Pseudo-parallel Contact Normal Submanifolds in Kenmotsu Manifolds

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Abstract

In this paper, we studies n-dimensional f-pseudo-parallel anti-invariant contact normal submanifolds M, with parallel mean curvature in a $2n+1$-dimensional Kenmotsu space form $\tilde{M}(c)$ with $c<-1$. We prove that if M is not totally umbilical, then $f \geq -\frac{1}{n}P_{H,c}(S-n|H|^2)$ where $P_{H,c}(x) = \frac{3}{2}x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x - \left(\frac{n(c-3)+(c+1)}{4} + n|H|^2\right)$ and that if $f \leq 0$, then M is totally umbilical or $|H|^2 > -\frac{D(c)}{2Q(n)}$, where $Q(n) = n[(n-2)^2+6(n-1)]$ and $D(c) = 3(n-1)[n(c-3)+(c+1)]$. Finally, we prove that if M has constant scalar curvature and $f \geq 0$ then M is parallel.

1 Submanifold f-Pseudo-parallels

An n-dimensional submanifold M in a $\widetilde{M}^m$-dimensional is *f-pseudo-parallel* if its second fundamental form σ satisfies the following condition

$$\overline{R}(X,Y)\cdot\sigma = f\ X\wedge Y\cdot\sigma,$$

for some real valued smooth function f on M and for any X and Y vectors tangent to M, where $\overline{R}(X,Y)$ is the curvature operator of the Van der Waerden-Bortolotti connection $\overline{\nabla}$ of M, which with the operator $X\wedge Y$ act on σ as a derivation [1].

Pseudo-parallel immersions have been studied in [1], [2], [6], and [8]. Pseudo-parallel immersions are a generalization of semi-parallel immersions and consequently of parallel immersions, in particular of totally geodesic immersions.

2 Anti-invariant Contact Normal Submanifolds in Kenmotsu Manifolds

Let $\tilde{M}$ be a $(2m+1)$-dimensional manifold and $\Gamma(\tilde{M})$ the Lie algebra of vector fields on $\tilde{M}$. An almost contact structure on $\tilde{M}$ is defined by a (1,1)-tensor φ, a vector field ξ and a 1-form η on $\tilde{M}$ such that for any $p\in\tilde{M}$, we have

$$\varphi_p^2 = -I+\eta_p\otimes\xi_p,\quad \eta_p(\xi_p)=1,$$

where I denote the identity transformation of the tangent space $T_p\tilde{M}$ at p. Then $\varphi(\xi)=0$ and $\eta\circ\varphi=0$. Manifolds equipped whit an almost contact structure are called almost contact manifolds.

A Riemannian manifold $\tilde{M}$ with metric tensor $\langle\ ,\ \rangle$ and an almost contact structure (φ,ξ,η) such that

$$\langle\varphi\widetilde{X},\varphi\widetilde{Y}\rangle = \langle\widetilde{X},\widetilde{Y}\rangle - \eta(\widetilde{X})\eta(\widetilde{Y}),$$

or equivalently

$$\langle\widetilde{X},\varphi\widetilde{Y}\rangle = -\langle\varphi\widetilde{X},\widetilde{Y}\rangle \quad\text{and}\quad \langle\widetilde{X},\xi\rangle=\eta(\widetilde{X}),$$

for all $\widetilde{X},\widetilde{Y}\in\Gamma(\tilde{M})$, is an almost contact metric manifold.

$\widetilde{M}$ is called a Kenmotsu manifold if [3]

$$(\widetilde{\nabla}_{\widetilde{X}}\varphi)\widetilde{Y} = \langle\widetilde{X},\widetilde{Y}\rangle\xi - \eta(\widetilde{Y})\varphi\widetilde{X} \quad\text{and} \tag{1}$$

$$\widetilde{\nabla}_{\widetilde{X}}\xi = \widetilde{X}-\eta(\widetilde{X})\xi, \tag{2}$$

where $\widetilde{\nabla}$ is the covariant differentiation on $\widetilde{M}$.

