag{6}$$

for all $X\in\mathbf{\Gamma}(TM)$, where A_f,A_g denotes the shape operators associated to f and g, respectively. As the set of points $x\in M$ such that $\nu_f(x)\neq 0$ is dense in M, equality (6) implies that $A_f=A_g$. This immediately implies that $\alpha_g=\phi\circ\alpha_f$ for one of the two vector bundle isometries $\phi:TM_f^\perp\to TM_g^\perp$, and the proof is completed by the Theorem 1. □

References

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