An n-dimensional Riemannian manifold M isometrically immersed in $\tilde{M}$ is said to be a contact normal anti-invariant submanifold in $\tilde{M}$ if the structure vector field ξ is normal to M and $\varphi T_pM\subset T_pM^\perp$ for each p of M, where T_pM and $T_pM^\perp$ denote respectively the tangent and the normal space to M at p.

For an anti-invariant contact normal submanifold in a Kenmotsu manifold we obtain the following result:

Proposition. *Let M be an anti-invariant contact normal submanifold in a Kenmotsu manifold. Then:*

1. $A_{\varphi Y}X = -\varphi(\sigma(X,Y))$;
2. $\nabla^\perp_X\varphi Y - \varphi(\nabla_X Y) = \langle X,Y\rangle\xi$.

In [?], N.Papaghiuc show that M is never minimal.

2.1 Pseudo-parallel Anti-invariant Contact Normal Submanifolds

From now on, we assume that M^n is anti-invariant contact normal submanifold in a $2n+1$-dimensional Kenmotsu space form $M(c)$.

The curvature tensor $\tilde{R}$ of $\tilde{M}(c)$ is given by [4]:

$$\begin{aligned}\tilde{R}(X,Y)Z = \tfrac{c-3}{4}(X\wedge Y)Z + \tfrac{c+1}{4}\{&\eta(X)\eta(Z)Y\\ &-\eta(Y)\eta(Z)X+\langle X,Z\rangle\eta(Y)\xi\\ &-\langle Y,Z\rangle\eta(X)\xi+\langle X,\varphi Z\rangle\varphi Y\\ &-\langle\varphi Z,Y\rangle\varphi X+2\langle X,\varphi Y\rangle\varphi Z\},\end{aligned} \tag{3}$$

where $X\wedge Y$ is the operator defined by $(X\wedge Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y$.

Let S be the squared norm of second fundamental form, H denote the mean curvature vector field and $|H|$ the mean curvature of M, that is

$$S=\sum_{\alpha,i,j}(h^\alpha_{ij})^2,\quad H=\frac{1}{n}\sum_\alpha\left(\sum_i h^\alpha_{ii}\right)e_\alpha. \tag{4}$$

The Ricci curvature tensor $\{R_{kl}\}$ and the scalar curvature κ are expressed, respectively, as follows:

$$R_{kl} = \frac{c-3}{4}(n-1)\delta_{kl}+\sum_\alpha\left(\sum_i h^\alpha_{ii}\right)h^\alpha_{kl}-\sum_{\alpha,i}h^\alpha_{ki}h^\alpha_{il}, \tag{5}$$

$$\kappa = \frac{c-3}{4}n(n-1)+(n^2|H|^2-S). \tag{6}$$

The components of the covariant derivative of σ are given by

$$h^\alpha_{ijk} = \left\langle\left(\overline{\nabla}_{e_k}\sigma\right)(e_i,e_j),e_\alpha\right\rangle = \overline{\nabla}_{e_k}h^\alpha_{ij}, \tag{7}$$

hence, the square of the length of third fundamental form of M is given

$$|\overline{\nabla}\sigma|^2 = \sum_{\alpha,i,j,k}\left(h^\alpha_{ijk}\right)^2. \tag{8}$$

The components of the second covariant derivative of σ are given by

$$h^\alpha_{ijkl} = \left\langle\left(\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}\sigma\right)(e_i,e_j),e_\alpha\right\rangle = \overline{\nabla}_{e_l}h^\alpha_{ijk} = \overline{\nabla}_{e_l}\overline{\nabla}_{e_k}h^\alpha_{ij}. \tag{9}$$

Hence, we get

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij}-\sum_r h^\alpha_{jr}\omega_{ri}-\sum_r h^\alpha_{ir}\omega_{rj}+\sum_\beta h^\beta_{ij}\omega_{\alpha\beta}, \tag{10}$$

The Laplacian $\triangle h^\alpha_{ij}$ of h^α_{ij} is defined by $\triangle h^\alpha_{ij} = \sum_k h^\alpha_{ijkk} = \sum_k h^\alpha_{kijk}$.

It is we need considerer $\Phi : T_pM\times T_pM\to T_pM^\perp$ a bilinear map defined as follows: choose an orthonormal frame $\{e_{n+1},...,e_{2m+1}\}$ of $T_pM^\perp$ and for each $\alpha = n+1,\dots,2m+1$, define maps $\Phi_\alpha : T_pM\to T_pM$ by

$$\Phi_\alpha X = \langle H,e_\alpha\rangle X - A_{e_\alpha}X, \tag{11}$$

where H is the mean curvature vector and A_{e_α}'s are the shape operators. Then Φ is given by

$$\Phi(X,Y) = \sum_\alpha\langle\Phi_\alpha X,Y\rangle e_\alpha. \tag{12}$$

Since

$$\frac{1}{2}\Delta S = \sum_{\alpha,i,j}h^\alpha_{ij}\Delta h^\alpha_{ij}+\sum_{\alpha,i,j,k}\left(h^\alpha_{ijk}\right)^2, \tag{13}$$

we have the following result:

Proposition. *Let $\tilde{M}(c)$ an $(2n+1)$-dimensional Kenmotsu space form with structure $(\varphi,\xi,\eta,\langle\ ,\ \rangle)$ and M an n-dimensional anti-invariant contact normal submanifold with parallel mean curvature vector in $\tilde{M}(c)$. If $c\leq 1$,then*

$$\begin{aligned}\tfrac{1}{2}\Delta S \geq\ &|\overline{\nabla}\sigma|^2-\tfrac{3}{2}|\Phi|^4-\tfrac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3\\ &+\left(\tfrac{n(c-3)+c+1}{4}+n|H|^2\right)|\Phi|^2.\end{aligned} \tag{14}$$

Now, for any $H\in\mathbb{R}$, we define the polynomial $P_{H,c}$ by

$$\begin{aligned}P_{H,c}(x) = \tfrac{3}{2}x^2+\tfrac{n(n-2)}{\sqrt{n(n-1)}}|H|x\\ -\left(\tfrac{n(c-3)+(c+1)}{4}+n|H|^2\right).\end{aligned} \tag{15}$$

Now we give our main results:

Theorem 1. Let M be an n-dimensional($n>2$) complete f-pseudo-parallel anti-invariant contact normal submanifold with parallel mean curvature in a $2n+1$-dimensional Kenmotsu space form $\tilde{M}(c)$ with $c\leq 1$. If M is not totally umbilical, then

$$f\geq -\frac{1}{n}P_{H,c}(|\Phi|). \tag{16}$$

Theorem 2. Let M be an n-dimensional($n>2$) complete f-pseudo-parallel anti-invariant contact normal submanifold with parallel mean curvature in a $2n+1$-dimensional Kenmotsu space form $\tilde{M}(c)$ with $c\leq 1$. If $f\leq 0$ on M, then

$$|\Phi|^2P_{H,c}(|\Phi|)\geq 0. \tag{17}$$

The equality in (17) holds if only if

1. M is totally umbilical or
2. $|H|^2 > -\frac{D(c)}{2Q(n)}$. In this case, we have:

$$\begin{aligned}S = -\tfrac{n(n-2)}{9(n-1)}\Big\{&\tfrac{2Q(n)+3(n-1)}{n(n-2)}|H|^2\\ &+2|H|\sqrt{\tfrac{2Q(n)|H|^2+D(c)}{2n}}\Big\}+\tfrac{D(c)}{18(n-1)},\end{aligned}$$

where $Q(n) = n[(n-2)^2+6(n-1)]$ and $D(c) = 3(n-1)[n(c-3)+(c+1)]$.

Theorem 3. Let M be an n-dimensional complete f-pseudo-parallel anti-invariant contact normal submanifolds with parallel mean curvature in a $2n+1$-dimensional Kenmotsu space form $\tilde{M}(c)$. If M has constant scalar curvature κ and $f\geq 0$ then M is parallel.

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